Section 2-5 : Computing Limits

1. Evaluate $\lim\limits_{x\to 2}\left(8-3x+12x^2\right)$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. We know that the first thing that we should try to do is simply plug in the value and see if we can compute the limit.

$$\lim_{x\to 2}\left(8-3x+12x^2\right)=8-3(2)+12(4)=\boxed{50}$$

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2. Evaluate $\lim\limits_{t\to -3}\dfrac{6+4t}{t^2+1}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. We know that the first thing that we should try to do is simply plug in the value and see if we can compute the limit.

$$\lim_{t\to -3}\frac{6+4t}{t^2+1}=\frac{-6}{10}=\boxed{-\frac{3}{5}}$$

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3. Evaluate $\lim\limits_{x\to -5}\dfrac{x^2-25}{x^2+2x-15}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{x\to -5}\frac{x^2-25}{x^2+2x-15}=\lim_{x\to -5}\frac{(x-5)(x+5)}{(x-3)(x+5)}=\lim_{x\to -5}\frac{x-5}{x-3}=\boxed{\frac{5}{4}}$$

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4. Evaluate $\lim\limits_{z\to 8}\dfrac{2z^2-17z+8}{8-z}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{z\to 8}\frac{2z^2-17z+8}{8-z}=\lim_{z\to 8}\frac{(2z-1)(z-8)}{-(z-8)}=\lim_{z\to 8}\frac{2z-1}{-1}=\boxed{-15}$$

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5. Evaluate $\lim\limits_{y\to 7}\dfrac{y^2-4y-21}{3y^2-17y-28}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{y\to 7}\frac{y^2-4y-21}{3y^2-17y-28}=\lim_{y\to 7}\frac{(y-7)(y+3)}{(3y+4)(y-7)}=\lim_{y\to 7}\frac{y+3}{3y+4}=\frac{10}{25}=\boxed{\frac{2}{5}}$$

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6. Evaluate $\lim\limits_{h\to 0}\dfrac{(6+h)^2-36}{h}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{h\to 0}\frac{(6+h)^2-36}{h}=\lim_{h\to 0}\frac{36+12h+h^2-36}{h}=\lim_{h\to 0}\frac{h(12+h)}{h}=\lim_{h\to 0}(12+h)=\boxed{12}$$

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7. Evaluate $\lim\limits_{z\to 4}\dfrac{\sqrt{z}-2}{z-4}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. If you're really good at factoring you can factor this and simplify. Another method that can be used however is to rationalize the numerator, so let's do that for this problem.

$$\lim_{z \to 4} \frac{\sqrt{z}-2}{z-4} = \lim_{z \to 4} \frac{\left(\sqrt{z}-2\right)}{\left(z-4\right)} \frac{\left(\sqrt{z}+2\right)}{\left(\sqrt{z}+2\right)} = \lim_{z \to 4} \frac{z-4}{\left(z-4\right)\left(\sqrt{z}+2\right)} = \lim_{z \to 4} \frac{1}{\sqrt{z}+2} = \boxed{\frac{1}{4}}$$

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8. Evaluate $\lim\limits_{x \to -3} \dfrac{\sqrt{2x+22}-4}{x+3}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. Simply factoring will not do us much good here so in this case it looks like we'll need to rationalize the numerator.

$$\lim_{x \to -3} \frac{\sqrt{2x+22}-4}{x+3} = \lim_{x \to -3} \frac{\left(\sqrt{2x+22}-4\right)}{\left(x+3\right)} \frac{\left(\sqrt{2x+22}+4\right)}{\left(\sqrt{2x+22}+4\right)} = \lim_{x \to -3} \frac{2x+22-16}{\left(x+3\right)\left(\sqrt{2x+22}+4\right)}$$

$$= \lim_{x \to -3} \frac{2\left(x+3\right)}{\left(x+3\right)\left(\sqrt{2x+22}+4\right)} = \lim_{x \to -3} \frac{2}{\sqrt{2x+22}+4} = \frac{2}{8} = \boxed{\frac{1}{4}}$$

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9. Evaluate $\lim\limits_{x \to 0} \dfrac{x}{3-\sqrt{x+9}}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. Simply factoring will not do us much good here so in this case it looks like we'll need to rationalize the denominator.

$$\lim_{x\to 0}\frac{x}{3-\sqrt{x+9}}=\lim_{x\to 0}\frac{x}{\left(3-\sqrt{x+9}\right)}\frac{\left(3+\sqrt{x+9}\right)}{\left(3+\sqrt{x+9}\right)}=\lim_{x\to 0}\frac{x\left(3+\sqrt{x+9}\right)}{9-\left(x+9\right)}$$

$$=\lim_{x\to 0}\frac{x\left(3+\sqrt{x+9}\right)}{-x}=\lim_{x\to 0}\frac{3+\sqrt{x+9}}{-1}=\boxed{-6}$$

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10. Given the function

$$f\left(x\right)=\begin{cases}7-4x & x<1\\ x^2+2 & x\geq 1\end{cases}$$

Evaluate the following limits, if they exist.

(a) $\lim\limits_{x\to -6} f\left(x\right)$ (b) $\lim\limits_{x\to 1} f\left(x\right)$

Hint : Recall that when looking at overall limits (as opposed to one-sided limits) we need to make sure that the value of the function must be approaching the same value from both sides. In other words, the two one sided limits must both exist and be equal.

(a) $\lim\limits_{x\to -6} f\left(x\right)$ Solution

For this part we know that $-6<1$ and so there will be values of *x* on both sides of -6 in the range $x<1$ and so we can assume that, in the limit, we will have $x<1$. This will allow us to use the piece of the function in that range and then just use standard limit techniques to compute the limit.

$$\lim_{x\to -6} f\left(x\right)=\lim_{x\to -6}\left(7-4x\right)=\boxed{31}$$

(b) $\lim\limits_{x\to 1} f\left(x\right)$ Solution

This part is going to be different from the previous part. We are looking at the limit at $x=1$ and that is the “cut–off” point in the piecewise functions. Recall from the discussion in the section, that this means that we are going to have to look at the two one sided limits.

$$\lim_{x\to 1^-} f\left(x\right)=\lim_{x\to 1^-}\left(7-4x\right)=\underline{3} \qquad \text{because } x\to 1^- \text{ implies that } x<1$$

$$\lim_{x\to 1^+} f\left(x\right)=\lim_{x\to 1^+}\left(x^2+2\right)=\underline{3} \qquad \text{because } x\to 1^+ \text{ implies that } x>1$$

So, in this case, we can see that,

$$\lim_{x\to 1^-} f\left(x\right)=\lim_{x\to 1^+} f\left(x\right)=3$$

and so we know that the overall limit must exist and,

$$\lim_{x\to 1} f\left(x\right)=\boxed{3}$$

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11. Given the function

$$h(z) = \begin{cases} 6z & z \le -4 \\ 1-9z & z > -4 \end{cases}$$

Evaluate the following limits, if they exist.

(a) $\lim\limits_{z \to 7} h(z)$ (b) $\lim\limits_{z \to -4} h(z)$

Hint : Recall that when looking at overall limits (as opposed to one-sided limits) we need to make sure that the value of the function must be approaching the same value from both sides. In other words, the two one sided limits must both exist and be equal.

(a) $\lim\limits_{z \to 7} h(z)$ Solution

For this part we know that $7 > -4$ and so there will be values of *z* on both sides of 7 in the range $z > -4$ and so we can assume that, in the limit, we will have $z > -4$. This will allow us to use the piece of the function in that range and then just use standard limit techniques to compute the limit.

$$\lim_{z \to 7} h(z) = \lim_{z \to 7}(1-9z) = \boxed{-62}$$

(b) $\lim\limits_{z \to -4} h(z)$ Solution

This part is going to be different from the previous part. We are looking at the limit at $z = -4$ and that is the "cut–off" point in the piecewise functions. Recall from the discussion in the section, that this means that we are going to have to look at the two one sided limits.

$$\lim_{z \to -4^-} h(z) = \lim_{z \to -4^-} 6z = \underline{-24} \qquad \text{because } z \to -4^- \text{ implies that } z < -4$$

$$\lim_{z \to -4^+} h(z) = \lim_{z \to -4^+} (1-9z) = \underline{37} \qquad \text{because } z \to -4^+ \text{ implies that } z > -4$$

So, in this case, we can see that,

$$\lim_{z \to -4^-} h(z) = -24 \ne 37 = \lim_{z \to -4^+} h(z)$$

and so we know that the overall limit **does not exist**.

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12. Evaluate $\lim\limits_{x \to 5}\left(10 + |x-5|\right)$, if it exists.

Hint : Recall the mathematical definition of the absolute value function and that it is in fact a piecewise function.

Solution

Recall the definition of the absolute value function.

$$|p| = \begin{cases} p & p \ge 0 \\ -p & p < 0 \end{cases}$$

So, because the function inside the absolute value is zero at $x = 5$ we can see that,

$$|x-5| = \begin{cases} x-5 & x \geq 5 \\ -(x-5) & x < 5 \end{cases}$$

This means that we are being asked to compute the limit at the "cut–off" point in a piecewise function and so, as we saw in this section, we'll need to look at two one-sided limits in order to determine if this limit exists (and its value if it does exist).

$$\lim_{x\to 5^-} \left(10+|x-5|\right) = \lim_{x\to 5^-} \left(10-(x-5)\right) = \lim_{x\to 5^-} (15-x) = 10 \qquad \text{recall } x \to 5^- \text{ implies } x < 5$$

$$\lim_{x\to 5^+} \left(10+|x-5|\right) = \lim_{x\to 5^+} \left(10+(x-5)\right) = \lim_{x\to 5^+} (5+x) = 10 \qquad \text{recall } x \to 5^+ \text{ implies } x > 5$$

So, for this problem, we can see that,

$$\lim_{x\to 5^-} \left(10+|x-5|\right) = \lim_{x\to 5+} \left(10+|x-5|\right) = 10$$

and so the overall limit must exist and,

$$\lim_{x\to 5} \left(10+|x-5|\right) = \boxed{10}$$

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13. Evaluate $\lim\limits_{t\to -1} \dfrac{t+1}{|t+1|}$, if it exists.

Hint : Recall the mathematical definition of the absolute value function and that it is in fact a piecewise function.

Solution

Recall the definition of the absolute value function.

$$|p| = \begin{cases} p & p \geq 0 \\ -p & p < 0 \end{cases}$$

So, because the function inside the absolute value is zero at $t=-1$ we can see that,

$$|t+1| = \begin{cases} t+1 & t \geq -1 \\ -(t+1) & t < -1 \end{cases}$$

This means that we are being asked to compute the limit at the "cut–off" point in a piecewise function and so, as we saw in this section, we'll need to look at two one-sided limits in order to determine if this limit exists (and its value if it does exist).

$$\lim_{t\to -1^-} \frac{t+1}{|t+1|} = \lim_{t\to -1^-} \frac{t+1}{-(t+1)} = \lim_{t\to -1^-} -1 = -1 \qquad \text{recall } t \to -1^- \text{ implies } t < -1$$

$$\lim_{t\to -1^+} \frac{t+1}{|t+1|} = \lim_{t\to -1^+} \frac{t+1}{t+1} = \lim_{t\to -1^+} 1 = 1 \qquad \text{recall } t \to -1^+ \text{ implies } t > -1$$

So, for this problem, we can see that,

$$\lim_{t \to -1^-} \frac{t+1}{|t+1|} = -1 \neq 1 = \lim_{t \to -1^+} \frac{t+1}{|t+1|}$$

and so the overall limit **does not exist**.

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14. Given that $7x \le f(x) \le 3x^2 + 2$ for all *x* determine the value of $\lim_{x \to 2} f(x)$.

Hint : Recall the Squeeze Theorem.

Solution

This problem is set up to use the Squeeze Theorem. First, we already know that $f(x)$ is always between two other functions. Now all that we need to do is verify that the two "outer" functions have the same limit at $x = 2$ and if they do we can use the Squeeze Theorem to get the answer.

$$\lim_{x \to 2} 7x = 14 \qquad\qquad \lim_{x \to 2} \left(3x^2 + 2\right) = 14$$

So, we have,

$$\lim_{x \to 2} 7x = \lim_{x \to 2} \left(3x^2 + 2\right) = 14$$

and so by the Squeeze Theorem we must also have,

$$\lim_{x \to 2} f(x) = 14$$

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15. Use the Squeeze Theorem to determine the value of $\lim_{x \to 0} x^4 \sin\left(\frac{\pi}{x}\right)$.

Hint : Recall how we worked the Squeeze Theorem problem in this section to find the lower and upper functions we need in order to use the Squeeze Theorem.

Solution

We first need to determine lower/upper functions. We'll start off by acknowledging that provided $x \neq 0$ (which we know it won't be because we are looking at the limit as $x \to 0$) we will have,

$$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$$

Now, simply multiply through this by x^4 to get,

$$-x^4 \le x^4 \sin\left(\frac{\pi}{x}\right) \le x^4$$

Before proceeding note that we can only do this because we know that $x^4 > 0$ for $x \neq 0$. Recall that if we multiply through an inequality by a negative number we would have had to switch the signs. So, for instance, had we multiplied through by x^3 we would have had issues because this is positive if $x > 0$ and negative if $x < 0$.

Now, let's get back to the problem. We have a set of lower/upper functions and clearly,

$$\lim_{x\to 0} x^4 = \lim_{x\to 0}\left(-x^4\right) = 0$$

Therefore, by the Squeeze Theorem we must have,

$$\lim_{x\to 0} x^4 \sin\left(\frac{\pi}{x}\right) = 0$$
