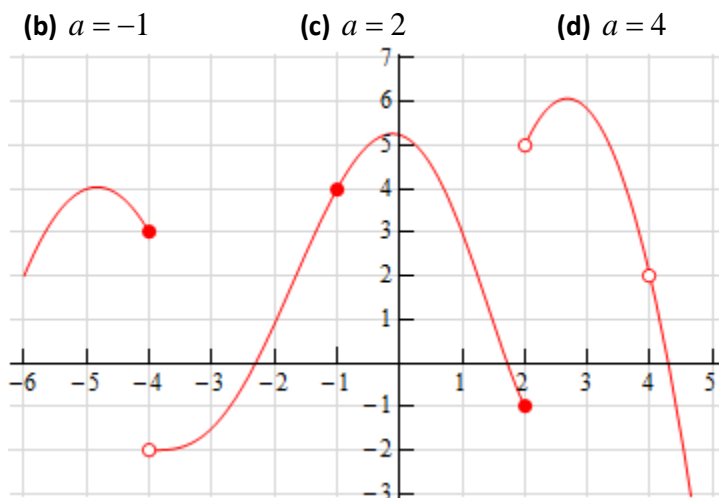


Section 2-3 : One-Sided Limits

1. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$, $\lim_{x \rightarrow a^-} f(x)$, $\lim_{x \rightarrow a^+} f(x)$, and $\lim_{x \rightarrow a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -4$



(b) $a = -1$

(c) $a = 2$

(d) $a = 4$

(a) $a = -4$

From the graph we can see that,

$$\boxed{f(-4) = 3}$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -4$ from the left the graph is approaching a value of 3 and as we approach from the right the graph is approaching a value of -2. Therefore, we get,

$$\boxed{\lim_{x \rightarrow -4^-} f(x) = 3 \quad \& \quad \lim_{x \rightarrow -4^+} f(x) = -2}$$

Now, because the two one-sided limits are different we know that,

$$\boxed{\lim_{x \rightarrow -4} f(x) \text{ does not exist}}$$

(b) $a = -1$

From the graph we can see that,

$$\boxed{f(-1) = 4}$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = -1$ from both sides the graph is approaching the same value, 4, and so we get,

$$\lim_{x \rightarrow -1^-} f(x) = 4 \quad \& \quad \lim_{x \rightarrow -1^+} f(x) = 4$$

The two one-sided limits are the same so we know,

$$\lim_{x \rightarrow -1} f(x) = 4$$

(c) $a = 2$

From the graph we can see that,

$$f(2) = -1$$

because the closed dot is at the value of $y = -1$.

We can also see that as we approach $x = 2$ from the left the graph is approaching a value of -1 and as we approach from the right the graph is approaching a value of 5. Therefore, we get,

$$\lim_{x \rightarrow 2^-} f(x) = -1 \quad \& \quad \lim_{x \rightarrow 2^+} f(x) = 5$$

Now, because the two one-sided limits are different we know that,

$$\lim_{x \rightarrow 2} f(x) \text{ does not exist}$$

(d) $a = 4$

Because there is no closed dot for $x = 4$ we can see that,

$$f(4) \text{ does not exist}$$

We can also see that as we approach $x = 4$ from both sides the graph is approaching the same value, 2, and so we get,

$$\lim_{x \rightarrow 4^-} f(x) = 2 \quad \& \quad \lim_{x \rightarrow 4^+} f(x) = 2$$

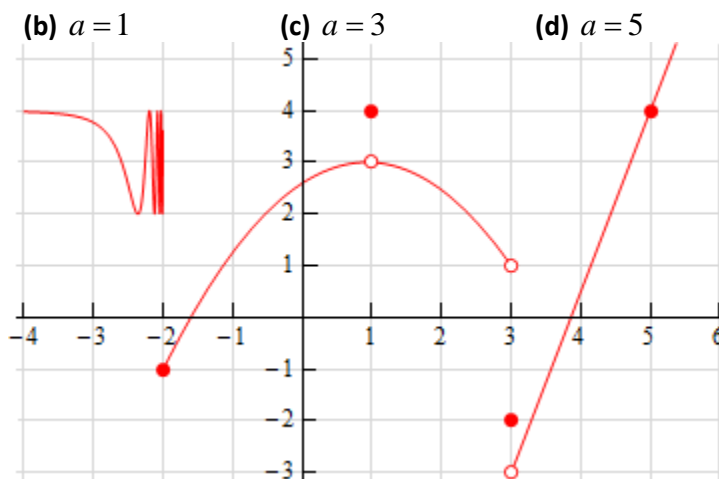
The two one-sided limits are the same so we know,

$$\lim_{x \rightarrow 4} f(x) = 2$$

Always recall that the value of a limit (including one-sided limits) does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Therefore, even though the function doesn't exist at this point the limit and one-sided limits can still have a value.

2. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$, $\lim_{x \rightarrow a^-} f(x)$, $\lim_{x \rightarrow a^+} f(x)$, and $\lim_{x \rightarrow a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -2$



(a) $a = -2$

From the graph we can see that,

$$\boxed{f(-2) = -1}$$

because the closed dot is at the value of $y = -1$.

We can also see that as we approach $x = -2$ from the left the graph is not approaching a single value, but instead oscillating wildly, and as we approach from the right the graph is approaching a value of -1. Therefore, we get,

$$\boxed{\lim_{x \rightarrow -2^-} f(x) \text{ does not exist} \quad \& \quad \lim_{x \rightarrow -2^+} f(x) = -1}$$

Recall that in order for limit to exist the function must be approaching a single value and so, in this case, because the graph to the left of $x = -2$ is not approaching a single value the left-hand limit will not exist. This does not mean that the right-hand limit will not exist. In this case the graph to the right of $x = -2$ is approaching a single value the right-hand limit will exist.

Now, because the two one-sided limits are different we know that,

$$\boxed{\lim_{x \rightarrow -2} f(x) \text{ does not exist}}$$

(b) $a = 1$

From the graph we can see that,

$$\boxed{f(1) = 4}$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = 1$ from both sides the graph is approaching the same value, 3, and so we get,

$$\lim_{x \rightarrow 1^-} f(x) = 3 \quad \& \quad \lim_{x \rightarrow 1^+} f(x) = 3$$

The two one-sided limits are the same so we know,

$$\lim_{x \rightarrow 1} f(x) = 3$$

(c) $a = 3$

From the graph we can see that,

$$f(3) = -2$$

because the closed dot is at the value of $y = -2$.

We can also see that as we approach $x = 2$ from the left the graph is approaching a value of 1 and as we approach from the right the graph is approaching a value of -3. Therefore, we get,

$$\lim_{x \rightarrow 3^-} f(x) = 1 \quad \& \quad \lim_{x \rightarrow 3^+} f(x) = -3$$

Now, because the two one-sided limits are different we know that,

$$\lim_{x \rightarrow 3} f(x) \text{ does not exist}$$

(d) $a = 5$

From the graph we can see that,

$$f(5) = 4$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = 5$ from both sides the graph is approaching the same value, 4, and so we get,

$$\lim_{x \rightarrow 5^-} f(x) = 4 \quad \& \quad \lim_{x \rightarrow 5^+} f(x) = 4$$

The two one-sided limits are the same so we know,

$$\lim_{x \rightarrow 5} f(x) = 4$$

3. Sketch a graph of a function that satisfies each of the following conditions.

$$\lim_{x \rightarrow 2^-} f(x) = 1$$

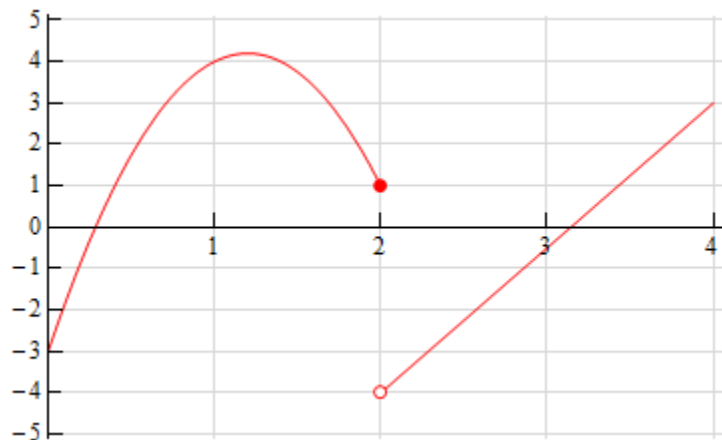
$$\lim_{x \rightarrow 2^+} f(x) = -4$$

$$f(2) = 1$$

Solution

There are literally an infinite number of possible graphs that we could give here for an answer. However, all of them must have a closed dot on the graph at the point $(2, 1)$, the graph must be approaching a value of 1 as it approaches $x = 2$ from the left (as indicated by the left-hand limit) and it must be approaching a value of -4 as it approaches $x = 2$ from the right (as indicated by the right-hand limit).

Here is a sketch of one possible graph that meets these conditions.



4. Sketch a graph of a function that satisfies each of the following conditions.

$$\lim_{x \rightarrow 3^-} f(x) = 0$$

$$\lim_{x \rightarrow 3^+} f(x) = 4$$

$$f(3) \text{ does not exist}$$

$$\lim_{x \rightarrow -1} f(x) = -3$$

$$f(-1) = 2$$

Solution

There are literally an infinite number of possible graphs that we could give here for an answer. However, all of them must the following two sets of criteria.

First, at $x = 3$ there cannot be a closed dot on the graph (as indicated by the fact that the function does not exist here), the graph must be approaching a value of 0 as it approaches $x = 3$ from the left (as indicated by the left-hand limit) and it must be approaching a value of 4 as it approaches $x = 3$ from the right (as indicated by the right-hand limit).

Next, the graph must have a closed dot at the point $(-1, 2)$ and the graph must be approaching a value of -3 as it approaches $x = -1$ from both sides (as indicated by the fact that value of the overall limit is -3 at this point).

Here is a sketch of one possible graph that meets these conditions.

