Section 2-1 : Tangent Lines and Rates of Change

1. For the function $f\left(x\right)=3\left(x+2\right)^2$ and the point *P* given by $x=-3$ answer each of the following questions.

(a) For the points *Q* given by the following values of *x* compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points *P* and *Q*.

(i) -3.5 **(ii)** -3.1 **(iii)** -3.01 **(iv)** -3.001 **(v)** -3.0001
(vi) -2.5 **(vii)** -2.9 **(viii)** -2.99 **(ix)** -2.999 **(x)** -2.9999

(b) Use the information from **(a)** to estimate the slope of the tangent line to $f\left(x\right)$ at $x=-3$ and write down the equation of the tangent line.

(a) For the points *Q* given by the following values of *x* compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points *P* and *Q*.

(i) -3.5 **(ii)** -3.1 **(iii)** -3.01 **(iv)** -3.001 **(v)** -3.0001
(vi) -2.5 **(vii)** -2.9 **(viii)** -2.99 **(ix)** -2.999 **(x)** -2.9999

Solution
The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$m_{PQ}=\frac{f\left(x\right)-f\left(-3\right)}{x-\left(-3\right)}=\frac{3\left(x+2\right)^2-3}{x+3}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of *x*. All of the values in the table below are accurate to 8 decimal places, but in this case the values terminated prior to 8 decimal places and so the "trailing" zeros are not shown.

x	m_{PQ}	x	m_{PQ}
-3.5	-7.5	-2.5	-4.5
-3.1	-6.3	-2.9	-5.7
-3.01	-6.03	-2.99	-5.97
-3.001	-6.003	-2.999	-5.997
-3.0001	-6.0003	-2.9999	-5.9997

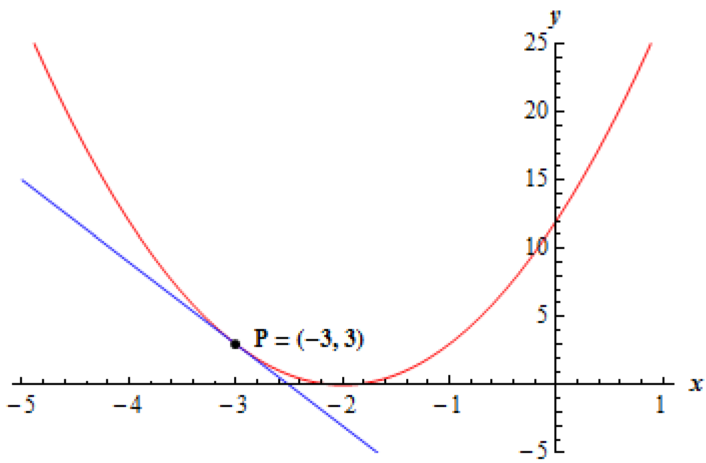
(b) Use the information from **(a)** to estimate the slope of the tangent line to $f\left(x\right)$ at $x=-3$ and write down the equation of the tangent line.
Solution

From the table of values above we can see that the slope of the secant lines appears to be moving towards a value of -6 from both sides of $x=-3$ and so we can estimate that the slope of the tangent line is : $\boxed{m=-6}$.

The equation of the tangent line is then,

$$y = f(-3) + m(x-(-3)) = 3 - 6(x+3) \qquad \Rightarrow \qquad \boxed{y=-6x-15}$$

Here is a graph of the function and the tangent line.



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2. For the function $g(x)=\sqrt{4x+8}$ and the point *P* given by $x=2$ answer each of the following questions.

(a) For the points *Q* given by the following values of *x* compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points *P* and *Q*.

(i) 2.5	**(ii)** 2.1	**(iii)** 2.01	**(iv)** 2.001	**(v)** 2.0001
(vi) 1.5	**(vii)** 1.9	**(viii)** 1.99	**(ix)** 1.999	**(x)** 1.9999

(b) Use the information from **(a)** to estimate the slope of the tangent line to $g(x)$ at $x=2$ and write down the equation of the tangent line.

(a) For the points *Q* given by the following values of *x* compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points *P* and *Q*.

(i) 2.5	**(ii)** 2.1	**(iii)** 2.01	**(iv)** 2.001	**(v)** 2.0001
(vi) 1.5	**(vii)** 1.9	**(viii)** 1.99	**(ix)** 1.999	**(x)** 1.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$m_{PQ} = \frac{g(x) - g(2)}{x-2} = \frac{\sqrt{4x+8}-4}{x-2}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of *x*. All of the values in the table below are accurate to 8 decimal places.

x	m_{PQ}	x	m_{PQ}
2.5	0.48528137	1.5	0.51668523
2.1	0.49691346	1.9	0.50316468
2.01	0.49968789	1.99	0.50031289
2.001	0.49996875	1.999	0.50003125
2.0001	0.49999688	1.9999	0.50000313

(b) Use the information from **(a)** to estimate the slope of the tangent line to $g(x)$ at $x=2$ and write down the equation of the tangent line.

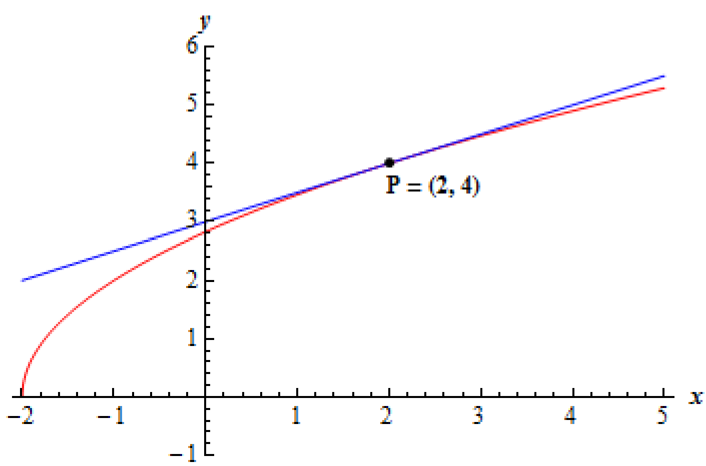
Solution

From the table of values above we can see that the slope of the secant lines appears to be moving towards a value of 0.5 from both sides of $x=2$ and so we can estimate that the slope of the tangent line is : $\boxed{m = 0.5 = \frac{1}{2}}$.

The equation of the tangent line is then,

$$y = g(2) + m(x-2) = 4 + \frac{1}{2}(x-2) \qquad \Rightarrow \qquad \boxed{y = \frac{1}{2}x + 3}$$

Here is a graph of the function and the tangent line.



3. For the function $W(x) = \ln(1+x^4)$ and the point *P* given by $x = 1$ answer each of the following questions.

(a) For the points *Q* given by the following values of *x* compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points *P* and *Q*.

(i) 1.5	**(ii)** 1.1	**(iii)** 1.01	**(iv)** 1.001	**(v)** 1.0001
(vi) 0.5	**(vii)** 0.9	**(viii)** 0.99	**(ix)** 0.999	**(x)** 0.9999

(b) Use the information from **(a)** to estimate the slope of the tangent line to $W(x)$ at $x = 1$ and write down the equation of the tangent line.

(a) For the points *Q* given by the following values of *x* compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points *P* and *Q*.

(i) 1.5	**(ii)** 1.1	**(iii)** 1.01	**(iv)** 1.001	**(v)** 1.0001
(vi) 0.5	**(vii)** 0.9	**(viii)** 0.99	**(ix)** 0.999	**(x)** 0.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$m_{PQ} = \frac{W(x) - W(1)}{x-1} = \frac{\ln(1+x^4) - \ln(2)}{x-1}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of *x*. All of the values in the table below are accurate to 8 decimal places.

x	m_{PQ}	*x*	m_{PQ}
1.5	2.21795015	0.5	1.26504512
1.1	2.08679449	0.9	1.88681740
1.01	2.00986668	0.99	1.98986668
1.001	2.00099867	0.999	1.99899867
1.0001	2.00009999	0.9999	1.99989999

(b) Use the information from **(a)** to estimate the slope of the tangent line to $W(x)$ at $x = 1$ and write down the equation of the tangent line.

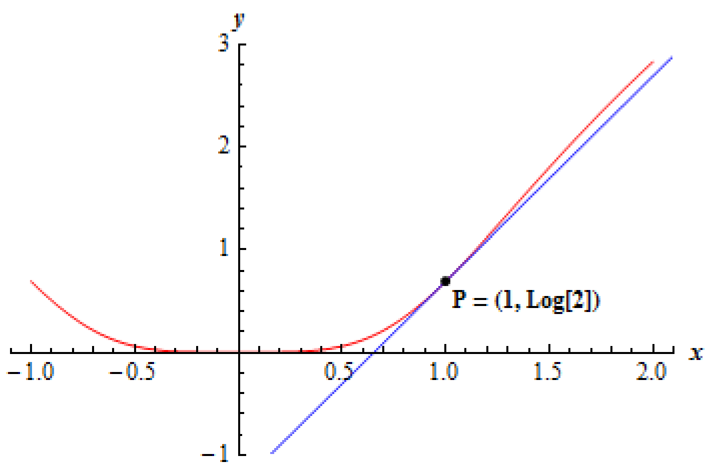
Solution

From the table of values above we can see that the slope of the secant lines appears to be moving towards a value of 2 from both sides of $x = 1$ and so we can estimate that the slope of the tangent line is : $\boxed{m = 2}$.

The equation of the tangent line is then,

$$y = W(1) + m(x-1) = \boxed{\ln(2) + 2(x-1)}$$

Here is a graph of the function and the tangent line.



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4. The volume of air in a balloon is given by $V(t) = \dfrac{6}{4t+1}$ answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the volume of air in the balloon between $t = 0.25$ and the following values of t.

(i) 1	**(ii)** 0.5	**(iii)** 0.251	**(iv)** 0.2501	**(v)** 0.25001
(vi) 0	**(vii)** 0.1	**(viii)** 0.249	**(ix)** 0.2499	**(x)** 0.24999

(b) Use the information from **(a)** to estimate the instantaneous rate of change of the volume of air in the balloon at $t = 0.25$.

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the volume of air in the balloon between $t = 0.25$ and the following values of t.

(i) 1	**(ii)** 0.5	**(iii)** 0.251	**(iv)** 0.2501	**(v)** 0.25001
(vi) 0	**(vii)** 0.1	**(viii)** 0.249	**(ix)** 0.2499	**(x)** 0.24999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$ARC = \frac{V(t) - V(0.25)}{t - 0.25} = \frac{\frac{6}{4t+1} - 3}{t - 0.25}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of *x*. All of the values in the table below are accurate to 8 decimal places. In several of the initial values in the table the values terminated and so the "trailing" zeroes were not shown.

x	*ARC*	*x*	*ARC*
1	-2.4	0	-12
0.5	-4	0.1	-8.57142857
0.251	-5.98802395	0.249	-6.01202405
0.2501	-5.99880024	0.2499	-6.00120024
0.25001	-5.99988000	0.24999	-6.00012000

(b) Use the information from **(a)** to estimate the instantaneous rate of change of the volume of air in the balloon at $t = 0.25$.
Solution
From the table of values above we can see that the average rate of change of the volume of air is moving towards a value of -6 from both sides of $t = 0.25$ and so we can estimate that the instantaneous rate of change of the volume of air in the balloon is -6.

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5. The population (in hundreds) of fish in a pond is given by $P(t) = 2t + \sin(2t - 10)$ answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the population of fish between $t = 5$ and the following values of *t*. Make sure your calculator is set to radians for the computations.

(i) 5.5 **(ii)** 5.1 **(iii)** 5.01 **(iv)** 5.001 **(v)** 5.0001
(vi) 4.5 **(vii)** 4.9 **(viii)** 4.99 **(ix)** 4.999 **(x)** 4.9999

(b) Use the information from **(a)** to estimate the instantaneous rate of change of the population of the fish at $t = 5$.

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the population of fish between $t = 5$ and the following values of *t*. Make sure your calculator is set to radians for the computations.

(i) 5.5 **(ii)** 5.1 **(iii)** 5.01 **(iv)** 5.001 **(v)** 5.0001

(vi) 4.5 **(vii)** 4.9 **(viii)** 4.99 **(ix)** 4.999 **(x)** 4.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$ARC = \frac{P(t) - P(5)}{t-5} = \frac{2t + \sin(2t-10) - 10}{t-5}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of *x*. All of the values in the table below are accurate to 8 decimal places.

x	*ARC*	*x*	*ARC*
5.5	3.68294197	4.5	3.68294197
5.1	3.98669331	4.9	3.98669331
5.01	3.99986667	4.99	3.99986667
5.001	3.99999867	4.999	3.99999867
5.0001	3.99999999	4.9999	3.99999999

(b) Use the information from **(a)** to estimate the instantaneous rate of change of the population of the fish at $t = 5$.

Solution

From the table of values above we can see that the average rate of change of the population of fish is moving towards a value of 4 from both sides of $t = 5$ and so we can estimate that the instantaneous rate of change of the population of the fish is 400 (remember the population is in hundreds).

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6. The position of an object is given by $s(t) = \cos^2\left(\frac{3t-6}{2}\right)$ answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 2$ and the following values of *t*. Make sure your calculator is set to radians for the computations.

(i) 2.5 **(ii)** 2.1 **(iii)** 2.01 **(iv)** 2.001 **(v)** 2.0001
(vi) 1.5 **(vii)** 1.9 **(viii)** 1.99 **(ix)** 1.999 **(x)** 1.9999

(b) Use the information from **(a)** to estimate the instantaneous velocity of the object at $t = 2$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 2$ and the following values of *t*. Make sure your calculator is set to radians for the computations.

(i) 2.5 **(ii)** 2.1 **(iii)** 2.01 **(iv)** 2.001 **(v)** 2.0001
(vi) 1.5 **(vii)** 1.9 **(viii)** 1.99 **(ix)** 1.999 **(x)** 1.9999

Solution
The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$AV = \frac{s(t)-s(2)}{t-2} = \frac{\cos^2\left(\frac{3t-6}{2}\right)-1}{t-2}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of *x*. All of the values in the table below are accurate to 8 decimal places.

t	*AV*	*t*	*AV*
2.5	-0.92926280	1.5	0.92926280
2.1	-0.22331755	1.9	0.22331755
2.01	-0.02249831	1.99	0.02249831
2.001	-0.00225000	1.999	0.00225000
2.0001	-0.00022500	1.9999	0.00022500

(b) Use the information from **(a)** to estimate the instantaneous velocity of the object at $t = 2$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).
Solution
From the table of values above we can see that the average velocity of the object is moving towards a value of 0 from both sides of $t = 2$ and so we can estimate that the instantaneous velocity is 0 and so the object will not be moving at $t = 2$.

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7. The position of an object is given by $s(t) = (8-t)(t+6)^{\frac{3}{2}}$. Note that a negative position here simply means that the position is to the left of the "zero position" and is perfectly acceptable. Answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 10$ and the following values of *t*.
(i) 10.5 **(ii)** 10.1 **(iii)** 10.01 **(iv)** 10.001 **(v)** 10.0001
(vi) 9.5 **(vii)** 9.9 **(viii)** 9.99 **(ix)** 9.999 **(x)** 9.9999

(b) Use the information from **(a)** to estimate the instantaneous velocity of the object at $t = 10$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 10$ and the following values of *t*.

(i) 10.5 **(ii)** 10.1 **(iii)** 10.01 **(iv)** 10.001 **(v)** 10.0001
(vi) 9.5 **(vii)** 9.9 **(viii)** 9.99 **(ix)** 9.999 **(x)** 9.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$AV = \frac{s(t) - s(10)}{t - 10} = \frac{(8-t)(t+6)^{\frac{3}{2}} + 128}{t - 10}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of *x*. All of the values in the table below are accurate to 8 decimal places.

t	*AV*	*t*	*AV*
10.5	-79.11658419	9.5	-72.92931693
10.1	-76.61966704	9.9	-75.38216890
10.01	-76.06188418	9.99	-75.93813418
10.001	-76.00618759	9.999	-75.99381259
10.0001	-76.00061875	9.9999	-75.99938125

(b) Use the information from **(a)** to estimate the instantaneous velocity of the object at $t = 10$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

Solution

From the table of values above we can see that the average velocity of the object is moving towards a value of -76 from both sides of $t = 10$ and so we can estimate that the instantaneous velocity is -76 and so the object will be moving to the left at $t = 10$.