Section 5-3 : Substitution Rule for Indefinite Integrals

1. Evaluate $\int (8x-12)(4x^2-12x)^4\,dx$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 4x^2 - 12x$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *x*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dx* as well as the remaining *x*'s in the integrand.

$$du = (8x-12)\,dx$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that.

Step 3
Doing the substitution and evaluating the integral gives,

$$\int (8x-12)(4x^2-12x)^4\,dx = \int u^4\,du = \tfrac{1}{5}u^5 + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int (8x-12)(4x^2-12x)^4\,dx = \boxed{\tfrac{1}{5}(4x^2-12x)^5 + c}$$

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2. Evaluate $\int 3t^{-4}(2+4t^{-3})^{-7}\,dt$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 2 + 4t^{-3}$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *t*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dt* as well as the remaining *t*'s in the integrand.

$$du = -12t^{-4}\,dt$$

To help with the substitution let's do a little rewriting of this to get,

$$3t^{-4}\,dt = -\tfrac{1}{4}\,du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int 3t^{-4}\left(2+4t^{-3}\right)^{-7} dt = -\tfrac{1}{4}\int u^{-7}\,du = \tfrac{1}{24}u^{-6} + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int 3t^{-4}\left(2+4t^{-3}\right)^{-7} dt = \boxed{\tfrac{1}{24}\left(2+4t^{-3}\right)^{-6} + c}$$

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3. Evaluate $\int (3-4w)\left(4w^2-6w+7\right)^{10} dw$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 4w^2 - 6w + 7$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *w*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dw* as well as the remaining *w*'s in the integrand.

$$du = \left(8w - 6\right)dw$$

To help with the substitution let's do a little rewriting of this to get,

$$du = -2\left(3 - 4w\right)dw \qquad \Rightarrow \qquad \left(3 - 4w\right)dw = -\tfrac{1}{2}\,du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \left(3 - 4w\right)\left(4w^2 - 6w + 7\right)^{10} dw = -\tfrac{1}{2}\int u^{10}\,du = -\tfrac{1}{22}u^{11} + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \left(3 - 4w\right)\left(4w^2 - 6w + 7\right)^{10} dw = \boxed{-\tfrac{1}{22}\left(4w^2 - 6w + 7\right)^{11} + c}$$

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4. Evaluate $\int 5\left(z - 4\right)\,\sqrt[3]{z^2 - 8z}\,dz$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = z^2 - 8z$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *z*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dz* as well as the remaining *z*'s in the integrand.

$$du = (2z - 8)\,dz$$

To help with the substitution let's do a little rewriting of this to get,

$$du = (2z - 8)\,dz = 2(z - 4)\,dz \qquad \Rightarrow \qquad (z - 4)\,dz = \tfrac{1}{2}\,du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int 5(z-4)\ \sqrt[3]{z^2 - 8z}\,dz = \tfrac{5}{2}\int u^{\frac{1}{3}}\,du = \tfrac{15}{8}u^{\frac{4}{3}} + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int 5(z-4)\ \sqrt[3]{z^2 - 8z}\,dz = \boxed{\tfrac{15}{8}\left(z^2 - 8z\right)^{\frac{4}{3}} + c}$$

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5. Evaluate $\int 90x^2 \sin\left(2 + 6x^3\right)dx$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 2 + 6x^3$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *x*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dx* as well as the remaining *x*'s in the integrand.

$$du = 18x^2dx$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that. When doing the substitution just notice that 90 = (18)(5).

Step 3
Doing the substitution and evaluating the integral gives,

$$\int 90x^2 \sin\left(2+6x^3\right) dx = \int 5\sin\left(u\right) du = -5\cos\left(u\right) + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int 90x^2 \sin\left(2+6x^3\right) dx = \boxed{-5\cos\left(2+6x^3\right) + c}$$

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6. Evaluate $\int \sec\left(1-z\right)\tan\left(1-z\right) dz$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 1-z$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *z*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dz* as well as the remaining *z*'s in the integrand.

$$du = -dz$$

To help with the substitution let's do a little rewriting of this to get,

$$dz = -du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \sec\left(1-z\right)\tan\left(1-z\right) dz = -\int \sec\left(u\right)\tan\left(u\right) du = -\sec\left(u\right) + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4

Finally, don't forget to go back to the original variable!

$$\int \sec(1-z)\tan(1-z)\,dz = \boxed{-\sec(1-z)+c}$$

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7. Evaluate $\int \left(15t^{-2}-5t\right)\cos\left(6t^{-1}+t^{2}\right)dt$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 6t^{-1}+t^{2}$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *t*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dt* as well as the remaining *t*'s in the integrand.

$$du = \left(-6t^{-2}+2t\right)dt$$

To help with the substitution let's do a little rewriting of this to get,

$$du = \left(-6t^{-2}+2t\right)dt = -2\left(\tfrac{5}{5}\right)\left(3t^{-2}-t\right)dt \qquad \Rightarrow \qquad \left(15t^{-2}-5t\right)dt = -\tfrac{5}{2}\,du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \left(15t^{-2}-5t\right)\cos\left(6t^{-1}+t^{2}\right)dt = -\tfrac{5}{2}\int \cos(u)\,du = -\tfrac{5}{2}\sin(u)+c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \left(15t^{-2}-5t\right)\cos\left(6t^{-1}+t^{2}\right)dt = \boxed{-\tfrac{5}{2}\sin\left(6t^{-1}+t^{2}\right)+c}$$

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8. Evaluate $\int \left(7y-2y^3\right)\mathbf{e}^{y^4-7y^2}\,dy$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = y^4 - 7y^2$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *y*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dy* as well as the remaining *y*'s in the integrand.

$$du = \left(4y^3 - 14y\right)dy$$

To help with the substitution let's do a little rewriting of this to get,

$$du = \left(4y^3 - 14y\right)dy = -2\left(7y-2y^3\right)dy \qquad \Rightarrow \qquad \left(7y-2y^3\right)dy = -\tfrac{1}{2}\,du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \left(7y-2y^3\right)\mathbf{e}^{y^4-7y^2}\,dy = -\tfrac{1}{2}\int \mathbf{e}^u\,du = -\tfrac{1}{2}\mathbf{e}^u + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \left(7y-2y^3\right)\mathbf{e}^{y^4-7y^2}\,dy = \boxed{-\tfrac{1}{2}\mathbf{e}^{y^4-7y^2} + c}$$

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9. Evaluate $\int \frac{4w+3}{4w^2+6w-1}\,dw$.

Hint : What is the derivative of the denominator?

Step 1

In this case it looks like we should use the following as our substitution.

$$u = 4w^2 + 6w - 1$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *w*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dw* as well as the remaining *w*'s in the integrand.

$$du = (8w + 6)\,dw$$

To help with the substitution let's do a little rewriting of this to get,

$$du = 2(4w + 3)\,dw \qquad \Rightarrow \qquad (4w + 3)\,dw = \tfrac{1}{2}du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \frac{4w+3}{4w^2+6w-1}\,dw = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \frac{4w+3}{4w^2+6w-1}\,dw = \boxed{\frac{1}{2}\ln\left|4w^2+6w-1\right| + c}$$

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10. Evaluate $\int \left(\cos(3t) - t^2\right)\left(\sin(3t) - t^3\right)^5\,dt$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = \sin(3t) - t^3$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *t*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dt* as well as the remaining *t*'s in the integrand.

$$du = \left(3\cos\left(3t\right) - 3t^2\right)dt$$

To help with the substitution let's do a little rewriting of this to get,

$$du = 3\left(\cos\left(3t\right) - t^2\right)dt \qquad \Rightarrow \qquad \left(\cos\left(3t\right) - t^2\right)dt = \tfrac{1}{3}du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \left(\cos\left(3t\right) - t^2\right)\left(\sin\left(3t\right) - t^3\right)^5 dt = \tfrac{1}{3}\int u^5\, du = \tfrac{1}{18}u^6 + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \left(\cos\left(3t\right) - t^2\right)\left(\sin\left(3t\right) - t^3\right)^5 dt = \boxed{\tfrac{1}{18}\left(\sin\left(3t\right) - t^3\right)^6 + c}$$

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11. Evaluate $\displaystyle\int 4\left(\frac{1}{z} - \mathbf{e}^{-z}\right)\cos\left(\mathbf{e}^{-z} + \ln z\right)dz$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = \mathbf{e}^{-z} + \ln z$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *z*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dz* as well as the remaining *z*'s in the integrand.

$$du = \left(-\mathbf{e}^{-z} + \tfrac{1}{z}\right)dt$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that.

Step 3
Doing the substitution and evaluating the integral gives,

$$\int 4\left(\frac{1}{z} - \mathbf{e}^{-z}\right)\cos\left(\mathbf{e}^{-z} + \ln z\right)dz = \int 4\cos\left(u\right)du = 4\sin\left(u\right) + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int 4\left(\frac{1}{z} - \mathbf{e}^{-z}\right)\cos\left(\mathbf{e}^{-z} + \ln z\right)dz = \boxed{4\sin\left(\mathbf{e}^{-z} + \ln z\right) + c}$$

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12. Evaluate $\int \sec^2\left(v\right)\mathbf{e}^{1+\tan\left(v\right)}\,dv$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 1 + \tan\left(v\right)$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *v*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dv* as well as the remaining *v*'s in the integrand.

$$du = \sec^2\left(v\right)dv$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that.

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \sec^2(v)\,\mathbf{e}^{1+\tan(v)}\,dv = \int \mathbf{e}^u\,du = \mathbf{e}^u + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \sec^2(v)\,\mathbf{e}^{1+\tan(v)}\,dv = \boxed{\mathbf{e}^{1+\tan(v)} + c}$$

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13. Evaluate $\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x)-5}\,dx$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = \cos^2(2x) - 5$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *x*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dx* as well as the remaining *x*'s in the integrand.

$$du = -4\cos(2x)\sin(2x)\,dx$$

To help with the substitution let's do a little rewriting of this to get,

$$du = -4\cos(2x)\sin(2x)\,dx = -2(2)\left(\tfrac{5}{5}\right)\cos(2x)\sin(2x)\,dx$$
$$\Rightarrow \quad 10\cos(2x)\sin(2x)\,dx = -\tfrac{5}{2}\,du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x)-5}\,dx = -\tfrac{5}{2}\int u^{\frac{1}{2}}\,du = -\tfrac{5}{3}u^{\frac{3}{2}} + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int 10\sin\left(2x\right)\cos\left(2x\right)\sqrt{\cos^2\left(2x\right)-5}\,dx = \boxed{-\tfrac{5}{3}\left(\cos^2\left(2x\right)-5\right)^{\frac{3}{2}}+c}$$

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14. Evaluate $\displaystyle\int \frac{\csc\left(x\right)\cot\left(x\right)}{2-\csc\left(x\right)}\,dx$.

Hint : What is the derivative of the denominator?

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 2-\csc\left(x\right)$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *x*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dx* as well as the remaining *x*'s in the integrand.

$$du = \csc\left(x\right)\cot\left(x\right)dx$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that.

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \frac{\csc\left(x\right)\cot\left(x\right)}{2-\csc\left(x\right)}\,dx = \int \frac{1}{u}\,du = \ln\left|u\right|+c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \frac{\csc(x)\cot(x)}{2-\csc(x)}\,dx = \boxed{\ln\left|2-\csc(x)\right|+c}$$

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15. Evaluate $\displaystyle\int \frac{6}{7+y^2}\,dy$.

Hint : Be careful with this substitution. The integrand should look somewhat familiar, so maybe we should try to put it into a more familiar form.

Step 1
The integrand looks an awful lot like the derivative of the inverse tangent.

$$\frac{d}{du}\left(\tan^{-1}(u)\right) = \frac{1}{1+u^2}$$

So, let's do a little rewrite to make the integrand look more like this.

$$\int \frac{6}{7+y^2}\,dy = \int \frac{6}{7\left(1+\frac{1}{7}y^2\right)}\,dy = \frac{6}{7}\int \frac{1}{1+\frac{1}{7}y^2}\,dy$$

Hint : One more little rewrite of the integrand should make this look almost exactly like the derivative the inverse tangent and the substitution should then be fairly obvious.

Step 2
Let's do one more rewrite of the integrand.

$$\int \frac{6}{7+y^2}\,dy = \frac{6}{7}\int \frac{1}{1+\left(\frac{y}{\sqrt{7}}\right)^2}\,dy$$

At this point we can see that the following substitution will work for us.

$$u = \tfrac{y}{\sqrt{7}} \qquad \to \qquad du = \tfrac{1}{\sqrt{7}}\,dy \qquad \to \qquad dy = \sqrt{7}\,du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \frac{6}{7+y^2}\,dy = \frac{6}{7}\left(\sqrt{7}\right)\int \frac{1}{1+u^2}\,du = \frac{6}{\sqrt{7}}\tan^{-1}(u)+c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \frac{6}{7+y^2}\,dy = \frac{6}{7}\left(\sqrt{7}\right)\int \frac{1}{1+u^2}\,du = \boxed{\frac{6}{\sqrt{7}}\tan^{-1}\left(\tfrac{y}{\sqrt{7}}\right)+c}$$

Substitutions for inverse trig functions can be a little tricky to spot when you are first start doing them. Once you do enough of them however they start to become a little easier to spot.

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16. Evaluate $\displaystyle\int \frac{1}{\sqrt{4-9w^2}}\,dw$.

Hint : Be careful with this substitution. The integrand should look somewhat familiar, so maybe we should try to put it into a more familiar form.

Step 1
The integrand looks an awful lot like the derivative of the inverse sine.

$$\frac{d}{du}\left(\sin^{-1}(u)\right) = \frac{1}{\sqrt{1-u^2}}$$

So, let's do a little rewrite to make the integrand look more like this.

$$\int \frac{1}{\sqrt{4-9w^2}}\,dw = \int \frac{1}{\sqrt{4\left(1-\frac{9}{4}w^2\right)}}\,dw = \frac{1}{2}\int \frac{1}{\sqrt{1-\frac{9}{4}w^2}}\,dw$$

Hint : One more little rewrite of the integrand should make this look almost exactly like the derivative the inverse sine and the substitution should then be fairly obvious.

Step 2
Let's do one more rewrite of the integrand.

$$\int \frac{1}{\sqrt{4-9w^2}}\,dw = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{3w}{2}\right)^2}}\,dw$$

At this point we can see that the following substitution will work for us.

$$u = \tfrac{3w}{2} \qquad \to \qquad du = \tfrac{3}{2}\,dw \qquad \to \qquad dw = \tfrac{2}{3}\,du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int \frac{1}{\sqrt{4-9w^2}}\,dw = \frac{1}{2}\left(\frac{2}{3}\right)\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{3}\sin^{-1}\left(u\right)+c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \frac{1}{\sqrt{4-9w^2}}\,dw = \boxed{\frac{1}{3}\sin^{-1}\left(\tfrac{3w}{2}\right)+c}$$

Substitutions for inverse trig functions can be a little tricky to spot when you are first start doing them. Once you do enough of them however they start to become a little easier to spot.

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17. Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{3x}{1+9x^2}\,dx$

(b) $\displaystyle\int \frac{3x}{\left(1+9x^2\right)^4}\,dx$

(c) $\displaystyle\int \frac{3}{1+9x^2}\,dx$

Hint : Make sure you pay attention to each of these and note the differences between each integrand and how that will affect the substitution and/or answer.

(a) $\displaystyle\int \frac{3x}{1+9x^2}\,dx$

Solution
In this case it looks like the substitution should be

$$u = 1+9x^2$$

Here is the differential for this substitution.

$$du = 18x\,dx \qquad \Rightarrow \qquad 3x\,dx = \tfrac{1}{6}\,du$$

The integral is then,

$$\int \frac{3x}{1+9x^2}\,dx = \frac{1}{6}\int \frac{1}{u}\,du = \frac{1}{6}\ln|u| + c = \boxed{\frac{1}{6}\ln\left|1+9x^2\right| + c}$$

(b) $\displaystyle\int \frac{3x}{\left(1+9x^2\right)^4}\,dx$

Solution
The substitution and differential work for this part are identical to the previous part.

$$u = 1+9x^2 \qquad du = 18x\,dx \qquad \Rightarrow \qquad 3x\,dx = \tfrac{1}{6}\,du$$

Here is the integral for this part,

$$\int \frac{3x}{\left(1+9x^2\right)^4}\,dx = \frac{1}{6}\int \frac{1}{u^4}\,du = \frac{1}{6}\int u^{-4}\,du = -\frac{1}{18}u^{-3} + c = \boxed{-\frac{1}{18}\frac{1}{\left(1+9x^2\right)^3} + c}$$

Be careful to not just turn every integral of functions of the form of 1/(something) into logarithms! This is one of the more common mistakes that students often make.

(c) $\displaystyle\int \frac{3}{1+9x^2}\,dx$

Solution
Because we no longer have an *x* in the numerator this integral is very different from the previous two. Let's do a quick rewrite of the integrand to make the substitution clearer.

$$\int \frac{3}{1+9x^2}\,dx = \int \frac{3}{1+(3x)^2}\,dx$$

So, this looks like an inverse tangent problem that will need the substitution.

$$u = 3x \qquad \to \qquad du = 3dx$$

The integral is then,

$$\int \frac{3}{1+9x^2}\,dx = \int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + c = \boxed{\tan^{-1}(3x) + c}$$