Chapter 5 : Integrals

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Indefinite Integrals – In this section we will start off the chapter with the definition and properties of indefinite integrals. We will not be computing many indefinite integrals in this section. This section is devoted to simply defining what an indefinite integral is and to give many of the properties of the indefinite integral. Actually computing indefinite integrals will start in the next section.

Computing Indefinite Integrals – In this section we will compute some indefinite integrals. The integrals in this section will tend to be those that do not require a lot of manipulation of the function we are integrating in order to actually compute the integral. As we will see starting in the next section many integrals do require some manipulation of the function before we can actually do the integral. We will also take a quick look at an application of indefinite integrals.

Substitution Rule for Indefinite Integrals – In this section we will start using one of the more common and useful integration techniques – The Substitution Rule. With the substitution rule we will be able integrate a wider variety of functions. The integrals in this section will all require some manipulation of the function prior to integrating unlike most of the integrals from the previous section where all we really needed were the basic integration formulas.

More Substitution Rule – In this section we will continue to look at the substitution rule. The problems in this section will tend to be a little more involved than those in the previous section.

Area Problem – In this section we start off with the motivation for definite integrals and give one of the interpretations of definite integrals. We will be approximating the amount of area that lies between a function and the \(x\)-axis. As we will see in the next section this problem will lead us to the definition of the definite integral and will be one of the main interpretations of the definite integral that we'll be looking at in this material.

Definition of the Definite Integral – In this section we will formally define the definite integral, give many of its properties and discuss a couple of interpretations of the definite integral. We will also look at the first part of the Fundamental Theorem of Calculus which shows the very close relationship between derivatives and integrals.

Computing Definite Integrals – In this section we will take a look at the second part of the Fundamental Theorem of Calculus. This will show us how we compute definite integrals without using (the often very unpleasant) definition. The examples in this section can all be done with a basic knowledge of indefinite integrals and will not require the use of the substitution rule. Included in the examples in this section are computing definite integrals of piecewise and absolute value functions.

Substitution Rule for Definite Integrals – In this section we will revisit the substitution rule as it applies to definite integrals. The only real requirements to being able to do the examples in this section are being able to do the substitution rule for indefinite integrals and understanding how to compute definite integrals in general.

Section 5-1 : Indefinite Integrals

1. Evaluate each of the following indefinite integrals.

(a) $\int 6x^5 - 18x^2 + 7\,dx$

(b) $\int 6x^5\,dx - 18x^2 + 7$

Hint : As long as you recall your derivative rules and the fact that all this problem is really asking is the for us to determine the function that we differentiated to get the integrand (*i.e*. the function inside the integral….) this problem shouldn't be too difficult.

(a) $\int 6x^5 - 18x^2 + 7\,dx$

All we are being asked to do here is "undo" a differentiation and if you recall the basic differentiation rules for polynomials this shouldn't be too difficult. As we saw in the notes for this section all we really need to do is increase the exponent by one (so upon differentiation we get the correct exponent) and then fix up the coefficient to make sure that we will get the correct coefficient upon differentiation.

Here is the answer for this part.

$$\int 6x^5 - 18x^2 + 7\,dx = \boxed{x^6 - 6x^3 + 7x + c}$$

Don't forget the "+c"! Remember that the original function may have had a constant on it and the "+c" is there to remind us of that.

Also, don't forget that you can easily check your answer by differentiating your answer and making sure that the result is the same as the integrand.

(b) $\int 6x^5\,dx - 18x^2 + 7$

This part is not really all that different from the first part. The only difference is the placement of the *dx*. Recall that one of the things that the *dx* tells us where to end the integration. So, in the part we are only going to integrate the first term.

Here is the answer for this part.

$$\int 6x^5\,dx - 18x^2 + 7 = \boxed{x^6 + c - 18x^2 + 7}$$

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2. Evaluate each of the following indefinite integrals.

(a) $\int 40x^3 + 12x^2 - 9x + 14\,dx$

(b) $\int 40x^3 + 12x^2 - 9x\,dx + 14$

(c) $\int 40x^3 + 12x^2 \, dx - 9x + 14$

Hint : As long as you recall your derivative rules and the fact that all this problem is really asking is the for us to determine the function that we differentiated to get the integrand (*i.e*. the function inside the integral….) this problem shouldn't be too difficult.

(a) $\int 40x^3 + 12x^2 - 9x + 14 \, dx$

All we are being asked to do here is "undo" a differentiation and if you recall the basic differentiation rules for polynomials this shouldn't be too difficult. As we saw in the notes for this section all we really need to do is increase the exponent by one (so upon differentiation we get the correct exponent) and then fix up the coefficient to make sure that we will get the correct coefficient upon differentiation.

Here is that answer for this part.

$$\int 40x^3 + 12x^2 - 9x + 14 \, dx = \boxed{10x^4 + 4x^3 - \tfrac{9}{2}x^2 + 14x + c}$$

Don't forget the "+c"! Remember that the original function may have had a constant on it and the "+c" is there to remind us of that.

Also, don't forget that you can easily check your answer by differentiating your answer and making sure that the result is the same as the integrand.

(b) $\int 40x^3 + 12x^2 - 9x \, dx + 14$

This part is not really all that different from the first part. The only difference is the placement of the *dx*. Recall that one of the things that the *dx* tells us where to end the integration. So, in the part we are only going to integrate the first term.

Here is the answer for this part.

$$\int 40x^3 + 12x^2 - 9x \, dx + 14 = \boxed{10x^4 + 4x^3 - \tfrac{9}{2}x^2 + c + 14}$$

(c) $\int 40x^3 + 12x^2 \, dx - 9x + 14$

The only difference between this part and the previous part is that the location of the *dx* moved.

Here is the answer for this part.

$$\int 40x^3 + 12x^2 \, dx - 9x + 14 = \boxed{10x^4 + 4x^3 + c - 9x + 14}$$

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3. Evaluate $\int 12t^7 - t^2 - t + 3 \, dt$.

Hint : As long as you recall your derivative rules and the fact that all this problem is really asking is the for us to determine the function that we differentiated to get the integrand (*i.e*. the function inside the integral….) this problem shouldn't be too difficult.

Solution
All we are being asked to do here is "undo" a differentiation and if you recall the basic differentiation rules for polynomials this shouldn't be too difficult. As we saw in the notes for this section all we really need to do is increase the exponent by one (so upon differentiation we get the correct exponent) and then fix up the coefficient to make sure that we will get the correct coefficient upon differentiation.

Here is the answer.

$$\int 12t^7 - t^2 - t + 3\,dt = \boxed{\tfrac{3}{2}t^8 - \tfrac{1}{3}t^3 - \tfrac{1}{2}t^2 + 3t + c}$$

Don't forget the "+c"! Remember that the original function may have had a constant on it and the "+c" is there to remind us of that.

Also, don't forget that you can easily check your answer by differentiating your answer and making sure that the result is the same as the integrand.

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4. Evaluate $\int 10w^4 + 9w^3 + 7w\ dw$.

Hint : As long as you recall your derivative rules and the fact that all this problem is really asking is the for us to determine the function that we differentiated to get the integrand (*i.e*. the function inside the integral….) this problem shouldn't be too difficult.

Solution
All we are being asked to do here is "undo" a differentiation and if you recall the basic differentiation rules for polynomials this shouldn't be too difficult. As we saw in the notes for this section all we really need to do is increase the exponent by one (so upon differentiation we get the correct exponent) and then fix up the coefficient to make sure that we will get the correct coefficient upon differentiation.

Here is the answer.

$$\int 10w^4 + 9w^3 + 7w\ dw = \boxed{2w^5 + \tfrac{9}{4}w^4 + \tfrac{7}{2}w^2 + c}$$

Don't forget the "+c"! Remember that the original function may have had a constant on it and the "+c" is there to remind us of that.

Also, don't forget that you can easily check your answer by differentiating your answer and making sure that the result is the same as the integrand.

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5. Evaluate $\int z^6 + 4z^4 - z^2 \, dz$.

Hint : As long as you recall your derivative rules and the fact that all this problem is really asking is the for us to determine the function that we differentiated to get the integrand (*i.e*. the function inside the integral….) this problem shouldn't be too difficult.

Solution
All we are being asked to do here is "undo" a differentiation and if you recall the basic differentiation rules for polynomials this shouldn't be too difficult. As we saw in the notes for this section all we really need to do is increase the exponent by one (so upon differentiation we get the correct exponent) and then fix up the coefficient to make sure that we will get the correct coefficient upon differentiation.

Here is the answer.

$$\int z^6 + 4z^4 - z^2 \, dz = \boxed{\tfrac{1}{7} z^7 + \tfrac{4}{5} z^5 - \tfrac{1}{3} z^3 + c}$$

Don't forget the "+c"! Remember that the original function may have had a constant on it and the "+c" is there to remind us of that.

Also, don't forget that you can easily check your answer by differentiating your answer and making sure that the result is the same as the integrand.

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6. Determine $f(x)$ given that $f'(x) = 6x^8 - 20x^4 + x^2 + 9$.

Hint : Remember that all indefinite integrals are asking us to do is "undo" a differentiation.

Solution
We know that indefinite integrals are asking us to undo a differentiation to so all we are really being asked to do here is evaluate the following indefinite integral.

$$f(x) = \int f'(x) \, dx = \int 6x^8 - 20x^4 + x^2 + 9 \, dx = \boxed{\tfrac{2}{3} x^9 - 4x^5 + \tfrac{1}{3} x^3 + 9x + c}$$

Don't forget the "+c"! Remember that the original function may have had a constant on it and the "+c" is there to remind us of that.

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7. Determine $h(t)$ given that $h'(t) = t^4 - t^3 + t^2 + t - 1$.

Hint : Remember that all indefinite integrals are asking us to do is "undo" a differentiation.

Solution
We know that indefinite integrals are asking us to undo a differentiation to so all we are really being asked to do here is evaluate the following indefinite integral.

$$h\left(t\right) = \int h'\left(t\right)dt = \int t^4 - t^3 + t^2 + t - 1\,dt = \boxed{\tfrac{1}{5}t^5 - \tfrac{1}{4}t^4 + \tfrac{1}{3}t^3 + \tfrac{1}{2}t^2 - t + c}$$

Don't forget the "+c"! Remember that the original function may have had a constant on it and the "+c" is there to remind us of that.

Section 5-2 : Computing Indefinite Integrals

1. Evaluate $\int 4x^6 - 2x^3 + 7x - 4\,dx$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int 4x^6 - 2x^3 + 7x - 4\,dx = \tfrac{4}{7}x^7 - \tfrac{2}{4}x^4 + \tfrac{7}{2}x^2 - 4x + c = \boxed{\tfrac{4}{7}x^7 - \tfrac{1}{2}x^4 + \tfrac{7}{2}x^2 - 4x + c}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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2. Evaluate $\int z^7 - 48z^{11} - 5z^{16}\,dz$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int z^7 - 48z^{11} - 5z^{16}\,dz = \tfrac{1}{8}z^8 - \tfrac{48}{12}z^{12} - \tfrac{5}{17}z^{17} + c = \boxed{\tfrac{1}{8}z^8 - 4z^{12} - \tfrac{5}{17}z^{17} + c}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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3. Evaluate $\int 10t^{-3} + 12t^{-9} + 4t^3\,dt$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int 10t^{-3} + 12t^{-9} + 4t^3\,dt = \tfrac{10}{-2}t^{-2} + \tfrac{12}{-8}t^{-8} + \tfrac{4}{4}t^4 + c = \boxed{-5t^{-2} - \tfrac{3}{2}t^{-8} + t^4 + c}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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4. Evaluate $\int w^{-2} + 10w^{-5} - 8\,dw$.

Solution
There really isn't too much to do other than to evaluate the integral.

$$\int w^{-2} + 10w^{-5} - 8\,dw = \tfrac{1}{-1}w^{-1} + \tfrac{10}{-4}w^{-4} - 8w + c = \boxed{-w^{-1} - \tfrac{5}{2}w^{-4} - 8w + c}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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5. Evaluate $\int 12\,dy$.

Solution
There really isn't too much to do other than to evaluate the integral.

$$\int 12\,dy = \boxed{12y + c}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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6. Evaluate $\int \sqrt[3]{w} + 10\sqrt[5]{w^3}\,dw$.

Hint : Don't forget to convert the roots to fractional exponents.

Step 1
We first need to convert the roots to fractional exponents.

$$\int \sqrt[3]{w} + 10\sqrt[5]{w^3}\,dw = \int w^{\frac{1}{3}} + 10\left(w^3\right)^{\frac{1}{5}}\,dw = \int w^{\frac{1}{3}} + 10w^{\frac{3}{5}}\,dw$$

Step 2
Once we've gotten the roots converted to fractional exponents there really isn't too much to do other than to evaluate the integral.

$$\int \sqrt[3]{w} + 10\sqrt[5]{w^3}\,dw = \int w^{\frac{1}{3}} + 10w^{\frac{3}{5}}\,dw = \tfrac{3}{4}w^{\frac{4}{3}} + 10\left(\tfrac{5}{8}\right)w^{\frac{8}{5}} + c = \boxed{\tfrac{3}{4}w^{\frac{4}{3}} + \tfrac{25}{4}w^{\frac{8}{5}} + c}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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7. Evaluate $\int \sqrt{x^7} - 7\sqrt[6]{x^5} + 17\sqrt[3]{x^{10}}\,dx$.

Hint : Don't forget to convert the roots to fractional exponents.

Step 1
We first need to convert the roots to fractional exponents.

$$\int \sqrt{x^7} - 7\sqrt[6]{x^5} + 17\sqrt[3]{x^{10}}\,dx = \int x^{\frac{7}{2}} - 7\left(x^5\right)^{\frac{1}{6}} + 17\left(x^{10}\right)^{\frac{1}{3}}\,dx = \int x^{\frac{7}{2}} - 7x^{\frac{5}{6}} + 17x^{\frac{10}{3}}\,dx$$

Step 2
Once we've gotten the roots converted to fractional exponents there really isn't too much to do other than to evaluate the integral.

$$\begin{aligned}\int \sqrt{x^7} - 7\sqrt[6]{x^5} + 17\sqrt[3]{x^{10}}\,dx &= \int x^{\frac{7}{2}} - 7x^{\frac{5}{6}} + 17x^{\frac{10}{3}}\,dx \\ &= \tfrac{2}{9}x^{\frac{9}{2}} - 7\left(\tfrac{6}{11}\right)x^{\frac{11}{6}} + 17\left(\tfrac{3}{13}\right)x^{\frac{13}{3}} + c = \boxed{\tfrac{2}{9}x^{\frac{9}{2}} - \tfrac{42}{11}x^{\frac{11}{6}} + \tfrac{51}{13}x^{\frac{13}{3}} + c}\end{aligned}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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8. Evaluate $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}\,dx$.

Hint : Don't forget to move the *x*'s in the denominator to the numerator with negative exponents.

Step 1
We first need to move the *x*'s in the denominator to the numerator with negative exponents.

$$\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}\,dx = \int 4x^{-2} + 2 - \frac{1}{8}x^{-3}\,dx$$

Remember that the "8" in the denominator of the third term stays in the denominator and does not move up with the *x*.

Step 2

Once we've gotten the *x*'s out of the denominator there really isn't too much to do other than to evaluate the integral.

$$\begin{aligned}\int \frac{4}{x^2}+2-\frac{1}{8x^3}\,dx &= \int 4x^{-2}+2-\frac{1}{8}x^{-3}\,dx\\ &= 4\left(\frac{1}{-1}\right)x^{-1}+2x-\frac{1}{8}\left(\frac{1}{-2}\right)x^{-2}+c = \boxed{-4x^{-1}+2x+\frac{1}{16}x^{-2}+c}\end{aligned}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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9. Evaluate $\displaystyle\int \frac{7}{3y^6}+\frac{1}{y^{10}}-\frac{2}{\sqrt[3]{y^4}}\,dy$.

Hint : Don't forget to convert the root to a fractional exponents and move the *y*'s in the denominator to the numerator with negative exponents.

Step 1

We first need to convert the root to a fractional exponent and move the *y*'s in the denominator to the numerator with negative exponents.

$$\int \frac{7}{3y^6}+\frac{1}{y^{10}}-\frac{2}{\sqrt[3]{y^4}}\,dy = \int \frac{7}{3y^6}+\frac{1}{y^{10}}-\frac{2}{y^{\frac{4}{3}}}\,dy = \int \frac{7}{3}y^{-6}+y^{-10}-2y^{-\frac{4}{3}}\,dy$$

Remember that the "3" in the denominator of the first term stays in the denominator and does not move up with the *y*.

Step 2

Once we've gotten the root converted to a fractional exponent and the *y*'s out of the denominator there really isn't too much to do other than to evaluate the integral.

$$\begin{aligned}\int \frac{7}{3y^6}+\frac{1}{y^{10}}-\frac{2}{\sqrt[3]{y^4}}\,dy &= \int \frac{7}{3}y^{-6}+y^{-10}-2y^{-\frac{4}{3}}\,dy\\ &= \frac{7}{3}\left(\frac{1}{-5}\right)y^{-5}+\left(\frac{1}{-9}\right)y^{-9}-2\left(-\frac{3}{1}\right)y^{-\frac{1}{3}}+c\\ &= \boxed{-\frac{7}{15}y^{-5}-\frac{1}{9}y^{-9}+6y^{-\frac{1}{3}}+c}\end{aligned}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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10. Evaluate $\int \left(t^2 - 1\right)\left(4 + 3t\right) dt$.

Hint : Remember that there is no "Product Rule" for integrals and so we'll need to eliminate the product before integrating.

Step 1
Since there is no "Product Rule" for integrals we'll need to multiply the terms out prior to integration.

$$\int \left(t^2 - 1\right)\left(4 + 3t\right) dt = \int 3t^3 + 4t^2 - 3t - 4\, dt$$

Step 2
At this point there really isn't too much to do other than to evaluate the integral.

$$\int \left(t^2 - 1\right)\left(4 + 3t\right) dt = \int 3t^3 + 4t^2 - 3t - 4\, dt = \boxed{\tfrac{3}{4}t^4 + \tfrac{4}{3}t^3 - \tfrac{3}{2}t^2 - 4t + c}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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11. Evaluate $\int \sqrt{z}\left(z^2 - \frac{1}{4z}\right) dz$.

Hint : Remember that there is no "Product Rule" for integrals and so we'll need to eliminate the product before integrating.

Step 1
Since there is no "Product Rule" for integrals we'll need to multiply the terms out prior to integration.

$$\int \sqrt{z}\left(z^2 - \frac{1}{4z}\right) dz = \int z^{\frac{5}{2}} - \frac{1}{4z^{\frac{1}{2}}}\, dz = \int z^{\frac{5}{2}} - \frac{1}{4} z^{-\frac{1}{2}}\, dz$$

Don't forget to convert the root to a fractional exponent and move the *z*'s out of the denominator.

Step 2
At this point there really isn't too much to do other than to evaluate the integral.

$$\int \sqrt{z}\left(z^2 - \frac{1}{4z}\right)dz = \int z^{\frac{5}{2}} - \frac{1}{4}z^{-\frac{1}{2}}\,dz = \boxed{\frac{2}{7}z^{\frac{7}{2}} - \frac{1}{2}z^{\frac{1}{2}} + c}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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12. Evaluate $\displaystyle\int \frac{z^8 - 6z^5 + 4z^3 - 2}{z^4}\,dz$.

Hint : Remember that there is no "Quotient Rule" for integrals and so we'll need to eliminate the quotient before integrating.

Step 1
Since there is no "Quotient Rule" for integrals we'll need to break up the integrand and simplify a little prior to integration.

$$\int \frac{z^8 - 6z^5 + 4z^3 - 2}{z^4}\,dz = \int \frac{z^8}{z^4} - \frac{6z^5}{z^4} + \frac{4z^3}{z^4} - \frac{2}{z^4}\,dz = \int z^4 - 6z + \frac{4}{z} - 2z^{-4}\,dz$$

Step 2
At this point there really isn't too much to do other than to evaluate the integral.

$$\int \frac{z^8 - 6z^5 + 4z^3 - 2}{z^4}\,dz = \int z^4 - 6z + \frac{4}{z} - 2z^{-4}\,dz = \boxed{\frac{1}{5}z^5 - 3z^2 + 4\ln|z| + \frac{2}{3}z^{-3} + c}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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13. Evaluate $\displaystyle\int \frac{x^4 - \sqrt[3]{x}}{6\sqrt{x}}\,dx$.

Hint : Remember that there is no "Quotient Rule" for integrals and so we'll need to eliminate the quotient before integrating.

Step 1
Since there is no "Quotient Rule" for integrals we'll need to break up the integrand and simplify a little prior to integration.

$$\int \frac{x^4 - \sqrt[3]{x}}{6\sqrt{x}}\,dx = \int \frac{x^4}{6x^{\frac{1}{2}}} - \frac{x^{\frac{1}{3}}}{6x^{\frac{1}{2}}}\,dx = \int \frac{1}{6}x^{\frac{7}{2}} - \frac{1}{6}x^{-\frac{1}{6}}\,dx$$

Don't forget to convert the roots to fractional exponents!

Step 2
At this point there really isn't too much to do other than to evaluate the integral.

$$\int \frac{x^4 - \sqrt[3]{x}}{6\sqrt{x}}\,dx = \int \frac{1}{6}x^{\frac{7}{2}} - \frac{1}{6}x^{-\frac{1}{6}}\,dx = \boxed{\frac{1}{27}x^{\frac{9}{2}} - \frac{1}{5}x^{\frac{5}{6}} + c}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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14. Evaluate $\int \sin\left(x\right) + 10\csc^2\left(x\right)dx$.

Solution
There really isn't too much to do other than to evaluate the integral.

$$\int \sin\left(x\right) + 10\csc^2\left(x\right)dx = \boxed{-\cos\left(x\right) - 10\cot\left(x\right) + c}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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15. Evaluate $\int 2\cos\left(w\right) - \sec\left(w\right)\tan\left(w\right)dw$.

Solution
There really isn't too much to do other than to evaluate the integral.

$$\int 2\cos\left(w\right) - \sec\left(w\right)\tan\left(w\right)dw = \boxed{2\sin\left(w\right) - \sec\left(w\right) + c}$$

Don't forget to add on the "+c" since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

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16. Evaluate $\int 12+\csc(\theta)\left[\sin(\theta)+\csc(\theta)\right]d\theta$.

Hint : From previous problems in this set we should know how to deal with the product in the integrand.

Step 1
Before doing the integral we need to multiply out the product and don't forget the definition of cosecant in terms of sine.

$$\begin{aligned}\int 12+\csc(\theta)\left[\sin(\theta)+\csc(\theta)\right]d\theta &= \int 12+\csc(\theta)\sin(\theta)+\csc^2(\theta)\,d\theta \\ &= \int 13+\csc^2(\theta)\,d\theta\end{aligned}$$

Recall that,

$$\csc(\theta)=\frac{1}{\sin(\theta)}$$

and so,

$$\csc(\theta)\sin(\theta)=1$$

Doing this allows us to greatly simplify the integrand and, in fact, allows us to actually do the integral. Without this simplification we would not have been able to integrate the second term with the knowledge that we currently have.

Step 2
At this point there really isn't too much to do other than to evaluate the integral.

$$\int 12+\csc(\theta)\left[\sin(\theta)+\csc(\theta)\right]d\theta = \int 13+\csc^2(\theta)\,d\theta = \boxed{13\theta-\cot(\theta)+c}$$

Don't forget that with trig functions some terms can be greatly simplified just by recalling the definition of the trig functions and/or their relationship with the other trig functions.

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17. Evaluate $\displaystyle\int 4\mathbf{e}^{z}+15-\frac{1}{6z}\,dz$.

Solution
There really isn't too much to do other than to evaluate the integral.

$$\int 4\mathbf{e}^{z}+15-\frac{1}{6z}\,dz = \int 4\mathbf{e}^{z}+15-\frac{1}{6}\frac{1}{z}\,dz = \boxed{4\mathbf{e}^{z}+15z-\frac{1}{6}\ln|z|+c}$$

Be careful with the "6" in the denominator of the third term. The "best" way of dealing with it in this case is to split up the third term as we've done above and then integrate.

Note that the "best" way to do a problem is always relative for many Calculus problems. There are other ways of dealing with this term (later section material) and so what one person finds the best another may not. For us, this seems to be an easy way to deal with the 6 and not overly complicate the integration process.

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18. Evaluate $\int t^3 - \frac{\mathbf{e}^{-t} - 4}{\mathbf{e}^{-t}}\, dt$.

Hint : From previous problems in this set we should know how to deal with the quotient in the integrand.

Step 1
Before doing the integral we need to break up the quotient and do some simplification.

$$\int t^3 - \frac{\mathbf{e}^{-t} - 4}{\mathbf{e}^{-t}}\, dt = \int t^3 - \frac{\mathbf{e}^{-t}}{\mathbf{e}^{-t}} + \frac{4}{\mathbf{e}^{-t}}\, dt = \int t^3 - 1 + 4\mathbf{e}^{t}\, dt$$

Make sure that you correctly distribute the minus sign when breaking up the second term and don't forget to move the exponential in the denominator of the third term (after splitting up the integrand) to the numerator and changing the sign on the *t* to a "+" in the process.

Step 2
At this point there really isn't too much to do other than to evaluate the integral.

$$\int t^3 - \frac{\mathbf{e}^{-t} - 4}{\mathbf{e}^{-t}}\, dt = \int t^3 - 1 + 4\mathbf{e}^{t}\, dt = \boxed{\frac{1}{4}t^4 - t + 4\mathbf{e}^{t} + c}$$

---

19. Evaluate $\int \frac{6}{w^3} - \frac{2}{w}\, dw$.

Solution
There really isn't too much to do other than to evaluate the integral.

$$\int \frac{6}{w^3} - \frac{2}{w}\, dw = \int 6w^{-3} - \frac{2}{w}\, dw = \boxed{-3w^{-2} - 2\ln|w| + c}$$

---

20. Evaluate $\int \frac{1}{1+x^2} + \frac{12}{\sqrt{1-x^2}}\,dx$.

Solution
There really isn't too much to do other than to evaluate the integral.

$$\int \frac{1}{1+x^2} + \frac{12}{\sqrt{1-x^2}}\,dx = \boxed{\tan^{-1}(x) + 12\sin^{-1}(x) + c}$$

Note that because of the similarity of the derivative of inverse sine and inverse cosine an alternate answer is,

$$\int \frac{1}{1+x^2} + \frac{12}{\sqrt{1-x^2}}\,dx = \boxed{\tan^{-1}(x) - 12\cos^{-1}(x) + c}$$

---

21. Evaluate $\int 6\cos(z) + \frac{4}{\sqrt{1-z^2}}\,dz$.

Solution
There really isn't too much to do other than to evaluate the integral.

$$\int 6\cos(z) + \frac{4}{\sqrt{1-z^2}}\,dz = \boxed{6\sin(z) + 4\sin^{-1}(z) + c}$$

Note that because of the similarity of the derivative of inverse sine and inverse cosine an alternate answer is,

$$\int 6\cos(z) + \frac{4}{\sqrt{1-z^2}}\,dz = \boxed{6\sin(z) - 4\cos^{-1}(z) + c}$$

---

22. Determine $f(x)$ given that $f'(x) = 12x^2 - 4x$ and $f(-3) = 17$.

Hint : We know that integration is simply asking what function we differentiated to get the integrand and so we should be able to use this idea to arrive at a general formula for the function.

Step 1
Recall from the notes in this section that we saw,

$$f(x) = \int f'(x)\, dx$$

and so to arrive at a general formula for $f(x)$ all we need to do is integrate the derivative that we've been given in the problem statement.

$$f(x) = \int 12x^2 - 4x\, dx = 4x^3 - 2x^2 + c$$

Don't forget the "+c"!

Hint : To determine the value of the constant of integration, *c*, we have the value of the function at $x = -3$.

Step 2
Because we have the condition that $f(-3) = 17$ we can just plug $x = -3$ into our answer from the previous step, set the result equal to 17 and solve the resulting equation for *c*.

Doing this gives,

$$17 = f(-3) = -126 + c \qquad \Rightarrow \qquad c = 143$$

The function is then,

$$\boxed{f(x) = 4x^3 - 2x^2 + 143}$$

---

23. Determine $g(z)$ given that $g'(z) = 3z^3 + \dfrac{7}{2\sqrt{z}} - \mathbf{e}^z$ and $g(1) = 15 - \mathbf{e}$.

Hint : We know that integration is simply asking what function we differentiated to get the integrand and so we should be able to use this idea to arrive at a general formula for the function.

Step 1
Recall from the notes in this section that we saw,

$$g(z) = \int g'(z)\, dz$$

and so to arrive at a general formula for $g(z)$ all we need to do is integrate the derivative that we've been given in the problem statement.

$$g(z) = \int 3z^3 + \tfrac{7}{2} z^{-\frac{1}{2}} - \mathbf{e}^z\, dz = \tfrac{3}{4} z^4 + 7z^{\frac{1}{2}} - \mathbf{e}^z + c$$

Don't forget the "+c"!

Hint : To determine the value of the constant of integration, *c*, we have the value of the function at $z = 1$.

Step 2
Because we have the condition that $g(1) = 15 - \mathbf{e}$ we can just plug $z = 1$ into our answer from the previous step, set the result equal to 15 – **e** and solve the resulting equation for *c*.

Doing this gives,

$$15 - \mathbf{e} = g(1) = \tfrac{31}{4} - \mathbf{e} + c \qquad \Rightarrow \qquad c = \tfrac{29}{4}$$

The function is then,

$$\boxed{g(z) = \tfrac{3}{4}z^4 + 7z^{\frac{1}{2}} - \mathbf{e}^z + \tfrac{29}{4}}$$

---

24. Determine $h(t)$ given that $h''(t) = 24t^2 - 48t + 2$, $h(1) = -9$ and $h(-2) = -4$.

Hint : We know how to find $h(t)$ from $h'(t)$ but we don't have that. We should however be able to determine the general formula for $h'(t)$ from $h''(t)$ which we are given.

Step 1
Because we know that the 2nd derivative is just the derivative of the 1st derivative we know that,

$$h'(t) = \int h''(t)\, dt$$

and so to arrive at a general formula for $h'(t)$ all we need to do is integrate the 2nd derivative that we've been given in the problem statement.

$$h'(t) = \int 24t^2 - 48t + 2 \, dt = 8t^3 - 24t^2 + 2t + c$$

Don't forget the "+c"!

Hint : From the previous two problems you should be able to determine a general formula for $h(t)$. Just don't forget that *c* is just a constant!

Step 2
Now, just as we did in the previous two problems, all that we need to do is integrate the 1st derivative (which we found in the first step) to determine a general formula for $h(t)$.

$$h(t) = \int 8t^3 - 24t^2 + 2t + c \; dt = 2t^4 - 8t^3 + t^2 + ct + d$$

Don't forget that *c* is just a constant and so it will integrate just like we were integrating 2 or 4 or any other number. Also, the constant of integration from this step is liable to be different that the constant of integration from the first step and so we'll need to make sure to call it something different, *d* in this case.

Hint : To determine the value of the constants of integration, *c* and *d*, we have the value of the function at two values that should help with that.

Step 3

Now, we know the value of the function at two values of *z*. So let's plug both of these into the general formula for $h(t)$ that we found in the previous step to get,

$$\begin{aligned} -9 &= h(1) = -5 + c + d \\ -4 &= h(-2) = 100 - 2c + d \end{aligned}$$

Solving this system of equations (you do remember your Algebra class right?) for *c* and *d* gives,

$$c = \tfrac{100}{3} \qquad\qquad d = -\tfrac{112}{3}$$

The function is then,

$$\boxed{h(t) = 2t^4 - 8t^3 + t^2 + \tfrac{100}{3}t - \tfrac{112}{3}}$$

Section 5-3 : Substitution Rule for Indefinite Integrals

1. Evaluate $\int (8x-12)(4x^2-12x)^4\,dx$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 4x^2 - 12x$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *x*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dx* as well as the remaining *x*'s in the integrand.

$$du = (8x-12)\,dx$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that.

Step 3
Doing the substitution and evaluating the integral gives,

$$\int (8x-12)(4x^2-12x)^4\,dx = \int u^4\,du = \tfrac{1}{5}u^5 + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int (8x-12)(4x^2-12x)^4\,dx = \boxed{\tfrac{1}{5}(4x^2-12x)^5 + c}$$

---

2. Evaluate $\int 3t^{-4}(2+4t^{-3})^{-7}\,dt$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 2 + 4t^{-3}$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *t*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dt* as well as the remaining *t*'s in the integrand.

$$du = -12t^{-4}\,dt$$

To help with the substitution let's do a little rewriting of this to get,

$$3t^{-4}\,dt = -\tfrac{1}{4}\,du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int 3t^{-4}\left(2+4t^{-3}\right)^{-7} dt = -\tfrac{1}{4}\int u^{-7}\,du = \tfrac{1}{24}u^{-6} + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int 3t^{-4}\left(2+4t^{-3}\right)^{-7} dt = \boxed{\tfrac{1}{24}\left(2+4t^{-3}\right)^{-6} + c}$$

---

3. Evaluate $\int \left(3-4w\right)\left(4w^2-6w+7\right)^{10} dw$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 4w^2 - 6w + 7$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *w*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dw* as well as the remaining *w*'s in the integrand.

$$du = \left(8w - 6\right)dw$$

To help with the substitution let's do a little rewriting of this to get,

$$du = -2\left(3-4w\right)dw \qquad \Rightarrow \qquad \left(3-4w\right)dw = -\tfrac{1}{2}\,du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \left(3-4w\right)\left(4w^2 - 6w + 7\right)^{10} dw = -\tfrac{1}{2}\int u^{10}\,du = -\tfrac{1}{22}u^{11} + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \left(3-4w\right)\left(4w^2 - 6w + 7\right)^{10} dw = \boxed{-\tfrac{1}{22}\left(4w^2 - 6w + 7\right)^{11} + c}$$

---

4. Evaluate $\int 5\left(z-4\right)\,\sqrt[3]{z^2 - 8z}\,dz$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = z^2 - 8z$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *z*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dz* as well as the remaining *z*'s in the integrand.

$$du = \left(2z - 8\right)dz$$

To help with the substitution let's do a little rewriting of this to get,

$$du = \left(2z - 8\right)dz = 2\left(z - 4\right)dz \qquad \Rightarrow \qquad \left(z - 4\right)dz = \tfrac{1}{2}du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int 5\left(z-4\right)\ \sqrt[3]{z^2 - 8z}\,dz = \tfrac{5}{2}\int u^{\frac{1}{3}}\,du = \tfrac{15}{8}u^{\frac{4}{3}} + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int 5\left(z-4\right)\ \sqrt[3]{z^2 - 8z}\,dz = \boxed{\tfrac{15}{8}\left(z^2 - 8z\right)^{\frac{4}{3}} + c}$$

---

5. Evaluate $\int 90x^2 \sin\left(2 + 6x^3\right)dx$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 2 + 6x^3$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *x*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dx* as well as the remaining *x*'s in the integrand.

$$du = 18x^2 dx$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that. When doing the substitution just notice that 90 = (18)(5).

Step 3
Doing the substitution and evaluating the integral gives,

$$\int 90x^2 \sin\left(2+6x^3\right)dx = \int 5\sin\left(u\right)du = -5\cos\left(u\right)+c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int 90x^2 \sin\left(2+6x^3\right)dx = \boxed{-5\cos\left(2+6x^3\right)+c}$$

---

6. Evaluate $\int \sec\left(1-z\right)\tan\left(1-z\right)dz$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 1-z$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *z*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dz* as well as the remaining *z*'s in the integrand.

$$du = -dz$$

To help with the substitution let's do a little rewriting of this to get,

$$dz = -du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \sec\left(1-z\right)\tan\left(1-z\right)dz = -\int \sec\left(u\right)\tan\left(u\right)du = -\sec\left(u\right)+c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4

Finally, don't forget to go back to the original variable!

$$\int \sec\left(1-z\right)\tan\left(1-z\right)dz = \boxed{-\sec\left(1-z\right)+c}$$

---

7. Evaluate $\int \left(15t^{-2} - 5t\right)\cos\left(6t^{-1} + t^2\right)dt$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 6t^{-1} + t^2$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *t*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dt* as well as the remaining *t*'s in the integrand.

$$du = \left(-6t^{-2} + 2t\right)dt$$

To help with the substitution let's do a little rewriting of this to get,

$$du = \left(-6t^{-2} + 2t\right)dt = -2\left(\tfrac{5}{5}\right)\left(3t^{-2} - t\right)dt \qquad \Rightarrow \qquad \left(15t^{-2} - 5t\right)dt = -\tfrac{5}{2}du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \left(15t^{-2} - 5t\right)\cos\left(6t^{-1} + t^2\right)dt = -\tfrac{5}{2}\int \cos\left(u\right)du = -\tfrac{5}{2}\sin\left(u\right) + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \left(15t^{-2} - 5t\right)\cos\left(6t^{-1} + t^2\right)dt = \boxed{-\tfrac{5}{2}\sin\left(6t^{-1} + t^2\right) + c}$$

---

8. Evaluate $\int \left(7y-2y^3\right)\mathbf{e}^{y^4-7y^2}\,dy$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = y^4 - 7y^2$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *y*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dy* as well as the remaining *y*'s in the integrand.

$$du = \left(4y^3 - 14y\right)dy$$

To help with the substitution let's do a little rewriting of this to get,

$$du = \left(4y^3 - 14y\right)dy = -2\left(7y - 2y^3\right)dy \qquad \Rightarrow \qquad \left(7y-2y^3\right)dy = -\tfrac{1}{2}\,du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \left(7y-2y^3\right)\mathbf{e}^{y^4-7y^2}\,dy = -\tfrac{1}{2}\int \mathbf{e}^u\,du = -\tfrac{1}{2}\mathbf{e}^u + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \left(7y-2y^3\right)\mathbf{e}^{y^4-7y^2}\,dy = \boxed{-\tfrac{1}{2}\mathbf{e}^{y^4-7y^2} + c}$$

---

9. Evaluate $\displaystyle\int \frac{4w+3}{4w^2+6w-1}\,dw$.

Hint : What is the derivative of the denominator?

Step 1

In this case it looks like we should use the following as our substitution.

$$u = 4w^2 + 6w - 1$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *w*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dw* as well as the remaining *w*'s in the integrand.

$$du = \left(8w + 6\right)dw$$

To help with the substitution let's do a little rewriting of this to get,

$$du = 2\left(4w + 3\right)dw \qquad \Rightarrow \qquad \left(4w + 3\right)dw = \tfrac{1}{2}du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \frac{4w+3}{4w^2 + 6w - 1}\,dw = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln\left|u\right| + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \frac{4w+3}{4w^2 + 6w - 1}\,dw = \boxed{\frac{1}{2}\ln\left|4w^2 + 6w - 1\right| + c}$$

---

10. Evaluate $\int \left(\cos\left(3t\right) - t^2\right)\left(\sin\left(3t\right) - t^3\right)^5\,dt$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = \sin\left(3t\right) - t^3$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *t*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dt* as well as the remaining *t*'s in the integrand.

$$du = \left(3\cos\left(3t\right) - 3t^2\right)dt$$

To help with the substitution let's do a little rewriting of this to get,

$$du = 3\left(\cos\left(3t\right) - t^2\right)dt \qquad \Rightarrow \qquad \left(\cos\left(3t\right) - t^2\right)dt = \tfrac{1}{3}du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \left(\cos\left(3t\right) - t^2\right)\left(\sin\left(3t\right) - t^3\right)^5 dt = \tfrac{1}{3}\int u^5\,du = \tfrac{1}{18}u^6 + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \left(\cos\left(3t\right) - t^2\right)\left(\sin\left(3t\right) - t^3\right)^5 dt = \boxed{\tfrac{1}{18}\left(\sin\left(3t\right) - t^3\right)^6 + c}$$

---

11. Evaluate $\displaystyle\int 4\left(\frac{1}{z} - \mathbf{e}^{-z}\right)\cos\left(\mathbf{e}^{-z} + \ln z\right)dz$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = \mathbf{e}^{-z} + \ln z$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *z*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dz* as well as the remaining *z*'s in the integrand.

$$du = \left(-\mathbf{e}^{-z} + \tfrac{1}{z}\right)dt$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that.

Step 3
Doing the substitution and evaluating the integral gives,

$$\int 4\left(\frac{1}{z} - \mathbf{e}^{-z}\right)\cos\left(\mathbf{e}^{-z} + \ln z\right)dz = \int 4\cos\left(u\right)du = 4\sin\left(u\right) + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int 4\left(\frac{1}{z} - \mathbf{e}^{-z}\right)\cos\left(\mathbf{e}^{-z} + \ln z\right)dz = \boxed{4\sin\left(\mathbf{e}^{-z} + \ln z\right) + c}$$

---

12. Evaluate $\int \sec^2\left(v\right)\mathbf{e}^{1+\tan\left(v\right)}\,dv$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 1 + \tan\left(v\right)$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *v*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dv* as well as the remaining *v*'s in the integrand.

$$du = \sec^2\left(v\right)dv$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that.

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \sec^2(v)\,\mathbf{e}^{1+\tan(v)}\,dv = \int \mathbf{e}^u\,du = \mathbf{e}^u + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \sec^2(v)\,\mathbf{e}^{1+\tan(v)}\,dv = \boxed{\mathbf{e}^{1+\tan(v)} + c}$$

---

13. Evaluate $\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x)-5}\,dx$.

Hint : Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1
In this case it looks like we should use the following as our substitution.

$$u = \cos^2(2x) - 5$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *x*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dx* as well as the remaining *x*'s in the integrand.

$$du = -4\cos(2x)\sin(2x)\,dx$$

To help with the substitution let's do a little rewriting of this to get,

$$du = -4\cos(2x)\sin(2x)\,dx = -2(2)\left(\tfrac{5}{5}\right)\cos(2x)\sin(2x)\,dx$$
$$\Rightarrow\ 10\cos(2x)\sin(2x)\,dx = -\tfrac{5}{2}\,du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x)-5}\,dx = -\tfrac{5}{2}\int u^{\frac{1}{2}}\,du = -\tfrac{5}{3}u^{\frac{3}{2}} + c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int 10\sin\left(2x\right)\cos\left(2x\right)\sqrt{\cos^2\left(2x\right)-5}\,dx = \boxed{-\tfrac{5}{3}\left(\cos^2\left(2x\right)-5\right)^{\frac{3}{2}}+c}$$

---

14. Evaluate $\displaystyle\int \frac{\csc\left(x\right)\cot\left(x\right)}{2-\csc\left(x\right)}\,dx$.

Hint : What is the derivative of the denominator?

Step 1
In this case it looks like we should use the following as our substitution.

$$u = 2-\csc\left(x\right)$$

Hint : Recall that after the substitution all the original variables in the integral should be replaced with *u*'s.

Step 2
Because we need to make sure that all the *x*'s are replaced with *u*'s we need to compute the differential so we can eliminate the *dx* as well as the remaining *x*'s in the integrand.

$$du = \csc\left(x\right)\cot\left(x\right)dx$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that.

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \frac{\csc\left(x\right)\cot\left(x\right)}{2-\csc\left(x\right)}\,dx = \int \frac{1}{u}\,du = \ln\left|u\right|+c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \frac{\csc(x)\cot(x)}{2-\csc(x)}\,dx = \boxed{\ln\left|2-\csc(x)\right|+c}$$

---

15. Evaluate $\displaystyle\int \frac{6}{7+y^2}\,dy$.

Hint : Be careful with this substitution. The integrand should look somewhat familiar, so maybe we should try to put it into a more familiar form.

Step 1
The integrand looks an awful lot like the derivative of the inverse tangent.

$$\frac{d}{du}\left(\tan^{-1}(u)\right)=\frac{1}{1+u^2}$$

So, let's do a little rewrite to make the integrand look more like this.

$$\int \frac{6}{7+y^2}\,dy = \int \frac{6}{7\left(1+\frac{1}{7}y^2\right)}\,dy = \frac{6}{7}\int \frac{1}{1+\frac{1}{7}y^2}\,dy$$

Hint : One more little rewrite of the integrand should make this look almost exactly like the derivative the inverse tangent and the substitution should then be fairly obvious.

Step 2
Let's do one more rewrite of the integrand.

$$\int \frac{6}{7+y^2}\,dy = \frac{6}{7}\int \frac{1}{1+\left(\frac{y}{\sqrt{7}}\right)^2}\,dy$$

At this point we can see that the following substitution will work for us.

$$u=\tfrac{y}{\sqrt{7}} \qquad \to \qquad du=\tfrac{1}{\sqrt{7}}\,dy \qquad \to \qquad dy=\sqrt{7}\,du$$

Step 3
Doing the substitution and evaluating the integral gives,

$$\int \frac{6}{7+y^2}\,dy = \frac{6}{7}\left(\sqrt{7}\right)\int \frac{1}{1+u^2}\,du = \frac{6}{\sqrt{7}}\tan^{-1}(u)+c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \frac{6}{7+y^2}\,dy = \frac{6}{7}\left(\sqrt{7}\right)\int \frac{1}{1+u^2}\,du = \boxed{\frac{6}{\sqrt{7}}\tan^{-1}\left(\tfrac{y}{\sqrt{7}}\right)+c}$$

Substitutions for inverse trig functions can be a little tricky to spot when you are first start doing them. Once you do enough of them however they start to become a little easier to spot.

---

16. Evaluate $\displaystyle\int \frac{1}{\sqrt{4-9w^2}}\,dw$.

Hint : Be careful with this substitution. The integrand should look somewhat familiar, so maybe we should try to put it into a more familiar form.

Step 1
The integrand looks an awful lot like the derivative of the inverse sine.

$$\frac{d}{du}\left(\sin^{-1}\left(u\right)\right) = \frac{1}{\sqrt{1-u^2}}$$

So, let's do a little rewrite to make the integrand look more like this.

$$\int \frac{1}{\sqrt{4-9w^2}}\,dw = \int \frac{1}{\sqrt{4\left(1-\frac{9}{4}w^2\right)}}\,dw = \frac{1}{2}\int \frac{1}{\sqrt{1-\frac{9}{4}w^2}}\,dw$$

Hint : One more little rewrite of the integrand should make this look almost exactly like the derivative the inverse sine and the substitution should then be fairly obvious.

Step 2
Let's do one more rewrite of the integrand.

$$\int \frac{1}{\sqrt{4-9w^2}}\,dw = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{3w}{2}\right)^2}}\,dw$$

At this point we can see that the following substitution will work for us.

$$u = \tfrac{3w}{2} \qquad \to \qquad du = \tfrac{3}{2}\,dw \qquad \to \qquad dw = \tfrac{2}{3}\,du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int \frac{1}{\sqrt{4-9w^2}}\,dw = \frac{1}{2}\left(\frac{2}{3}\right)\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{3}\sin^{-1}(u)+c$$

Hint : Don't forget that the original variable in the integrand was not *u*!

Step 4
Finally, don't forget to go back to the original variable!

$$\int \frac{1}{\sqrt{4-9w^2}}\,dw = \boxed{\frac{1}{3}\sin^{-1}\left(\tfrac{3w}{2}\right)+c}$$

Substitutions for inverse trig functions can be a little tricky to spot when you are first start doing them. Once you do enough of them however they start to become a little easier to spot.

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17. Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{3x}{1+9x^2}\,dx$

(b) $\displaystyle\int \frac{3x}{\left(1+9x^2\right)^4}\,dx$

(c) $\displaystyle\int \frac{3}{1+9x^2}\,dx$

Hint : Make sure you pay attention to each of these and note the differences between each integrand and how that will affect the substitution and/or answer.

(a) $\displaystyle\int \frac{3x}{1+9x^2}\,dx$

Solution
In this case it looks like the substitution should be

$$u = 1+9x^2$$

Here is the differential for this substitution.

$$du = 18x\,dx \qquad \Rightarrow \qquad 3x\,dx = \tfrac{1}{6}\,du$$

The integral is then,

$$\int \frac{3x}{1+9x^2}\,dx = \frac{1}{6}\int \frac{1}{u}\,du = \frac{1}{6}\ln|u| + c = \boxed{\frac{1}{6}\ln\left|1+9x^2\right| + c}$$

(b) $\displaystyle\int \frac{3x}{\left(1+9x^2\right)^4}\,dx$

Solution
The substitution and differential work for this part are identical to the previous part.

$$u = 1+9x^2 \qquad\qquad du = 18x\,dx \qquad\qquad \Rightarrow \qquad\qquad 3x\,dx = \tfrac{1}{6}\,du$$

Here is the integral for this part,

$$\int \frac{3x}{\left(1+9x^2\right)^4}\,dx = \frac{1}{6}\int \frac{1}{u^4}\,du = \frac{1}{6}\int u^{-4}\,du = -\frac{1}{18}u^{-3} + c = \boxed{-\frac{1}{18}\frac{1}{\left(1+9x^2\right)^3} + c}$$

Be careful to not just turn every integral of functions of the form of 1/(something) into logarithms! This is one of the more common mistakes that students often make.

(c) $\displaystyle\int \frac{3}{1+9x^2}\,dx$

Solution
Because we no longer have an *x* in the numerator this integral is very different from the previous two. Let's do a quick rewrite of the integrand to make the substitution clearer.

$$\int \frac{3}{1+9x^2}\,dx = \int \frac{3}{1+\left(3x\right)^2}\,dx$$

So, this looks like an inverse tangent problem that will need the substitution.

$$u = 3x \qquad\qquad \to \qquad\qquad du = 3dx$$

The integral is then,

$$\int \frac{3}{1+9x^2}\,dx = \int \frac{1}{1+u^2}\,du = \tan^{-1}\left(u\right) + c = \boxed{\tan^{-1}\left(3x\right) + c}$$

Section 5-4 : More Substitution Rule

1. Evaluate $\int 4\sqrt{5+9t} + 12\left(5+9t\right)^7 \, dt$.

Hint : Each term seems to require the same substitution and recall that the same substitution can be used in multiple terms of an integral if we need to.

Step 1
Don't get too excited about the fact that there are two terms in this integrand. Each term requires the same substitution,

$$u = 5 + 9t$$

so we'll simply use that in both terms.

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 2
Here is the differential work for the substitution.

$$du = 9\,dt \qquad \to \qquad dt = \tfrac{1}{9}du$$

Doing the substitution and evaluating the integral gives,

$$\int \left[4u^{\frac{1}{2}} + 12u^7\right]\left(\tfrac{1}{9}\right)du = \tfrac{1}{9}\left[\tfrac{8}{3}u^{\frac{3}{2}} + \tfrac{3}{2}u^8\right] + c = \boxed{\tfrac{1}{9}\left[\tfrac{8}{3}\left(5+9t\right)^{\frac{3}{2}} + \tfrac{3}{2}\left(5+9t\right)^8\right] + c}$$

Be careful when dealing with the *dt* substitution here. Make sure that the $\frac{1}{9}$ gets multiplied times the whole integrand and not just one of the terms. You can do this either by using parenthesis (as we've done here) or pulling the $\frac{1}{9}$ completely out of the integral.

Do not forget to go back to the original variable after evaluating the integral!

---

2. Evaluate $\int 7x^3 \cos\left(2 + x^4\right) - 8x^3 \mathbf{e}^{2+x^4} \, dx$.

Hint : Each term seems to require the same substitution and recall that the same substitution can be used in multiple terms of an integral if we need to.

Step 1
Don't get too excited about the fact that there are two terms in this integrand. Each term requires the same substitution,

$$u = 2 + x^4$$

so we'll simply use that in both terms.

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 2
Here is the differential work for the substitution.

$$du = 4x^3\,dx \qquad \to \qquad x^3 dx = \tfrac{1}{4}du$$

Before doing the actual substitution it might be convenient to factor an x^3 out of the integrand as follows.

$$\int 7x^3\cos\left(2+x^4\right) - 8x^3\mathbf{e}^{2+x^4}\,dx = \int\left[7\cos\left(2+x^4\right) - 8\mathbf{e}^{2+x^4}\right]x^3dx$$

Doing this should make the differential part (*i.e.* the *du* part) of the substitution clearer.

Now, doing the substitution and evaluating the integral gives,

$$\begin{aligned}\int 7x^3\cos\left(2+x^4\right) - 8x^3\mathbf{e}^{2+x^4}\,dx &= \tfrac{1}{4}\int 7\cos\left(u\right) - 8\mathbf{e}^{u}du \\ &= \tfrac{1}{4}\left[7\sin\left(u\right) - 8\mathbf{e}^{u}\right] + c = \boxed{\tfrac{1}{4}\left[7\sin\left(2+x^4\right) - 8\mathbf{e}^{2+x^4}\right] + c}\end{aligned}$$

Be careful when dealing with the *dx* substitution here. Make sure that the $\frac{1}{4}$ gets multiplied times the whole integrand and not just one of the terms. You can do this either by using parenthesis around the whole integrand or pulling the $\frac{1}{4}$ completely out of the integral (as we've done here).

Do not forget to go back to the original variable after evaluating the integral!

---

3. Evaluate $\displaystyle\int \frac{6\mathbf{e}^{7w}}{\left(1-8\mathbf{e}^{7w}\right)^3} + \frac{14\mathbf{e}^{7w}}{1-8\mathbf{e}^{7w}}\,dw$.

Hint : Each term seems to require the same substitution and recall that the same substitution can be used in multiple terms of an integral if we need to.

Step 1
Don't get too excited about the fact that there are two terms in this integrand. Each term requires the same substitution,

$$u = 1 - 8\mathbf{e}^{7w}$$

so we'll simply use that in both terms.

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 2
Here is the differential work for the substitution.

$$du = -56\mathbf{e}^{7w}dw \qquad \to \qquad \mathbf{e}^{7w}dw = -\tfrac{1}{56}du$$

Before doing the actual substitution it might be convenient to factor an $\mathbf{e}^{7w}$ out of the integrand as follows.

$$\int \frac{6\mathbf{e}^{7w}}{\left(1-8\mathbf{e}^{7w}\right)^3} + \frac{14\mathbf{e}^{7w}}{1-8\mathbf{e}^{7w}}\,dw = \int \left[\frac{6}{\left(1-8\mathbf{e}^{7w}\right)^3} + \frac{14}{1-8\mathbf{e}^{7w}}\right]\mathbf{e}^{7w}\,dw$$

Doing this should make the differential part (*i.e.* the *du* part) of the substitution clearer.

Now, doing the substitution and evaluating the integral gives,

$$\int \frac{6\mathbf{e}^{7w}}{\left(1-8\mathbf{e}^{7w}\right)^3} + \frac{14\mathbf{e}^{7w}}{1-8\mathbf{e}^{7w}}\,dw = -\tfrac{1}{56}\int 6u^{-3} + \frac{14}{u}\,du = -\tfrac{1}{56}\left(-3u^{-2} + 14\ln\left|u\right|\right) + c$$

$$= \boxed{-\tfrac{1}{56}\left(-3\left(1-8\mathbf{e}^{7w}\right)^{-2} + 14\ln\left|1-8\mathbf{e}^{7w}\right|\right) + c}$$

Be careful when dealing with the *dw* substitution here. Make sure that the $-\frac{1}{56}$ gets multiplied times the whole integrand and not just one of the terms. You can do this either by using parenthesis around the whole integrand or pulling the $-\frac{1}{56}$ completely out of the integral (as we've done here).

Do not forget to go back to the original variable after evaluating the integral!

---

4. Evaluate $\int x^4 - 7x^5\cos\left(2x^6 + 3\right)dx$.

Hint : Recall that terms that do not need substitutions should not be in the integral when the substitution is being done. At this point we should know how to "break" integrals up so that we can get the terms that require a substitution into a one integral and those that don't into another integral.

Step 1
Clearly the first term does not need a substitution while the second term does need a substitution. So, we'll first need to split up the integral as follows.

$$\int x^4 - 7x^5 \cos\left(2x^6+3\right) dx = \int x^4\, dx - \int 7x^5 \cos\left(2x^6+3\right) dx$$

Step 2
The substitution needed for the second integral is then,

$$u = 2x^6 + 3$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 3
Here is the differential work for the substitution.

$$du = 12x^5 dx \qquad \to \qquad x^5 dx = \tfrac{1}{12}\,du$$

Now, doing the substitution and evaluating the integrals gives,

$$\begin{aligned}\int x^4 - 7x^5 \cos\left(2x^6+3\right) dx &= \int x^4\, dx - \tfrac{7}{12}\int \cos\left(u\right) du = \tfrac{1}{5}x^5 - \tfrac{7}{12}\sin\left(u\right) + c \\ &= \boxed{\tfrac{1}{5}x^5 - \tfrac{7}{12}\sin\left(2x^6+3\right) + c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

---

5. Evaluate $\displaystyle\int \mathbf{e}^z + \frac{4\sin\left(8z\right)}{1+9\cos\left(8z\right)}\,dz$.

Hint : Recall that terms that do not need substitutions should not be in the integral when the substitution is being done. At this point we should know how to "break" integrals up so that we can get the terms that require a substitution into a one integral and those that don't into another integral.

Step 1
Clearly the first term does not need a substitution while the second term does need a substitution. So, we'll first need to split up the integral as follows.

$$\int \mathbf{e}^z + \frac{4\sin\left(8z\right)}{1+9\cos\left(8z\right)}\,dz = \int \mathbf{e}^z\, dz + \int \frac{4\sin\left(8z\right)}{1+9\cos\left(8z\right)}\,dz$$

Step 2
The substitution needed for the second integral is then,

$$u = 1 + 9\cos\left(8z\right)$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 3
Here is the differential work for the substitution.

$$du = -72\sin\left(8z\right)dz \qquad \to \qquad \sin\left(8z\right)dz = -\tfrac{1}{72}\,du$$

Now, doing the substitution and evaluating the integrals gives,

$$\int \mathbf{e}^{z} + \frac{4\sin\left(8z\right)}{1+9\cos\left(8z\right)}\,dz = \int \mathbf{e}^{z}\,dz - \frac{4}{72}\int\frac{1}{u}\,du = \boxed{\mathbf{e}^{z} - \tfrac{1}{18}\ln\left|1+9\cos\left(8z\right)\right| + c}$$

Do not forget to go back to the original variable after evaluating the integral!

---

6. Evaluate $\int 20\mathbf{e}^{2-8w}\sqrt{1+\mathbf{e}^{2-8w}} + 7w^{3} - 6\sqrt[3]{w}\,dw$.

Hint : Recall that terms that do not need substitutions should not be in the integral when the substitution is being done. At this point we should know how to "break" integrals up so that we can get the terms that require a substitution into a one integral and those that don't into another integral.

Step 1
Clearly the first term needs a substitution while the second and third terms don't. So, we'll first need to split up the integral as follows.

$$\int 20\mathbf{e}^{2-8w}\sqrt{1+\mathbf{e}^{2-8w}} + 7w^{3} - 6\sqrt[3]{w}\,dw = \int 20\mathbf{e}^{2-8w}\sqrt{1+\mathbf{e}^{2-8w}}\,dw + \int 7w^{3} - 6\sqrt[3]{w}\,dw$$

Step 2
The substitution needed for the first integral is then,

$$u = 1 + \mathbf{e}^{2-8w}$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 3
Here is the differential work for the substitution.

$$du = -8\mathbf{e}^{2-8w}dw \qquad \to \qquad \mathbf{e}^{2-8w}dw = -\tfrac{1}{8}du$$

Now, doing the substitutions and evaluating the integrals gives,

$$\begin{aligned}\int 20\mathbf{e}^{2-8w}\sqrt{1+\mathbf{e}^{2-8w}} + 7w^3 - 6\sqrt[3]{w}\,dw &= -\tfrac{20}{8}\int u^{\frac{1}{2}}\,du + \int 7w^3 - 6w^{\frac{1}{3}}\,dw \\ &= -\tfrac{5}{3}u^{\frac{3}{2}} + \tfrac{7}{4}w^4 - \tfrac{9}{2}w^{\frac{4}{3}} + c \\ &= \boxed{-\tfrac{5}{3}\left(1+\mathbf{e}^{2-8w}\right)^{\frac{3}{2}} + \tfrac{7}{4}w^4 - \tfrac{9}{2}w^{\frac{4}{3}} + c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

---

7. Evaluate $\int \left(4+7t\right)^3 - 9t\,\sqrt[4]{5t^2+3}\,dt$.

Hint : You can only do one substitution per integral. At this point we should know how to "break" integrals up so that we can get the terms that require different substitutions into different integrals.

Step 1
Clearly each term needs a separate substitution. So, we'll first need to split up the integral as follows.

$$\int \left(4+7t\right)^3 - 9t\,\sqrt[4]{5t^2+3}\,dt = \int \left(4+7t\right)^3 dt - \int 9t\,\sqrt[4]{5t^2+3}\,dt$$

Step 2
The substitutions needed for each integral are then,

$$u = 4+7t \qquad\qquad v = 5t^2+3$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 3
Here is the differential work for each substitution.

$$du = 7\,dt \qquad \to \qquad dt = \tfrac{1}{7}du \qquad\qquad dv = 10t\,dt \qquad \to \qquad t\,dt = \tfrac{1}{10}dv$$

Now, doing the substitutions and evaluating the integrals gives,

$$\int (4+7t)^3\,dt - \int 9t\,\sqrt[4]{5t^2+3}\,dt = \tfrac{1}{7}\int u^3\,du - \tfrac{9}{10}\int v^{\frac{1}{4}}\,dv = \tfrac{1}{28}u^4 - \tfrac{18}{25}v^{\frac{5}{4}} + c$$
$$= \boxed{\tfrac{1}{28}(4+7t)^4 - \tfrac{18}{25}\left(5t^2+3\right)^{\frac{5}{4}} + c}$$

Do not forget to go back to the original variable after evaluating the integral!

---

8. Evaluate $\displaystyle\int \frac{6x-x^2}{x^3-9x^2+8} - \csc^2\left(\frac{3x}{2}\right)dx$.

Hint : You can only do one substitution per integral. At this point we should know how to "break" integrals up so that we can get the terms that require different substitutions into different integrals.

Step 1
Clearly each term needs a separate substitution. So, we'll first need to split up the integral as follows.

$$\int \frac{6x-x^2}{x^3-9x^2+8} - \csc^2\left(\frac{3x}{2}\right)dx = \int \frac{6x-x^2}{x^3-9x^2+8}\,dx - \int \csc^2\left(\frac{3x}{2}\right)dx$$

Step 2
The substitutions needed for each integral are then,

$$u = x^3 - 9x^2 + 8 \qquad\qquad v = \frac{3x}{2}$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 3
Here is the differential work for each substitution.

$$du = \left(3x^2 - 18x\right)dx = -3\left(6x - x^2\right)dx \qquad \to \qquad \left(6x - x^2\right)dx = -\tfrac{1}{3}\,du$$
$$dv = \tfrac{3}{2}\,dx \qquad \to \qquad dx = \tfrac{2}{3}\,dv$$

Now, doing the substitutions and evaluating the integrals gives,

$$\int \frac{6x-x^2}{x^3-9x^2+8} - \csc^2\left(\frac{3x}{2}\right)dx = -\frac{1}{3}\int \frac{1}{u}\,du - \frac{2}{3}\int \csc^2(v)\,dv = -\frac{1}{3}\ln|u| + \frac{2}{3}\cot(v) + c$$
$$= \boxed{-\tfrac{1}{3}\ln\left|x^3 - 9x^2 + 8\right| + \tfrac{2}{3}\cot\left(\tfrac{3x}{2}\right) + c}$$

Do not forget to go back to the original variable after evaluating the integral!

---

9. Evaluate $\int 7\left(3y+2\right)\left(4y+3y^2\right)^3+\sin\left(3+8y\right)dy$.

Hint : You can only do one substitution per integral. At this point we should know how to "break" integrals up so that we can get the terms that require different substitutions into different integrals.

Step 1
Clearly each term needs a separate substitution. So, we'll first need to split up the integral as follows.

$$\int 7\left(3y+2\right)\left(4y+3y^2\right)^3+\sin\left(3+8y\right)dy=\int 7\left(3y+2\right)\left(4y+3y^2\right)^3\,dy+\int\sin\left(3+8y\right)dy$$

Step 2
The substitutions needed for each integral are then,

$$u=4y+3y^2 \qquad\qquad v=3+8y$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 3
Here is the differential work for each substitution.

$$\begin{aligned} du&=\left(4+6y\right)dy=2\left(3y+2\right)dy && \to && \left(3y+2\right)dy=\tfrac{1}{2}du\\ dv&=8\,dy && \to && dy=\tfrac{1}{8}dv \end{aligned}$$

Now, doing the substitutions and evaluating the integrals gives,

$$\begin{aligned}\int 7\left(3y+2\right)\left(4y+3y^2\right)^3+\sin\left(3+8y\right)dy&=\tfrac{7}{2}\int u^3du+\tfrac{1}{8}\int\sin\left(v\right)dv=\tfrac{7}{8}u^4-\tfrac{1}{8}\cos\left(v\right)+c\\ &=\boxed{\tfrac{7}{8}\left(4y+3y^2\right)^4-\tfrac{1}{8}\cos\left(3+8y\right)+c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

---

10. Evaluate $\int\sec^2\left(2t\right)\left[9+7\tan\left(2t\right)-\tan^2\left(2t\right)\right]dt$.

Hint : Don't let this one fool you. This is simply an integral that requires you to use the same substitution more than once.

Step 1
This integral can be a little daunting at first glance. To do it all we need to notice is that the derivative of $\tan\left(x\right)$ is $\sec^2\left(x\right)$ and we can notice that there is a $\sec^2\left(2t\right)$ times the remaining portion of the integrand and that portion only contains constants and tangents.

So, it looks like the substitution is then,

$$u = \tan\left(2t\right)$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 2
Here is the differential work for the substitution.

$$du = 2\sec^2\left(2t\right)dt \qquad \to \qquad \sec^2\left(2t\right)dt = \tfrac{1}{2}du$$

Now, doing the substitution and evaluating the integrals gives,

$$\begin{aligned}\int \sec^2\left(2t\right)\left[9+7\tan\left(2t\right)-\tan^2\left(2t\right)\right]dt &= \tfrac{1}{2}\int 9+7u-u^2\,du = \tfrac{1}{2}\left(9u+\tfrac{7}{2}u^2-\tfrac{1}{3}u^3\right)+c\\ &= \boxed{\tfrac{1}{2}\left(9\tan\left(2t\right)+\tfrac{7}{2}\tan^2\left(2t\right)-\tfrac{1}{3}\tan^3\left(2t\right)\right)+c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

---

11. Evaluate $\displaystyle\int \frac{8-w}{4w^2+9}\,dw$.

Hint : With the integrand written as it is here this problem can't be done.

Step 1
As written we can't do this problem. In order to do this integral we'll need to rewrite the integral as follows.

$$\int \frac{8-w}{4w^2+9}\,dw = \int \frac{8}{4w^2+9}\,dw - \int \frac{w}{4w^2+9}\,dw$$

Step 2
Now, the first integral looks like it might be an inverse tangent (although we'll need to do a rewrite of that integral) and the second looks like it's a logarithm (with a quick substitution).

So, here is the rewrite on the first integral.

$$\int \frac{8-w}{4w^2+9}\,dw = \frac{8}{9}\int \frac{1}{\frac{4}{9}w^2+1}\,dw - \int \frac{w}{4w^2+9}\,dw$$

Step 3

Now we'll need a substitution for each integral. Here are the substitutions we'll need for each integral.

$$u = \tfrac{2}{3}w \quad \left(\text{so } u^2 = \tfrac{4}{9}w^2\right) \qquad\qquad v = 4w^2+9$$

Step 4

Here is the differential work for the substitution.

$$du = \tfrac{2}{3}\,dw \quad \to \quad dw = \tfrac{3}{2}\,du \qquad\qquad dv = 8w\,dw \quad \to \quad w\,dw = \tfrac{1}{8}\,dv$$

Now, doing the substitutions and evaluating the integrals gives,

$$\begin{aligned}\int \frac{8-w}{4w^2+9}\,dw &= \frac{8}{9}\left(\frac{3}{2}\right)\int \frac{1}{u^2+1}\,du - \frac{1}{8}\int \frac{1}{v}\,dv = \tfrac{4}{3}\tan^{-1}\left(u\right) - \tfrac{1}{8}\ln\left|v\right| + c\\ &= \boxed{\tfrac{4}{3}\tan^{-1}\left(\tfrac{2}{3}w\right) - \tfrac{1}{8}\ln\left|4w^2+9\right| + c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

---

12. Evaluate $\displaystyle\int \frac{7x+2}{\sqrt{1-25x^2}}\,dx$.

Hint : With the integrand written as it is here this problem can't be done.

Step 1

As written we can't do this problem. In order to do this integral we'll need to rewrite the integral as follows.

$$\int \frac{7x+2}{\sqrt{1-25x^2}}\,dx = \int \frac{7x}{\sqrt{1-25x^2}}\,dx + \int \frac{2}{\sqrt{1-25x^2}}\,dx$$

Step 2

Now, the second integral looks like it might be an inverse sine (although we'll need to do a rewrite of that integral) and the first looks like a simple substitution will work for us.

So, here is the rewrite on the second integral.

$$\int \frac{7x+2}{\sqrt{1-25x^2}}\,dx = \int \frac{7x}{\sqrt{1-25x^2}}\,dx + 2\int \frac{1}{\sqrt{1-(5x)^2}}\,dx$$

Step 3

Now we'll need a substitution for each integral. Here are the substitutions we'll need for each integral.

$$u = 1-25x^2 \qquad\qquad v = 5x \quad \left(\text{so } v^2 = 25x^2\right)$$

Step 4

Here is the differential work for the substitution.

$$du = -50x\,dx \quad \to \quad x\,dx = -\tfrac{1}{50}\,du \qquad\qquad dv = 5dx \quad \to \quad dx = \tfrac{1}{5}\,dv$$

Now, doing the substitutions and evaluating the integrals gives,

$$\begin{aligned}\int \frac{7x+2}{\sqrt{1-25x^2}}\,dx &= -\frac{7}{50}\int u^{-\frac{1}{2}}\,du + \frac{2}{5}\int \frac{1}{\sqrt{1-v^2}}\,dv = -\frac{7}{25}u^{\frac{1}{2}} + \frac{2}{5}\sin^{-1}(v) + c \\ &= \boxed{-\tfrac{7}{25}\left(1-25x^2\right)^{\frac{1}{2}} + \tfrac{2}{5}\sin^{-1}(5x) + c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

---

13. Evaluate $\int z^7\left(8+3z^4\right)^8 dz$.

Hint : Use the "obvious" substitution and don't forget that the substitution can be used more than once and in different ways.

Step 1

Okay, the "obvious" substitution here is probably,

$$u = 8+3z^4 \quad \to \quad du = 12z^3 dz \quad \to \quad z^3 dz = \tfrac{1}{12}\,du$$

however, that doesn't look like it might work because of the z^7 .

Step 2

Let's do a quick rewrite of the integrand.

$$\int z^7\left(8+3z^4\right)^8 dz = \int z^4 z^3\left(8+3z^4\right)^8 dz = \int z^4\left(8+3z^4\right)^8 z^3 dz$$

Step 3

Now, notice that we can convert all of the *z*'s in the integrand except apparently for the z^4 that is in the front. However, notice from the substitution that we can solve for z^4 to get,

$$z^4 = \tfrac{1}{3}\left(u-8\right)$$

Step 4
With this we can now do the substitution and evaluate the integral.

$$\begin{aligned}\int z^7\left(8+3z^4\right)^8 dz &= \tfrac{1}{12}\int \tfrac{1}{3}\left(u-8\right)u^8 du = \tfrac{1}{36}\int u^9 - 8u^8 du = \tfrac{1}{36}\left(\tfrac{1}{10}u^{10} - \tfrac{8}{9}u^9\right) + c \\ &= \boxed{\tfrac{1}{36}\left(\tfrac{1}{10}\left(8+3z^4\right)^{10} - \tfrac{8}{9}\left(8+3z^4\right)^9\right) + c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

Section 5-5 : Area Problem

1. Estimate the area of the region between $f\left(x\right)=x^3-2x^2+4$ the *x*-axis on $\left[1,4\right]$ using $n=6$ and using,

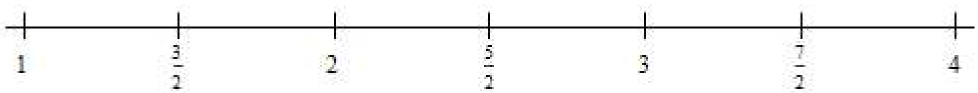
(a) the right end points of the subintervals for the height of the rectangles,
(b) the left end points of the subintervals for the height of the rectangles and,
(c) the midpoints of the subintervals for the height of the rectangles.

(a) the right end points of the subintervals for the height of the rectangles,

The widths of each of the subintervals for this problem are,

$$\Delta x=\frac{4-1}{6}=\frac{1}{2}$$

We don't need to actually graph the function to do this problem. It would probably help to have a number line showing subintervals however. Here is that number line.



In this case we're going to be using right end points of each of these subintervals to determine the height of each of the rectangles.

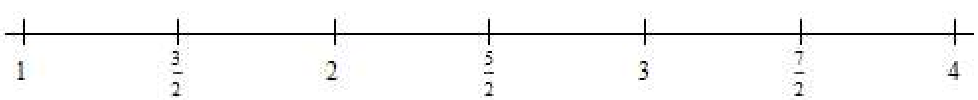
The area between the function and the *x*-axis is then approximately,

$$\begin{aligned}\text{Area} &\approx \tfrac{1}{2}f\left(\tfrac{3}{2}\right)+\tfrac{1}{2}f\left(2\right)+\tfrac{1}{2}f\left(\tfrac{5}{2}\right)+\tfrac{1}{2}f\left(3\right)+\tfrac{1}{2}f\left(\tfrac{7}{2}\right)+\tfrac{1}{2}f\left(4\right)\\ &=\tfrac{1}{2}\left(\tfrac{23}{8}\right)+\tfrac{1}{2}\left(4\right)+\tfrac{1}{2}\left(\tfrac{57}{8}\right)+\tfrac{1}{2}\left(13\right)+\tfrac{1}{2}\left(\tfrac{179}{8}\right)+\tfrac{1}{2}\left(36\right)=\boxed{\tfrac{683}{16}=42.6875}\end{aligned}$$

(b) the left end points of the subintervals for the height of the rectangles and,

As we found in the previous part the widths of each of the subintervals are $\Delta x=\frac{1}{2}$.

Here is a copy of the number line showing the subintervals to help with the problem.



In this case we're going to be using left end points of each of these subintervals to determine the height of each of the rectangles.

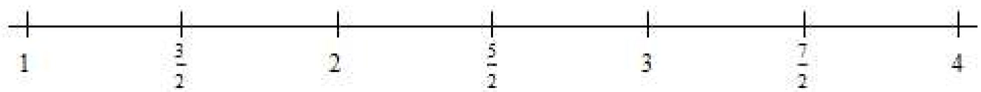
The area between the function and the *x*-axis is then approximately,

$$\begin{aligned}\text{Area} &\approx \tfrac{1}{2} f\left(1\right) + \tfrac{1}{2} f\left(\tfrac{3}{2}\right) + \tfrac{1}{2} f\left(2\right) + \tfrac{1}{2} f\left(\tfrac{5}{2}\right) + \tfrac{1}{2} f\left(3\right) + \tfrac{1}{2} f\left(\tfrac{7}{2}\right) \\ &= \tfrac{1}{2}\left(3\right) + \tfrac{1}{2}\left(\tfrac{23}{8}\right) + \tfrac{1}{2}\left(4\right) + \tfrac{1}{2}\left(\tfrac{57}{8}\right) + \tfrac{1}{2}\left(13\right) + \tfrac{1}{2}\left(\tfrac{179}{8}\right) = \boxed{\tfrac{419}{16} = 26.1875}\end{aligned}$$

(c) the midpoints of the subintervals for the height of the rectangles.

As we found in the first part the widths of each of the subintervals are $\Delta x = \frac{1}{2}$.

Here is a copy of the number line showing the subintervals to help with the problem.



In this case we're going to be using midpoints of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the *x*-axis is then approximately,

$$\begin{aligned}\text{Area} &\approx \tfrac{1}{2} f\left(\tfrac{5}{4}\right) + \tfrac{1}{2} f\left(\tfrac{7}{4}\right) + \tfrac{1}{2} f\left(\tfrac{9}{4}\right) + \tfrac{1}{2} f\left(\tfrac{11}{4}\right) + \tfrac{1}{2} f\left(\tfrac{13}{4}\right) + \tfrac{1}{2} f\left(\tfrac{15}{4}\right) \\ &= \tfrac{1}{2}\left(\tfrac{181}{64}\right) + \tfrac{1}{2}\left(\tfrac{207}{64}\right) + \tfrac{1}{2}\left(\tfrac{337}{64}\right) + \tfrac{1}{2}\left(\tfrac{619}{64}\right) + \tfrac{1}{2}\left(\tfrac{1101}{64}\right) + \tfrac{1}{2}\left(\tfrac{1831}{64}\right) = \boxed{\tfrac{1069}{32} = 33.40625}\end{aligned}$$

---

2. Estimate the area of the region between $g\left(x\right) = 4 - \sqrt{x^2 + 2}$ the *x*-axis on $\left[-1, 3\right]$ using $n = 6$ and using,

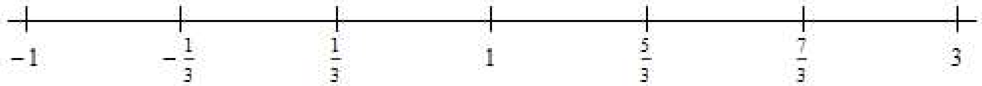
   (a) the right end points of the subintervals for the height of the rectangles,
   (b) the left end points of the subintervals for the height of the rectangles and,
   (c) the midpoints of the subintervals for the height of the rectangles.

(a) the right end points of the subintervals for the height of the rectangles,

The widths of each of the subintervals for this problem are,

$$\Delta x = \frac{3 - \left(-1\right)}{6} = \frac{2}{3}$$

We don't need to actually graph the function to do this problem. It would probably help to have a number line showing subintervals however. Here is that number line.



In this case we're going to be using right end points of each of these subintervals to determine the height of each of the rectangles.

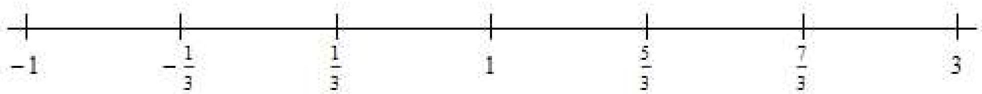
The area between the function and the *x*-axis is then approximately,

$$\begin{aligned}\text{Area} &\approx \tfrac{2}{3} f\left(-\tfrac{1}{3}\right) + \tfrac{2}{3} f\left(\tfrac{1}{3}\right) + \tfrac{2}{3} f\left(1\right) + \tfrac{2}{3} f\left(\tfrac{5}{3}\right) + \tfrac{2}{3} f\left(\tfrac{7}{3}\right) + \tfrac{2}{3} f\left(3\right) \\ &= \tfrac{2}{3}\left(4-\tfrac{\sqrt{19}}{3}\right) + \tfrac{2}{3}\left(4-\tfrac{\sqrt{19}}{3}\right) + \tfrac{2}{3}\left(4-\sqrt{3}\right) + \tfrac{2}{3}\left(4-\tfrac{\sqrt{43}}{3}\right) + \tfrac{2}{3}\left(4-\tfrac{\sqrt{67}}{3}\right) + \tfrac{2}{3}\left(4-\sqrt{11}\right) \\ &= \boxed{7.420752}\end{aligned}$$

(b) the left end points of the subintervals for the height of the rectangles and,

As we found in the previous part the widths of each of the subintervals are $\Delta x = \frac{2}{3}$.

Here is a copy of the number line showing the subintervals to help with the problem.



In this case we're going to be using left end points of each of these subintervals to determine the height of each of the rectangles.

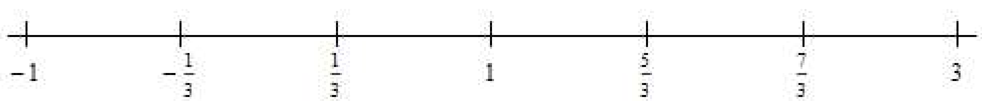
The area between the function and the *x*-axis is then approximately,

$$\begin{aligned}\text{Area} &\approx \tfrac{2}{3} f\left(-1\right) + \tfrac{2}{3} f\left(-\tfrac{1}{3}\right) + \tfrac{2}{3} f\left(\tfrac{1}{3}\right) + \tfrac{2}{3} f\left(1\right) + \tfrac{2}{3} f\left(\tfrac{5}{3}\right) + \tfrac{2}{3} f\left(\tfrac{7}{3}\right) \\ &= \tfrac{2}{3}\left(4-\sqrt{3}\right) + \tfrac{2}{3}\left(4-\tfrac{\sqrt{19}}{3}\right) + \tfrac{2}{3}\left(4-\tfrac{\sqrt{19}}{3}\right) + \tfrac{2}{3}\left(4-\sqrt{3}\right) + \tfrac{2}{3}\left(4-\tfrac{\sqrt{43}}{3}\right) + \tfrac{2}{3}\left(4-\tfrac{\sqrt{67}}{3}\right) \\ &= \boxed{8.477135}\end{aligned}$$

(c) the midpoints of the subintervals for the height of the rectangles.

As we found in the first part the widths of each of the subintervals are $\Delta x = \frac{2}{3}$.

Here is a copy of the number line showing the subintervals to help with the problem.



In this case we're going to be using midpoints of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the *x*-axis is then approximately,

$$\begin{aligned}\text{Area} &\approx \tfrac{2}{3}f\left(-\tfrac{2}{3}\right)+\tfrac{2}{3}f\left(0\right)+\tfrac{2}{3}f\left(\tfrac{2}{3}\right)+\tfrac{2}{3}f\left(\tfrac{4}{3}\right)+\tfrac{2}{3}f\left(2\right)+\tfrac{2}{3}f\left(\tfrac{8}{3}\right)\\ &=\tfrac{2}{3}\left(4-\tfrac{\sqrt{22}}{3}\right)+\tfrac{2}{3}\left(4-\sqrt{2}\right)+\tfrac{2}{3}\left(4-\tfrac{\sqrt{22}}{3}\right)+\tfrac{2}{3}\left(4-\tfrac{\sqrt{34}}{3}\right)+\tfrac{2}{3}\left(4-\sqrt{6}\right)+\tfrac{2}{3}\left(4-\tfrac{\sqrt{82}}{3}\right)\\ &=\boxed{8.031494}\end{aligned}$$

---

3. Estimate the area of the region between $h\left(x\right)=-x\cos\left(\frac{x}{3}\right)$ the *x*-axis on $\left[0,3\right]$ using $n=6$ and using,

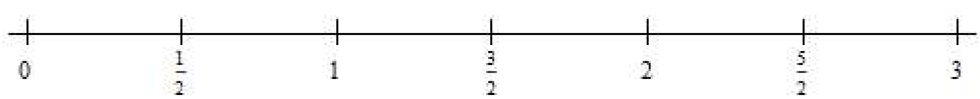
(a) the right end points of the subintervals for the height of the rectangles,
(b) the left end points of the subintervals for the height of the rectangles and,
(c) the midpoints of the subintervals for the height of the rectangles.

(a) the right end points of the subintervals for the height of the rectangles,

The widths of each of the subintervals for this problem are,

$$\Delta x=\frac{3-0}{6}=\frac{1}{2}$$

We don't need to actually graph the function to do this problem. It would probably help to have a number line showing subintervals however. Here is that number line.



In this case we're going to be using right end points of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the *x*-axis is then approximately,

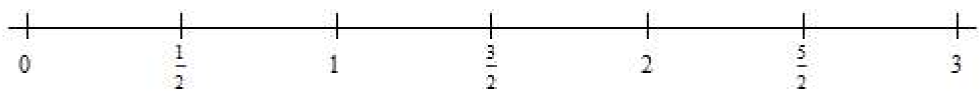
$$\begin{aligned}\text{Area} &\approx \tfrac{1}{2}f\left(\tfrac{1}{2}\right)+\tfrac{1}{2}f\left(1\right)+\tfrac{1}{2}f\left(\tfrac{3}{2}\right)+\tfrac{1}{2}f\left(2\right)+\tfrac{1}{2}f\left(\tfrac{5}{2}\right)+\tfrac{1}{2}f\left(3\right)\\ &=\tfrac{1}{2}\left(-\tfrac{1}{2}\cos\left(\tfrac{1}{6}\right)\right)+\tfrac{1}{2}\left(-\cos\left(\tfrac{1}{3}\right)\right)+\tfrac{1}{2}\left(-\tfrac{3}{2}\cos\left(\tfrac{1}{2}\right)\right)+\tfrac{1}{2}\left(-2\cos\left(\tfrac{2}{3}\right)\right)\\ &\qquad+\tfrac{1}{2}\left(-\tfrac{5}{2}\cos\left(\tfrac{5}{6}\right)\right)+\tfrac{1}{2}\left(-3\cos\left(1\right)\right)\\ &=\boxed{-3.814057}\end{aligned}$$

Do not get excited about the negative area here. As we discussed in this section this just means that the graph, in this case, is below the *x*-axis as you could verify if you'd like to.

(b) the left end points of the subintervals for the height of the rectangles and,

As we found in the previous part the widths of each of the subintervals are $\Delta x = \frac{2}{3}$.

Here is a copy of the number line showing the subintervals to help with the problem.



In this case we're going to be using left end points of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the *x*-axis is then approximately,

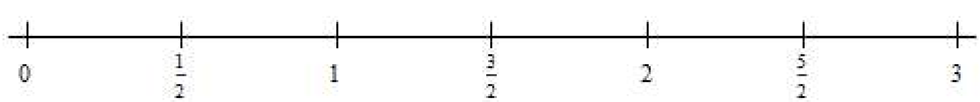
$$\begin{aligned}\text{Area} &\approx \tfrac{1}{2} f\left(0\right) + \tfrac{1}{2} f\left(\tfrac{1}{2}\right) + \tfrac{1}{2} f\left(1\right) + \tfrac{1}{2} f\left(\tfrac{3}{2}\right) + \tfrac{1}{2} f\left(2\right) + \tfrac{1}{2} f\left(\tfrac{5}{2}\right)\\ &= +\tfrac{1}{2}\left(0\right) + \tfrac{1}{2}\left(-\tfrac{1}{2}\cos\left(\tfrac{1}{6}\right)\right) + \tfrac{1}{2}\left(-\cos\left(\tfrac{1}{3}\right)\right) + \tfrac{1}{2}\left(-\tfrac{3}{2}\cos\left(\tfrac{1}{2}\right)\right) + \tfrac{1}{2}\left(-2\cos\left(\tfrac{2}{3}\right)\right)\\ &\qquad + \tfrac{1}{2}\left(-\tfrac{5}{2}\cos\left(\tfrac{5}{6}\right)\right)\\ &= \boxed{-3.003604}\end{aligned}$$

Do not get excited about the negative area here. As we discussed in this section this just means that the graph, in this case, is below the *x*-axis as you could verify if you'd like to.

(c) the midpoints of the subintervals for the height of the rectangles.

As we found in the first part the widths of each of the subintervals are $\Delta x = \frac{2}{3}$.

Here is a copy of the number line showing the subintervals to help with the problem.



In this case we're going to be using midpoints of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the *x*-axis is then approximately,

$$\begin{aligned}\text{Area} &\approx \tfrac{1}{2} f\left(\tfrac{1}{4}\right) + \tfrac{1}{2} f\left(\tfrac{3}{4}\right) + \tfrac{1}{2} f\left(\tfrac{5}{4}\right) + \tfrac{1}{2} f\left(\tfrac{7}{4}\right) + \tfrac{1}{2} f\left(\tfrac{9}{4}\right) + \tfrac{1}{2} f\left(\tfrac{11}{4}\right) \\ &= \tfrac{1}{2}\left(-\tfrac{1}{4}\cos\left(\tfrac{1}{12}\right)\right) + \tfrac{1}{2}\left(-\tfrac{3}{4}\cos\left(\tfrac{1}{4}\right)\right) + \tfrac{1}{2}\left(-\tfrac{5}{4}\cos\left(\tfrac{5}{12}\right)\right) + \tfrac{1}{2}\left(-\tfrac{7}{4}\cos\left(\tfrac{7}{12}\right)\right) \\ &\qquad + \tfrac{1}{2}\left(-\tfrac{9}{4}\cos\left(\tfrac{3}{4}\right)\right) + \tfrac{1}{2}\left(-\tfrac{11}{4}\cos\left(\tfrac{11}{12}\right)\right) \\ &= \boxed{-3.449532}\end{aligned}$$

Do not get excited about the negative area here. As we discussed in this section this just means that the graph, in this case, is below the *x*-axis as you could verify if you'd like to.

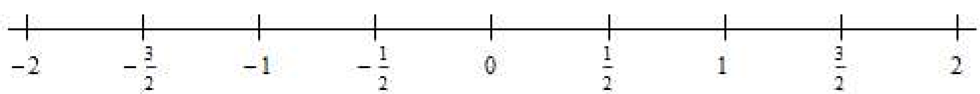
---

4. Estimate the net area between $f\left(x\right) = 8x^2 - x^5 - 12$ and the *x*-axis on $\left[-2, 2\right]$ using $n = 8$ and the midpoints of the subintervals for the height of the rectangles. Without looking at a graph of the function on the interval does it appear that more of the area is above or below the *x*-axis?

Step 1
First let's estimate the area between the function and the *x*-axis on the interval. The widths of each of the subintervals for this problem are,

$$\Delta x = \frac{2 - \left(-2\right)}{8} = \frac{1}{2}$$

We don't need to actually graph the function to do this problem. It would probably help to have a number line showing subintervals however. Here is that number line.



Now, we'll be using midpoints of each of these subintervals to determine the height of each of the rectangles.

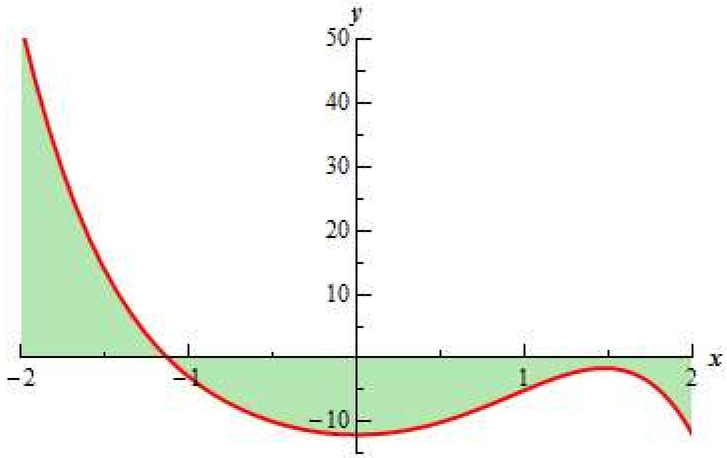
The area between the function and the *x*-axis is then approximately,

$$\text{Area} \approx \tfrac{1}{2} f\left(-\tfrac{7}{4}\right) + \tfrac{1}{2} f\left(-\tfrac{5}{4}\right) + \tfrac{1}{2} f\left(-\tfrac{3}{4}\right) + \tfrac{1}{2} f\left(-\tfrac{1}{4}\right) + \tfrac{1}{2} f\left(\tfrac{1}{4}\right) + \tfrac{1}{2} f\left(\tfrac{3}{4}\right) + \tfrac{1}{2} f\left(\tfrac{5}{4}\right) + \tfrac{1}{2} f\left(\tfrac{7}{4}\right) = \boxed{-6}$$

We'll leave it to you to check all the function evaluations. They get a little messy, but after all the arithmetic is done we get a net area of -6.

Step 2
Now, as we (hopefully) recall from the discussion in this section area above the *x*-axis is positive and area below the *x*-axis is negative. In this case we have estimated that the net area is -6 and so, assuming that our estimate is accurate, it looks like we should have more area is below the *x*-axis as above it.

Graph
For reference purposes here is the graph of the function with the area shaded in and as we can see it does appear that there is slightly more area below as above the *x*-axis.



Section 5-6 : Definition of the Definite Integral

1. Use the definition of the definite integral to evaluate the integral. Use the right end point of each interval for x_i^*.

$$\int_1^4 2x+3\,dx$$

Step 1
The width of each subinterval will be,

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

The subintervals for the interval $[1,4]$ are then,

$$\left[1,1+\frac{3}{n}\right],\left[1+\frac{3}{n},1+\frac{6}{n}\right],\left[1+\frac{6}{n},1+\frac{9}{n}\right],\ldots,\left[1+\frac{3(i-1)}{n},1+\frac{3i}{n}\right],\ldots,\left[1+\frac{3(n-1)}{n},4\right]$$

From this it looks like the right end point, and hence x_i^*, of the general subinterval is,

$$x_i^* = 1+\frac{3i}{n}$$

Step 2
The summation in the definition of the definite integral is then,

$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \sum_{i=1}^{n}\left[2\left(1+\frac{3i}{n}\right)+3\right]\left[\frac{3}{n}\right] = \sum_{i=1}^{n}\left[\frac{15}{n}+\frac{18i}{n^2}\right]$$

Step 3
Now we need to use the formulas from the **Summation Notation** section in the Extras chapter to "evaluate" the summation.

$$\begin{aligned}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x &= \sum_{i=1}^{n}\frac{15}{n}+\sum_{i=1}^{n}\frac{18i}{n^2} = \frac{1}{n}\sum_{i=1}^{n}15+\frac{18}{n^2}\sum_{i=1}^{n} i \\ &= \frac{1}{n}(15n)+\frac{18}{n^2}\left(\frac{n(n+1)}{2}\right) = 15+\frac{9n+9}{n}\end{aligned}$$

Step 4
Finally, we can use the definition of the definite integral to determine the value of the integral.

$$\int_1^4 2x+3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \lim_{n\to\infty}\left[15+\frac{9n+9}{n}\right] = \lim_{n\to\infty}\left[24+\frac{9}{n}\right] = \boxed{24}$$

2. Use the definition of the definite integral to evaluate the integral. Use the right end point of each interval for x_i^*.

$$\int_0^1 6x\left(x-1\right)dx$$

Step 1
The width of each subinterval will be,

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

The subintervals for the interval $\left[0,1\right]$ are then,

$$\left[0,\frac{1}{n}\right],\left[\frac{1}{n},\frac{2}{n}\right],\left[\frac{2}{n},\frac{3}{n}\right],\ldots,\left[\frac{i-1}{n},\frac{i}{n}\right],\ldots,\left[\frac{n-1}{n},1\right]$$

From this it looks like the right end point, and hence x_i^*, of the general subinterval is,

$$x_i^* = \frac{i}{n}$$

Step 2
The summation in the definition of the definite integral is then,

$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \sum_{i=1}^{n}\left[\left(\frac{6i}{n}\right)\left(\frac{i}{n}-1\right)\right]\left[\frac{1}{n}\right] = \sum_{i=1}^{n}\left[\frac{6i^2}{n^3}-\frac{6i}{n^2}\right]$$

Step 3
Now we need to use the formulas from the **Summation Notation** section in the Extras chapter to "evaluate" the summation.

$$\begin{aligned}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x &= \sum_{i=1}^{n}\left[\frac{6i^2}{n^3}\right]-\sum_{i=1}^{n}\left[\frac{6i}{n^2}\right] = \frac{6}{n^3}\sum_{i=1}^{n} i^2 - \frac{6}{n^2}\sum_{i=1}^{n} i \\ &= \frac{6}{n^3}\left(\frac{n\left(n+1\right)\left(2n+1\right)}{6}\right)-\frac{6}{n^2}\left(\frac{n\left(n+1\right)}{2}\right) = \frac{2n^2+3n+1}{n^2}-\frac{3n+3}{n}\end{aligned}$$

Step 4
Finally, we can use the definition of the definite integral to determine the value of the integral.

$$\int_0^1 6x\left(x-1\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \lim_{n\to\infty}\left[\frac{2n^2+3n+1}{n^2}-\frac{3n+3}{n}\right] = 2-3 = \boxed{-1}$$

3. Evaluate : $\displaystyle\int_4^4 \frac{\cos\left(\mathbf{e}^{3x} + x^2\right)}{x^4 + 1}\,dx$

Solution

There really isn't much to this problem other than use **Property 2** from the notes on this section.

$$\int_4^4 \frac{\cos\left(\mathbf{e}^{3x} + x^2\right)}{x^4 + 1}\,dx = \boxed{0}$$

---

4. Determine the value of $\int_{11}^{6} 9f(x)\,dx$ given that $\int_{6}^{11} f(x)\,dx = -7$.

Solution

There really isn't much to this problem other than use the **properties** from the notes of this section until we get the given interval at which point we use the given value.

$$\begin{aligned} \int_{11}^{6} 9f(x)\,dx &= 9\int_{11}^{6} f(x)\,dx && \text{Property 3} \\ &= -9\int_{6}^{11} f(x)\,dx && \text{Property 1} \\ &= -9(-7) = \boxed{63} \end{aligned}$$

---

5. Determine the value of $\int_{6}^{11} 6g(x) - 10f(x)\,dx$ given that $\int_{6}^{11} f(x)\,dx = -7$ and $\int_{6}^{11} g(x)\,dx = 24$.

Solution

There really isn't much to this problem other than use the **properties** from the notes of this section until we get the given intervals at which point we use the given values.

$$\begin{aligned} \int_{6}^{11} 6g(x) - 10f(x)\,dx &= \int_{6}^{11} 6g(x)\,dx - \int_{6}^{11} 10f(x)\,dx && \text{Property 4} \\ &= 6\int_{6}^{11} g(x)\,dx - 10\int_{6}^{11} f(x)\,dx && \text{Property 3} \\ &= 6(24) - 10(-7) = \boxed{214} \end{aligned}$$

---

6. Determine the value of $\int_{2}^{9} f(x)\,dx$ given that $\int_{5}^{2} f(x)\,dx = 3$ and $\int_{5}^{9} f(x)\,dx = 8$.

Step 1
First we need to use **Property 5** from the notes of this section to break up the integral into two integrals that use the same limits as the integrals given in the problem statement.

Note that we won't worry about whether the limits are in correct place at this point.

$$\int_{2}^{9} f(x)\,dx = \int_{2}^{5} f(x)\,dx + \int_{5}^{9} f(x)\,dx$$

Step 2
Finally, all we need to do is use **Property 1** from the notes of this section to interchange the limits on the first integral so they match up with the limits on the given integral. We can then use the given values to determine the value of the integral.

$$\int_{2}^{9} f(x)\,dx = -\int_{5}^{2} f(x)\,dx + \int_{5}^{9} f(x)\,dx = -(3) + 8 = \boxed{5}$$

---

7. Determine the value of $\int_{-4}^{20} f(x)\,dx$ given that $\int_{-4}^{0} f(x)\,dx = -2$, $\int_{31}^{0} f(x)\,dx = 19$ and $\int_{20}^{31} f(x)\,dx = -21$.

Step 1
First we need to use **Property 5** from the notes of this section to break up the integral into three integrals that use the same limits as the integrals given in the problem statement.

Note that we won't worry about whether the limits are in correct place at this point.

$$\int_{-4}^{20} f(x)\,dx = \int_{-4}^{0} f(x)\,dx + \int_{0}^{31} f(x)\,dx + \int_{31}^{20} f(x)\,dx$$

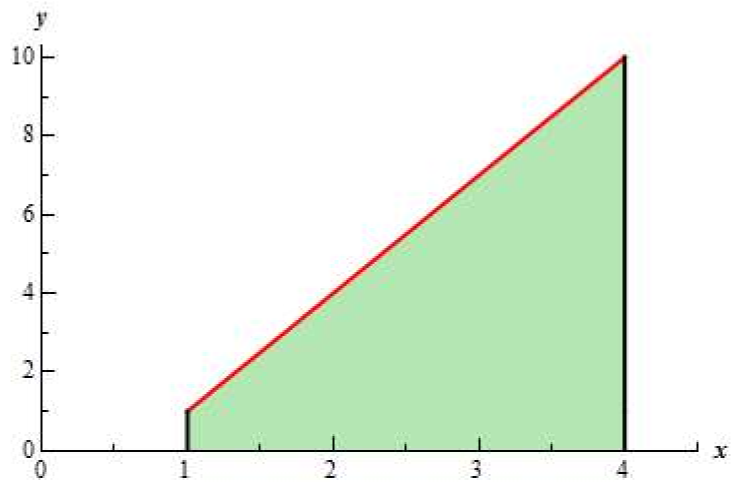
Step 2
Finally, all we need to do is use **Property 1** from the notes of this section to interchange the limits on the second and third integrals so they match up with the limits on the given integral. We can then use the given values to determine the value of the integral.

$$\int_{-4}^{20} f(x)\,dx = \int_{-4}^{0} f(x)\,dx - \int_{31}^{0} f(x)\,dx - \int_{20}^{31} f(x)\,dx = -2 - (19) - (-21) = \boxed{0}$$

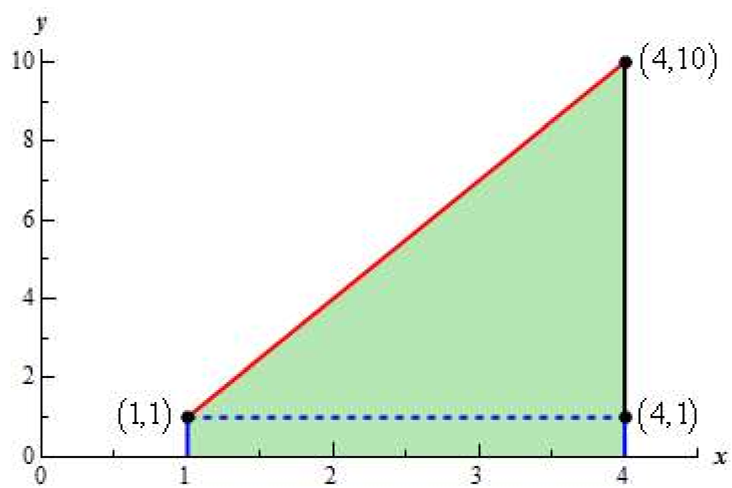
---

8. For $\int_1^4 3x-2\,dx$ sketch the graph of the integrand and use the area interpretation of the definite integral to determine the value of the integral.

Step 1
Here is the graph of the integrand, $f(x)=3x-2$, on the interval $[1,4]$.



Step 2
Now, we know that the integral is simply the area between the line and the *x*-axis and so we should be able to use basic area formulas to help us determine the value of the integral. Here is a "modified" graph that will help with this.



From this sketch we can see that we can think of this area as a rectangle with width 3 and height 1 and a triangle with base 3 and height 9. The value of the integral will then be the sum of the areas of the rectangle and the triangle.

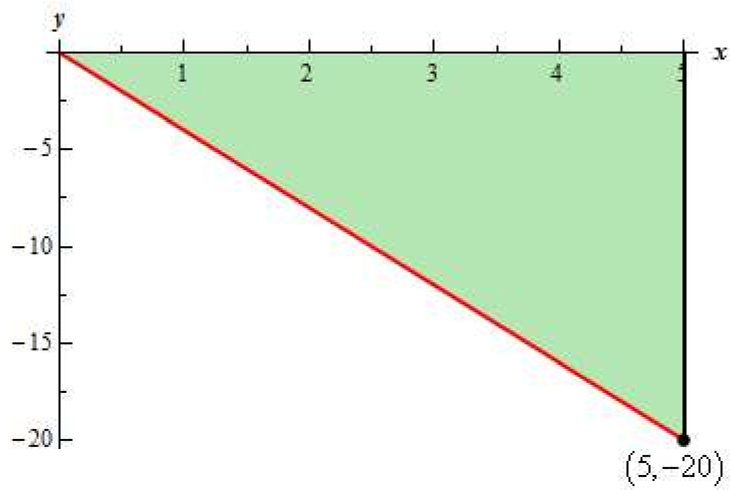
Here is the value of the integral,

$$\int_1^4 3x - 2\,dx = (3)(1) + \tfrac{1}{2}(3)(9) = \boxed{\tfrac{33}{2}}$$

---

9. For $\int_0^5 -4x\,dx$ sketch the graph of the integrand and use the area interpretation of the definite integral to determine the value of the integral.

Step 1
Here is the graph of the integrand, $f(x) = -4x$ on the interval $[0,5]$.



Step 2
Now, we know that the integral is simply the area between the line and the *x*-axis and so we should be able to use basic area formulas to help us determine the value of the integral.

In this case we can see the area is clearly a triangle with base 5 and height 20. However, we need to be a little careful here and recall that area that is below the *x*-axis is considered to be negative area and so we'll need to keep that in mind when we do the area computation.

Here is the value of the integral,

$$\int_0^5 -4x\,dx = -\tfrac{1}{2}(5)(20) = \boxed{-50}$$

---

10. Differentiate the following integral with respect to *x*.

$$\int_4^x 9\cos^2\left(t^2-6t+1\right)dt$$

Solution
This is nothing more than a quick application of the Fundamental Theorem of Calculus, Part I.

The derivative is,

$$\frac{d}{dx}\left[\int_4^x 9\cos^2\left(t^2-6t+1\right)dt\right]=\boxed{9\cos^2\left(x^2-6x+1\right)}$$

---

11. Differentiate the following integral with respect to *x*.

$$\int_7^{\sin(6x)}\sqrt{t^2+4}\,dt$$

Solution
This is nothing more than a quick application of the Fundamental Theorem of Calculus, Part I.

Note however, that because the upper limit is not just *x* we'll need to use the Chain Rule, with the "inner function" as $\sin(6x)$.

The derivative is,

$$\frac{d}{dx}\left[\int_7^{\sin(6x)}\sqrt{t^2+4}\,dt\right]=\boxed{6\cos(6x)\sqrt{\sin^2(6x)+4}}$$

---

12. Differentiate the following integral with respect to *x*.

$$\int_{3x^2}^{-1}\frac{\mathbf{e}^t-1}{t}\,dt$$

Solution
This is nothing more than a quick application of the Fundamental Theorem of Calculus, Part I.

Note however, that we'll need to interchange the limits to get the lower limit to a number and the *x*'s in the upper limit as required by the theorem. Also, note that because the upper limit is not just *x* we'll need to use the Chain Rule, with the "inner function" as $3x^2$.

The derivative is,

$$\frac{d}{dx}\left[\int_{3x^2}^{-1}\frac{\mathbf{e}^t-1}{t}\,dt\right]=\frac{d}{dx}\left[-\int_{-1}^{3x^2}\frac{\mathbf{e}^t-1}{t}\,dt\right]=-(6x)\frac{\mathbf{e}^{3x^2}-1}{3x^2}=\boxed{\frac{2-2\mathbf{e}^{3x^2}}{x}}$$

Section 5-7 : Computing Definite Integrals

1. Evaluate each of the following integrals.

a. $\displaystyle\int \cos(x) - \frac{3}{x^5}\,dx$

b. $\displaystyle\int_{-3}^{4} \cos(x) - \frac{3}{x^5}\,dx$

c. $\displaystyle\int_{1}^{4} \cos(x) - \frac{3}{x^5}\,dx$

a. $\displaystyle\int \cos(x) - \frac{3}{x^5}\,dx$

This is just an indefinite integral and by this point we should be comfortable doing them so here is the answer to this part.

$$\int \cos(x) - \frac{3}{x^5}\,dx = \int \cos(x) - 3x^{-5}\,dx = \sin(x) + \tfrac{3}{4}x^{-4} + c = \boxed{\sin(x) + \frac{3}{4x^4} + c}$$

Don’t forget to add on the “+c” since we are doing an indefinite integral!

b. $\displaystyle\int_{-3}^{4} \cos(x) - \frac{3}{x^5}\,dx$

Recall that in order to do a definite integral the integrand (*i.e.* the function we are integrating) must be continuous on the interval over which we are integrating, $\left[-3,4\right]$ in this case.

We can clearly see that the second term will have division by zero at $x = 0$ and $x = 0$ is in the interval over which we are integrating and so this function is not continuous on the interval over which we are integrating.

Therefore, this integral cannot be done.

c. $\displaystyle\int_{1}^{4} \cos(x) - \frac{3}{x^5}\,dx$

Now, the function still has a division by zero problem in the second term at $x = 0$. However, unlike the previous part $x = 0$ does not fall in the interval over which we are integrating, $\left[1,4\right]$ in this case.

This integral can therefore be done. Here is the work for this integral.

$$\int_1^4 \cos(x) - \frac{3}{x^5}\,dx = \int_1^4 \cos(x) - 3x^{-5}\,dx = \left.\left(\sin(x) + \frac{3}{4x^4}\right)\right|_1^4$$

$$= \sin(4) + \frac{3}{4(4^4)} - \left(\sin(1) - \frac{3}{4(1^4)}\right)$$

$$= \sin(4) + \frac{3}{1024} - \left(\sin(1) - \frac{3}{4}\right) = \boxed{\sin(4) - \sin(1) - \frac{765}{1024}}$$

---

2. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_1^6 12x^3 - 9x^2 + 2\,dx$$

Step 1
First we need to integrate the function.

$$\int_1^6 12x^3 - 9x^2 + 2\,dx = \left.\left(3x^4 - 3x^3 + 2x\right)\right|_1^6$$

Recall that we don't need to add the "+c" in the definite integral case as it will just cancel in the next step.

Step 2
The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_1^6 12x^3 - 9x^2 + 2\,dx = 3252 - 2 = \boxed{3250}$$

---

3. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-2}^1 5z^2 - 7z + 3\,dz$$

Step 1
First we need to integrate the function.

$$\int_{-2}^{1} 5z^2 - 7z + 3\,dz = \left(\tfrac{5}{3}z^3 - \tfrac{7}{2}z^2 + 3z\right)\Big|_{-2}^{1}$$

Recall that we don't need to add the "+c" in the definite integral case as it will just cancel in the next step.

Step 2
The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{-2}^{1} 5z^2 - 7z + 3\,dz = \tfrac{7}{6} - \left(-\tfrac{100}{3}\right) = \boxed{\tfrac{69}{2}}$$

---

4. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{3}^{0} 15w^4 - 13w^2 + w\,dw$$

Step 1
First, do not get excited about the fact that the lower limit of integration is a larger number than the upper limit of integration. The problem works in exactly the same way.

So, we need to integrate the function.

$$\int_{3}^{0} 15w^4 - 13w^2 + w\,dw = \left(3w^5 - \tfrac{13}{3}w^3 + \tfrac{1}{2}w^2\right)\Big|_{3}^{0}$$

Recall that we don't need to add the "+c" in the definite integral case as it will just cancel in the next step.

Step 2
The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{3}^{0} 15w^4 - 13w^2 + w\,dw = 0 - \tfrac{1233}{2} = \boxed{-\tfrac{1233}{2}}$$

---

5. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{1}^{4} \frac{8}{\sqrt{t}} - 12\sqrt{t^3}\,dt$$

Step 1
First we need to integrate the function.

$$\int_{1}^{4} \frac{8}{\sqrt{t}} - 12\sqrt{t^3}\,dt = \int_{1}^{4} 8t^{-\frac{1}{2}} - 12t^{\frac{3}{2}}\,dt = \left(16t^{\frac{1}{2}} - \tfrac{24}{5}t^{\frac{5}{2}}\right)\Big|_{1}^{4}$$

Recall that we don't need to add the "+c" in the definite integral case as it will just cancel in the next step.

Step 2
The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{1}^{4} \frac{8}{\sqrt{t}} - 12\sqrt{t^3}\,dt = -\tfrac{608}{5} - \tfrac{56}{5} = \boxed{-\tfrac{664}{5}}$$

---

6. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{1}^{2} \frac{1}{7z} + \frac{\sqrt[3]{z^2}}{4} - \frac{1}{2z^3}\,dz$$

Step 1
First we need to integrate the function.

$$\int_{1}^{2} \frac{1}{7z} + \frac{\sqrt[3]{z^2}}{4} - \frac{1}{2z^3}\,dz = \int_{1}^{2} \frac{1}{7}\frac{1}{z} + \frac{1}{4}z^{\frac{2}{3}} - \frac{1}{2}z^{-3}\,dz = \left(\tfrac{1}{7}\ln|z| + \tfrac{3}{20}z^{\frac{5}{3}} + \tfrac{1}{4}z^{-2}\right)\Big|_{1}^{2}$$

Recall that we don't need to add the "+c" in the definite integral case as it will just cancel in the next step.

Step 2
The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_1^2 \frac{1}{7z} + \frac{\sqrt[3]{z^2}}{4} - \frac{1}{2z^3}\,dz = \left(\tfrac{1}{7}\ln\left(2\right) + \tfrac{3}{20}\left(2^{\frac{5}{3}}\right) + \tfrac{1}{16}\right) - \left(\tfrac{1}{7}\ln\left(1\right) + \tfrac{2}{5}\right) = \boxed{\tfrac{1}{7}\ln\left(2\right) + \tfrac{3}{20}\left(2^{\frac{5}{3}}\right) - \tfrac{27}{80}}$$

Don't forget that $\ln\left(1\right) = 0$! Also, don't get excited about "messy" answers like this. They happen on occasion.

---

7. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-2}^{4} x^6 - x^4 + \frac{1}{x^2}\,dx$$

Solution

In this case note that the third term will have division by zero at $x = 0$ and this is in the interval we are integrating over, $\left[-2, 4\right]$ and hence is not continuous on this interval.

Therefore, this integral cannot be done.

---

8. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-4}^{-1} x^2\left(3 - 4x\right)dx$$

Step 1
In this case we'll first need to multiply out the integrand before we actually do the integration. Doing that integrating the function gives,

$$\int_{-4}^{-1} x^2 \left(3-4x\right) dx = \int_{-4}^{-1} 3x^2 - 4x^3 \, dx = \left(x^3 - x^4\right)\Big|_{-4}^{-1}$$

Recall that we don't need to add the "+c" in the definite integral case as it will just cancel in the next step.

Step 2
The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{-4}^{-1} x^2 \left(3-4x\right) dx = -2 - \left(-320\right) = \boxed{318}$$

---

9. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{2}^{1} \frac{2y^3 - 6y^2}{y^2} \, dy$$

Step 1
In this case we'll first need to simplify the integrand to remove the quotient before we actually do the integration. Doing that integrating the function gives,

$$\int_{2}^{1} \frac{2y^3 - 6y^2}{y^2} \, dy = \int_{2}^{1} 2y - 6 \, dy = \left(y^2 - 6y\right)\Big|_{2}^{1}$$

Do not get excited about the fact that the lower limit of integration is larger than the upper limit of integration. This will happen on occasion and the integral works in exactly the same manner as we've been doing them.

Also, recall that we don't need to add the "+c" in the definite integral case as it will just cancel in the next step.

Step 2
The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{2}^{1} \frac{2y^3 - 6y^2}{y^2}\,dy = -5 - (-8) = \boxed{3}$$

---

10. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{0}^{\frac{\pi}{2}} 7\sin(t) - 2\cos(t)\,dt$$

Step 1
First we need to integrate the function.

$$\int_{0}^{\frac{\pi}{2}} 7\sin(t) - 2\cos(t)\,dt = \left(-7\cos(t) - 2\sin(t)\right)\Big|_{0}^{\frac{\pi}{2}}$$

Recall that we don't need to add the "+c" in the definite integral case as it will just cancel in the next step.

Step 2
The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{0}^{\frac{\pi}{2}} 7\sin(t) - 2\cos(t)\,dt = -2 - (-7) = \boxed{5}$$

---

11. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{0}^{\pi} \sec(z)\tan(z) - 1\,dz$$

Solution

Be careful with this integral. Recall that,

$$\sec(z) = \frac{1}{\cos(z)} \qquad\qquad \tan(z) = \frac{\sin(z)}{\cos(z)}$$

Also recall that $\cos\left(\frac{\pi}{2}\right) = 0$ and that $x = \frac{\pi}{2}$ is in the interval we are integrating over, $\left[0, \pi\right]$ and hence is not continuous on this interval.

Therefore, this integral cannot be done.

It is often easy to overlook these kinds of division by zero problems in integrands when the integrand is not explicitly written as a rational expression. So, be careful and don't forget that division by zero can sometimes be "hidden" in the integrand!

---

12. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sec^2\left(w\right) - 8\csc\left(w\right)\cot\left(w\right)dw$$

Step 1
First notice that even though we do have some "hidden" rational expression here (in the definitions of the trig functions) neither cosine nor sine is zero in the interval we are integrating over and so both terms are continuous over the interval.

Therefore all we need to do integrate the function.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sec^2\left(w\right) - 8\csc\left(w\right)\cot\left(w\right)dw = \left(2\tan\left(w\right) + 8\csc\left(w\right)\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

Recall that we don't need to add the "+c" in the definite integral case as it will just cancel in the next step.

Step 2
The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sec^2\left(w\right) - 8\csc\left(w\right)\cot\left(w\right)dw = \left(\frac{16}{\sqrt{3}} + 2\sqrt{3}\right) - \left(16 + \frac{2}{\sqrt{3}}\right) = \boxed{\frac{14}{\sqrt{3}} + 2\sqrt{3} - 16}$$

---

13. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{0}^{2} \mathbf{e}^{x} + \frac{1}{x^{2}+1}\,dx$$

Step 1
First we need to integrate the function.

$$\int_{0}^{2} \mathbf{e}^{x} + \frac{1}{x^{2}+1}\,dx = \left(\mathbf{e}^{x} + \tan^{-1}\left(x\right)\right)\Big|_{0}^{2}$$

Recall that we don't need to add the "+c" in the definite integral case as it will just cancel in the next step.

Step 2
The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{0}^{2} \mathbf{e}^{x} + \frac{1}{x^{2}+1}\,dx = \left(\mathbf{e}^{2} + \tan^{-1}\left(2\right)\right) - \left(\mathbf{e}^{0} + \tan^{-1}\left(0\right)\right) = \boxed{\mathbf{e}^{2} + \tan^{-1}\left(2\right) - 1}$$

Note that $\tan^{-1}\left(0\right) = 0$ but $\tan^{-1}\left(2\right)$ doesn't have a "nice" answer and so was left as is.

---

14. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-5}^{-2} 7\mathbf{e}^{y} + \frac{2}{y}\,dy$$

Step 1
First we need to integrate the function.

$$\int_{-5}^{-2} 7\mathbf{e}^{y} + \frac{2}{y}\,dy = \left(7\mathbf{e}^{y} + 2\ln\left|y\right|\right)\Big|_{-5}^{-2}$$

Recall that we don't need to add the "+c" in the definite integral case as it will just cancel in the next step.

Step 2

The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{-5}^{-2} 7\mathbf{e}^{y} + \frac{2}{y}\,dy = \left(7\mathbf{e}^{-2} + 2\ln\left|-2\right|\right) - \left(7\mathbf{e}^{-5} + 2\ln\left|-5\right|\right) = \boxed{7\left(\mathbf{e}^{-2} - \mathbf{e}^{-5}\right) + 2\left(\ln\left(2\right) - \ln\left(5\right)\right)}$$

---

15. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{0}^{4} f\left(t\right) dt \quad \text{where} \quad f\left(t\right) = \begin{cases} 2t & t > 1 \\ 1 - 3t^{2} & t \le 1 \end{cases}$$

Hint : Recall that integrals we can always "break up" an integral as follows,

$$\int_{a}^{b} f\left(x\right) dx = \int_{a}^{c} f\left(x\right) dx + \int_{c}^{b} f\left(x\right) dx$$

See if you can find a good choice for "*c*" that will make this integral doable.

Step 1

This integral can't be done as a single integral give the obvious change of the function at $t = 1$ which is in the interval over which we are integrating. However, recall that we can always break up an integral at any point and $t = 1$ seems to be a good point to do this.

Breaking up the integral at $t = 1$ gives,

$$\int_{0}^{4} f\left(t\right) dt = \int_{0}^{1} f\left(t\right) dt + \int_{1}^{4} f\left(t\right) dt$$

So, in the first integral we have $0 \le t \le 1$ and so we can use $f\left(t\right) = 1 - 3t^{2}$ in the first integral.

Likewise, in the second integral we have $1 \le t \le 4$ and so we can use $f\left(t\right) = 2t$ in the second integral.

Making these function substitutions gives,

$$\int_{0}^{4} f\left(t\right) dt = \int_{0}^{1} 1 - 3t^{2}\,dt + \int_{1}^{4} 2t\,dt$$

Step 2

All we need to do at this point is evaluate each integral. Here is that work.

$$\int_0^4 f(t)\,dt = \int_0^1 1-3t^2\,dt + \int_1^4 2t\,dt = \left(t-t^3\right)\Big|_0^1 + t^2\Big|_1^4 = [0-0]+[16-1] = \boxed{15}$$

---

16. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-6}^{1} g(z)\,dz \quad \text{where} \quad g(z) = \begin{cases} 2-z & z>-2 \\ 4\mathbf{e}^{z} & z\le -2 \end{cases}$$

Hint : Recall that integrals we can always "break up" an integral as follows,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

See if you can find a good choice for "*c*" that will make this integral doable.

Step 1
This integral can't be done as a single integral give the obvious change of the function at $z=-2$ which is in the interval over which we are integrating. However, recall that we can always break up an integral at any point and $z=-2$ seems to be a good point to do this.

Breaking up the integral at $z=-2$ gives,

$$\int_{-6}^{1} g(z)\,dz = \int_{-6}^{-2} g(z)\,dz + \int_{-2}^{1} g(z)\,dz$$

So, in the first integral we have $-6\le z\le -2$ and so we can use $g(z)=4\mathbf{e}^{z}$ in the first integral. Likewise, in the second integral we have $-2\le z\le 1$ and so we can use $g(z)=2-z$ in the second integral.

Making these function substitutions gives,

$$\int_{-6}^{1} g(z)\,dz = \int_{-6}^{-2} 4\mathbf{e}^{z}\,dz + \int_{-2}^{1} 2-z\,dz$$

Step 2
All we need to do at this point is evaluate each integral. Here is that work.

$$\begin{aligned}\int_{-6}^{1} g(z)\,dz &= \int_{-6}^{-2} 4\mathbf{e}^{z}\,dz + \int_{-2}^{1} 2-z\,dz = \left(4\mathbf{e}^{z}\right)\Big|_{-6}^{-2} + \left(2z-\tfrac{1}{2}z^2\right)\Big|_{-2}^{1} \\ &= \left[4\mathbf{e}^{-2}-4\mathbf{e}^{-6}\right] + \left[\tfrac{3}{2}-(-6)\right] = \boxed{4\mathbf{e}^{-2}-4\mathbf{e}^{-6}+\tfrac{15}{2}}\end{aligned}$$

---

17. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_3^6 \left|2x-10\right| dx$$

Hint : In order to do this integral we need to "remove" the absolute value bars from the integrand and we should know how to do that by this point.

Step 1
We'll need to "remove" the absolute value bars in order to do this integral. However, in order to do that we'll need to know where $2x-10$ is positive and negative.

Since $2x-10$ is the equation of a line is should be fairly clear that we have the following positive/negative nature of the function.

$$x<5 \qquad \Rightarrow \qquad 2x-10<0$$
$$x>5 \qquad \Rightarrow \qquad 2x-10>0$$

Step 2
So, to remove the absolute value bars all we need to do then is break the integral up at $x=5$.

$$\int_3^6 \left|2x-10\right| dx = \int_3^5 \left|2x-10\right| dx + \int_5^6 \left|2x-10\right| dx$$

So, in the first integral we have $3 \le x \le 5$ and so we have $\left|2x-10\right| = -\left(2x-10\right)$ in the first integral. Likewise, in the second integral we have $5 \le x \le 6$ and so we have $\left|2x-10\right| = 2x-10$ in the second integral. Or,

$$\int_3^6 \left|2x-10\right| dx = \int_3^5 -\left(2x-10\right) dx + \int_5^6 2x-10\, dx$$

Step 3
All we need to do at this point is evaluate each integral. Here is that work.

$$\begin{aligned}\int_3^6 \left|2x-10\right| dx &= \int_3^5 -2x+10\, dx + \int_5^6 2x-10\, dx = \left(-x^2+10x\right)\Big|_3^5 + \left(x^2-10x\right)\Big|_5^6 \\ &= \left[25-21\right] + \left[-24-\left(-25\right)\right] = \boxed{5}\end{aligned}$$

18. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-1}^{0} \left|4w+3\right| dw$$

Hint : In order to do this integral we need to "remove" the absolute value bars from the integrand and we should know how to do that by this point.

Step 1
We'll need to "remove" the absolute value bars in order to do this integral. However, in order to do that we'll need to know where $4w+3$ is positive and negative.

Since $4w+3$ is the equation a line is should be fairly clear that we have the following positive/negative nature of the function.

$$w < -\tfrac{3}{4} \qquad \Rightarrow \qquad 4w+3 < 0$$
$$w > -\tfrac{3}{4} \qquad \Rightarrow \qquad 4w+3 > 0$$

Step 2
So, to remove the absolute value bars all we need to do then is break the integral up at $w = -\frac{3}{4}$.

$$\int_{-1}^{0} \left|4w+3\right| dw = \int_{-1}^{-\frac{3}{4}} \left|4w+3\right| dw + \int_{-\frac{3}{4}}^{0} \left|4w+3\right| dw$$

So, in the first integral we have $-1 \le w \le -\frac{3}{4}$ and so we have $\left|4w+3\right| = -\left(4w+3\right)$ in the first integral. Likewise, in the second integral we have $-\frac{3}{4} \le w \le 0$ and so we have $\left|4w+3\right| = 4w+3$ in the second integral. Or,

$$\int_{-1}^{0} \left|4w+3\right| dw = \int_{-1}^{-\frac{3}{4}} -\left(4w+3\right) dw + \int_{-\frac{3}{4}}^{0} 4w+3\, dw$$

Step 3
All we need to do at this point is evaluate each integral. Here is that work.

$$\begin{aligned}\int_{-1}^{0} \left|4w+3\right| dw &= \int_{-1}^{-\frac{3}{4}} -4w-3\, dw + \int_{-\frac{3}{4}}^{0} 4w+3\, dw = \left(-2w^2-3w\right)\Big|_{-1}^{-\frac{3}{4}} + \left(2w^2+3w\right)\Big|_{-\frac{3}{4}}^{0} \\ &= \left[\tfrac{9}{8}-1\right] + \left[0-\left(-\tfrac{9}{8}\right)\right] = \boxed{\tfrac{5}{4}}\end{aligned}$$

Section 5-8 : Substitution Rule for Definite Integrals

1. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_0^1 3\left(4x + x^4\right)\left(10x^2 + x^5 - 2\right)^6 \, dx$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,

$$u = 10x^2 + x^5 - 2$$

Step 2
Here is the actual substitution work for this problem.

$$du = \left(20x + 5x^4\right)dx = 5\left(4x + x^4\right)dx \qquad \to \qquad \left(4x + x^4\right)dx = \tfrac{1}{5}du$$
$$x = 0 : u = -2 \qquad\qquad x = 1 : u = 9$$

As we did in the notes for this section we are also going to convert the limits to *u*'s to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_0^1 3\left(4x + x^4\right)\left(10x^2 + x^5 - 2\right)^6 \, dx = \tfrac{3}{5}\int_{-2}^{9} u^6 \, du$$

Step 3
The integral is then,

$$\int_0^1 3\left(4x + x^4\right)\left(10x^2 + x^5 - 2\right)^6 \, dx = \tfrac{3}{35}u^7\Big|_{-2}^{9} = \tfrac{3}{35}\left(4,782,969 - \left(-128\right)\right) = \boxed{\frac{14,349,291}{35}}$$

Do not get excited about "messy" or "large" answers. They will happen on occasion so don't worry about them when the happen.

2. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_0^{\frac{\pi}{4}} \frac{8\cos(2t)}{\sqrt{9-5\sin(2t)}}\,dt$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,

$$u = 9 - 5\sin(2t)$$

Step 2
Here is the actual substitution work for this problem.

$$du = -10\cos(2t)\,dt \qquad \rightarrow \qquad \cos(2t)\,dt = -\tfrac{1}{10}\,du$$
$$t = 0 : u = 9 \qquad\qquad t = \tfrac{\pi}{4} : u = 4$$

As we did in the notes for this section we are also going to convert the limits to *u*'s to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_0^{\frac{\pi}{4}} \frac{8\cos(2t)}{\sqrt{9-5\sin(2t)}}\,dt = -\tfrac{8}{10}\int_9^4 u^{-\frac{1}{2}}\,du$$

Step 3
The integral is then,

$$\int_0^{\frac{\pi}{4}} \frac{8\cos(2t)}{\sqrt{9-5\sin(2t)}}\,dt = -\tfrac{8}{5}u^{\frac{1}{2}}\Big|_9^4 = -\tfrac{16}{5} - \left(-\tfrac{24}{5}\right) = \boxed{\tfrac{8}{5}}$$

---

3. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_\pi^0 \sin(z)\cos^3(z)\,dz$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,

$$u = \cos(z)$$

Step 2
Here is the actual substitution work for this problem.

$$du = -\sin(z)\,dz \qquad \to \qquad \sin(z)\,dz = -du$$
$$z = \pi : u = -1 \qquad\qquad z = 0 : u = 1$$

As we did in the notes for this section we are also going to convert the limits to *u*'s to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{\pi}^{0} \sin(z)\cos^3(z)\,dz = -\int_{-1}^{1} u^3\,du$$

Step 3
The integral is then,

$$\int_{\pi}^{0} \sin(z)\cos^3(z)\,dz = -\tfrac{1}{4}u^4\Big|_{-1}^{1} = -\tfrac{1}{4} - \left(-\tfrac{1}{4}\right) = \boxed{0}$$

---

4. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{1}^{4} \sqrt{w}\,\mathbf{e}^{1-\sqrt{w^3}}\,dw$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,

$$u = 1 - w^{\frac{3}{2}}$$

Step 2
Here is the actual substitution work for this problem.

$$du = -\tfrac{3}{2}w^{\frac{1}{2}}dw \qquad \to \qquad \sqrt{w}\,dw = -\tfrac{2}{3}du$$
$$w = 1 : u = 0 \qquad\qquad w = 4 : u = -7$$

As we did in the notes for this section we are also going to convert the limits to *u*'s to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_1^4 \sqrt{w}\,\mathbf{e}^{1-\sqrt{w^3}}\,dw = -\tfrac{2}{3}\int_0^{-7}\mathbf{e}^u\,du$$

Step 3
The integral is then,

$$\int_1^4 \sqrt{w}\,\mathbf{e}^{1-\sqrt{w^3}}\,dw = -\tfrac{2}{3}\mathbf{e}^u\Big|_0^{-7} = -\tfrac{2}{3}\mathbf{e}^{-7} - \left(-\tfrac{2}{3}\mathbf{e}^0\right) = \boxed{\tfrac{2}{3}\left(1-\mathbf{e}^{-7}\right)}$$

---

5. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-4}^{-1} \sqrt[3]{5-2y} + \frac{7}{5-2y}\,dy$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,

$$u = 5 - 2y$$

Step 2
Here is the actual substitution work for this problem.

$$du = -2\,dy \qquad \to \qquad dy = -\tfrac{1}{2}du$$
$$y = -4 : u = 13 \qquad\qquad y = -1 : u = 7$$

As we did in the notes for this section we are also going to convert the limits to *u*'s to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{-4}^{-1} \sqrt[3]{5-2y} + \frac{7}{5-2y}\,dy = -\tfrac{1}{2}\int_{13}^{7} u^{\frac{1}{3}} + \frac{7}{u}\,du$$

Step 3
The integral is then,

$$\begin{aligned}\int_{-4}^{-1} \sqrt[3]{5-2y} + \frac{7}{5-2y}\,dy &= \left(-\tfrac{1}{2}\left[\tfrac{3}{4}u^{\frac{4}{3}} + 7\ln|u|\right]\right)\Bigg|_{13}^{7} \\ &= -\tfrac{3}{8}7^{\frac{4}{3}} - \tfrac{7}{2}\ln|7| - \left(-\tfrac{3}{8}13^{\frac{4}{3}} - \tfrac{7}{2}\ln|13|\right) \\ &= \boxed{\tfrac{3}{8}\left(13^{\frac{4}{3}} - 7^{\frac{4}{3}}\right) + \tfrac{7}{2}\left(\ln(13) - \ln(7)\right)}\end{aligned}$$

---

6. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-1}^{2} x^3 + \mathbf{e}^{\frac{1}{4}x}\,dx$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

Before setting up the substitution we'll need to break up the integral because the first term doesn't need a substitution. Doing this gives,

$$\int_{-1}^{2} x^3 + \mathbf{e}^{\frac{1}{4}x}\,dx = \int_{-1}^{2} x^3\,dx + \int_{-1}^{2} \mathbf{e}^{\frac{1}{4}x}\,dx$$

The substitution for the second integral is then,

$$u = \tfrac{1}{4}x$$

Step 2
Here is the actual substitution work for this second integral.

$$du = \tfrac{1}{4}dx \qquad \to \qquad dx = 4du$$
$$x = -1 : u = -\tfrac{1}{4} \qquad\qquad x = 2 : u = \tfrac{1}{2}$$

As we did in the notes for this section we are also going to convert the limits to *u*'s to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{-1}^{2} x^3 + \mathbf{e}^{\frac{1}{4}x}\,dx = \int_{-1}^{2} x^3\,dx + 4\int_{-\frac{1}{4}}^{\frac{1}{2}} \mathbf{e}^{u}\,du$$

Step 3
The integral is then,

$$\int_{-1}^{2} x^3 + \mathbf{e}^{\frac{1}{4}x}\,dx = \left.\tfrac{1}{4}x^4\right|_{-1}^{2} + \left.4\mathbf{e}^{u}\right|_{-\frac{1}{4}}^{\frac{1}{2}} = \left(4-\tfrac{1}{4}\right)+\left(4\mathbf{e}^{\frac{1}{2}} - 4\mathbf{e}^{-\frac{1}{4}}\right) = \boxed{\tfrac{15}{4} + 4\mathbf{e}^{\frac{1}{2}} - 4\mathbf{e}^{-\frac{1}{4}}}$$

---

7. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{\pi}^{\frac{3\pi}{2}} 6\sin\left(2w\right) - 7\cos\left(w\right)dw$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

Before setting up the substitution we'll need to break up the integral because the second term doesn't need a substitution. Doing this gives,

$$\int_{\pi}^{\frac{3\pi}{2}} 6\sin\left(2w\right) - 7\cos\left(w\right)dw = \int_{\pi}^{\frac{3\pi}{2}} 6\sin\left(2w\right)dw - \int_{\pi}^{\frac{3\pi}{2}} 7\cos\left(w\right)dw$$

The substitution for the first integral is then,

$$u = 2w$$

Step 2
Here is the actual substitution work for this first integral.

$$\begin{array}{ccc} du = 2dw & \to & dw = \tfrac{1}{2}du \\ w = \pi : u = 2\pi & & w = \tfrac{3\pi}{2} : u = 3\pi \end{array}$$

As we did in the notes for this section we are also going to convert the limits to *u*'s to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{\pi}^{\frac{3\pi}{2}} 6\sin\left(2w\right) - 7\cos\left(w\right)dw = 3\int_{2\pi}^{3\pi} \sin\left(u\right)du - \int_{\pi}^{\frac{3\pi}{2}} 7\cos\left(w\right)dw$$

Step 3
The integral is then,

$$\int_{\pi}^{\frac{3\pi}{2}} 6\sin\left(2w\right) - 7\cos\left(w\right)dw = -3\cos\left(u\right)\Big|_{2\pi}^{3\pi} - 7\sin\left(w\right)\Big|_{\pi}^{\frac{3\pi}{2}} = \left(3-\left(-3\right)\right) + \left(7-0\right) = \boxed{13}$$

---

8. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{1}^{5} \frac{2x^3 + x}{x^4 + x^2 + 1} - \frac{x}{x^2 - 4}\,dx$$

Solution

Be very careful with this problem. Recall that we can only do definite integrals if the integrand (*i.e.* the function we are integrating) is continuous on the interval over which we are integrating.

In this case the second term has division by zero at $x = 2$ and so is not continuous on $\left[1,5\right]$ and therefore this integral can't be done.

---

9. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-2}^{0} t\sqrt{3+t^2} + \frac{3}{\left(6t-1\right)^2}\,dt$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

Before setting up the substitution we'll need to break up the integral because each term requires a different substitution. Doing this gives,

$$\int_{-2}^{0} t\sqrt{3+t^2} + \frac{3}{(6t-1)^2}\,dt = \int_{-2}^{0} t\sqrt{3+t^2}\,dt + \int_{-2}^{0} \frac{3}{(6t-1)^2}\,dt$$

The substitution for each integral is then,

$$u = 3 + t^2 \qquad\qquad v = 6t - 1$$

Step 2
Here is the actual substitution work for this first integral.

$$du = 2t\,dt \quad \to \quad t\,dt = \tfrac{1}{2}du$$
$$t = -2 : u = 7 \qquad t = 0 : u = 3$$

Here is the actual substitution work for the second integral.

$$dv = 6\,dt \quad \to \quad dt = \tfrac{1}{6}dv$$
$$t = -2 : v = -13 \qquad t = 0 : v = -1$$

As we did in the notes for this section we are also going to convert the limits to *u*'s to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{-2}^{0} t\sqrt{3+t^2} + \frac{3}{(6t-1)^2}\,dt = \tfrac{1}{2}\int_{7}^{3} u^{\frac{1}{2}}\,du + \tfrac{3}{6}\int_{-13}^{-1} v^{-2}\,dv$$

Step 3
The integral is then,

$$\int_{-2}^{0} t\sqrt{3+t^2} + \frac{3}{(6t-1)^2}\,dt = \tfrac{1}{3}u^{\frac{3}{2}}\Big|_{7}^{3} - \tfrac{1}{2}v^{-1}\Big|_{-13}^{-1} = \tfrac{1}{3}\left(3^{\frac{3}{2}} - 7^{\frac{3}{2}}\right) - \tfrac{1}{2}\left(-1-\left(-\tfrac{1}{13}\right)\right) = \boxed{\tfrac{1}{3}\left(3^{\frac{3}{2}} - 7^{\frac{3}{2}}\right) + \tfrac{6}{13}}$$

---

10. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-2}^{1} (2-z)^3 + \sin(\pi z)\left[3 + 2\cos(\pi z)\right]^3\,dz$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

Before setting up the substitution we'll need to break up the integral because each term requires a different substitution. Doing this gives,

$$\int_{-2}^{1} (2-z)^3 + \sin(\pi z)\left[3+2\cos(\pi z)\right]^3 dz = \int_{-2}^{1} (2-z)^3 \, dz + \int_{-2}^{1} \sin(\pi z)\left[3+2\cos(\pi z)\right]^3 dz$$

The substitution for each integral is then,

$$u = 2-z \qquad\qquad v = 3+2\cos(\pi z)$$

Step 2
Here is the actual substitution work for this first integral.

$$\begin{aligned} du &= -dz & \rightarrow \qquad dz &= -du \\ z = -2 &: u = 4 & z = 1 &: u = 1 \end{aligned}$$

Here is the actual substitution work for the second integral.

$$\begin{aligned} dv &= -2\pi\sin(\pi z)\,dz & \rightarrow \qquad \sin(\pi z)\,dz &= -\tfrac{1}{2\pi}\,dv \\ z = -2 &: v = 5 & z = 1 &: v = 1 \end{aligned}$$

As we did in the notes for this section we are also going to convert the limits to *u*'s to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{-2}^{1} (2-z)^3 + \sin(\pi z)\left[3+2\cos(\pi z)\right]^3 dz = -\int_{4}^{1} u^3 \, du - \tfrac{1}{2\pi}\int_{5}^{1} v^3 \, dv$$

Step 3
The integral is then,

$$\begin{aligned} \int_{-2}^{1} (2-z)^3 + \sin(\pi z)\left[3+2\cos(\pi z)\right]^3 dz &= -\tfrac{1}{4}u^4\Big|_4^1 - \tfrac{1}{8\pi}v^4\Big|_5^1 \\ &= -\tfrac{1}{4}(1-256) - \tfrac{1}{8\pi}(1-625) = \boxed{\tfrac{255}{4} + \tfrac{78}{\pi}} \end{aligned}$$