Chapter 3 : Derivatives

Here are a set of practice problems for the Derivatives chapter of the Calculus I notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a couple of places on the site.

7. If you'd like a pdf document containing the solutions the download tab on the website contains links to pdf's containing the solutions for the full book, chapter and section. At this time, I do not offer pdf's for solutions to individual problems.

8. If you'd like to view the solutions on the web go to the problem set web page, click the solution link for any problem and it will take you to the solution to that problem.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of all the sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

The Definition of the Derivative – In this section we define the derivative, give various notations for the derivative and work a few problems illustrating how to use the definition of the derivative to actually compute the derivative of a function.

Interpretation of the Derivative – In this section we give several of the more important interpretations of the derivative. We discuss the rate of change of a function, the velocity of a moving object and the slope of the tangent line to a graph of a function.

Differentiation Formulas – In this section we give most of the general derivative formulas and properties used when taking the derivative of a function. Examples in this section concentrate mostly on polynomials, roots and more generally variables raised to powers.

Product and Quotient Rule – In this section we will give two of the more important formulas for differentiating functions. We will discuss the Product Rule and the Quotient Rule allowing us to differentiate functions that, up to this point, we were unable to differentiate.

Derivatives of Trig Functions – In this section we will discuss differentiating trig functions. Derivatives of all six trig functions are given and we show the derivation of the derivative of sin(x) and tan(x).

Derivatives of Exponential and Logarithm Functions – In this section we derive the formulas for the derivatives of the exponential and logarithm functions.

Derivatives of Inverse Trig Functions – In this section we give the derivatives of all six inverse trig functions. We show the derivation of the formulas for inverse sine, inverse cosine and inverse tangent.

Derivatives of Hyperbolic Functions – In this section we define the hyperbolic functions, give the relationships between them and some of the basic facts involving hyperbolic functions. We also give the derivatives of each of the six hyperbolic functions and show the derivation of the formula for hyperbolic sine.

Chain Rule – In this section we discuss one of the more useful and important differentiation formulas, The Chain Rule. With the chain rule in hand we will be able to differentiate a much wider variety of functions. As you will see throughout the rest of your Calculus courses a great many of derivatives you take will involve the chain rule!

Implicit Differentiation – In this section we will discuss implicit differentiation. Not every function can be explicitly written in terms of the independent variable, e.g. y = f(x) and yet we will still need to know what f'(x) is. Implicit differentiation will allow us to find the derivative in these cases. Knowing implicit differentiation will allow us to do one of the more important applications of derivatives, Related Rates (the next section).

Related Rates – In this section we will discuss the only application of derivatives in this section, Related Rates. In related rates problems we are give the rate of change of one quantity in a problem and asked to determine the rate of one (or more) quantities in the problem. This is often one of the more difficult sections for students. We work quite a few problems in this section so hopefully by the end of this section you will get a decent understanding on how these problems work.

Higher Order Derivatives – In this section we define the concept of higher order derivatives and give a quick application of the second order derivative and show how implicit differentiation works for higher order derivatives.

Logarithmic Differentiation – In this section we will discuss logarithmic differentiation. Logarithmic differentiation gives an alternative method for differentiating products and quotients (sometimes easier than using product and quotient rule). More importantly, however, is the fact that logarithm differentiation allows us to differentiate functions that are in the form of one function raised to another function, i.e. there are variables in both the base and exponent of the function.

Section 3-1 : The Definition of the Derivative

Use the definition of the derivative to find the derivative of the following functions.

1. $f\left(x\right)=6$

2. $V\left(t\right)=3-14t$

3. $g\left(x\right)=x^{2}$

4. $Q\left(t\right)=10+5t-t^{2}$

5. $W\left(z\right)=4z^{2}-9z$

6. $f\left(x\right)=2x^{3}-1$

7. $g\left(x\right)=x^{3}-2x^{2}+x-1$

8. $R\left(z\right)=\dfrac{5}{z}$

9. $V\left(t\right)=\dfrac{t+1}{t+4}$

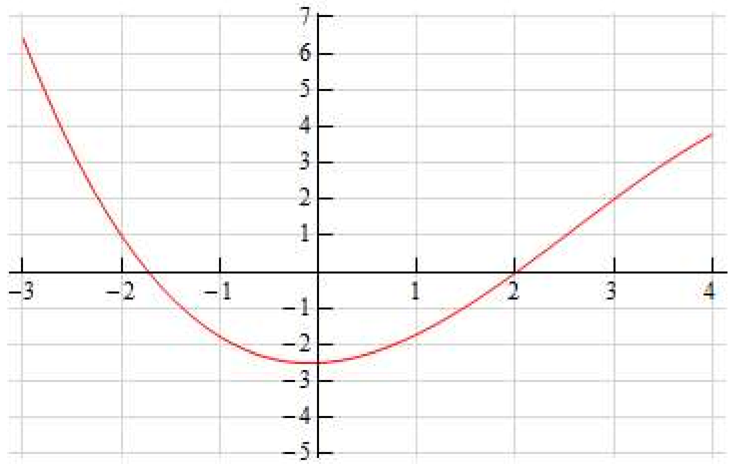
10. $Z\left(t\right)=\sqrt{3t-4}$

11. $f\left(x\right)=\sqrt{1-9x}$

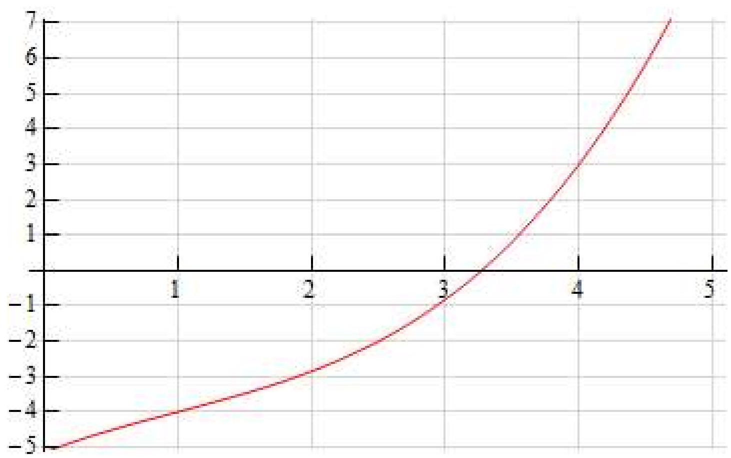
Section 3-2 : Interpretation of the Derivative

For problems 1 and 2 use the graph of the function, $f\left(x\right)$, estimate the value of $f'\left(a\right)$ for the given values of *a*.

1. **(a)** $a = -2$ **(b)** $a = 3$



2. **(a)** $a = 1$ **(b)** $a = 4$



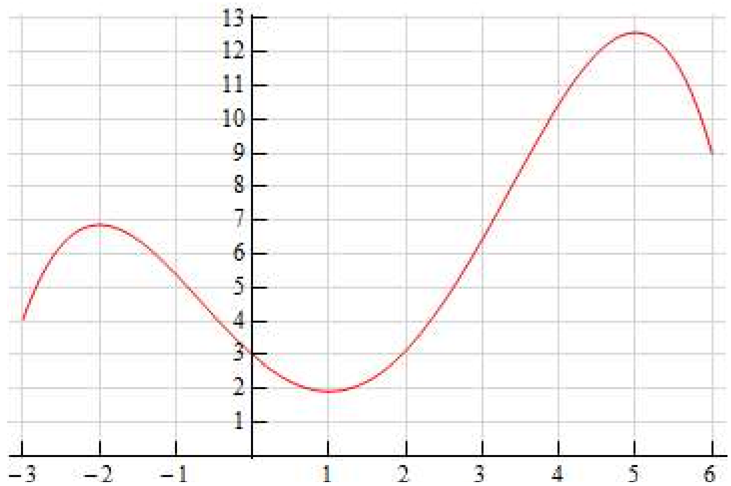
For problems 3 and 4 sketch the graph of a function that satisfies the given conditions.

3. $f\left(1\right) = 3$, $f'\left(1\right) = 1$, $f\left(4\right) = 5$, $f'\left(4\right) = -2$

4. $f\left(-3\right) = 5$, $f'\left(-3\right) = -2$, $f\left(1\right) = 2$, $f'\left(1\right) = 0$, $f\left(4\right) = -2$, $f'\left(4\right) = -3$

For problems 5 and 6 the graph of a function, $f\left(x\right)$, is given. Use this to sketch the graph of the derivative, $f'\left(x\right)$.

5.



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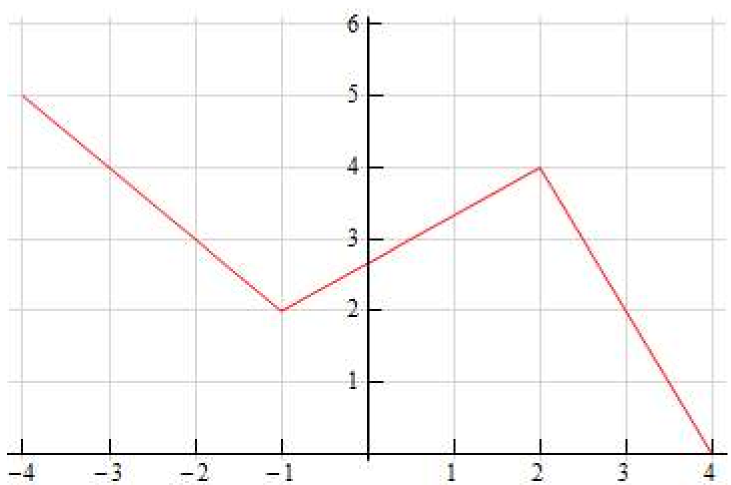


7. Answer the following questions about the function $W\left(z\right)=4z^2-9z$.

(a) Is the function increasing or decreasing at $z=-1$?
(b) Is the function increasing or decreasing at $z=2$?
(c) Does the function ever stop changing? If yes, at what value(s) of *z* does the function stop changing?

8. What is the equation of the tangent line to $f\left(x\right)=3-14x$ at $x=8$.

9. The position of an object at any time *t* is given by $s\left(t\right)=\dfrac{t+1}{t+4}$.

(a) Determine the velocity of the object at any time *t*.
(b) Does the object ever stop moving? If yes, at what time(s) does the object stop moving?

10. What is the equation of the tangent line to $f\left(x\right)=\dfrac{5}{x}$ at $x=\dfrac{1}{2}$?

11. Determine where, if anywhere, the function $g\left(x\right)=x^3-2x^2+x-1$ stops changing.

12. Determine if the function $Z\left(t\right)=\sqrt{3t-4}$ increasing or decreasing at the given points.

(a) $t=5$
(b) $t=10$
(c) $t=300$

13. Suppose that the volume of water in a tank for $0\le t\le 6$ is given by $Q\left(t\right)=10+5t-t^2$.

(a) Is the volume of water increasing or decreasing at $t=0$?
(b) Is the volume of water increasing or decreasing at $t=6$?
(c) Does the volume of water ever stop changing? If yes, at what times(s) does the volume stop changing?

Section 3-3 : Differentiation Formulas

For problems 1 – 12 find the derivative of the given function.

1. $f\left(x\right)=6x^3-9x+4$

2. $y=2t^4-10t^2+13t$

3. $g\left(z\right)=4z^7-3z^{-7}+9z$

4. $h\left(y\right)=y^{-4}-9y^{-3}+8y^{-2}+12$

5. $y=\sqrt{x}+8\sqrt[3]{x}-2\sqrt[4]{x}$

6. $f\left(x\right)=10\sqrt[5]{x^3}-\sqrt{x^7}+6\sqrt[3]{x^8}-3$

7. $f\left(t\right)=\dfrac{4}{t}-\dfrac{1}{6t^3}+\dfrac{8}{t^5}$

8. $R\left(z\right)=\dfrac{6}{\sqrt{z^3}}+\dfrac{1}{8z^4}-\dfrac{1}{3z^{10}}$

9. $z=x\left(3x^2-9\right)$

10. $g\left(y\right)=\left(y-4\right)\left(2y+y^2\right)$

11. $h\left(x\right)=\dfrac{4x^3-7x+8}{x}$

12. $f\left(y\right)=\dfrac{y^5-5y^3+2y}{y^3}$

13. Determine where, if anywhere, the function $f\left(x\right)=x^3+9x^2-48x+2$ is not changing.

14. Determine where, if anywhere, the function $y=2z^4-z^3-3z^2$ is not changing.

15. Find the tangent line to $g\left(x\right)=\dfrac{16}{x}-4\sqrt{x}$ at $x=4$.

16. Find the tangent line to $f\left(x\right)=7x^4+8x^{-6}+2x$ at $x=-1$.

17. The position of an object at any time *t* is given by $s\left(t\right)=3t^4-40t^3+126t^2-9$.

(a) Determine the velocity of the object at any time *t*.
(b) Does the object ever stop changing?
(c) When is the object moving to the right and when is the object moving to the left?

18. Determine where the function $h\left(z\right)=6+40z^3-5z^4-4z^5$ is increasing and decreasing.

19. Determine where the function $R\left(x\right)=\left(x+1\right)\left(x-2\right)^2$ is increasing and decreasing.

20. Determine where, if anywhere, the tangent line to $f\left(x\right)=x^3-5x^2+x$ is parallel to the line $y=4x+23$.

Section 3-4 : Product and Quotient Rule

For problems 1 – 6 use the Product Rule or the Quotient Rule to find the derivative of the given function.

1. $f(t) = (4t^2 - t)(t^3 - 8t^2 + 12)$

2. $y = \left(1 + \sqrt{x^3}\right)\left(x^{-3} - 2\sqrt[3]{x}\right)$

3. $h(z) = (1 + 2z + 3z^2)(5z + 8z^2 - z^3)$

4. $g(x) = \dfrac{6x^2}{2 - x}$

5. $R(w) = \dfrac{3w + w^4}{2w^2 + 1}$

6. $f(x) = \dfrac{\sqrt{x} + 2x}{7x - 4x^2}$

7. If $f(2) = -8$, $f'(2) = 3$, $g(2) = 17$ and $g'(2) = -4$ determine the value of $(f\,g)'(2)$.

8. If $f(x) = x^3 g(x)$, $g(-7) = 2$, $g'(-7) = -9$ determine the value of $f'(-7)$.

9. Find the equation of the tangent line to $f(x) = \left(1 + 12\sqrt{x}\right)\left(4 - x^2\right)$ at $x = 9$.

10. Determine where $f(x) = \dfrac{x - x^2}{1 + 8x^2}$ is increasing and decreasing.

11. Determine where $V(t) = (4 - t^2)(1 + 5t^2)$ is increasing and decreasing.

Section 3-5 : Derivatives of Trig Functions

For problems 1 – 3 evaluate the given limit.

1. $\displaystyle \lim_{z\to 0}\frac{\sin(10z)}{z}$

2. $\displaystyle \lim_{\alpha\to 0}\frac{\sin(12\alpha)}{\sin(5\alpha)}$

3. $\displaystyle \lim_{x\to 0}\frac{\cos(4x)-1}{x}$

For problems 4 – 10 differentiate the given function.

4. $f(x)=2\cos(x)-6\sec(x)+3$

5. $g(z)=10\tan(z)-2\cot(z)$

6. $f(w)=\tan(w)\sec(w)$

7. $h(t)=t^3-t^2\sin(t)$

8. $y=6+4\sqrt{x}\csc(x)$

9. $\displaystyle R(t)=\frac{1}{2\sin(t)-4\cos(t)}$

10. $\displaystyle Z(v)=\frac{v+\tan(v)}{1+\csc(v)}$

11. Find the tangent line to $f(x)=\tan(x)+9\cos(x)$ at $x=\pi$.

12. The position of an object is given by $s(t)=2+7\cos(t)$ determine all the points where the object is not moving.

13. Where in the range $[-2,7]$ is the function $f(x)=4\cos(x)-x$ is increasing and decreasing.

Section 3-6 : Derivatives of Exponential and Logarithm Functions

For problems 1 – 6 differentiate the given function.

1. $f\left(x\right)=2\mathbf{e}^{x}-8^{x}$

2. $g\left(t\right)=4\log_{3}\left(t\right)-\ln\left(t\right)$

3. $R\left(w\right)=3^{w}\log\left(w\right)$

4. $y=z^{5}-\mathbf{e}^{z}\ln\left(z\right)$

5. $h\left(y\right)=\dfrac{y}{1-\mathbf{e}^{y}}$

6. $f\left(t\right)=\dfrac{1+5t}{\ln\left(t\right)}$

7. Find the tangent line to $f\left(x\right)=7^{x}+4\mathbf{e}^{x}$ at $x=0$.

8. Find the tangent line to $f\left(x\right)=\ln\left(x\right)\log_{2}\left(x\right)$ at $x=2$.

9. Determine if $V\left(t\right)=\dfrac{t}{\mathbf{e}^{t}}$ is increasing or decreasing at the following points.

(a) $t=-4$ **(b)** $t=0$ **(c)** $t=10$

10. Determine if $G\left(z\right)=\left(z-6\right)\ln\left(z\right)$ is increasing or decreasing at the following points.

(a) $z=1$ **(b)** $z=5$ **(c)** $z=20$

Section 3-7 : Derivatives of Inverse Trig Functions

For each of the following problems differentiate the given function.

1. $T\left(z\right)=2\cos\left(z\right)+6\cos^{-1}\left(z\right)$

2. $g\left(t\right)=\csc^{-1}\left(t\right)-4\cot^{-1}\left(t\right)$

3. $y=5x^{6}-\sec^{-1}\left(x\right)$

4. $f\left(w\right)=\sin\left(w\right)+w^{2}\tan^{-1}\left(w\right)$

5. $h\left(x\right)=\dfrac{\sin^{-1}\left(x\right)}{1+x}$

Section 3-8 : Derivatives of Hyperbolic Functions

For each of the following problems differentiate the given function.

1. $f(x) = \sinh(x) + 2\cosh(x) - \operatorname{sech}(x)$

2. $R(t) = \tan(t) + t^2 \operatorname{csch}(t)$

3. $g(z) = \dfrac{z+1}{\tanh(z)}$

Section 3-9 : Chain Rule

For problems 1 – 27 differentiate the given function.

1. $f(x) = (6x^2 + 7x)^4$

2. $g(t) = (4t^2 - 3t + 2)^{-2}$

3. $y = \sqrt[3]{1 - 8z}$

4. $R(w) = \csc(7w)$

5. $G(x) = 2\sin(3x + \tan(x))$

6. $h(u) = \tan(4 + 10u)$

7. $f(t) = 5 + \mathbf{e}^{4t + t^7}$

8. $g(x) = \mathbf{e}^{1 - \cos(x)}$

9. $H(z) = 2^{1-6z}$

10. $u(t) = \tan^{-1}(3t - 1)$

11. $F(y) = \ln(1 - 5y^2 + y^3)$

12. $V(x) = \ln(\sin(x) - \cot(x))$

13. $h(z) = \sin(z^6) + \sin^6(z)$

14. $S(w) = \sqrt{7w} + \mathbf{e}^{-w}$

15. $g(z) = 3z^7 - \sin(z^2 + 6)$

16. $f(x) = \ln(\sin(x)) - (x^4 - 3x)^{10}$

17. $h(t) = t^6\sqrt{5t^2 - t}$

18. $q(t) = t^2 \ln\left(t^5\right)$

19. $g(w) = \cos(3w)\sec(1-w)$

20. $y = \dfrac{\sin(3t)}{1+t^2}$

21. $K(x) = \dfrac{1+\mathbf{e}^{-2x}}{x+\tan(12x)}$

22. $f(x) = \cos\left(x^2\mathbf{e}^x\right)$

23. $z = \sqrt{5x+\tan(4x)}$

24. $f(t) = \left(\mathbf{e}^{-6t} + \sin(2-t)\right)^3$

25. $g(x) = \left(\ln\left(x^2+1\right) - \tan^{-1}(6x)\right)^{10}$

26. $h(z) = \tan^4\left(z^2+1\right)$

27. $f(x) = \left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-1}$

28. Find the tangent line to $f(x) = 4\sqrt{2x} - 6\mathbf{e}^{2-x}$ at $x = 2$.

29. Determine where $V(z) = z^4(2z-8)^3$ is increasing and decreasing.

30. The position of an object is given by $s(t) = \sin(3t) - 2t + 4$. Determine where in the interval $[0,3]$ the object is moving to the right and moving to the left.

31. Determine where $A(t) = t^2\mathbf{e}^{5-t}$ is increasing and decreasing.

32. Determine where in the interval $[-1,20]$ the function $f(x) = \ln\left(x^4+20x^3+100\right)$ is increasing and decreasing.

Section 3-10 : Implicit Differentiation

For problems 1 – 3 do each of the following.

(a) Find y' by solving the equation for *y* and differentiating directly.

(b) Find y' by implicit differentiation.

(c) Check that the derivatives in **(a)** and **(b)** are the same.

1. $\dfrac{x}{y^3} = 1$

2. $x^2 + y^3 = 4$

3. $x^2 + y^2 = 2$

For problems 4 – 9 find y' by implicit differentiation.

4. $2y^3 + 4x^2 - y = x^6$

5. $7y^2 + \sin(3x) = 12 - y^4$

6. $\mathbf{e}^x - \sin(y) = x$

7. $4x^2y^7 - 2x = x^5 + 4y^3$

8. $\cos(x^2 + 2y) + x\,\mathbf{e}^{y^2} = 1$

9. $\tan(x^2y^4) = 3x + y^2$

For problems 10 & 11 find the equation of the tangent line at the given point.

10. $x^4 + y^2 = 3$ at $(1, -\sqrt{2})$.

11. $y^2\mathbf{e}^{2x} = 3y + x^2$ at $(0, 3)$.

For problems 12 & 13 assume that $x = x(t)$, $y = y(t)$ and $z = z(t)$ and differentiate the given equation with respect to *t*.

12. $x^2 - y^3 + z^4 = 1$

13. $x^2 \cos(y) = \sin(y^3 + 4z)$

Section 3-11 : Related Rates

1. In the following assume that *x* and *y* are both functions of *t*. Given $x = -2$, $y = 1$ and $x' = -4$ determine y' for the following equation.

$$6y^2 + x^2 = 2 - x^3\mathbf{e}^{4-4y}$$

2. In the following assume that *x*, *y* and *z* are all functions of *t*. Given $x = 4$, $y = -2$, $z = 1$, $x' = 9$ and $y' = -3$ determine z' for the following equation.

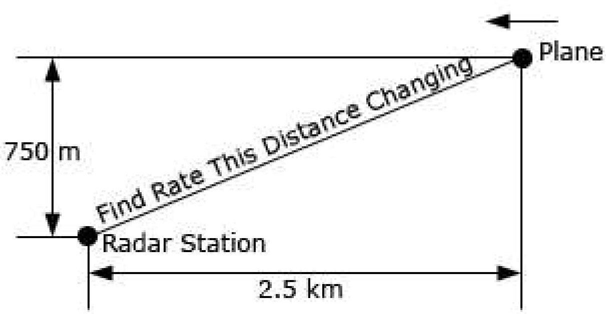
$$x(1-y) + 5z^3 = y^2z^2 + x^2 - 3$$

3. For a certain rectangle the length of one side is always three times the length of the other side.
   (a) If the shorter side is decreasing at a rate of 2 inches/minute at what rate is the longer side decreasing?
   (b) At what rate is the enclosed area decreasing when the shorter side is 6 inches long and is decreasing at a rate of 2 inches/minute?

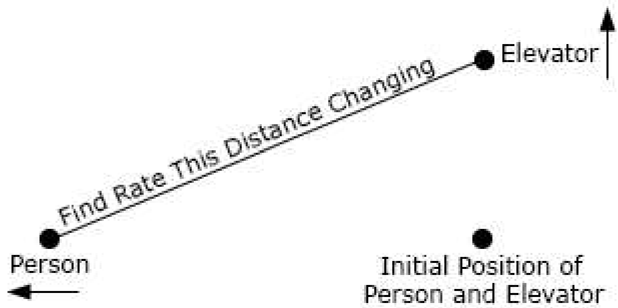
4. A thin sheet of ice is in the form of a circle. If the ice is melting in such a way that the area of the sheet is decreasing at a rate of 0.5 m^2/sec at what rate is the radius decreasing when the area of the sheet is 12 m^2 ?

5. A person is standing 350 feet away from a model rocket that is fired straight up into the air at a rate of 15 ft/sec. At what rate is the distance between the person and the rocket increasing **(a)** 20 seconds after liftoff? **(b)** 1 minute after liftoff?

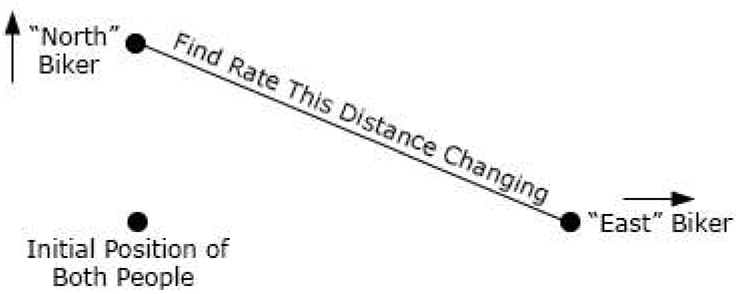
6. A plane is 750 meters in the air flying parallel to the ground at a speed of 100 m/s and is initially 2.5 kilometers away from a radar station. At what rate is the distance between the plane and the radar station changing **(a)** initially and **(b)** 30 seconds after it passes over the radar station?



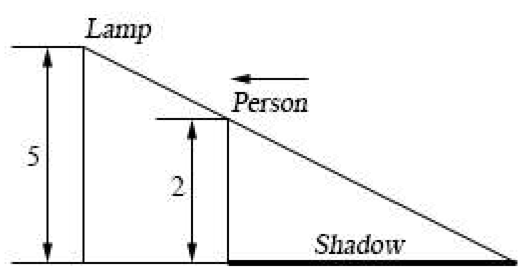
7. Two people are at an elevator. At the same time one person starts to walk away from the elevator at a rate of 2 ft/sec and the other person starts going up in the elevator at a rate of 7 ft/sec. What rate is the distance between the two people changing 15 seconds later?



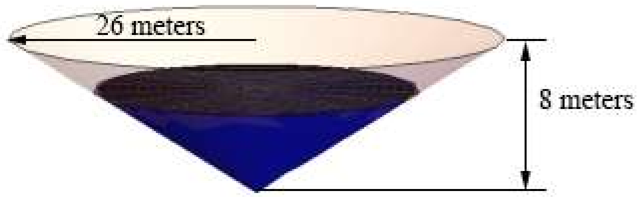
8. Two people on bikes are at the same place. One of the bikers starts riding directly north at a rate of 8 m/sec. Five seconds after the first biker started riding north the second starts to ride directly east at a rate of 5 m/sec. At what rate is the distance between the two riders increasing 20 seconds after the second person started riding?



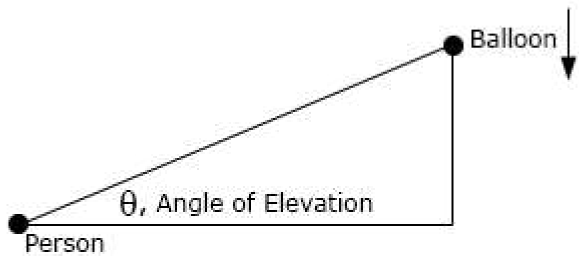
9. A light is mounted on a wall 5 meters above the ground. A 2 meter tall person is initially 10 meters from the wall and is moving towards the wall at a rate of 0.5 m/sec. After 4 seconds of moving is the tip of the shadow moving **(a)** towards or away from the person and **(b)** towards or away from the wall?



10. A tank of water in the shape of a cone is being filled with water at a rate of 12 m^3/sec. The base radius of the tank is 26 meters and the height of the tank is 8 meters. At what rate is the depth of the water in the tank changing when the radius of the top of the water is 10 meters? Note the image below is not completely to scale….



11. The angle of elevation is the angle formed by a horizontal line and a line joining the observer's eye to an object above the horizontal line. A person is 500 feet way from the launch point of a hot air balloon. The hot air balloon is starting to come back down at a rate of 15 ft/sec. At what rate is the angle of elevation, θ, changing when the hot air balloon is 200 feet above the ground.



Section 3-12 : Higher Order Derivatives

For problems 1 – 5 determine the fourth derivative of the given function.

1. $h(t) = 3t^7 - 6t^4 + 8t^3 - 12t + 18$

2. $V(x) = x^3 - x^2 + x - 1$

3. $f(x) = 4\sqrt[5]{x^3} - \dfrac{1}{8x^2} - \sqrt{x}$

4. $f(w) = 7\sin\left(\frac{w}{3}\right) + \cos(1 - 2w)$

5. $y = \mathbf{e}^{-5z} + 8\ln\left(2z^4\right)$

For problems 6 – 9 determine the second derivative of the given function.

6. $g(x) = \sin\left(2x^3 - 9x\right)$

7. $z = \ln\left(7 - x^3\right)$

8. $Q(v) = \dfrac{2}{\left(6 + 2v - v^2\right)^4}$

9. $H(t) = \cos^2(7t)$

For problems 10 & 11 determine the second derivative of the given function.

10. $2x^3 + y^2 = 1 - 4y$

11. $6y - xy^2 = 1$

Section 3-13 : Logarithmic Differentiation

For problems 1 – 3 use logarithmic differentiation to find the first derivative of the given function.

1. $f\left(x\right)=\left(5-3x^2\right)^7\ \sqrt{6x^2+8x-12}$

2. $y=\dfrac{\sin\left(3z+z^2\right)}{\left(6-z^4\right)^3}$

3. $h\left(t\right)=\dfrac{\sqrt{5t+8}\ \sqrt[3]{1-9\cos\left(4t\right)}}{\sqrt[4]{t^2+10t}}$

For problems 4 & 5 find the first derivative of the given function.

4. $g\left(w\right)=\left(3w-7\right)^{4w}$

5. $f\left(x\right)=\left(2x-\mathbf{e}^{8x}\right)^{\sin\left(2x\right)}$