Chapter 4 : Applications of Derivatives

Here are a set of practice problems for the Applications of Derivatives chapter of the Calculus I notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a couple of places on the site.

9. If you'd like a pdf document containing the solutions the download tab on the website contains links to pdf's containing the solutions for the full book, chapter and section. At this time, I do not offer pdf's for solutions to individual problems.

10. If you'd like to view the solutions on the web go to the problem set web page, click the solution link for any problem and it will take you to the solution to that problem.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of all the sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Rates of Change – In this section we review the main application/interpretation of derivatives from the previous chapter (i.e. rates of change) that we will be using in many of the applications in this chapter.

Critical Points – In this section we give the definition of critical points. Critical points will show up in most of the sections in this chapter, so it will be important to understand them and how to find them. We will work a number of examples illustrating how to find them for a wide variety of functions.

Minimum and Maximum Values – In this section we define absolute (or global) minimum and maximum values of a function and relative (or local) minimum and maximum values of a function. It is important to understand the difference between the two types of minimum/maximum (collectively called extrema) values for many of the applications in this chapter and so we use a variety of examples to help with this. We also give the Extreme Value Theorem and Fermat's Theorem, both of which are very important in the many of the applications we'll see in this chapter.

Finding Absolute Extrema – In this section we discuss how to find the absolute (or global) minimum and maximum values of a function. In other words, we will be finding the largest and smallest values that a function will have.

The Shape of a Graph, Part I – In this section we will discuss what the first derivative of a function can tell us about the graph of a function. The first derivative will allow us to identify the relative (or local) minimum and maximum values of a function and where a function will be increasing and decreasing. We will also give the First Derivative test which will allow us to classify critical points as relative minimums, relative maximums or neither a minimum or a maximum.

The Shape of a Graph, Part II – In this section we will discuss what the second derivative of a function can tell us about the graph of a function. The second derivative will allow us to determine where the graph of a function is concave up and concave down. The second derivative will also allow us to identify any inflection points (i.e. where concavity changes) that a function may have. We will also give the Second Derivative Test that will give an alternative method for identifying some critical points (but not all) as relative minimums or relative maximums.

The Mean Value Theorem – In this section we will give Rolle's Theorem and the Mean Value Theorem. With the Mean Value Theorem we will prove a couple of very nice facts, one of which will be very useful in the next chapter.

Optimization Problems – In this section we will be determining the absolute minimum and/or maximum of a function that depends on two variables given some constraint, or relationship, that the two variables must always satisfy. We will discuss several methods for determining the absolute minimum or maximum of the function. Examples in this section tend to center around geometric objects such as squares, boxes, cylinders, etc.

More Optimization Problems – In this section we will continue working optimization problems. The examples in this section tend to be a little more involved and will often involve situations that will be more easily described with a sketch as opposed to the 'simple' geometric objects we looked at in the previous section.

L'Hospital's Rule and Indeterminate Forms – In this section we will revisit indeterminate forms and limits and take a look at L'Hospital's Rule. L'Hospital's Rule will allow us to evaluate some limits we were not able to previously.

Linear Approximations – In this section we discuss using the derivative to compute a linear approximation to a function. We can use the linear approximation to a function to approximate values of the function at certain points. While it might not seem like a useful thing to do with when we have the function there really are reasons that one might want to do this. We give two ways this can be useful in the examples.

Differentials – In this section we will compute the differential for a function. We will give an application of differentials in this section. However, one of the more important uses of differentials will come in the next chapter and unfortunately we will not be able to discuss it until then.

Newton's Method – In this section we will discuss Newton's Method. Newton's Method is an application of derivatives will allow us to approximate solutions to an equation. There are many equations that cannot be solved directly and with this method we can get approximations to the solutions to many of those equations.

Business Applications – In this section we will give a cursory discussion of some basic applications of derivatives to the business field. We will revisit finding the maximum and/or minimum function value and we will define the marginal cost function, the average cost, the revenue function, the marginal revenue function and the marginal profit function. Note that this section is only intended to introduce these concepts and not teach you everything about them.

Section 4-1 : Rates of Change

As noted in the text for this section the purpose of this section is only to remind you of certain types of applications that were discussed in the previous chapter. As such there aren't any problems written for this section. Instead here is a list of links (note that these will only be active links in the web version and not the pdf version) to problems from the relevant sections from the previous chapter.

Each of the following sections has a selection of increasing/decreasing problems towards the bottom of the problem set.

Differentiation Formulas
Product & Quotient Rules
Derivatives of Trig Functions
Derivatives of Exponential and Logarithm Functions
Chain Rule

Related Rates problems are in the Related Rates section.

Section 4-2 : Critical Points

Determine the critical points of each of the following functions.

1. $f(x) = 8x^3 + 81x^2 - 42x - 8$

2. $R(t) = 1 + 80t^3 + 5t^4 - 2t^5$

3. $g(w) = 2w^3 - 7w^2 - 3w - 2$

4. $g(x) = x^6 - 2x^5 + 8x^4$

5. $h(z) = 4z^3 - 3z^2 + 9z + 12$

6. $Q(x) = (2 - 8x)^4 (x^2 - 9)^3$

7. $f(z) = \dfrac{z + 4}{2z^2 + z + 8}$

8. $R(x) = \dfrac{1 - x}{x^2 + 2x - 15}$

9. $r(y) = \sqrt[5]{y^2 - 6y}$

10. $h(t) = 15 - (3 - t)\left[t^2 - 8t + 7\right]^{\frac{1}{3}}$

11. $s(z) = 4\cos(z) - z$

12. $f(y) = \sin\left(\frac{y}{3}\right) + \frac{2y}{9}$

13. $V(t) = \sin^2(3t) + 1$

14. $f(x) = 5x\,\mathbf{e}^{9-2x}$

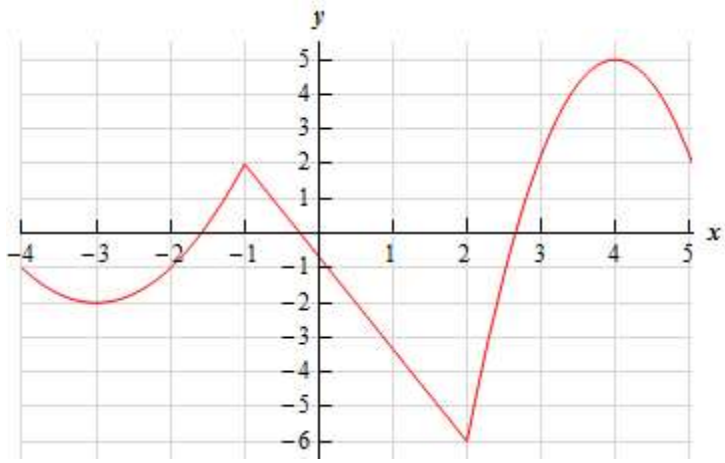
15. $g(w) = \mathbf{e}^{w^3 - 2w^2 - 7w}$

16. $R(x) = \ln\left(x^2 + 4x + 14\right)$

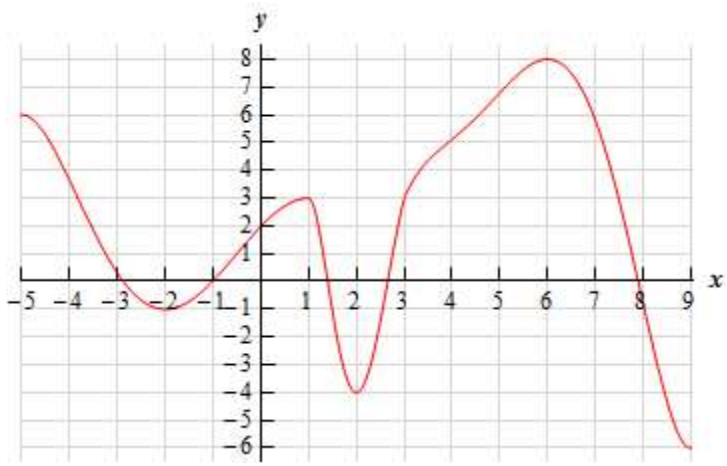
17. $A(t) = 3t - 7\ln(8t + 2)$

Section 4-3 : Minimum and Maximum Values

1. Below is the graph of some function, $f(x)$. Identify all of the relative extrema and absolute extrema of the function.



2. Below is the graph of some function, $f(x)$. Identify all of the relative extrema and absolute extrema of the function.



3. Sketch the graph of $g(x) = x^2 - 4x$ and identify all the relative extrema and absolute extrema of the function on each of the following intervals.

(a) $(-\infty, \infty)$

(b) $[-1, 4]$

(c) $[1, 3]$

(d) $[3, 5]$

(e) $(-1, 5]$

4. Sketch the graph of $h(x) = -(x+4)^3$ and identify all the relative extrema and absolute extrema of the function on each of the following intervals.

(a) $(-\infty, \infty)$

(b) $[-5.5, -2]$

(c) $[-4, -3)$

(d) $[-4, -3]$

5. Sketch the graph of some function on the interval $[1, 6]$ that has an absolute maximum at $x = 6$ and an absolute minimum at $x = 3$.

6. Sketch the graph of some function on the interval $[-4, 3]$ that has an absolute maximum at $x = -3$ and an absolute minimum at $x = 2$.

7. Sketch the graph of some function that meets the following conditions :

(a) The function is continuous.
(b) Has two relative minimums.
(c) One of relative minimums is also an absolute minimum and the other relative minimum is not an absolute minimum.
(d) Has one relative maximum.
(e) Has no absolute maximum.

Section 4-4 : Finding Absolute Extrema

For each of the following problems determine the absolute extrema of the given function on the specified interval.

1. $f(x) = 8x^3 + 81x^2 - 42x - 8$ on $[-8, 2]$

2. $f(x) = 8x^3 + 81x^2 - 42x - 8$ on $[-4, 2]$

3. $R(t) = 1 + 80t^3 + 5t^4 - 2t^5$ on $[-4.5,\ 4]$

4. $R(t) = 1 + 80t^3 + 5t^4 - 2t^5$ on $[0, 7]$

5. $h(z) = 4z^3 - 3z^2 + 9z + 12$ on $[-2, 1]$

6. $g(x) = 3x^4 - 26x^3 + 60x^2 - 11$ on $[1, 5]$

7. $Q(x) = (2 - 8x)^4 (x^2 - 9)^3$ on $[-3, 3]$

8. $h(w) = 2w^3 (w + 2)^5$ on $\left[-\frac{5}{2}, \frac{1}{2}\right]$

9. $f(z) = \dfrac{z + 4}{2z^2 + z + 8}$ on $[-10, 0]$

10. $A(t) = t^2 (10 - t)^{\frac{2}{3}}$ on $[2,\ 10.5]$

11. $f(y) = \sin\left(\frac{y}{3}\right) + \frac{2y}{9}$ on $[-10, 15]$

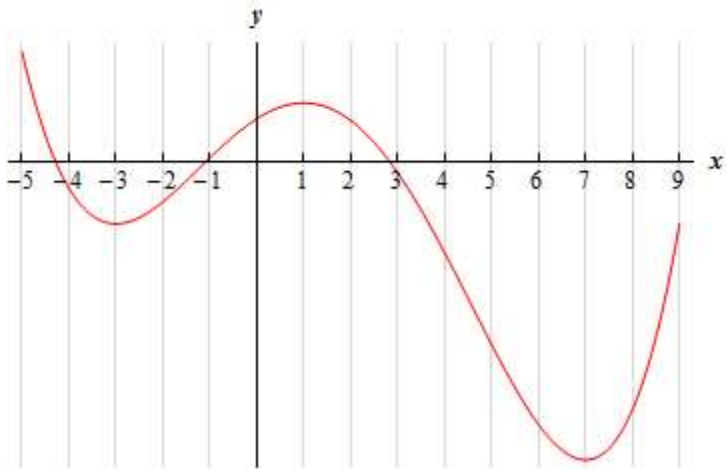
12. $g(w) = \mathbf{e}^{w^3 - 2w^2 - 7w}$ on $\left[-\frac{1}{2},\ \frac{5}{2}\right]$

13. $R(x) = \ln\left(x^2 + 4x + 14\right)$ on $[-4, 2]$

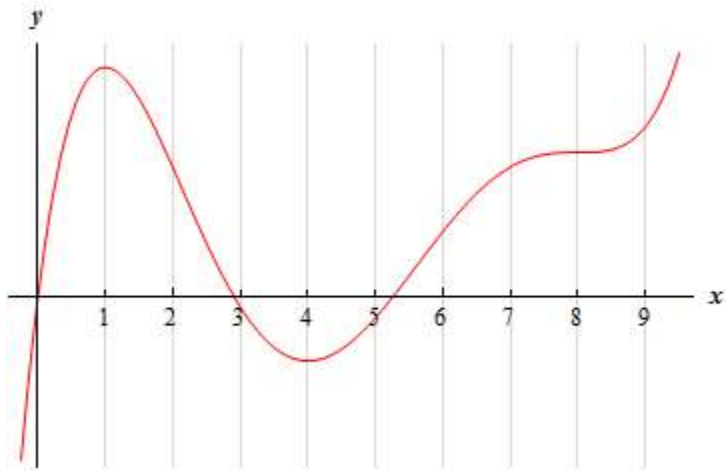
Section 4-5 : The Shape of a Graph, Part I

For problems 1 & 2 the graph of a function is given. Determine the intervals on which the function increases and decreases.

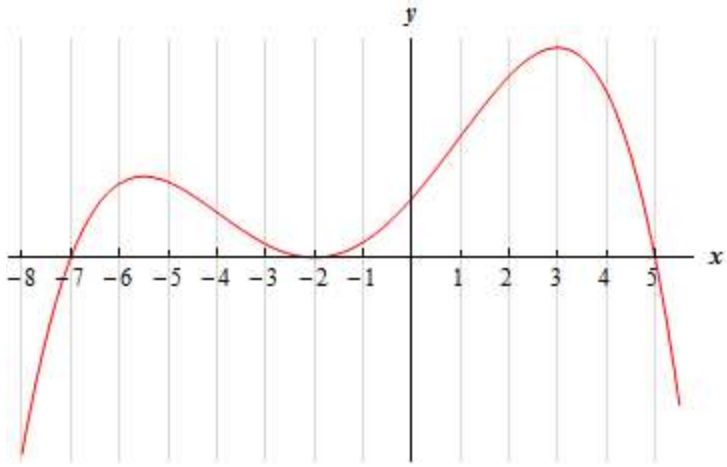
1.



2.



3. Below is the graph of the **derivative** of a function. From this graph determine the intervals in which the **function** increases and decreases.



4. This problem is about some function. All we know about the function is that it exists everywhere and we also know the information given below about the derivative of the function. Answer each of the following questions about this function.

(a) Identify the critical points of the function.
(b) Determine the intervals on which the function increases and decreases.
(c) Classify the critical points as relative maximums, relative minimums or neither.

$$f'(-5)=0 \qquad f'(-2)=0 \qquad f'(4)=0 \qquad f'(8)=0$$

$$f'(x)<0 \quad \text{on} \quad (-5,-2),\ (-2,4),\ (8,\infty) \qquad\qquad f'(x)>0 \quad \text{on} \quad (-\infty,-5),\ (4,8)$$

For problems 5 – 12 answer each of the following.

(a) Identify the critical points of the function.
(b) Determine the intervals on which the function increases and decreases.
(c) Classify the critical points as relative maximums, relative minimums or neither.

5. $f(x)=2x^3-9x^2-60x$

6. $h(t)=50+40t^3-5t^4-4t^5$

7. $y=2x^3-10x^2+12x-12$

8. $p(x)=\cos(3x)+2x$ on $\left[-\frac{3}{2},\ 2\right]$

9. $R(z)=2-5z-14\sin\left(\frac{z}{2}\right)$ on $[-10,7]$

10. $h(t)=t^2\sqrt[3]{t-7}$

11. $f(w) = w\mathbf{e}^{2-\frac{1}{2}w^2}$

12. $g(x) = x - 2\ln\left(1+x^2\right)$

13. For some function, $f(x)$, it is known that there is a relative maximum at $x = 4$. Answer each of the following questions about this function.

(a) What is the simplest form for the derivative of this function? Note : There really are many possible forms of the derivative so to make the rest of this problem as simple as possible you will want to use the simplest form of the derivative that you can come up with.

(b) Using your answer from **(a)** determine the most general form of the function.

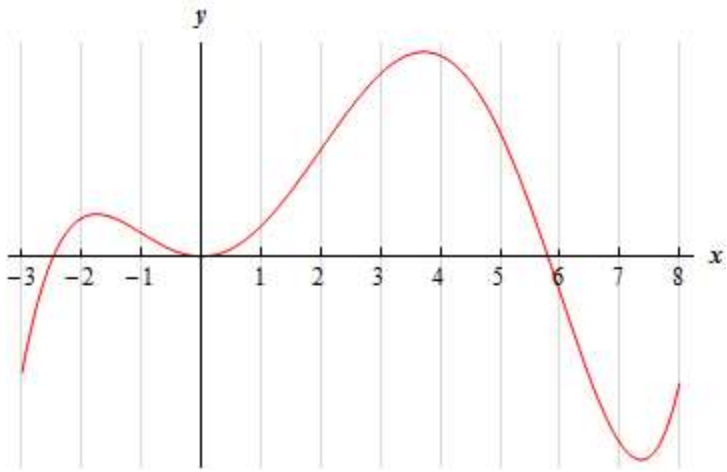
(c) Given that $f(4) = 1$ find a function that will have a relative maximum at $x = 4$. Note : You should be able to use your answer from **(b)** to determine an answer to this part.

14. Given that $f(x)$ and $g(x)$ are increasing functions. If we define $h(x) = f(x) + g(x)$ show that $h(x)$ is an increasing function.

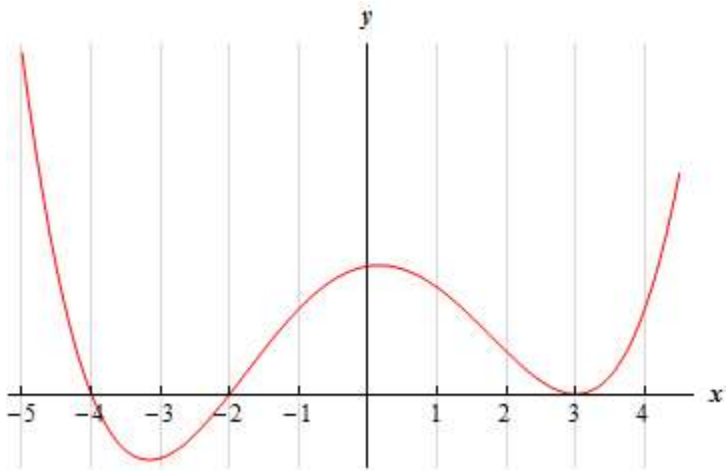
15. Given that $f(x)$ is an increasing function and define $h(x) = \left[f(x)\right]^2$. Will $h(x)$ be an increasing function? If yes, prove that $h(x)$ is an increasing function. If not, can you determine any other conditions needed on the function $f(x)$ that will guarantee that $h(x)$ will also increase?

Section 4-6 : The Shape of a Graph, Part I

1. The graph of a function is given below. Determine the intervals on which the function is concave up and concave down.



2. Below is the graph the **2nd derivative** of a function. From this graph determine the intervals in which the **function** is concave up and concave down.



For problems 3 – 8 answer each of the following.

(a) Determine a list of possible inflection points for the function.
(b) Determine the intervals on which the function is concave up and concave down.
(c) Determine the inflection points of the function.

3. $f(x) = 12 + 6x^2 - x^3$

4. $g(z) = z^4 - 12z^3 + 84z + 4$

5. $h(t) = t^4 + 12t^3 + 6t^2 - 36t + 2$

6. $h(w) = 8 - 5w + 2w^2 - \cos(3w)$ on $[-1, 2]$

7. $R(z) = z(z+4)^{\frac{2}{3}}$

8. $h(x) = \mathbf{e}^{4-x^2}$

For problems 9 – 14 answer each of the following.

(a) Identify the critical points of the function.
(b) Determine the intervals on which the function increases and decreases.
(c) Classify the critical points as relative maximums, relative minimums or neither.
(d) Determine the intervals on which the function is concave up and concave down.
(e) Determine the inflection points of the function.
(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

9. $g(t) = t^5 - 5t^4 + 8$

10. $f(x) = 5 - 8x^3 - x^4$

11. $h(z) = z^4 - 2z^3 - 12z^2$

12. $Q(t) = 3t - 8\sin\left(\frac{t}{2}\right)$ on $[-7, 4]$

13. $f(x) = x^{\frac{4}{3}}(x-2)$

14. $P(w) = w\mathbf{e}^{4w}$ on $\left[-2, \frac{1}{4}\right]$

15. Determine the minimum degree of a polynomial that has exactly one inflection point.

16. Suppose that we know that $f(x)$ is a polynomial with critical points $x = -1$, $x = 2$ and $x = 6$. If we also know that the 2nd derivative is $f''(x) = -3x^2 + 14x - 4$. If possible, classify each of the critical points as relative minimums, relative maximums. If it is not possible to classify the critical points clearly explain why they cannot be classified.

Section 4-7 : The Mean Value Theorem

For problems 1 & 2 determine all the number(s) *c* which satisfy the conclusion of Rolle's Theorem for the given function and interval.

1. $f(x) = x^2 - 2x - 8$ on $[-1,3]$

2. $g(t) = 2t - t^2 - t^3$ on $[-2,1]$

For problems 3 & 4 determine all the number(s) *c* which satisfy the conclusion of the Mean Value Theorem for the given function and interval.

3. $h(z) = 4z^3 - 8z^2 + 7z - 2$ on $[2,5]$

4. $A(t) = 8t + \mathbf{e}^{-3t}$ on $[-2,3]$

5. Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7,0]$, that $f(-7) = -3$ and that $f'(x) \le 2$. What is the largest possible value for $f(0)$?

6. Show that $f(x) = x^3 - 7x^2 + 25x + 8$ has exactly one real root.

Section 4-8 : Optimization

1. Find two positive numbers whose sum is 300 and whose product is a maximum.

2. Find two positive numbers whose product is 750 and for which the sum of one and 10 times the other is a minimum.

3. Let *x* and *y* be two positive numbers such that $x+2y=50$ and $(x+1)(y+2)$ is a maximum.

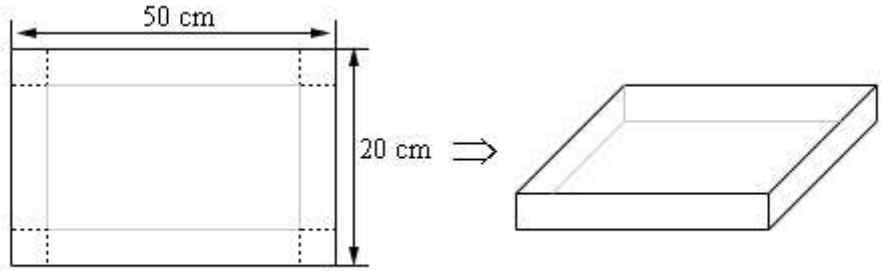
4. We are going to fence in a rectangular field. If we look at the field from above the cost of the vertical sides are \$10/ft, the cost of the bottom is \$2/ft and the cost of the top is \$7/ft. If we have \$700 determine the dimensions of the field that will maximize the enclosed area.

5. We have 45 m^2 of material to build a box with a square base and no top. Determine the dimensions of the box that will maximize the enclosed volume.

6. We want to build a box whose base length is 6 times the base width and the box will enclose 20 in^3. The cost of the material of the sides is \$3/in^2 and the cost of the top and bottom is \$15/in^2. Determine the dimensions of the box that will minimize the cost.

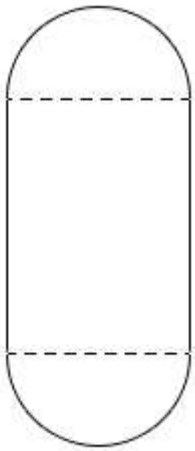
7. We want to construct a cylindrical can with a bottom but no top that will have a volume of 30 cm^3. Determine the dimensions of the can that will minimize the amount of material needed to construct the can.

8. We have a piece of cardboard that is 50 cm by 20 cm and we are going to cut out the corners and fold up the sides to form a box. Determine the height of the box that will give a maximum volume.

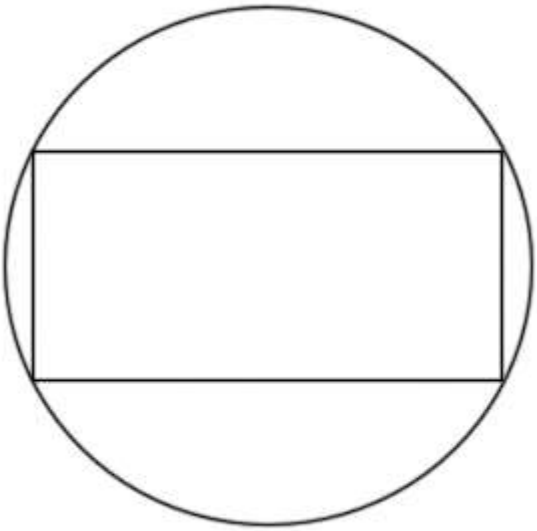


Section 4-9 : More Optimization

1. We want to construct a window whose middle is a rectangle and the top and bottom of the window are semi-circles. If we have 50 meters of framing material what are the dimensions of the window that will let in the most light?



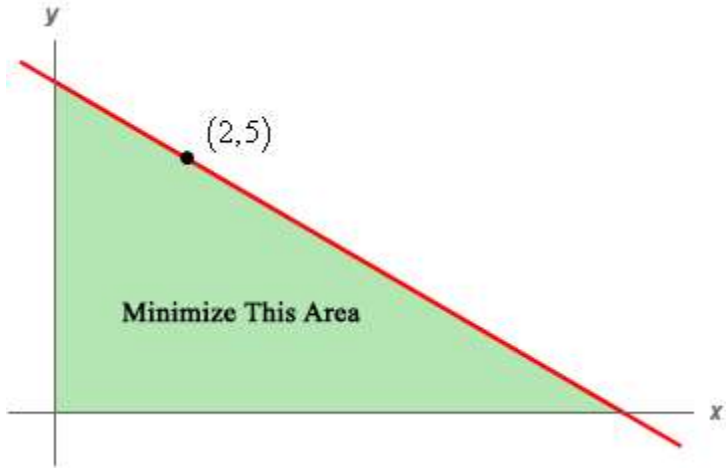
2. Determine the area of the largest rectangle that can be inscribed in a circle of radius 1.



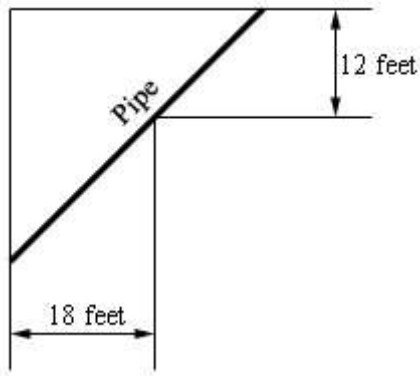
3. Find the point(s) on $x = 3 - 2y^2$ that are closest to $(-4, 0)$.

4. An 80 cm piece of wire is cut into two pieces. One piece is bent into an equilateral triangle and the other will be bent into a rectangle with one side 4 times the length of the other side. Determine where, if anywhere, the wire should be cut to maximize the area enclosed by the two figures.

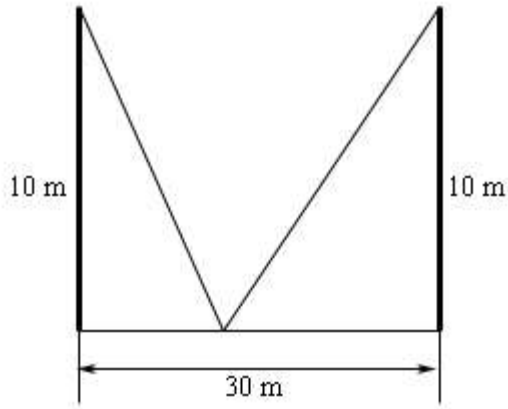
5. A line through the point $(2,5)$ forms a right triangle with the *x*-axis and *y*-axis in the 1st quadrant. Determine the equation of the line that will minimize the area of this triangle.



6. A piece of pipe is being carried down a hallway that is 18 feet wide. At the end of the hallway there is a right-angled turn and the hallway narrows down to 12 feet wide. What is the longest pipe (always keeping it horizontal) that can be carried around the turn in the hallway?



7. Two 10 meter tall poles are 30 meters apart. A length of wire is attached to the top of each pole and it is staked to the ground somewhere between the two poles. Where should the wire be staked so that the minimum amount of wire is used?



Section 4-10 : L'Hospital's Rule and Indeterminate Forms

Use L'Hospital's Rule to evaluate each of the following limits.

1. $\lim\limits_{x\to 2}\dfrac{x^3-7x^2+10x}{x^2+x-6}$

2. $\lim\limits_{w\to -4}\dfrac{\sin(\pi w)}{w^2-16}$

3. $\lim\limits_{t\to\infty}\dfrac{\ln(3t)}{t^2}$

4. $\lim\limits_{z\to 0}\dfrac{\sin(2z)+7z^2-2z}{z^2(z+1)^2}$

5. $\lim\limits_{x\to -\infty}\dfrac{x^2}{\mathbf{e}^{1-x}}$

6. $\lim\limits_{z\to\infty}\dfrac{z^2+\mathbf{e}^{4z}}{2z-\mathbf{e}^{z}}$

7. $\lim\limits_{t\to\infty}\left[t\ln\left(1+\dfrac{3}{t}\right)\right]$

8. $\lim\limits_{w\to 0^+}\left[w^2\ln\left(4w^2\right)\right]$

9. $\lim\limits_{x\to 1^+}\left[(x-1)\tan\left(\tfrac{\pi}{2}x\right)\right]$

10. $\lim\limits_{y\to 0^+}\left[\cos(2y)\right]^{1/y^2}$

11. $\lim\limits_{x\to\infty}\left[\mathbf{e}^{x}+x\right]^{1/x}$

Section 4-11 : Linear Approximations

For problems 1 & 2 find a linear approximation to the function at the given point.

1. $f(x) = 3x\,\mathbf{e}^{2x-10}$ at $x = 5$

2. $h(t) = t^4 - 6t^3 + 3t - 7$ at $t = -3$

3. Find the linear approximation to $g(z) = \sqrt[4]{z}$ at $z = 2$. Use the linear approximation to approximate the value of $\sqrt[4]{3}$ and $\sqrt[4]{10}$. Compare the approximated values to the exact values.

4. Find the linear approximation to $f(t) = \cos(2t)$ at $t = \frac{1}{2}$. Use the linear approximation to approximate the value of $\cos(2)$ and $\cos(18)$. Compare the approximated values to the exact values.

5. Without using any kind of computational aid use a linear approximation to estimate the value of $\mathbf{e}^{0.1}$.

Section 4-12 : Differentials

For problems 1 – 3 compute the differential of the given function.

1. $f(x) = x^2 - \sec(x)$

2. $w = \mathbf{e}^{x^4 - x^2 + 4x}$

3. $h(z) = \ln(2z)\sin(2z)$

4. Compute dy and Δy for $y = \mathbf{e}^{x^2}$ as *x* changes from 3 to 3.01.

5. Compute dy and Δy for $y = x^5 - 2x^3 + 7x$ as *x* changes from 6 to 5.9.

6. The sides of a cube are found to be 6 feet in length with a possible error of no more than 1.5 inches. What is the maximum possible error in the volume of the cube if we use this value of the length of the side to compute the volume?

Section 4-13 : Newton's Method

For problems 1 & 2 use Newton's Method to determine x_2 for the given function and given value of x_0.

1. $f(x) = x^3 - 7x^2 + 8x - 3$, $x_0 = 5$

2. $f(x) = x\cos(x) - x^2$, $x_0 = 1$

For problems 3 & 4 use Newton's Method to find the root of the given equation, accurate to six decimal places, that lies in the given interval.

3. $x^4 - 5x^3 + 9x + 3 = 0$ in $[4,6]$

4. $2x^2 + 5 = \mathbf{e}^x$ in $[3,4]$

For problems 5 & 6 use Newton's Method to find all the roots of the given equation accurate to six decimal places.

5. $x^3 - x^2 - 15x + 1 = 0$

6. $2 - x^2 = \sin(x)$

Section 4-14 : Business Applications

1. A company can produce a maximum of 1500 widgets in a year. If they sell *x* widgets during the year then their profit, in dollars, is given by,

$$P(x) = 30,000,000 - 360,000x + 750x^2 - \tfrac{1}{3}x^3$$

How many widgets should they try to sell in order to maximize their profit?

2. A management company is going to build a new apartment complex. They know that if the complex contains *x* apartments the maintenance costs for the building, landscaping etc. will be,

$$C(x) = 4000 + 14x - 0.04x^2$$

The land they have purchased can hold a complex of at most 500 apartments. How many apartments should the complex have in order to minimize the maintenance costs?

3. The production costs, in dollars, per day of producing *x* widgets is given by,

$$C(x) = 1750 + 6x - 0.04x^2 + 0.0003x^3$$

What is the marginal cost when $x = 175$ and $x = 300$? What do your answers tell you about the production costs?

4. The production costs, in dollars, per month of producing *x* widgets is given by,

$$C(x) = 200 + 0.5x + \frac{10000}{x}$$

What is the marginal cost when $x = 200$ and $x = 500$? What do your answers tell you about the production costs?

5. The production costs, in dollars, per week of producing *x* widgets is given by,

$$C(x) = 4000 - 32x + 0.08x^2 + 0.00006x^3$$

and the demand function for the widgets is given by,

$$p(x) = 250 + 0.02x - 0.001x^2$$

What is the marginal cost, marginal revenue and marginal profit when $x = 200$ and $x = 400$? What do these numbers tell you about the cost, revenue and profit?