Trigonometric Limits

more examples of limits

Substitution Theorem for Trigonometric Functions

laws for evaluating limits

Theorem A. **For each point c in function's domain:**

$$\lim_{x\to c}\sin x = \sin c, \qquad \lim_{x\to c}\cos x = \cos c,$$

$$\lim_{x\to c}\tan x = \tan c, \qquad \lim_{x\to c}\cot x = \cot c,$$

$$\lim_{x\to c}\csc x = \csc c, \qquad \lim_{x\to c}\sec x = \sec c.$$

Theorem A. For each point c in function's domain:

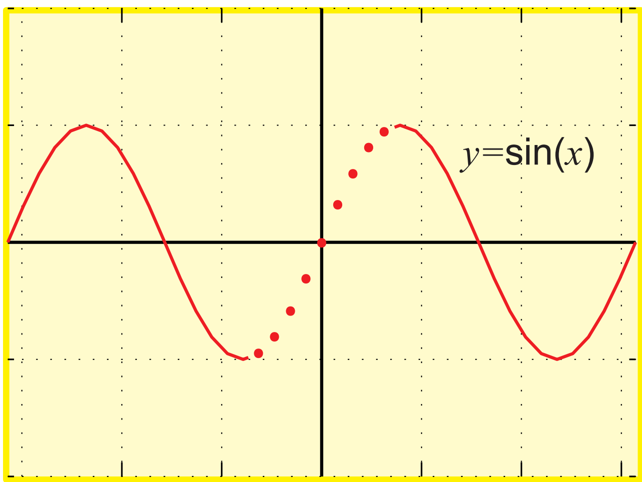
$$\lim_{x\to c}\sin x = \sin c, \qquad \lim_{x\to c}\cos x = \cos c,$$
$$\lim_{x\to c}\tan x = \tan c, \qquad \lim_{x\to c}\cot x = \cot c,$$
$$\lim_{x\to c}\csc x = \csc c, \qquad \lim_{x\to c}\sec x = \sec c.$$

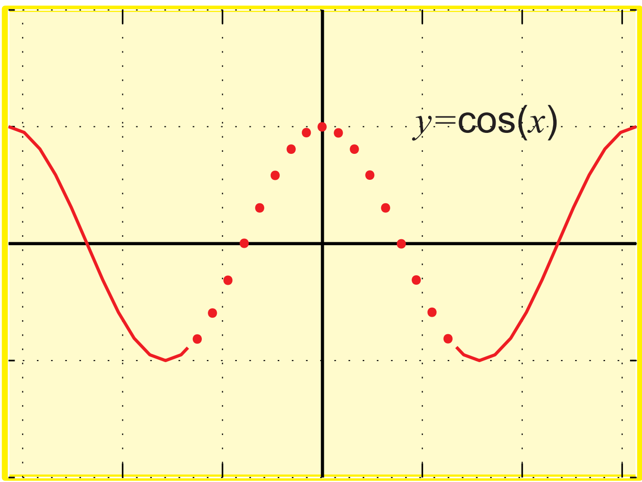
Proof. Prove first that

$$\lim_{x\to 0}\sin x = 0, \qquad \lim_{x\to 0}\cos x = 1.$$

Is it obvious?

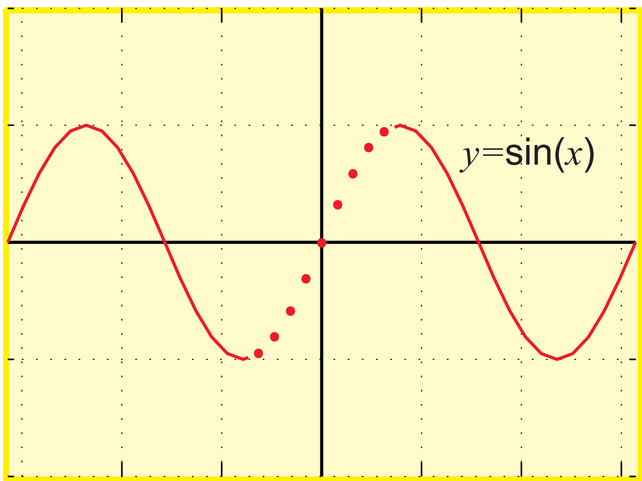
$$\lim_{x \to 0} \sin x = 0, \qquad \lim_{x \to 0} \cos x = 1.$$

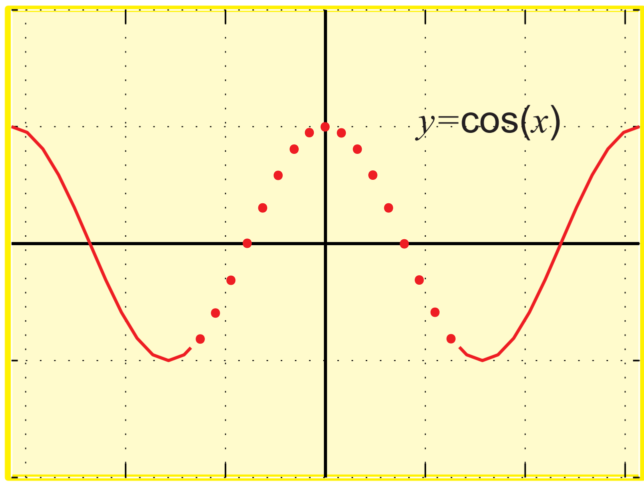




Is it obvious?

$$\lim_{x \to 0} \sin x = 0, \qquad \lim_{x \to 0} \cos x = 1.$$

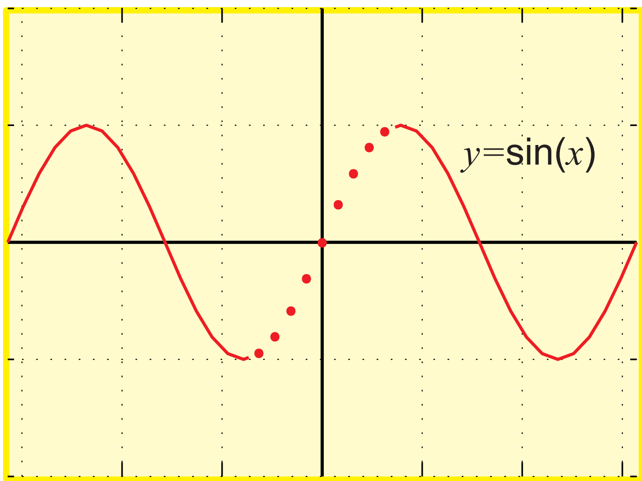


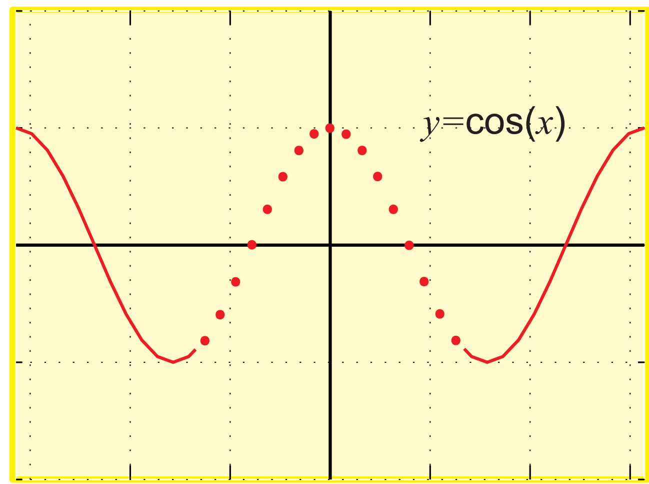


No. The picture is not precise.

Is it obvious?

$$\lim_{x \to 0} \sin x = 0, \qquad \lim_{x \to 0} \cos x = 1.$$



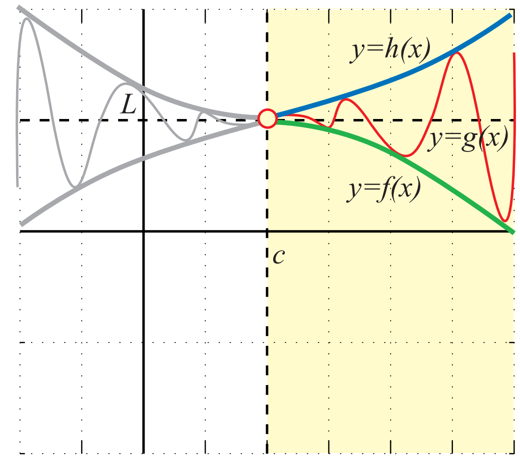


No. The picture is not precise.

Use definitions of $\sin(x)$ **and** $\cos(x)$.

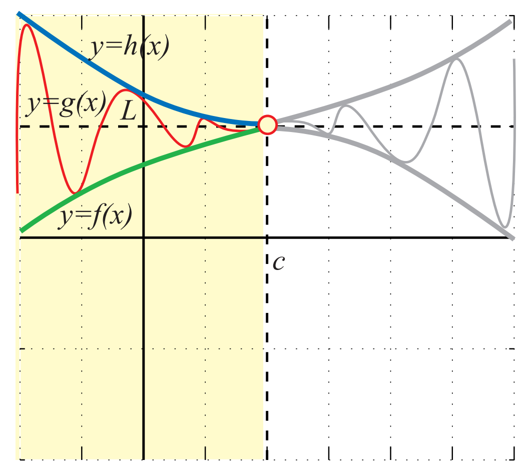
Use The One-Sided Squeeze Theorem. **If** $f(x) \le g(x) \le h(x)$ **near** c **and** $\lim_{x \to c^+} f(x) = \lim_{x \to c^+} h(x) = L_{\text{right}}$, **then**

$$\lim_{x \to c^+} g(x) = L_{\text{right}}$$



Also, if $\lim_{x \to c^-} f(x) = \lim_{x \to c^-} h(x) = L_{\text{left}}$, **then**

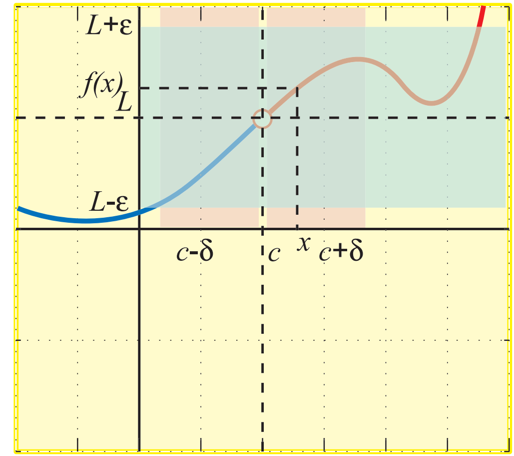
$$\lim_{x \to c^-} g(x) = L_{\text{left}}$$



Use The One-Sided Limits.

$$\lim_{x \to c} g(x) = L$$

$$\Leftrightarrow \lim_{x \to c^-} g(x) = \lim_{x \to c^+} g(x) = L$$



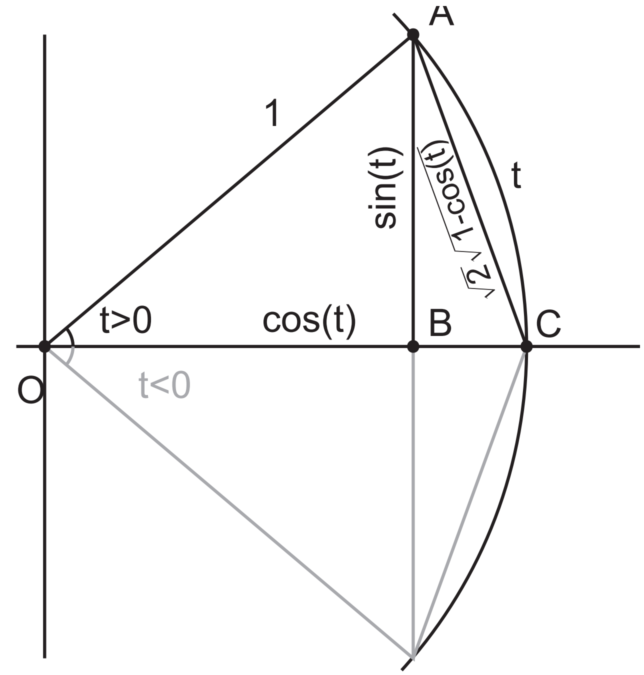
And other Limits Theorems.

$$\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x),$$

$$\lim_{x \to c} [f(x) g(x)] = [\lim_{x \to c} f(x)][\lim_{x \to c} g(x)],$$ e.t.c.

An estimate from geometry:

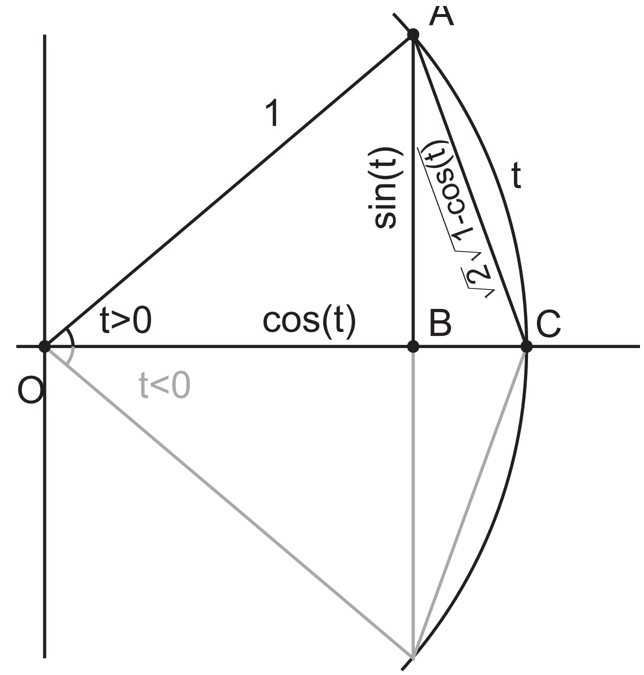
$$0 < AB < AC < \text{arc}AC$$



or,

$$\mathbf{0 < \sin(t) < \sqrt{2}\sqrt{1-\cos(t)} < t}$$

An estimate from geometry:



$$0 < AB < AC < \text{arc}AC$$

or,

$$0 < \sin(t) < \sqrt{2}\sqrt{1-\cos(t)} < t$$

By the Right-Sided Squeeze Theorem

$$\lim_{x\to 0^+} \sin(x) = 0, \quad \lim_{x\to 0^+} (1-\cos(x)) = 0,$$

Similarly,

$$\lim_{x \to 0^-} \sin(x) = 0, \qquad \lim_{x \to 0^-} (1 - \cos(x)) = 0.$$

Similarly,

$$\lim_{x\to 0^-} \sin(x) = 0, \quad \lim_{x\to 0^-} (1 - \cos(x)) = 0.$$

The left and the right limits are equal, thus

$$\lim_{x\to 0} \sin(x) = 0, \quad \lim_{x\to 0}(1 - \cos(x)) = 0$$

Similarly,

$$\lim_{x \to 0^-} \sin(x) = 0, \qquad \lim_{x \to 0^-} (1 - \cos(x)) = 0.$$

The left and the right limits are equal, thus

$$\lim_{x \to 0} \sin(x) = 0, \qquad \lim_{x \to 0} (1 - \cos(x)) = 0$$

or,

$$\lim_{x \to 0} \sin(x) = 0, \qquad \lim_{x \to 0} \cos(x) = 1.$$

EXAMPLES

EXAMPLE 1. Evaluate limit

$$\lim_{\theta \to \pi/4} \theta \tan(\theta)$$

Since $\theta = \pi/4$ is in the domain of the function $\theta \tan(\theta)$

EXAMPLE 1. Evaluate limit

$$\lim_{\theta \to \pi/4} \theta \tan(\theta)$$

Since $\theta = \pi/4$ is in the domain of the function $\theta \tan(\theta)$ we use Substitution Theorem to substitute $\pi/4$ for θ in the limit expression:

$$\lim_{\theta \to \pi/4} \theta \tan \theta = \frac{\pi}{4} \tan\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \cdot 1 = \frac{\pi}{4}.$$

EXAMPLE 2. Evaluate limit

$$\lim_{\theta \to \pi/2} \frac{\cos^2(\theta)}{1 - \sin(\theta)}.$$

Since at $\theta = \pi/2$ the denominator of $\cos^2(\theta)/(1 - \sin(\theta))$ turns to zero, we can not substitute $\pi/2$ for θ immediately.

EXAMPLE 2. Evaluate limit

$$\lim_{\theta\to\pi/2} \frac{\cos^2(\theta)}{1-\sin(\theta)}.$$

Since at $\theta = \pi/2$ the denominator of $\cos^2(\theta)/(1 - \sin(\theta))$ turns to zero, we can not substitute $\pi/2$ for θ immediately. Instead, we rewrite the expression using $\sin^2(\theta) + \cos^2(\theta) = 1$:

$$\lim_{\theta\to\pi/2} \frac{1-\sin^2(\theta)}{1-\sin(\theta)} = \lim_{\theta\to\pi/2} \frac{(1-\sin(\theta))(1+\sin(\theta))}{(1-\sin(\theta))}$$

Finally,

$$\lim_{\theta \to \pi/2} \frac{(1 - \sin(\theta))(1 + \sin(\theta))}{(1 - \sin(\theta))}$$

Finally,

$$\lim_{\theta\to\pi/2} \frac{(1-\sin(\theta))(1+\sin(\theta))}{(1-\sin(\theta))}$$

$$= \lim_{\theta\to\pi/2}\left(\frac{1-\sin(\theta)}{1-\sin(\theta)}\right) \lim_{\theta\to\pi/2}(1+\sin(\theta))$$

Finally,

$$\lim_{\theta\to\pi/2} \frac{(1-\sin(\theta))(1+\sin(\theta))}{(1-\sin(\theta))}$$

$$= \lim_{\theta\to\pi/2}\left(\frac{1-\sin(\theta)}{1-\sin(\theta)}\right) \lim_{\theta\to\pi/2}(1+\sin(\theta))$$

$$= 1\cdot(1+\sin(\pi/2)) = 2.$$

Special Trigonometric Limits

$\sin(x)/x \to ?$ **as** $x \to 0$

Theorem B1.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Theorem B2.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$

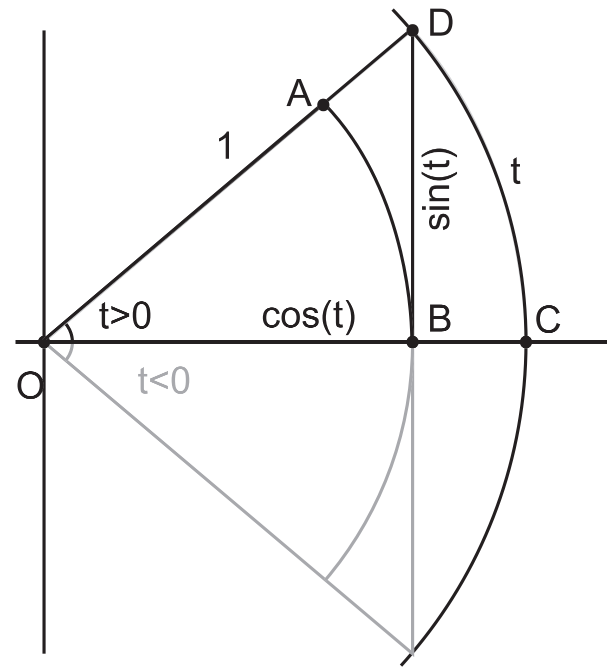
Proof B1. A fact from geometry: $(t > 0)$

area(OAB)≤area(ODB)≤area(ODC)

$$\cos^2(t)t/2 \leq \sin(t)\cos(t)/2 \leq t/2.$$

dividing by $\cos(t)t/2$ get

$$\cos t \leq \frac{\sin t}{t} \leq \frac{1}{\cos t}$$



Right-Sided Squeeze Theorem:

$$\lim_{t\to 0^+} \frac{\sin t}{t} = 1$$

The same inequality holds for $t < 0$:

$$\cos t \leq \frac{\sin t}{t} \leq \frac{1}{\cos t}$$

Left-Sided Squeeze Theorem:

$$\lim_{t \to 0^-} \frac{\sin t}{t} = 1$$

The left and the right limits are equal, thus,

$$\lim_{t \to 0} \frac{\sin t}{t} = 1$$

Proof B2. By multiplying numerator and denominator with $(1 + \cos x)$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{(1 - \cos x)}{x} \frac{(1 + \cos x)}{(1 + \cos x)}$$

Proof B2. By multiplying numerator and denominator with $(1+\cos x)$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{(1-\cos x)}{x}\frac{(1+\cos x)}{(1+\cos x)}$$

$$= \lim_{x\to 0}\frac{(1-\cos^2 x)}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$$

Proof B2. By multiplying numerator and denominator with $(1+\cos x)$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{(1-\cos x)}{x}\frac{(1+\cos x)}{(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{(1-\cos^2 x)}{x(1+\cos x)}=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$$

Using B1 write

$$=\left[\lim_{x\to 0}\frac{\sin x}{x}\right]\frac{\lim_{x\to 0}[\sin x]}{\lim_{x\to 0}[1+\cos x]}=0.$$

EXAMPLES

EXAMPLE 3. Evaluate limit

$$\lim_{t \to 0} \frac{\tan t}{t}$$

EXAMPLE 3. Evaluate limit

$$\lim_{t \to 0} \frac{\tan t}{t}$$

Recalling $\tan t = \sin t / \cos t$, and using B1:

$$= \lim_{t \to 0} \frac{\sin t}{(\cos t)t}$$

EXAMPLE 3. Evaluate limit

$$\lim_{t \to 0} \frac{\tan t}{t}$$

Recalling $\tan t = \sin t / \cos t$, and using B1:

$$= \lim_{t \to 0} \frac{\sin t}{(\cos t)t} = \left[\lim_{t \to 0} \frac{\sin t}{t}\right] \lim_{t \to 0} \frac{1}{\cos t}$$

EXAMPLE 3. Evaluate limit

$$\lim_{t \to 0} \frac{\tan t}{t}$$

Recalling $\tan t = \sin t / \cos t$, and using B1:

$$= \lim_{t \to 0} \frac{\sin t}{(\cos t)t} = \left[\lim_{t \to 0} \frac{\sin t}{t}\right] \lim_{t \to 0} \frac{1}{\cos t} = 1 \cdot \frac{1}{1} = 1$$

EXAMPLE 4. Evaluate limit (Can't use B1 !):

$$\lim_{t \longrightarrow 0} \frac{\sin(3t)}{t}$$

EXAMPLE 4. Evaluate limit (Can't use B1 !):

$$\lim_{t\to 0}\frac{\sin(3t)}{t}$$

multiply both numerator and denominator with 3:

$$=\lim_{t\to 0}3\frac{\sin(3t)}{3t}$$

EXAMPLE 4. Evaluate limit (Can't use B1 !):

$$\lim_{t\to 0}\frac{\sin(3t)}{t}$$

multiply both numerator and denominator with 3:

$$= \lim_{t\to 0} 3\frac{\sin(3t)}{3t}$$

Now, $t \longrightarrow 0$ as $3t \longrightarrow 0$, so

$$= \lim_{3t\to 0} 3\frac{\sin(3t)}{3t} = 3. \qquad \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

B1 applies (with a substitution $x = 3t$).

EXAMPLE 5. Evaluate limit

$$\lim_{t\to 0}\frac{1-\cos t}{\sin t}$$

EXAMPLE 5. Evaluate limit

$$\lim_{t\to 0}\frac{1-\cos t}{\sin t}$$

Divide both numerator and denominator with t:

$$=\lim_{t\to 0}\frac{\dfrac{1-\cos t}{t}}{\dfrac{\sin t}{t}}$$

EXAMPLE 5. Evaluate limit

$$\lim_{t\to 0}\frac{1-\cos t}{\sin t}$$

Divide both numerator and denominator with t:

$$=\lim_{t\to 0}\frac{\dfrac{1-\cos t}{t}}{\dfrac{\sin t}{t}}$$

Use B1 and B2:

$$=\frac{0}{1}=0.$$