2.3 PROPERTIES OF LIMITS

This section presents results which make it easier to calculate limits of combinations of functions or to show that a limit does not exist. The main result says we can determine the limit of "elementary combinations" of functions by calculating the limit of each function separately and recombining these results for our final answer.

Main Limit Theorem:

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$,

then

(a) $\lim_{x\to a} \{f(x) + g(x)\} = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$

(b) $\lim_{x\to a} \{f(x) - g(x)\} = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$

(c) $\lim_{x\to a} k\, f(x) = k \lim_{x\to a} f(x) = kL$

(d) $\lim_{x\to a} f(x)\cdot g(x) = \{\lim_{x\to a} f(x)\}\cdot\{\lim_{x\to a} g(x)\} = L\cdot M$

(e) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$ (if $M \neq 0$).

(f) $\lim_{x\to a} \{ f(x) \}^n = \left\{\lim_{x\to a} f(x)\right\}^n = L^n$

(g) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ (if $L > 0$ when n is even)

The Main Limit Theorem says we get the same result if we first perform the algebra and then take the limit or if we take the limits first and then perform the algebra: e.g., (a) the limit of the sum equals the sum of the limits. A proof of the Main Limit Theorem is not inherently difficult, but it requires a more precise definition of the limit concept than we have given, and it then involves a number of technical difficulties.

Practice 1: For $f(x) = x^2 - x - 6$ and $g(x) = x^2 - 2x - 3$, evaluate the following limits:

(a) $\lim_{x\to 1} \{f(x) + g(x)\}$ (b) $\lim_{x\to 1} f(x)g(x)$ (c) $\lim_{x\to 1} f(x)/g(x)$ (d) $\lim_{x\to 3} \{f(x) + g(x)\}$

(e) $\lim_{x\to 3} f(x)g(x)$ (f) $\lim_{x\to 3} f(x)/g(x)$ (g) $\lim_{x\to 2} \{ f(x) \}^3$ (h) $\lim_{x\to 2} \sqrt{1 - g(x)}$

Limits of Some Very Nice Functions: Substitution

As you may have noticed in the previous example, for some functions f(x) it is possible to calculate the limit as x approaches a simply by substituting x = a into the function and then evaluating f(a), but sometimes this method does not work. The Substitution Theorem uses the following Two Easy Limits and the Main Limit Theorem to partially answer when such a substitution is valid.

Two Easy Limits: $\lim_{x \to a} k = k$ and $\lim_{x \to a} x = a$.

Substitution Theorem For Polynomial and Rational Functions:

If P(x) and Q(x) are **polynomials** and a is any number,

then $\lim_{x \to a} P(x) = P(a)$ and $\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$ if $Q(a) \neq 0$.

The Substitution Theorem says that we can calculate the limits of polynomials and rational functions by substituting as long as the substitution does not result in a division by zero.

Practice 2: Evaluate (a) $\lim_{x \to 2} 5x^3 - x^2 + 3$ (b) $\lim_{x \to 2} \frac{x^3 - 7x}{x^2 + 3x}$ (c) $\lim_{x \to 2} \frac{x^2 - 2x}{x^2 - x - 2}$

Limits of Other Combinations of Functions

So far we have concentrated on limits of single functions and elementary combinations of functions. If we are working with limits of other combinations or compositions of functions, the situation is slightly more difficult, but sometimes these more complicated limits have useful geometric interpretations.

Example 1: Use the function defined by the graph in Fig. 1 to evaluate

(a) $\lim_{x \to 1} \{ 3 + f(x) \}$ (b) $\lim_{x \to 1} f(2+x)$ (c) $\lim_{x \to 0} f(3-x)$ (d) $\lim_{x \to 2} f(x+1) - f(x)$

Solution: (a) $\lim_{x \to 1} \{ 3 + f(x) \}$ is a straightforward application of part (a) of the Main Limit Theorem:

$$\lim_{x \to 1} \{ 3 + f(x) \} = \lim_{x \to 1} 3 + \lim_{x \to 1} f(x) = 3 + 2 = 5 .$$

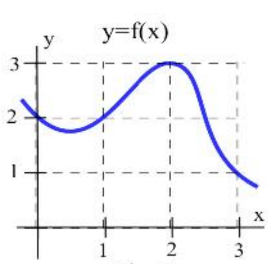

Fig. 1

(b) We first need to examine what happens to the quantity 2+x , as $x \to 1$, before we can consider the limit of f(2+x). When x is very close to 1, the value of 2+x is very close to 3, so the limit of f(2+x) as $x \to 1$ is equivalent to the limit of f(w) as $w \to 3$ (w=2+x), and it is clear from the graph that $\lim_{x \to 3} f(w) = 1$: $\lim_{x \to 1} f(2+x) = \lim_{x \to 3} f(w) = 1$ (w represents 2+x).

In most cases it is not necessary to formally substitute a new variable w for the quantity 2+x, but it is still necessary to think about what happens to the quantity 2+x as $x \to 1$.

(c) As $x \to 0$, the quantity 3–x will approach 3 so we want to know what happens to the values of f when the variable is approaching 3: $\lim_{x \to 0} f(3-x) = 1$.

(d) $\lim_{x \to 2} \{ f(x+1) - f(x) \} = \lim_{x \to 2} f(x+1) - \lim_{x \to 2} f(x)$ replace x+1 with w

$$= \lim_{x \to 3} f(w) - \lim_{x \to 2} f(x) = 1 - 3 = -2 .$$

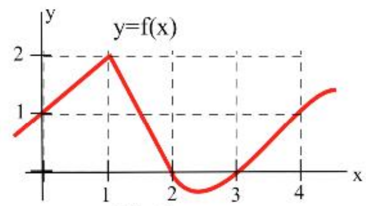

Fig. 2

Practice 3: Use the function defined by the graph in Fig. 2 to evaluate

(a) $\lim_{x \to 1} f(2x)$ (b) $\lim_{x \to 2} f(x-1)$

(c) $\lim_{x \to 0} 3 \cdot f(4+x)$ (d) $\lim_{x \to 2} f(3x-2)$.

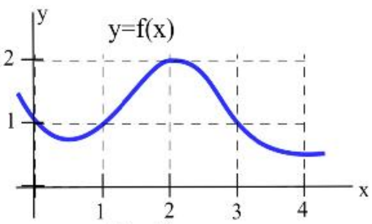

Fig. 3

Example 2: Use the function defined by the graph in Fig. 3 to evaluate

(a) $\lim_{h \to 0} f(3+h)$ (b) $\lim_{h \to 0} f(3)$

(c) $\lim_{h \to 0} \{ f(3+h) - f(3) \}$ (d) $\lim_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$

Solution: Part (d) is a common form of limit, and parts (a) – (c) are the steps we need to evaluate (d).

(a) As $h \to 0$, the quantity $w = 3+h$ will approach 3 so $\lim_{h \to 0} f(3+h) = \lim_{x \to 3} f(w) = 1$.

(b) f(3) is the constant 1 and f(3) does not depend on h in any way so $\lim_{h \to 0} f(3) = 1$.

(c) The limit in part (c) is just an algebraic combination of the limits in (a) and (b):

$$\lim_{h \to 0} \{ f(3+h) - f(3) \} = \lim_{h \to 0} f(3+h) - \lim_{h \to 0} f(3) = 1 - 1 = 0 .$$

The quantity $f(3+h) - f(3)$ also has a geometric interpretation — it is the change in the y–coordinates, the Δy, between the points (3,f(3)) and (3+h,f(3+h)). (Fig. 4)

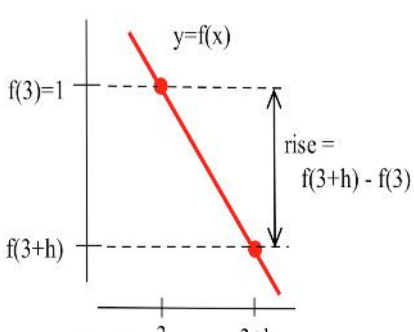

Fig. 4

(d) As $h \to 0$, the numerator and denominator of $\dfrac{f(3+h) - f(3)}{h}$ both approach 0 so we cannot immediately determine the value of the limit. But if we recognize that $f(3+h) - f(3) = \Delta y$ for the two points (3, f(3)) and (3+h, f(3+h)) and that $h = \Delta x$ for the same two points, then we can interpret $\dfrac{f(3+h) - f(3)}{h}$ as $\dfrac{\Delta y}{\Delta x}$ which is the slope of the secant line through the two points. So

$$\lim_{h \to 0} \frac{f(3+h)-f(3)}{h} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \{ \text{ slope of the secant line } \}$$

$$= \text{slope of the tangent line at } (3, f(3)) \approx -2 .$$

This limit, representing the slope of line tangent to the graph of f at the point (3, f(3)) , is a pattern we will see often in the future.

Tangent Lines as Limits

If we have two points on the graph of a function, (x, f(x)) and (x+h, f(x+h)), then $\Delta y = f(x+h) - f(x)$ and $\Delta x = (x+h) - (x) = h$ so the slope of the secant line through those points is $m_{secant} = \frac{\Delta y}{\Delta x}$ and the slope of the line tangent to the graph of f at the point (x, f(x)) is, by definition,

$$m_{tangent} = \lim_{\Delta x \to 0} \{ \text{ slope of the secant line } \} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} .$$

Example 3: Give a geometric interpretation for the following limits and **estimate** their values for the function in Fig. 5: (a) $\lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$ (b) $\lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$

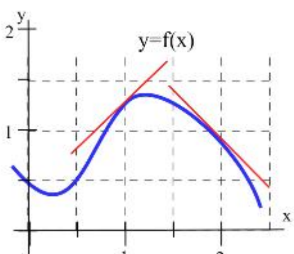

Fig. 5

Solution: Part (a) represents the slope of the line tangent to the graph of f(x) at the point (1, f(1)) so

$\lim_{h \to 0} \frac{f(1+h)-f(1)}{h} \approx 1$. Part (b) represents the slope of the line tangent to the graph of f(x) at the point (2, f(2)) so $\lim_{h \to 0} \frac{f(2+h)-f(2)}{h} \approx -1$.

Practice 4: Give a geometric interpretation for the following limits and estimate their values for the function in Fig. 6:

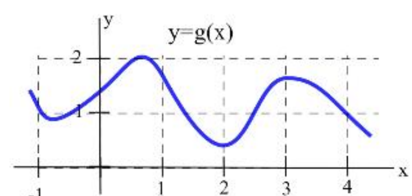

Fig. 6

$$\lim_{h \to 0} \frac{g(1+h)-g(1)}{h} \qquad \lim_{h \to 0} \frac{g(3+h)-g(3)}{h} \qquad \lim_{h \to 0} \frac{g(h)-g(0)}{h} .$$

Comparing the Limits of Functions

Sometimes it is difficult to work directly with a function. However, if we can compare our difficult function with easier ones, then we can use information about the easier functions to draw conclusions about the difficult one. If the complicated function is always between two functions whose limits are equal, then we know the limit of the complicated function.

Squeezing Theorem (Fig. 7):

If $g(x) \le f(x) \le h(x)$ for all x near c (for all x close to but not equal to c)

and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$

then for x near c, f(x) will be squeezed between g(x) and h(x), and $\lim_{x\to c} f(x) = L$.

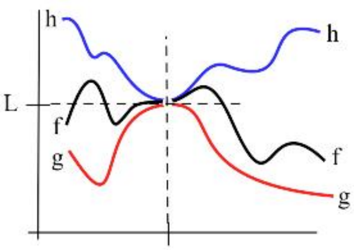

Fig. 7

Example 4: Use the inequality $-|x| \le \sin(x) \le |x|$ to determine $\lim_{x\to 0} \sin(x)$ and $\lim_{x\to 0} \cos(x)$.

Solution: $\lim_{x\to 0} |x| = 0$ and $\lim_{x\to 0} -|x| = 0$ so, by the Squeezing Theorem, $\lim_{x\to 0} \sin(x) = 0$. If $-\pi/2 < x < \pi/2$ then $\cos(x) = +\sqrt{1-\sin^2(x)}$ so $\lim_{x\to 0} \cos(x) = \lim_{x\to 0} +\sqrt{1-\sin^2(x)} = +\sqrt{1-0} = 1$.

Example 5: Evaluate $\lim_{x\to 0} x \cdot \sin(\frac{1}{x})$.

Solution: The graph of $y = \sin(\frac{1}{x})$ for values of x near 0 is shown in Fig. 8. The y–values of this graph change very rapidly for values of x near 0, but they all lie between –1 and +1: $-1 \le \sin(\frac{1}{x}) \le +1$. The fact that $\sin(\frac{1}{x})$ is bounded between –1 and +1 implies that $x \sin(\frac{1}{x})$ is stuck between $-x$ and $+x$, so the function we are interested in , $x \cdot \sin(\frac{1}{x})$, is squeezed between two "easy" functions, $-x$ and x (Fig. 9). Both "easy" functions approach 0 as $x \to 0$, so $x \cdot \sin(\frac{1}{x})$ must also approach 0 as $x \to 0$: $\lim_{x\to 0} x \cdot \sin(\frac{1}{x}) = 0$.

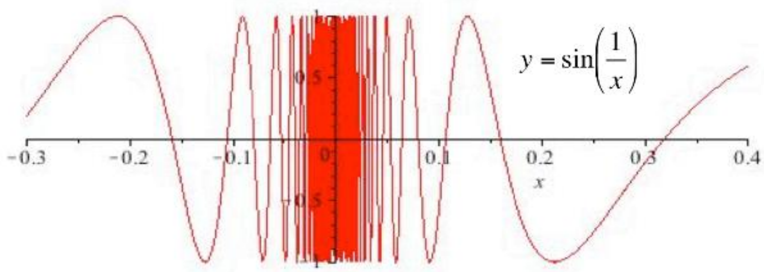

Fig. 8

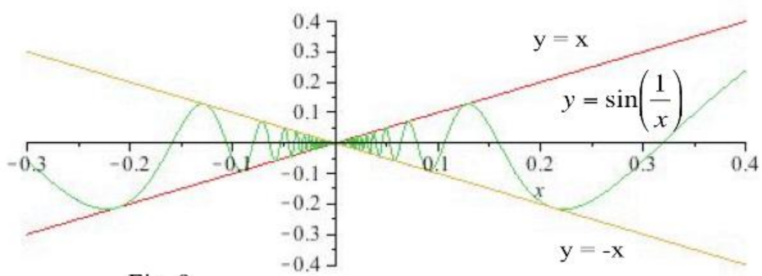

Fig. 9

Practice 5: If f(x) is always between $x^2 + 2$ and $2x + 1$, then $\lim_{x\to 1} f(x) = ?$

Practice 6: Use the relation $\cos(x) \le \frac{\sin(x)}{x} \le 1$ to show that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. (The steps for deriving the inequalities are shown in problem 19.)

List Method for Showing that a Limit Does Not Exist

If the limit, as x approaches c, exists and equals L, then we can guarantee that the values of f(x) are as close to L as we want by restricting the values of x to be very, very close to c. To show that a limit, as x approaches c, does not exist, we need to show that no matter how closely we restrict the values of x to c, the values of f(x) are not all close to a single, finite value L. One way to demonstrate that $\lim_{x \to c} f(x)$ does not exist is to show that the left and right limits exist but are not equal.

Another method of showing that $\lim_{x \to c} f(x)$ does not exist is to find two infinite lists of numbers, $\{a_1, a_2, a_3, a_4, \ldots\}$ and $\{b_1, b_2, b_3, b_4, \ldots\}$, which approach arbitrarily close to the value c as the subscripts get larger, but so that the lists of function values, $\{f(a_1), f(a_2), f(a_3), f(a_4), \ldots\}$ and $\{f(b_1), f(b_2), f(b_3), f(b_4), \ldots\}$, approach two different numbers as the subscripts get larger.

Example 6: For $f(x) = \begin{cases} 1 & \text{if } x < 1 \\ x & \text{if } 1 < x < 3 \\ 2 & \text{if } 3 < x \end{cases}$, show that $\lim_{x \to 3} f(x)$ does not exist.

Solution: We can use one–sided limits to show that this limit does not exist, or we can use the list method by selecting values for one list to approach 3 from the right and values for the other list to approach 3 from the left.

One way to define values of $\{a_1, a_2, a_3, a_4, \ldots\}$ which approach 3 from the right is to define $a_1 = 3 + 1$, $a_2 = 3 + \frac{1}{2}$, $a_3 = 3 + \frac{1}{3}$, $a_4 = 3 + \frac{1}{4}$ and, in general, $a_n = 3 + \frac{1}{n}$. Then $a_n > 3$ so $f(a_n) = 2$ for all subscripts n, and the values in the list $\{f(a_1), f(a_2), f(a_3), f(a_4), \ldots\}$ are approaching 2. In fact, all of the $f(a_n) = 2$.

We can define values of $\{b_1, b_2, b_3, b_4, \ldots\}$ which approach 3 from the left by $b_1 = 3 - 1$, $b_2 = 3 - \frac{1}{2}$, $b_3 = 3 - \frac{1}{3}$, $b_4 = 3 - \frac{1}{4}$ and, in general, $b_n = 3 - \frac{1}{n}$. Then $b_n < 3$ so $f(b_n) = b_n = 3 - \frac{1}{n}$ for each subscript n, and the values in the list $\{f(b_1), f(b_2), f(b_3), f(b_4), \ldots\} = \{2, 2.5, 2.67, 2.75, 2.8, \ldots, 3 - \frac{1}{n}, \ldots\}$ approach 3.

Since the values in the lists $\{f(a_1), f(a_2), f(a_3), f(a_4), \ldots\}$ and $\{f(b_1), f(b_2), f(b_3), f(b_4), \ldots\}$ approach two different numbers, we can conclude that $\lim_{x \to 3} f(x)$ does not exist.

Example 7: Let $h(x) = \begin{cases} 2 & \text{if } x \text{ is a \textbf{rational} number} \\ 1 & \text{if } x \text{ is an \textbf{irrational} number} \end{cases}$ be the "holey" function introduced in Section 0.4 . Use the list method to show that $\lim_{x \to 3} h(x)$ does not exist.

Solution: Let $\{ a_1, a_2, a_3, a_4, \ldots \}$ be a list of rational numbers which approach 3, for example, $a_1 = 3 + 1$, $a_2 = 3 + 1/2, \ldots, a_n = 3 + 1/n$. Then $f(a_n)$ always equals 2 so $\{ f(a_1), f(a_2), f(a_3), f(a_4), \ldots \} = \{ 2, 2, 2, \ldots \}$ and the $f(a_n)$ values "approach" 2. If $\{b_1, b_2, b_3, b_4, \ldots \}$ is a list of irrational numbers which approach 3, for example, $b_1 = 3 + \pi$, $b_2 = 3 + \pi/2, \ldots, b_n = 3 + \pi/n$. then $\{ f(b_1), f(b_2), f(b_3), f(b_4), \ldots \} = \{ 1, 1, 1, \ldots \}$ and the $f(b_n)$ "approach" 1. Since the $f(a_n)$ and $f(b_n)$ values approach different numbers, the limit as $x \to 3$ does not exist.

A similar argument will work as x approaches any number c, so for every c we have that $\lim_{x \to c} h(x)$ does not exist. The "holey" function does not have a limit as x approaches any value c.

PROBLEMS

1. Use the functions f and g defined by the graphs in Fig. 10 to determine the following limits.

   (a) $\lim_{x \to 1} \{ f(x) + g(x) \}$ (b) $\lim_{x \to 1} f(x) \cdot g(x)$

   (c) $\lim_{x \to 1} f(x)/g(x)$ (d) $\lim_{x \to 1} f(g(x))$

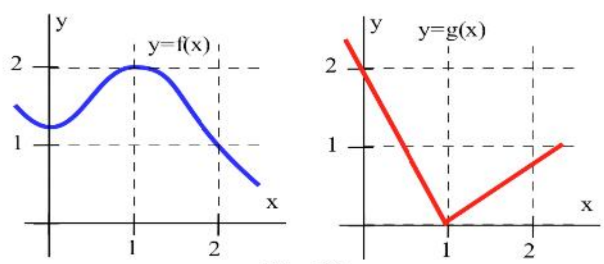

Fig. 10

2. Use the functions f and g defined by the graphs in Fig. 10 to determine the following limits.

   (a) $\lim_{x \to 2} \{ f(x) + g(x) \}$ (b) $\lim_{x \to 2} f(x) \cdot g(x)$

   (c) $\lim_{x \to 2} f(x)/g(x)$ (d) $\lim_{x \to 2} f(g(x))$

3. Use the function h defined by the graph in Fig. 11 to determine the following limits.

   (a) $\lim_{x \to 2} h(2x - 2)$ (b) $\lim_{x \to 2} \{ x + h(x) \}$

   (c) $\lim_{x \to 2} h(1 + x)$ (d) $\lim_{x \to 3} h(x/2)$

4. Use the function h defined by the graph in Fig. 11 to determine the following limits.

   (a) $\lim_{x \to 2} h(5 - x)$ (b) $\lim_{x \to 2} x \cdot h(x - 1)$

   (c) $\lim_{x \to 0} \{ h(3 + x) - h(3) \}$ (d) $\lim_{x \to 0} \dfrac{h(3+x) - h(3)}{x}$

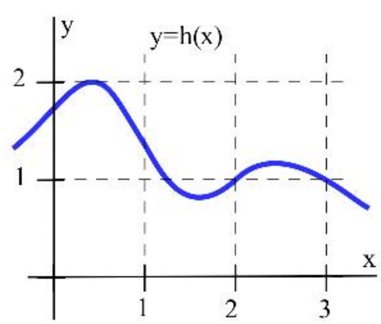

Fig. 11

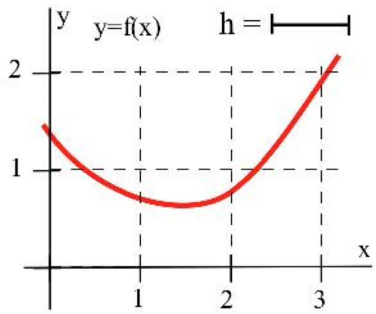

Fig. 12

5. Label the parts of the graph of f (Fig. 12) which are described by

(a) $2 + h$ (b) $f(2)$ (c) $f(2 + h)$

(d) $f(2 + h) - f(2)$ (e) $\dfrac{f(2 + h) - f(2)}{(2 + h) - (2)}$ (f) $\dfrac{f(2 - h) - f(2)}{(2 - h) - (2)}$

6. Label the parts of the graph of f (Fig. 13) which are described by

(a) $a + h$ (b) $g(a)$ (c) $g(a + h)$

(d) $g(a + h) - g(a)$ (e) $\dfrac{g(a + h) - g(a)}{(a + h) - (a)}$ (f) $\dfrac{g(a - h) - g(a)}{(a - h) - (a)}$

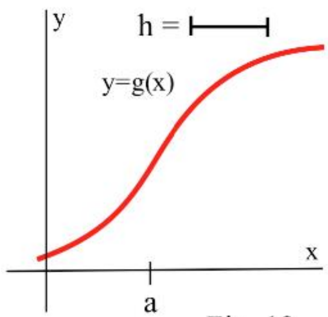

Fig. 13

7. Use the function f defined by the graph in Fig. 14 to determine the following limits.

(a) $\lim_{x \to 1^+} f(x)$ (b) $\lim_{x \to 1^-} f(x)$ (c) $\lim_{x \to 1} f(x)$

(d) $\lim_{x \to 3^+} f(x)$ (e) $\lim_{x \to 3^-} f(x)$ (f) $\lim_{x \to 3} f(x)$

(g) $\lim_{x \to -1^+} f(x)$ (h) $\lim_{x \to -1^-} f(x)$ (i) $\lim_{x \to -1} f(x)$

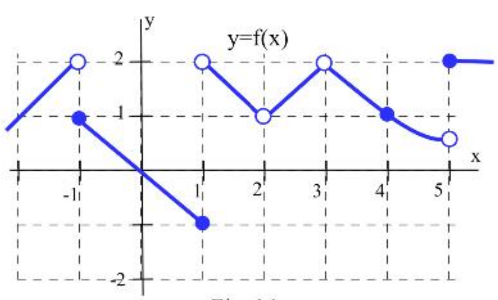

Fig. 14

8. Use the function f defined by the graph in Fig. 14 to determine the following limits.

(a) $\lim_{x \to 2^+} f(x)$ (b) $\lim_{x \to 2^-} f(x)$ (c) $\lim_{x \to 2} f(x)$

(d) $\lim_{x \to 4^+} f(x)$ (e) $\lim_{x \to 4^-} f(x)$ (f) $\lim_{x \to 4} f(x)$

(g) $\lim_{x \to -2^+} f(x)$ (h) $\lim_{x \to -2^-} f(x)$ (i) $\lim_{x \to -2} f(x)$

9. The Lorentz Contraction Formula in relativity theory says the length L of an object moving at v miles per second with respect to an observer is $L = A \cdot \sqrt{1 - \frac{v^2}{c^2}}$ where c is the speed of light (a constant).

a) Determine the "rest length" of the object $(v = 0)$. b) Determine $\lim_{v \to c^-} L$.

10. (a) $\lim_{x \to 2^+} INT(x)$ (b) $\lim_{x \to 2^-} INT(x)$ (c) $\lim_{x \to -2^+} INT(x)$ (d) $\lim_{x \to -2^-} INT(x)$

(e) $\lim_{x \to -2.3} INT(x)$ (f) $\lim_{x \to 3} INT(x/2)$ (g) $\lim_{x \to 3} INT(x)/2$ (h) $\lim_{x \to 0^+} \dfrac{INT(2 + x) - INT(2)}{x}$

11. $f(x) = \begin{cases} 1 & \text{if } x < 1 \\ x & \text{if } 1 < x \end{cases}$ and $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$.

(a) $\lim_{x\to 2} \{ f(x) + g(x) \}$ (b) $\lim_{x\to 2} f(x)/g(x)$ (c) $\lim_{x\to 2} f(g(x))$

(d) $\lim_{x\to 0} g(x)/f(x)$ (e) $\lim_{x\to 1} f(x)/g(x)$ (f) $\lim_{x\to 1} g(f(x))$

Problems 12 – 15 require a calculator.

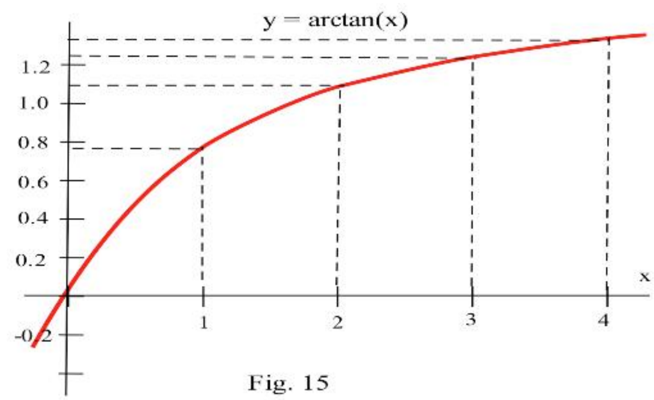

Fig. 15

12. Give geometric interpretations for the following limits and use a calculator to estimate their values.

(a) $\lim_{h\to 0} \dfrac{\arctan(0+h) - \arctan(0)}{h}$

(b) $\lim_{h\to 0} \dfrac{\arctan(1+h) - \arctan(1)}{h}$

(c) $\lim_{h\to 0} \dfrac{\arctan(2+h) - \arctan(2)}{h}$

13. (a) What does $\lim_{h\to 0} \dfrac{\cos(h) - 1}{h}$ represent on the graph of $y = \cos(x)$?

(It may help to recognize that $\dfrac{\cos(h) - 1}{h} = \dfrac{\cos(0+h) - \cos(0)}{h}$)

(b) Graphically and using your calculator, determine $\lim_{h\to 0} \dfrac{\cos(h) - 1}{h}$.

14. (a) What does the ratio $\dfrac{\ln(1+h)}{h}$ represent on the graph of $y = \ln(x)$?

(It may help to recognize that $\dfrac{\ln(1+h)}{h} = \dfrac{\ln(1+h) - \ln(1)}{h}$.)

(b) Graphically and using your calculator, determine $\lim_{h\to 0} \dfrac{\ln(1+h)}{h}$.

15. Use your calculator (to generate a table of values) to help you estimate

(a) $\lim_{h\to 0} \dfrac{e^h - 1}{h}$ (b) $\lim_{c\to 0} \dfrac{\tan(1+c) - \tan(1)}{c}$ (c) $\lim_{t\to 0} \dfrac{g(2+t) - g(2)}{t}$ when $g(t) = t^2 - 5$.

16. (a) For $h > 0$, find the slope of the line through the points $(h, |h|)$ and $(0,0)$.

(b) For $h < 0$, find the slope of the line through the points $(h, |h|)$ and $(0,0)$.

(c) Evaluate $\lim_{h\to 0^-} \dfrac{|h|}{h}$, $\lim_{h\to 0^+} \dfrac{|h|}{h}$, and $\lim_{h\to 0} \dfrac{|h|}{h}$

17. Describe the behavior of the function y = f(x) in Fig. 16 at each integer using one of the phrases:

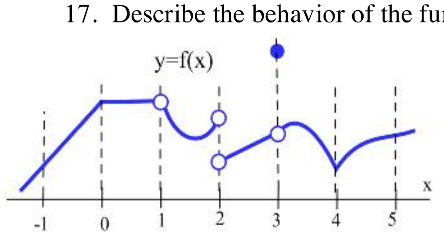

Fig. 16

(a) "connected and smooth", (b) "connected with a corner",

(c) "not connected because of a simple hole which could be plugged by adding or moving one point", or

(d) "not connected because of a vertical jump which could not be plugged by moving one point."

18. Describe the behavior of the function y = f(x) in Fig. 17 at each integer using one of the phrases: (a) "connected and smooth", (b) "connected with a corner", (c) "not connected because of a simple hole which could be plugged by adding or moving one point", or
(d) "not connected because of a vertical jump which could not be plugged by moving one point."

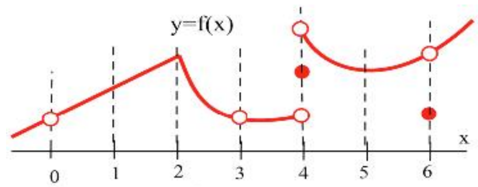

Fig. 17

19. This problem outlines the steps of a proof that $\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1$. Statements (a) – (h) below refer to Fig. 18. Assume that $0 < \theta < \frac{\pi}{2}$ and justify why each statement is true.

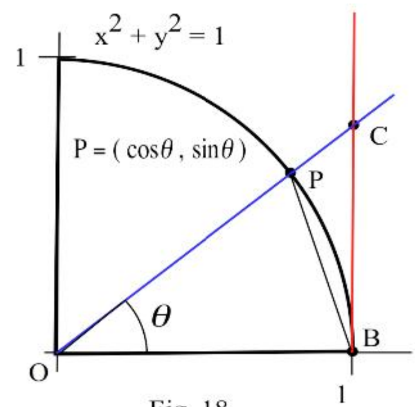

Fig. 18

(a) Area of $\Delta OPB = \frac{1}{2}$ (base)(height) $= \frac{1}{2} \sin(\theta)$.

(b) $\frac{\text{area of the sector (the pie shaped region) OPB}}{\text{area of the whole circle}}$

$= \frac{\text{angle defining sector OPB}}{\text{angle of the whole circle}} = \frac{\theta}{2\pi}$

so (area of the sector OPB) $= \frac{\theta\pi}{2\pi} = \frac{\theta}{2}$.

(c) The line L through the points (0,0) and P = (cos(θ),sin(θ)) has slope $m = \frac{\sin(\theta)}{\cos(\theta)}$, so

$C = (1, \frac{\sin(\theta)}{\cos(\theta)})$ and the area of $\Delta OCB = \frac{1}{2}$ (base)(height) $= \frac{1}{2} (1) \frac{\sin(\theta)}{\cos(\theta)}$.

(d) Area of ΔOPB < area of sector OPB < area of ΔOCB .

(e) $\frac{1}{2} \sin(\theta) < \frac{\theta}{2} < \frac{1}{2} (1) \frac{\sin(\theta)}{\cos(\theta)}$ and $\sin(\theta) < \theta < \frac{\sin(\theta)}{\cos(\theta)}$.

(f) $1 < \frac{\theta}{\sin(\theta)} < \frac{1}{\cos(\theta)}$ and $1 > \frac{\sin(\theta)}{\theta} > \cos(\theta)$.

(g) $\lim_{\theta \to 0^+} 1 = 1$ and $\lim_{\theta \to 0^+} \cos(\theta) = 1$.

(h) $\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1$.

20. Use the list method to show that $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist .

21. Show that $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

(Suggestion: Let $f(x) = \sin(\frac{1}{x})$ and pick $a_n = \frac{1}{n\pi}$ so $f(a_n) = \sin(\frac{1}{a_n}) = \sin(n\pi) = 0$ for every n.

Then pick $b_n = \frac{1}{2n\pi + \pi/2}$ so $f(b_n) = \sin(\frac{1}{b_n}) = \sin(2n\pi + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$ for every n.)

Section 2.3 **PRACTICE Answers**

Practice 1: (a) **–10** (b) **24** (c) **3/2** (d) **0**
(e) **0** (f) **5/4** (g) **–64** (h) **2**

Practice 2: (a) **39** (b) **–3/5** (c) **2/3**

Practice 3: (a) **0** (b) **2** (c) **3** (d) **1**

Practice 4: (a) slope of the line tangent to the graph of g at the point (1, g(1)): estimated slope ≈ **–2**

(b) slope of the line tangent to the graph of g at the point (3, g(3)): estimated slope ≈ **0**

(c) slope of the line tangent to the graph of g at the point (0, g(0)): estimated slope ≈ **1**

Practice 5: $\lim_{x\to 1} x^2 + 2 = 3$ and $\lim_{x\to 1} 2x + 1 = 3$ so $\lim_{x\to 1} f(x) = \mathbf{3}$.

Practice 6: $\lim_{x\to 0} \cos(x) = 1$ and $\lim_{x\to 0} 1 = 1$ so $\lim_{x\to 0} \frac{\sin(x)}{x} = \mathbf{1}$.