Continuity Practice

Some students say they have trouble with multipart functions. Other say they have issues with continuity problems. Here is a random assortment of old midterm questions that pertain to continuity and multipart functions. See if you can complete these problems. Solutions are posted online.

Remember a function $f(x)$ is *continuous* at $x = a$ if $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^-} f(x)$, $f(a)$ are all defined and are all the same.

1. Let $f(x) = \begin{cases} \cos(x) + 1 & , \text{ if } x \leq 0; \\ 2 - 3x & , \text{ if } x > 0. \end{cases}$ Determine if this function is continuous at $x = 0$.

2. Let $f(x) = \begin{cases} \frac{\sqrt{9x^4 + x^2}}{5x^2 + 3x + 1} & , \text{ if } x \leq 0; \\ x & , \text{ if } x < 0. \end{cases}$. Is f continuous at $x = 0$?

3. Let $f(x) = \begin{cases} e^x & , \text{ if } x < 0; \\ 9x^2 + x + 1 & , \text{ if } x \geq 0. \end{cases}$. Is f continuous at $x = 0$?

4. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + 3 & , \text{ if } x \neq 0; \\ 1 & , \text{ if } x = 0. \end{cases}$. Is f continuous at $x = 0$?

5. Let $f(x) = \begin{cases} -x + c & , \text{if } x \leq 1; \\ 6 - 2x^2 & , \text{if } x > 1. \end{cases}$ Find a value of c so that $f(x)$ is continuous at $x = 1$.

6. Let $f(x) = \begin{cases} \frac{x^2-9}{x-3} & , \text{if } x < 3; \\ cx^2 + 10 & , \text{if } x \geq 3. \end{cases}$ Find the value of c so that $f(x)$ is continuous at $x = 3$.

7. Let $G(x) = \begin{cases} \frac{1}{(x+3)^2} & , \text{if } x \leq -1; \\ 2 - x & , \text{if } -1 < x \leq 1; \\ \frac{3}{x+2} & , \text{if } x > 1. \end{cases}$. Find all values of x where G is not continuous.

Solutions:

1. Let $f(x) = \begin{cases} \cos(x) + 1 & , \text{if } x \leq 0; \\ 2 - 3x & , \text{if } x > 0. \end{cases}$ Determine if this function is continuous at $x = 0$.

   Solution:
   1. The function is defined at $x = 0$ and the value is $f(0) = \cos(0) + 1 = 2$.
   2. Since $y = \cos(x) + 1$ is continuous at $x = 0$, we have:
   $$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos(x) + 1 = \cos(0) + 1 = 2.$$
   3. Since $y = 2 - 3x$ is continuous at $x = 0$, we have:
   $$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2 - 3x = 2 - 3(0) = 2.$$
   Since all three of these values are the same, the function is continuous at $x = 0$.

2. Let $f(x) = \begin{cases} \frac{\sqrt{9x^4+x^2}}{5x^2+3x+1} & , \text{if } x \leq 0; \\ x & , \text{if } x > 0. \end{cases}$. Is f continuous at $x = 0$?

   Solution:
   1. The function is defined at $x = 0$ and its value is $f(0) = \frac{\sqrt{9(0)^4+(0)^2}}{5(0)^2+3(0)+1} = 0$.
   2. Since $y = \frac{\sqrt{9x^4+x^2}}{5x^2+3x+1}$ is continuous at $x = 0$, we have:
   $$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sqrt{9x^4 + x^2}}{5x^2 + 3x + 1} = 0.$$
   3. Since $y = x$ is continuous at $x = 0$, we have:
   $$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0.$$
   Since all three of these values are the same, the function is continuous at $x = 0$.

3. Let $f(x) = \begin{cases} e^x & , \text{if } x < 0; \\ 9x^2 + x + 1 & , \text{if } x \geq 0. \end{cases}$. Is f continuous at $x = 0$?

   Solution:
   1. The function is defined at $x = 0$ and its value is $f(0) = 9(0)^2 + (0) + 1 = 1$.
   2. Since $y = e^x$ is continuous at $x = 0$, we have:
   $$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^x = e^0 = 1.$$
   3. Since $y = x$ is continuous at $x = 0$, we have:
   $$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 9x^2 + x + 1 = 1.$$
   Since all three of these values are the same, the function is continuous at $x = 0$.

4. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + 3 & , \text{if } x \neq 0; \\ 1 & , \text{if } x = 0. \end{cases}$. Is f continuous at $x = 0$?

   Solution:
   1. The function is defined at $x = 0$ and its value is $f(0) = 1$.
   2. Now we use the squeeze theorem to find the value of the limit.
   Since $-1 \leq \sin\left(t\frac{1}{x}\right) \leq 1$ for all values of x, we can multiply by x^2 to get $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$ for all values of x. Since $\lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2$, we conclude that the function between them also approaches zero. Therefore $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$, which implies $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) + 3 = 3$.

   Since the value of limit does NOT equal the value of the function, $f(x)$ is NOT continuous at $x = 0$.

5. Let $f(x) = \begin{cases} -x+c & \text{, if } x \leq 1; \\ 6-2x^2 & \text{, if } x > 1. \end{cases}$ Find a value of c so that $f(x)$ is continuous at $x = 1$.

Solution:
1. The function is defined at $x = 1$ and its value is $f(1) = -1 + c$.
2. Since $y = -x + c$ is continuous at $x = 1$, we have:
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -x + c = -1 + c.$$
3. Since $y = 6 - 2x^2$ is continuous at $x = 1$, we have:
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 6 - 2x^2 = 6 - 2(1)^2 = 4.$$
In order to make all three of these the same, we need $-1 + c = 4$. Thus, $c = 5$.

6. Let $f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{, if } x < 3; \\ cx^2 + 10 & \text{, if } x \geq 3. \end{cases}$ Find the value of c so that $f(x)$ is continuous at $x = 3$.

Solution:
1. The function is defined at $x = 3$ and its value is $f(3) = c(3)^2 + 10 = 9c + 10$.
2. $\displaystyle\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3^-} \frac{(x-3)(x+3)}{x-3} = 6.$
3. Since $y = cx^2 + 10$ is continuous at $x = 3$, we have:
$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} cx^2 + 10 = 9c + 10.$$
In order to make all three of these the same, we need $9c + 10 = 6$. Thus, $c = -\frac{4}{9}$.

7. Let $G(x) = \begin{cases} \frac{1}{(x+3)^2} & \text{, if } x \leq -1; \\ 2 - x & \text{, if } -1 < x \leq 1; \\ \frac{3}{x+2} & \text{, if } x > 1. \end{cases}$. Find all values of x where G is not continuous.

Solution: There are four points to immediately consider: $x = -3$ and $x = -2$ because they make a denominator zero as well as $x = -1$ and $x = 1$ because the function rule changes at these values.

$\mathbf{x = -3}$: Since $y = \frac{1}{(x+3)^2}$ is discontinuous at $x = -3$ and $G(x)$ uses this rule for $x < -1$, we see that $G(x)$ is NOT continuous at $x = -3$.

$\mathbf{x = -2}$: Even through $y = \frac{3}{x+2}$ is discontinuous at $x = -2$, the function $G(x)$ only uses the rule $y = \frac{3}{x+2}$ for values where $x > 1$ and the rule it does use at $x = -2$ is continuous at that value. So $G(x)$ is continuous at $x = -2$.

$\mathbf{x = -1}$: $\displaystyle\lim_{x \to -1^-} G(x) = \frac{1}{(-1+3)^2} = \frac{1}{4}$ and $\displaystyle\lim_{x \to -1^+} G(x) = 2 - (-1) = 3$. Since these are not the same, the function $G(x)$ is NOT continuous at $x = -1$.

$\mathbf{x = 1}$: $\displaystyle\lim_{x \to 1^-} G(x) = 2 - (1) = 1$ and $\displaystyle\lim_{x \to 1^+} G(x) = \frac{3}{1+3} = 1$. Since these ARE the same and they equal the value of the function at $x = 1$, the function $G(x)$ is continuous at $x = 1$.

Therefore, the function $G(x)$ is continuous everywhere except $x = -3$ and $x = -1$.