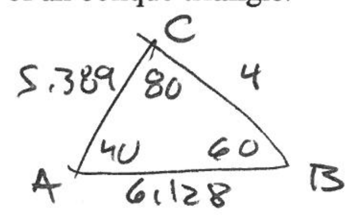
Law of Cosines, Law of Sines, Area of Triangle

Theorem: Law of Sines

For a triangle with sides *a, b, c* with opposite angles *A, B, C* respectively

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ **Case 1** (SSA OR ASA) and **Case 2** (SSA) of an oblique triangle.

1) Solve the triangle: A = 40°, B = 60° , a = 4



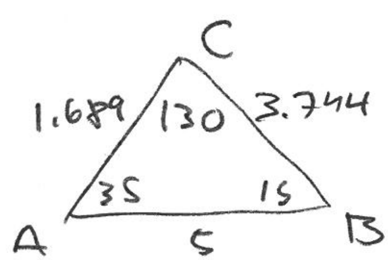
$$\frac{a}{\sin A} = \frac{b}{\sin B} \longrightarrow \frac{4}{\sin 40} = \frac{b}{\sin 60}$$

$$\boxed{b = \frac{4\sin 60}{\sin 40} \approx 5.389}$$

$$\frac{4}{\sin 40} = \frac{c}{\sin 80}$$

$$\boxed{c = \frac{4\sin 80}{\sin 40} \approx 6.128}$$

2) Solve the triangle: A = 35° , B = 15° , c = 5



$$\frac{a}{\sin 35} = \frac{5}{\sin 130}$$

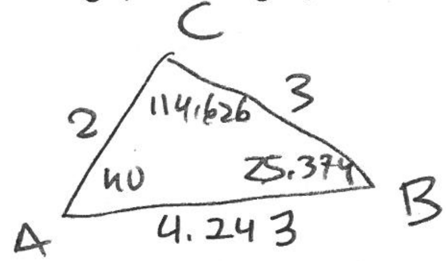
$$\boxed{a = \frac{5\sin 35}{\sin 130} \approx 3.744}$$

$$\frac{5}{\sin 130} = \frac{b}{\sin 15}$$

$$\boxed{b = \frac{5\sin 15}{\sin 130} \approx 1.689}$$

The Ambiguous Case – (SSA) which applies to triangles where two sides and the angle opposite one of them are known, is referred to as the ambiguous case, because the known information may result in one triangle, two triangles, or no triangles at all.

3) Solve the triangle: a = 3, b = 2 , A = 40°



$$\frac{\sin 40}{3} = \frac{\sin B}{2}$$

$$\sin B = \frac{2\sin 40}{3}$$

Note $\frac{2\sin 40}{3} \approx 0.429$

$$m\angle B = \sin^{-1}\left(\frac{2\sin 40}{3}\right)$$

$$\boxed{m\angle B \approx 25.374}$$

Note Supplement to $m\angle B = 154.626$

Note $154.626 + 40 > 180$

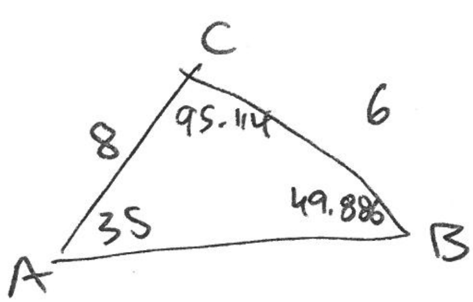
only 1 triangle

$$\boxed{m\angle C = 180 - 40 - \sin^{-1}\left(\frac{2\sin 40}{3}\right) \quad m\angle C \approx 114.626}$$

$$\frac{c}{\sin 114.626} = \frac{3}{\sin 40}$$

$$\boxed{c = \frac{3\sin 114.626}{\sin 40} \approx 4.243}$$

4) Solve the triangle: a = 6, b = 8, A = 35°



$$\frac{\sin B}{8} = \frac{\sin 35}{6}$$

$$\sin B = \frac{8\sin 35}{6} \approx 0.765$$

$$\boxed{m\angle B = \sin^{-1}\left(\frac{8\sin 35}{6}\right) \quad m\angle B \approx 49.886}$$

Note suppl to $\angle B$

130.114

$130.114 + 35 < 180$

and $\triangle$ exists

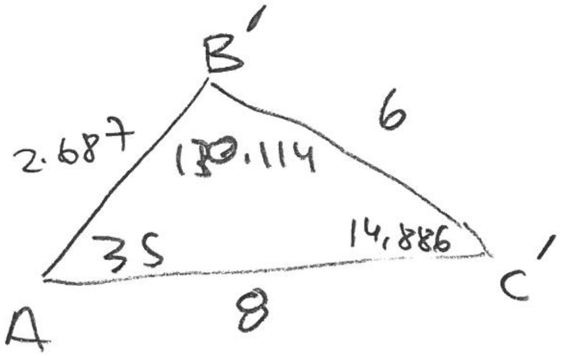
$$\boxed{m\angle B' = 130.114}$$

$$\boxed{m\angle C' = 180 - 165.114 = 14.886}$$

$$\boxed{m\angle C = 180 - 35 - \sin^{-1}\left(\frac{8\sin 35}{6}\right) \quad m\angle C \approx 95.114}$$

$$\boxed{\frac{c}{\sin 95.114} = \frac{6}{\sin 35} \quad c = \frac{6\sin 95.114}{\sin 35} \approx 10.419}$$

④ Second Δ



$m\angle A = 35$

$m\angle B' = 180 - 49.886$

$= 130.114$

$m\angle C' = 180 - 165.114$

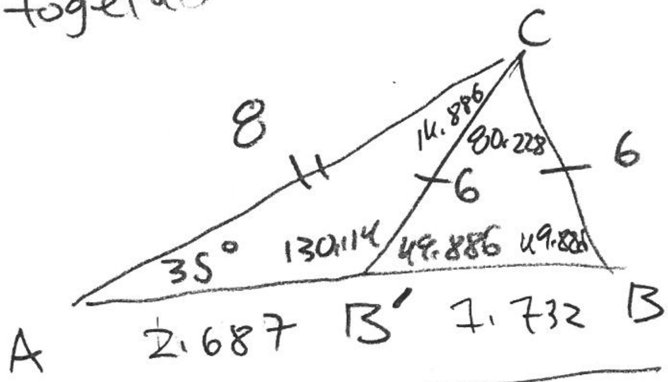
$= 14.886$

$$\frac{c}{\sin 14.886} = \frac{6}{\sin 35}$$

$$c = \frac{6 \sin 14.886}{\sin 35}$$

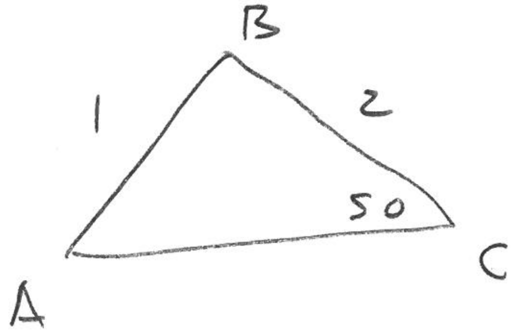
$$c \approx 2.687$$

two Δ's together



10.419

5) Solve the triangle: a = 2, c = 1, C = 50°



$$\frac{\sin 50}{1} = \frac{\sin A}{2}$$

$$2 \sin 50 = \sin A$$

$$1.532 = \sin A$$

not possible $\longrightarrow$ $\boxed{-1 \leq \sin A \leq 1}$

$$\cos B = \frac{2^2 + 2.646^2 - 3^2}{2(2)(2.646)} \rightarrow m\angle B = \cos^{-1}\left(\frac{2^2+2.646^2-3^2}{2(2)(2.646)}\right) \approx 79.107^\circ$$

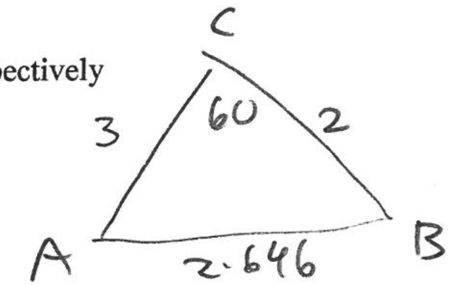
Law of Cosines

Case 3 = Two sides and the included angle are known (SAS)

Case 4 – Three sides are known (SSS).

Theorem – Law of Cosines

For a triangle with sides a, b, c and opposite angles, A, B, C, respectively



$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

1) Solve the triangle: a = 2, b = 3, C = 60° (SAS)

$$c^2 = 2^2 + 3^2 - 2(2)(3)\cos 60$$

$$\boxed{c = \sqrt{2^2+3^2-2(2)(3)\cos 60} = \sqrt{13 - 12\cos 60} \quad c \approx 2.646}$$

$$\cos A = \frac{3^2 + \left(\sqrt{13-12\cos 60}\right)^2 - 2^2}{2(3)\left(\sqrt{13-12\cos 60}\right)} \approx \frac{3^2 + 2.646^2 - 2^2}{2(3)(2.646)}$$

$$\boxed{\cos^{-1}\left(\frac{3^2+2.646^2-2^2}{2(3)(2.646)}\right) \approx m\angle A \approx 40.893} \rightarrow \boxed{m\angle B = 79.107}$$

2) Solve the triangle: a = 4, b = 3, c = 6 (SSS)

$$\cos A = \frac{3^2+6^2-4^2}{2(3)(6)} \qquad \cos B = \frac{6^2+4^2-3^2}{2(6)(4)}$$

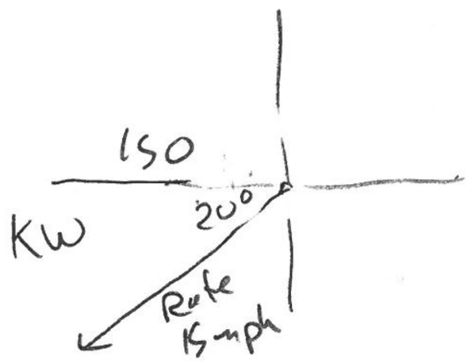
$$\boxed{m\angle A = \cos^{-1}\left(\frac{3^2+6^2-4^2}{2(3)(6)}\right) \approx 36.336^\circ}$$

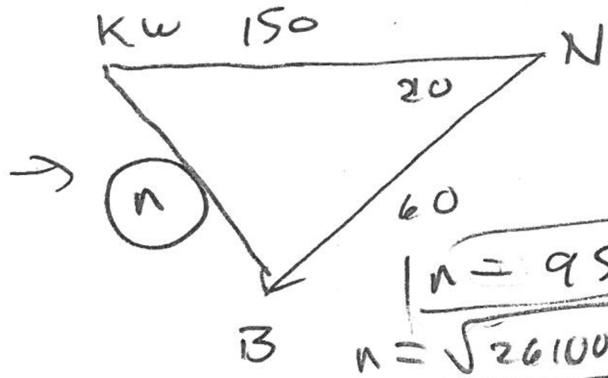
$$\boxed{m\angle B = \cos^{-1}\left(\frac{6^2+4^2-3^2}{2(6)(4)}\right) \quad m\angle B = 26.384}$$

$$\cos C = \frac{4^2+3^2-6^2}{2(4)(3)} \rightarrow \boxed{m\angle C = \cos^{-1}\left(\frac{4^2+3^2-6^2}{2(4)(3)}\right) \approx 117.280}$$

3) A motorized sailboat leaves Naples, Florida, bound for Key West, 150 miles away. Maintaining a constant speed of 15 miles per hour, but encountering heavy cross-winds, currents, the crew finds, after 4 hours, that the sailboat is off course by 20°.

a) How far is the sailboat from Key West at this time?



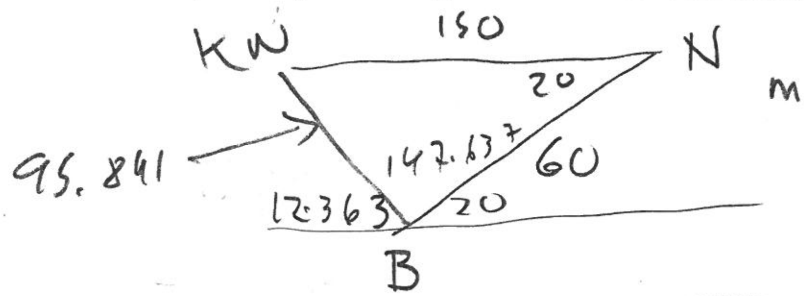


$$\boxed{n = 95.841}$$

$$n = \sqrt{26100 - 18000\cos 20}$$

$$n = \sqrt{150^2 + 60^2 - 2(150)(60)\cos(20)}$$

b) Through what angle should the sailboat turn to correct its course?



$$m\angle B = \cos^{-1}\left(\frac{95.841^2 + 60^2 - 150^2}{2(95.841)(60)}\right)$$

$$\approx 147.637°$$

$\boxed{12.363° \text{ N of W}}$ $\boxed{167.637° \text{ W of N}}$

c) How much time has been added to the trip because of this? (Assume that the speed remains at 15 miles per hour.)

$$\boxed{\frac{95.841}{15} = 6.389 \text{ hrs}}$$

Area of a Triangle

The area A of a triangle is:

$$A=\frac{1}{2}ab\sin C$$

$$A=\frac{1}{2}bc\sin A$$

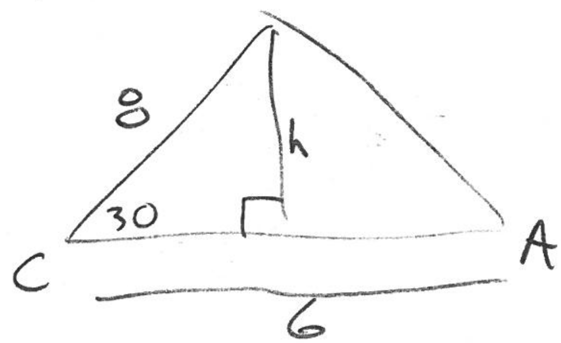
$$A=\frac{1}{2}ac\sin B$$

1) Find the area A of the triangle for which a = 8, b = 6, and <C = 30°

$A = \frac{1}{2}(8)(6)\sin(30)$

$= 24(0.5)$

$\boxed{A = 12}$



$\boxed{h = 8\sin 30}$

Theorem: Heron's Formula

The area A of a triangle with sides a, b, and c is $A=\sqrt{s(s-a)(s-b)(s-c)}$

2) Find the area of a triangle whose sides are 4, 5, and 7.

$S = 4+5+7$

$S = 16$

$A = \sqrt{16(16-4)(16-5)(16-7)}$

$= \sqrt{16(12)(11)(9)}$

$= \sqrt{19008}$

$\boxed{A \approx 137.870}$