7-2 Ellipses and Circles

Graph the ellipse given by each equation.

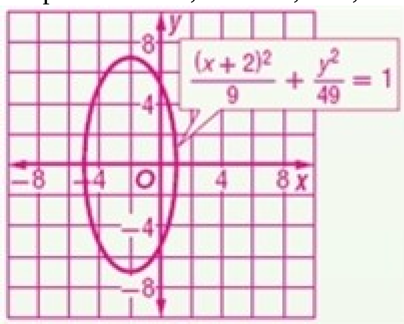
1. $\frac{(x+2)^2}{9} + \frac{y^2}{49} = 1$

SOLUTION:

The ellipse is in standard form, where $h = -2$, $k = 0$, $a = \sqrt{49}$ or 7, $b = \sqrt{9}$ or 3, and $c = \sqrt{40}$ or about 6.3. The orientation is vertical because the y–term contains a^2.

center: $(h, k) = (-2, 0)$
foci: $(h, k \pm c) = (-2, 6.3)$ and $(-2, -6.3)$
vertices: $(h, k \pm a) = (-2, 7)$ and $(-2, -7)$
covertices: $(h \pm b, k) = (1, 0)$ and $(-5, 0)$

Graph the center, vertices, foci, and axes. Then use a table of values to sketch the ellipse.



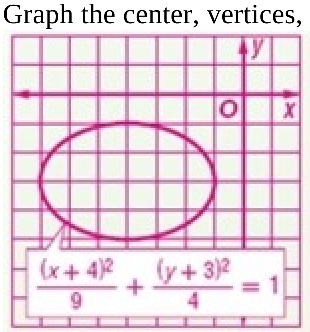
2. $\frac{(x+4)^2}{9} + \frac{(y+3)^2}{4} = 1$

SOLUTION:

The ellipse is in standard form, where $h = -4$, $k = -3$, $a = \sqrt{9}$ or 3, $b = \sqrt{4}$ or 2, and $c = \sqrt{9-4} \approx 2.2$. The orientation is horizontal because the x–term contains a^2.

center: $(h, k) = (-4, -3)$
foci: $(h \pm c, k) = (-1.8, -3)$ and $(-6.2, -3)$
vertices: $(h \pm a, k) = (-1, -3)$ and $(-7, -3)$
covertices: $(h, k \pm b) = (-4, -1)$ and $(-4, -5)$

Graph the center, vertices, foci, and axes. Then use a table of values to sketch the ellipse.



7-2 Ellipses and Circles

3. $x^2 + 9y^2 - 14x + 36y + 49 = 0$

SOLUTION:

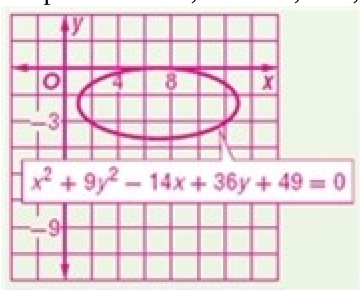
Complete the square for each variable to write the equation in standard form.

$x^2 + 9y^2 - 14x + 36y + 49 = 0$
$x^2 - 14x + 9y^2 + 36y + 49 = 0$
$x^2 - 14x + 49 + 9(y^2 + 4y + 4) = 0 + 36$
$(x - 7)^2 + 9(y + 2)^2 = 36$
$\frac{(x-7)^2}{36} + \frac{(y+2)^2}{4} = 1$

The ellipse is in standard form, where $h = 7$, $k = -2$, $a = \sqrt{36}$ or 6, $b = \sqrt{4}$ or 2, and $c = \sqrt{36-4} \approx 5.7$. The orientation is horizontal because the x–term contains a^2.

center: $(h, k) = (7, -2)$
foci: $(h \pm c, k) = (12.7, -2)$ and $(1.3, -2)$
vertices: $(h \pm a, k) = (13, -2)$ and $(1, -2)$
covertices: $(h, k \pm b) = (7, 0)$ and $(7, -4)$

Graph the center, vertices, foci, and axes. Then use a table of values to sketch the ellipse.



7-2 Ellipses and Circles

4. $4x^2 + y^2 - 64x - 12y + 276 = 0$

SOLUTION:

Complete the square for each variable to write the equation in standard form.

$4x^2 + y^2 - 64x - 12y + 276 = 0$

$4x^2 - 64x + y^2 - 12y + 276 = 0$

$4x^2 - 64x + 256 + y^2 - 12y + 36 + 276 = 0 + 256 + 36$

$4(x^2 - 16x + 64) + y^2 - 12y + 36 = 0 + 256 + 36 - 276$

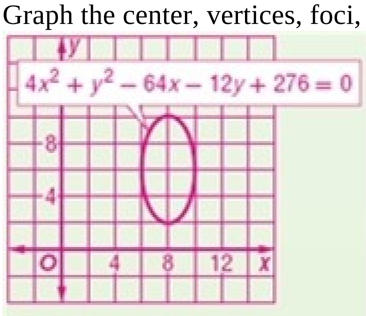
$4(x - 8)^2 + (y - 6)^2 = 16$

$$\frac{(x-8)^2}{4} + \frac{(y-6)^2}{16} = 1$$

The ellipse is in standard form, where $h = 8$, $k = 6$, $a = \sqrt{16}$ or 4, $b = \sqrt{4}$ or 2, and $c = \sqrt{16 - 4} \approx 3.5$. The orientation is vertical because the y–term contains a^2.

center: $(h, k) = (8, 6)$
foci: $(h, k \pm c) = (8, 9.5)$ and $(8, 2.5)$
vertices: $(h, k \pm a) = (8, 10)$ and $(8, 2)$
covertices: $(h \pm b, k) = (10, 6)$ and $(6, 6)$

Graph the center, vertices, foci, and axes. Then use a table of values to sketch the ellipse.



7-2 Ellipses and Circles

5. $9x^2 + y^2 + 126x + 2y + 433 = 0$

SOLUTION:

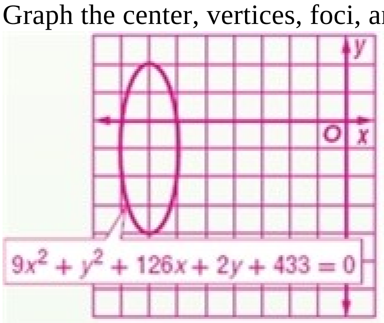
Complete the square for each variable to write the equation in standard form.

$9x^2 + y^2 + 126x + 2y + 433 = 0$
$9x^2 + 126x + y^2 + 2y + 433 = 0$
$9(x^2 + 14x + 49) + y^2 + 2y + 1 + 433 - 9(49) - 1 = 0$
$9(x + 7)^2 + (y + 1)^2 - 9 = 0$
$(x + 7)^2 + \frac{(y+1)^2}{9} = 1$

The ellipse is in standard form, where $h = -7$, $k = -1$, $a = \sqrt{9}$ or 3, $b = 1$, and $c = \sqrt{9-1} \approx 2.8$. The orientation is vertical because the y–term contains a^2.

center: $(h, k) = (-7, -1)$
foci: $(h, k \pm c) = (-7, 1.8)$ and $(-7, -3.8)$
vertices: $(h, k \pm a) = (-7, 2)$ and $(-7, -4)$
covertices: $(h \pm b, k) = (-6, -1)$ and $(-8, -1)$

Graph the center, vertices, foci, and axes. Then use a table of values to sketch the ellipse.



6. $x^2 + 25y^2 - 12x - 100y + 111 = 0$

SOLUTION:

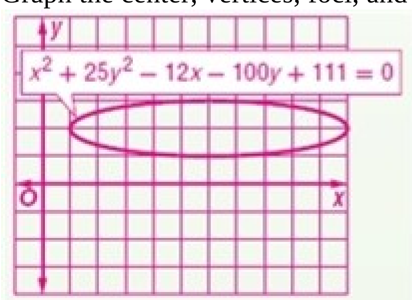
Complete the square for each variable to write the equation in standard form.

$x^2 + 25y^2 - 12x - 100y + 111 = 0$
$x^2 - 12x + 25y^2 - 100y + 111 = 0$
$x^2 - 12x + 36 + 25y^2 - 100y + 100 + 111 = 0 + 36 + 100$
$x^2 - 12x + 36 + 25(y^2 - 4y + 4) + 111 = 136$
$(x - 6)^2 + 25(y - 2)^2 = 25$
$\frac{(x-6)^2}{25} + (y - 2)^2 = 1$

The ellipse is in standard form, where $h = 6$, $k = 2$, $a = \sqrt{25}$ or 5, $b = 1$, and $c = \sqrt{25-1} \approx 4.9$. The orientation is horizontal because the x–term contains a^2.

center: $(h, k) = (6, 2)$
foci: $(h \pm c, k) = (10.9, 2)$ and $(1.1, 2)$
vertices: $(h \pm a, k) = (11, 2)$ and $(1, 2)$
covertices: $(h, k \pm b) = (6, 3)$ and $(6, 1)$

Graph the center, vertices, foci, and axes. Then use a table of values to sketch the ellipse.



7-2 Ellipses and Circles

Write an equation for the ellipse with each set of characteristics.

7. vertices (–7, –3), (13, –3); foci (–5, –3), (11, –3)

SOLUTION:

Because the y–coordinates of the vertices are the same, the major axis is horizontal, and the standard form of the equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

The center is the midpoint of the segment between the vertices, or (3, –3). So, $h = 3$ and $k = -3$.

The distance between the vertices is equal to $2a$ units. So, $2a = 20$, $a = 10$, and $a^2 = 100$. The distance between the foci is equal to $2c$ units. So, $2c = 16$ and $c = 8$.

Use the values of a and c to find b.

$$c^2 = a^2 - b^2$$
$$8^2 = 10^2 - b^2$$
$$b = 6$$

So, $b = 6$ and $b^2 = 36$.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x-3)^2}{100}+\frac{(y+3)^2}{36}=1$.

8. vertices (4, 3), (4, –9); length of minor axis is 8

SOLUTION:

Because the x–coordinates of the vertices are the same, the major axis is vertical, and the standard form of the equation is $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$.

The center is the midpoint of the segment between the vertices, or (4, –3). So, $h = 4$ and $k = -3$.

The distance between the vertices is equal to $2a$ units. So, $2a = 12$, $a = 6$, and $a^2 = 36$. The length of the minor axis is equal to $2b$. So, $2b = 8$, $b = 4$, and $b^2 = 16$.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x-4)^2}{16}+\frac{(y+3)^2}{36}=1$

7-2 Ellipses and Circles

9. vertices (7, 2), (–3, 2); foci (6, 2), (–2, 2)

SOLUTION:

Because the y–coordinates of the vertices are the same, the major axis is horizontal, and the standard form of the equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

The center is the midpoint of the segment between the vertices, or (2, 2). So, $h = 2$ and $k = 2$.

The distance between the vertices is equal to $2a$ units. So, $2a = 10$, $a = 5$, and $a^2 = 25$. The distance between the foci is equal to $2c$ units. So, $2c = 8$ and $c = 4$.

Use the values of a and c to find b.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ 4^2 &= 5^2 - b^2 \\ b &= 3 \end{aligned}$$

So, $b = 3$ and $b^2 = 9$.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x-2)^2}{25}+\frac{(y-2)^2}{9}=1$.

10. major axis (–13, 2) to (1, 2); minor axis (–6, 4) to (–6, 0)

SOLUTION:

Because the y–coordinates of the major axis are the same, the major axis is horizontal, and the standard form of the equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

The length of the major axis is equal to $2a$ units. So, $2a = 14$, $a = 7$, and $a^2 = 49$. The length of the minor axis is equal to $2b$ units. So, $2b = 4$, $b = 2$, and $b^2 = 4$. The center of the ellipse is at the midpoint of the major axis, or (–6, 2). So, $h = –6$ and $k = 2$.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x+6)^2}{49}+\frac{(y-2)^2}{4}=1$.

11. foci (–6, 9), (–6, –3); length of major axis is 20

SOLUTION:

Because the x–coordinates of the foci are the same, the major axis is vertical, and the standard form of the equation is $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$.

The center is located at the midpoint of the foci, or (–6, 3). So, $h = –6$ and $k = 3$. The length of the major axis is $2a$ units. So, $2a = 20$, $a = 10$, and $a^2 = 100$. The distance between the foci is equal to $2c$ units. So, $2c = 12$ and $c = 6$.

Use the values of a and c to find b.

$$c^2 = a^2 - b^2$$
$$6^2 = 10^2 - b^2$$
$$b = 8$$

So, $b = 8$ and $b^2 = 64$.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x+6)^2}{64}+\frac{(y-3)^2}{100}=1$.

12. co–vertices (–13, 7), (–3, 7); length of major axis is 16

SOLUTION:

Because the y–coordinates of the co–vertices are the same, the major axis is vertical, and the standard form of the equation is $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$.

The center is located at the midpoint of the co–vertices, or (–8, 7). So, $h = –8$ and $k = 7$. The distance between the co–vertices is equal to $2b$ units. So, $2b = 10$, $b = 5$, and $b^2 = 25$. The length of the major axis is equal to $2a$ units. So, $2a = 16$, $a = 8$, and $a^2 = 64$.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x+8)^2}{25}+\frac{(y-7)^2}{64}=1$.

7-2 Ellipses and Circles

13. foci (–10, 8), (14, 8); length of major axis is 30

SOLUTION:

Because the y–coordinates of the foci are the same, the major axis is horizontal, and the standard form of the equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

The midpoint is the distance between the foci, or (2, 8). So, $h = 2$ and $k = 8$. The distance between the foci is equal to $2c$ units. So, $2c = 24$ and $c = 12$. The length of the major axis is equal to $2a$. So, $2a = 30$, $a = 15$, and $a^2 = 225$.

Use the values of a and c to find b.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ 12^2 &= 15^2 - b^2 \\ b &= 9 \end{aligned}$$

So, $b = 9$ and $b^2 = 81$.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x-2)^2}{225}+\frac{(y-8)^2}{81}=1$.

Determine the eccentricity of the ellipse given by each equation.

14. $\frac{(x+5)^2}{72}+\frac{(y-3)^2}{54}=1$

SOLUTION:

The ellipse is in standard form. So, $a = \sqrt{72}$ or about 8.49 and $b = \sqrt{54}$.

First, determine the value of c.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ c^2 &= 72 - 54 \\ c &= \sqrt{18} \text{ or about } 4.24 \end{aligned}$$

Use the values of a and c to find the eccentricity of the ellipse.

$$\begin{aligned} e &= \frac{c}{a} \\ e &= \frac{4.24}{8.49} \text{ or about } 0.5 \end{aligned}$$

7-2 Ellipses and Circles

15. $\dfrac{(x+6)^2}{40} + \dfrac{(y-2)^2}{12} = 1$

SOLUTION:

The ellipse is in standard form. So, $a = \sqrt{40}$ or about 6.32 and $b = \sqrt{12}$.

First, determine the value of c.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ c^2 &= 40 - 12 \\ c &= \sqrt{28} \text{ or about } 5.29 \end{aligned}$$

Use the values of a and c to find the eccentricity of the ellipse.

$$\begin{aligned} e &= \frac{c}{a} \\ e &= \frac{5.29}{6.32} \text{ or about } 0.837 \end{aligned}$$

16. $\dfrac{(x-8)^2}{14} + \dfrac{(y+3)^2}{57} = 1$

SOLUTION:

The ellipse is in standard form. So, $a = \sqrt{57}$ or about 7.55 and $b = \sqrt{14}$.

First, determine the value of c.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ c^2 &= 57 - 14 \\ c &= \sqrt{43} \text{ or about } 6.56 \end{aligned}$$

Use the values of a and c to find the eccentricity of the ellipse.

$$\begin{aligned} e &= \frac{c}{a} \\ e &= \frac{6.56}{7.55} \text{ or about } 0.869 \end{aligned}$$

17. $\frac{(x+8)^2}{27} + \frac{(y-7)^2}{33} = 1$

SOLUTION:

The ellipse is in standard form. So, $a = \sqrt{33}$ or about 5.74 and $b = \sqrt{27}$.

First, determine the value of c.

$$c^2 = a^2 - b^2$$
$$c^2 = 33 - 27$$
$$c = \sqrt{6} \text{ or about } 2.45$$

Use the values of a and c to find the eccentricity of the ellipse.

$$e = \frac{c}{a}$$
$$e = \frac{2.45}{5.74} \text{ or about } 0.426$$

18. $\frac{(x-1)^2}{12} + \frac{(y+2)^2}{9} = 1$

SOLUTION:

The ellipse is in standard form. So, $a = \sqrt{12}$ or about 3.46 and $b = 3$.

First, determine the value of c.

$$c^2 = a^2 - b^2$$
$$c^2 = 12 - 9$$
$$c = \sqrt{3} \text{ or about } 1.73$$

Use the values of a and c to find the eccentricity of the ellipse.

$$e = \frac{c}{a}$$
$$e = \frac{1.73}{3.46} \text{ or about } 0.5$$

19. $\dfrac{(x-11)^2}{17}+\dfrac{(y+15)^2}{23}=1$

SOLUTION:

The ellipse is in standard form. So, $a = \sqrt{23}$ or about 4.8 and $b = \sqrt{17}$.

First, determine the value of c.

$$c^2 = a^2 - b^2$$
$$c^2 = 23 - 17$$
$$c = \sqrt{6} \text{ or about } 2.45$$

Use the values of a and c to find the eccentricity of the ellipse.

$$e = \frac{c}{a}$$
$$e = \frac{2.45}{4.8} \text{ or about } 0.511$$

20. $\dfrac{x^2}{38}+\dfrac{(y-12)^2}{13}=1$

SOLUTION:

The ellipse is in standard form. So, $a = \sqrt{38}$ or about 6.16 and $b = \sqrt{13}$.

First, determine the value of c.

$$c^2 = a^2 - b^2$$
$$c^2 = 38 - 13$$
$$c = \sqrt{25} \text{ or } 5$$

Use the values of a and c to find the eccentricity of the ellipse.

$$e = \frac{c}{a}$$
$$e = \frac{5}{6.16} \text{ or about } 0.811$$

21. $\frac{(x+9)^2}{10}+\frac{(y+11)^2}{8}=1$

SOLUTION:

The ellipse is in standard form. So, $a = \sqrt{10}$ or about 3.16 and $b = \sqrt{8}$.

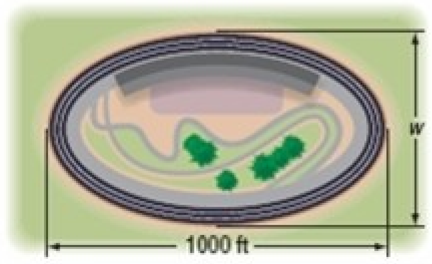
First, determine the value of c.

$c^2 = a^2 - b^2$
$c^2 = 10 - 8$
$c = \sqrt{2}$ or 1.41

Use the values of a and c to find the eccentricity of the ellipse.

$e = \frac{c}{a}$
$e = \frac{1.41}{3.16}$ or about 0.447

22. **RACING** The design of an elliptical racetrack with an eccentricity of 0.75 is shown.
a. What is the maximum width *w* of the track?
b. Write an equation for the ellipse if the origin *x* is located at the center of the racetrack.



SOLUTION:

a. The length of the track is 1000 feet. So, $2a = 1000$, $a = 500$, and $a^2 = 250{,}000$. Because the eccentricty is given, you can use the eccentricty equation to find c.

$$e = \frac{c}{a}$$
$$0.75 = \frac{c}{500}$$
$$375 = c$$

Use the values of a and c to find b.

$$b^2 = a^2 - c^2$$
$$b^2 = 500^2 - 375^2$$
$$b = \sqrt{500^2 - 375^2}$$
$$b \approx 330.72$$

So, the maximum width of the track is 2(330.72) or about 661.44 feet.

b. Use the values of a and b to write the equation of the ellipse.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{x^2}{250,000} + \frac{y^2}{109,375} = 1$$

7-2 Ellipses and Circles

23. **CARPENTRY** A carpenter has been hired to construct a sign for a pet grooming business. The plans for the sign call for an elliptical shape with an eccentricity of 0.60 and a length of 36 inches.

a. What is the height of the sign?

b. Write an equation for the ellipse if the origin is located at the center of the sign.

SOLUTION:

a. The length of the sign is 36 inches. So, $2a = 36$, $a = 18$, and $a^2 = 324$. Because the eccentricty is given, you can use the eccentricty equation to find c.

$$e = \frac{c}{a}$$
$$0.60 = \frac{c}{18}$$
$$10.8 = c$$

Use the values of a and c to find b.

$$b^2 = a^2 - c^2$$
$$b^2 = 18^2 - 10.8^2$$
$$b = \sqrt{18^2 - 10.8^2}$$
$$b \approx 14.4$$

So, the height of the sign is 2(14.4) or about 28.8 inches.

b. Use the values of a and b to write the equation of the ellipse. $b^2 = 14.4^2$.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
$$\frac{x^2}{324} + \frac{y^2}{207} = 1$$

Write each equation in standard form. Identify the related conic.

24. $x^2 + y^2 + 6x - 4y - 3 = 0$

SOLUTION:

$$x^2 + y^2 + 6x - 4y - 3 = 0$$
$$x^2 + 6x + y^2 - 4y - 3 = 0$$
$$x^2 + 6x + 9 + y^2 - 4y + 4 - 3 = 0 + 9 + 4$$
$$(x + 3)^2 + (y - 2)^2 = 16$$

Because $a = b$ and the graph is of the form $(x - h)^2 + (y - k)^2 = r^2$, the conic is a circle.

7-2 Ellipses and Circles

25. $4x^2 + 8y^2 - 8x + 48y + 44 = 0$

SOLUTION:

$4x^2 + 8y^2 - 8x + 48y + 44 = 0$
$4x^2 - 8x + 8y^2 + 48y + 4 = 0$
$4(x^2 - 2x + 1) + 8(y^2 + 6y + 9) + 44 = 0 + 4 + 72$
$4(x - 1)^2 + 8(y + 3)^2 = 32$
$$\frac{(x-1)^2}{8} + \frac{(y+3)^2}{4} = 1$$

The related conic is an ellipse because $a \neq b$ and the graph is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

26. $x^2 - 8x - 8y - 40 = 0$

SOLUTION:

$x^2 - 8x - 8y - 40 = 0$
$x^2 - 8x + 16 - 8y - 40 = 0 + 16$
$(x - 4)^2 = 8y + 56$
$(x - 4)^2 = 8(y + 7)$

Because only one term is squared, the graph is a parabola.

27. $y^2 - 12x + 18y + 153 = 0$

SOLUTION:

$y^2 - 12x + 18y + 153 = 0$
$y^2 + 18y + 153 = 12x$
$y^2 + 18y + 81 + 153 = 12x + 81$
$(y + 9)^2 = 12x - 72$
$(y + 9)^2 = 12(x - 6)$

Because only one term is squared, the graph is a parabola.

28. $x^2 + y^2 - 8x - 6y - 39 = 0$

SOLUTION:

$x^2 + y^2 - 8x - 6y - 39 = 0$
$x^2 - 8x + y^2 - 6y - 39 = 0$
$x^2 - 8x + 16 + y^2 - 6y + 9 - 39 = 0 + 16 + 9$
$(x - 4)^2 + (y - 3)^2 = 64$

Because $a = b$, and the graph is of the form $(x - h)^2 + (y - k)^2 = r^2$, the conic is a circle.

7-2 Ellipses and Circles

29. $3x^2 + y^2 - 42x + 4y + 142 = 0$

SOLUTION:

$3x^2 + y^2 - 42x + 4y + 142 = 0$
$3x^2 - 42x + y^2 + 4y + 142 = 0$
$3(x^2 - 14x + 49) + y^2 + 4y + 4 + 142 = 0 + 147 + 4$
$3(x - 7)^2 + (y + 2)^2 = 9$
$\frac{(x-7)^2}{3} + \frac{(y+2)^2}{9} = 1$

The related conic is an ellipse because $a \neq b$ and the graph is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

30. $5x^2 + 2y^2 + 30x - 16y + 27 = 0$

SOLUTION:

$5x^2 + 2y^2 + 30x - 16y + 27 = 0$
$5x^2 + 30x + 2y^2 - 16y + 27 = 0$
$5(x^2 + 6x + 9) + 2(y^2 - 8y + 16) + 27 = 0 + 45 + 32$
$5(x + 3)^2 + 2(y - 4)^2 = 50$
$\frac{(x+3)^2}{10} + \frac{(y-4)^2}{25} = 1$

The related conic is an ellipse because $a \neq b$, and the graph is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

31. $2x^2 + 7y^2 + 24x + 84y + 310 = 0$

SOLUTION:

$$2x^2 + 7y^2 + 24x + 84y + 310 = 0$$
$$2x^2 + 24x + 7y^2 + 84y + 310 = 0$$
$$2\left(x^2 + 12x + 36\right) + 7\left(y^2 + 12y + 36\right) + 310 = 0 + 72 + 252$$
$$2(x + 6)^2 + 7(y + 6)^2 = 14$$
$$\frac{(y+6)}{2} + \frac{(x+6)}{7} = 1$$

The related conic is an ellipse because $a \neq b$, and the graph is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

32. **HISTORY** The United States Capitol has a room with an elliptical ceiling. This type of room is called a *whispering gallery* because sound that is projected from one focus of an ellipse reflects off the ceiling and back to the other focus. The room in the Capitol is 96 feet in length, 45 feet wide, and has a ceiling that is 23 feet high.

a. Write an equation modeling the shape of the room. Assume that it is centered at the origin and that the major axis is horizontal.

b. Find the location of the two foci.

c. How far from one focus would one have to stand to be able to hear the sound reflecting from the other focus?

SOLUTION:

a. The length of the room is 96 feet. So, $2a = 96$, $a = 48$, and $a^2 = 2304$. The width of the room is 45 feet. So, $2b = 45$, $b = 22.5$, and $b^2 = 506.25$.

Use the values of a and b to write the equation of the ellipse.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
$$\frac{x^2}{2304} + \frac{y^2}{506.25} = 1$$

b. To find the location of the foci, find c.

$$c^2 = a^2 - b^2$$
$$c^2 = 2304 - 506.25$$
$$c = \sqrt{2304 - 506.25}$$
$$c \approx 42.4$$

So, the foci are about 42 ft on either side of the center along the major axis.

c. The length of the segment between the foci is $2c$ units. So, 2(42) is about 84 feet.

Write an equation for a circle that satisfies each set of conditions. Then graph the circle.

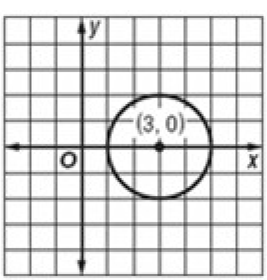
33. center at (3, 0), radius 2

SOLUTION:

The center is located at (3, 0), so $h = 3$ and $k = 0$. The radius is 2, so $r^2 = 4$.

Use the values of h, k, and r to write the equation of the circle.

$$(x-h)^2 + (y-k)^2 = r^2$$
$$(x-3)^2 + y^2 = 4$$



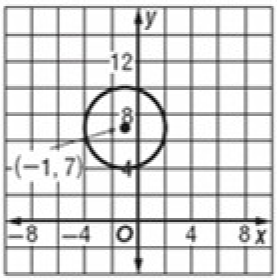
34. center at (–1, 7), diameter 6

SOLUTION:

The center is located at (–1, 7), so h = –1 and k = 7. The diameter is 6, so r = 3, and r^2 = 9.

Use the values of h, k, and r to write the equation of the circle.

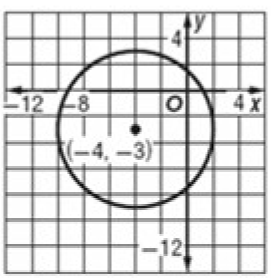
$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x + 1)^2 + (y - 7)^2 = 9$$



35. center at (–4, –3), tangent to y = 3

SOLUTION:

The center is located at (–4, –3), so h = –4 and k = –3. The line y = 3 is a horizontal line. Because the circle is tangent to this line and its center is at (–4, –3), one endpoint of the circle is at (–4, 3). So, the radius is 3 – (–3) = 6, and r^2 = 36.



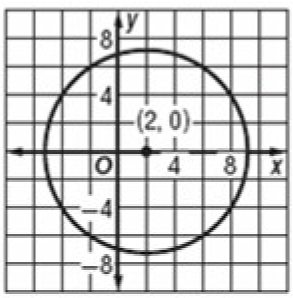
36. center at (2, 0), endpoints of diameter at (–5, 0) and (9, 0)

SOLUTION:

The center is located at (2, 0), so $h = 2$ and $k = 0$. The diameter is $|-5 - 9|$ or 14, so $r = 7$, and $r^2 = 49$.

Use the values of h, k, and r to write the equation of the circle.

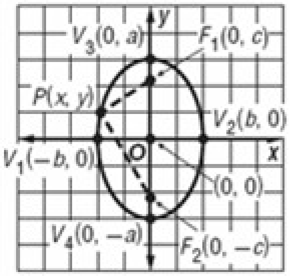
$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x - 2)^2 + y^2 = 49$$



7-2 Ellipses and Circles

37. **FORMULA** Derive the general form of the equation for an ellipse with a vertical major axis centered at the origin.

SOLUTION:



$$PF_1 + PF_2 = 2a$$
$$\sqrt{(x-0)^2 + (y-c)^2} + \sqrt{(x-0)^2 + (y-(-c))^2} = 2a$$
$$\sqrt{x^2 + (y-c)^2} + \sqrt{x^2 + (y+c)^2} = 2a$$
$$\sqrt{x^2 + (y-c)^2} = 2a - \sqrt{x^2 + (y+c)^2}$$
$$x^2 + y^2 - 2cy + c^2 = 4a^2 - 4a\sqrt{x^2 + (y+c)^2} + x^2 + y^2 + 2cy + c^2$$
$$4a\sqrt{x^2 + (y+c)^2} = 4a^2 + 4cy$$
$$a\sqrt{x^2 + (y+c)^2} = a^2 + cy$$
$$a^2\left(x^2 + y^2 + 2cy + c^2\right) = a^4 + 2a^2cy + c^2y^2$$
$$a^2x^2 + a^2y^2 + 2a^2cy + a^2c^2 = a^4 + 2a^2cy + c^2y^2$$
$$a^2y^2 - c^2y^2 + a^2x^2 = a^4 - a^2c^2$$
$$y^2\left(a^2 - c^2\right) + a^2x^2 = a^2\left(a^2 - c^2\right)$$
$$y^2b^2 + a^2x^2 = a^2b^2$$
$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$

7-2 Ellipses and Circles

38. **MEDICAL TECHNOLOGY** Indoor Positioning Systems (IPS) use ultrasound waves to detect tags that are linked to digital files containing information regarding a person or item being monitored. Hospitals often use IPS to detect the location of moveable equipment and patients.

a. If the tracking system receiver must be centrally located for optimal functioning, where should a receiver be situated in a hospital complex that is 800 meters by 942 meters?

b. Write an equation that models the sonar range of the IPS.

SOLUTION:

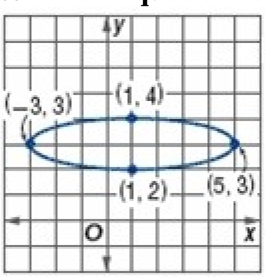
a. The receiver should be situated at the central point of the hospital, which is 471 meters within the long side of the building.

b. The building is shaped like a rectangle. In order to reach every part of the building, a circle must be drawn with a center at the central point of the building. From the central point, the farthest point of the building is at a corner of the rectangle. Use the Pythagorean Theorem to find this distance.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 400^2 + 471^2 &= c^2 \\ 381{,}841 &= c^2 \end{aligned}$$

This distance is the radius of the circle that encloses the entire building. So, an equation that models the sonar range of the IPS is $x^2 + y^2 = 381{,}841$.

Write an equation for each ellipse.

39. 

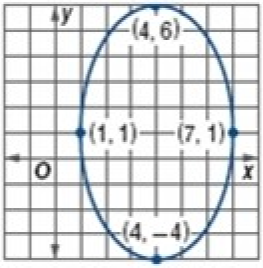
SOLUTION:

From the graph, you can see that the vertices are located at (–3, 3) and (5, 3) and the co–vertices are located at (1, 4) and (1, 2). The major axis is horizontal, so the standard form of the equation is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

The center is the midpoint of the segment between the vertices, or (1, 3). So, $h = 1$ and $k = 3$.

The distance between the vertices is equal to $2a$ units. So, $2a = 8$, $a = 4$, and $a^2 = 16$. The distance between the co–vertices is equal to $2b$ units. So, $2b = 2$, $b = 1$, and $b^2 = 1$.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x-1)^2}{16} + \frac{(y-3)^2}{1} = 1$.

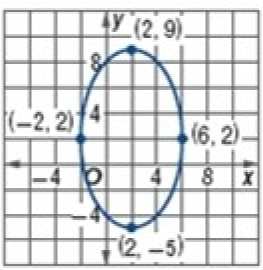
40. 

SOLUTION:

From the graph, you can see that the vertices are located at (4, 6) and (4, –4) and the co–vertices are located at (1, 1) and (7, 1). The major axis is vertical, so the standard form of the equation is $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$.

The center is the midpoint of the segment between the vertices, or (4, 1). So, h = 4 and k = 1.
The distance between the vertices is equal to 2a units. So, 2a = 10, a = 5, and a^2 = 25. The distance between the co–vertices is equal to 2b units. So, 2b = 6, b = 3, and b^2 = 9.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x-4)^2}{9}+\frac{(y-1)^2}{25}=1$.

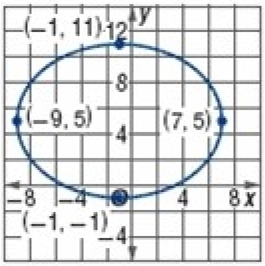
41. 

SOLUTION:

From the graph, you can see that the vertices are located at (2, 9) and (2, –5) and the co–vertices are located at (–2, 2) and (6, 2). The major axis is vertical, so the standard form of the equation is $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$.

The center is the midpoint of the segment between the vertices, or (2, 2). So, h = 2 and k = 2.
The distance between the vertices is equal to 2a units. So, 2a = 14, a = 7, and a^2 = 49. The distance between the co–vertices is equal to 2b units. So, 2b = 8, b = 4, and b^2 = 16.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x-2)^2}{16}+\frac{(y-2)^2}{49}=1$.

42. 

SOLUTION:

From the graph, you can see that the vertices are located at (–9, 5) and (7, 5) and the co–vertices are located at (–1, 11) and (–1, 1). The major axis is horizontal, so the standard form of the equation is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

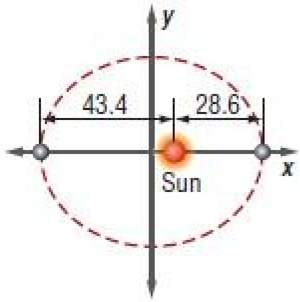
The center is the midpoint of the segment between the vertices, or (–1, 5). So, $h = -1$ and $k = 5$.

The distance between the vertices is equal to 2*a* units. So, $2a = 16$, $a = 8$, and $a^2 = 64$. The distance between the co–vertices is equal to 2*b* units. So, $2b = 12$, $b = 6$, and $b^2 = 36$.

Using the values of *h*, *k*, *a*, and *b*, the equation for the ellipse is $\frac{(x+1)^2}{64} + \frac{(y-5)^2}{36} = 1$.

7-2 Ellipses and Circles

43. **PLANETARY MOTION** Each of the planets in the solar system move around the Sun in an elliptical orbit, where the Sun is one focus of the ellipse. Mercury is 43.4 million miles from the Sun at its farthest point and 28.6 million miles at its closest, as shown below. The diameter of the Sun is 870,000 miles.



a. Find the length of the minor axis.
b. Find the eccentricity of the elliptical orbit.

SOLUTION:

a. The length of the major axis is 43.4 + 28.6 + 0.87 = 72.87.

The value of a is 72.87 ÷ 2 = 36.435. The distance from the focus (the sun) to the vertex is $28.6 + \frac{0.87}{2} = 29.035$.

Therefore, the value of c, the focus to the center, is 36.435 − 29.035 = 7.4.

In an ellipse, $c^2 = a^2 - b^2$. Use the values of a and c to find b.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ 7.4^2 &= 36.435^2 - b^2 \\ 36.435^2 - 7.4^2 &= b^2 \\ \sqrt{36.435^2 - 7.4^2} &= b \\ 35.676 &= b \end{aligned}$$

The value of $2b$, the length of the minor axis, is about $35.676 \cdot 2 \approx 71.35$ million miles.

b. Use the values of a and c to find the eccentricity of the orbit.

$$\begin{aligned} e &= \frac{c}{a} \\ &= \frac{7.4}{36.435} \\ &\approx 0.203 \end{aligned}$$

So, the orbit has an eccentricity of about 0.203.

7-2 Ellipses and Circles

Find the center, foci, and vertices of each ellipse.

44. $\frac{(x+5)^2}{16} + \frac{y^2}{7} = 1$

SOLUTION:

The ellipse is in standard form, where $h = -5$ and $k = 0$. So, the center is located at (–5, 0). The ellipse has a horizontal orientation, so $a^2 = 16$, $a = 4$, and $b^2 = 7$.

Use the values of a and b to find c.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ c^2 &= 16 - 7 \\ c^2 &= 9 \\ c &= 3 \end{aligned}$$

The foci are c units from the center, so they are located at (–8, 0) and (–2, 0). The vertices are a units from the center, so they are located at (–9, 0) and (–1, 0).

45. $\frac{x^2}{100} + \frac{(y+6)^2}{25} = 1$

SOLUTION:

The ellipse is in standard form, where $h = 0$ and $k = -6$. So, the center is located at (0, –6). The ellipse has a horizontal orientation, so $a^2 = 100$, $a = 10$, and $b^2 = 25$.

Use the values of a and b to find c.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ c^2 &= 100 - 25 \\ c^2 &= 75 \\ c &= \sqrt{75} \text{ or } 5\sqrt{3} \end{aligned}$$

The foci are c units from the center, so they are located at ($\pm 5\sqrt{3}$, –6). The vertices are a units from the center, so they are located at (±10, –6).

46. $9y^2 - 18y + 25x^2 + 100x - 116 = 0$

SOLUTION:

First, write the equation in standard form.

$$\begin{aligned} 9y^2 - 18y + 25x^2 + 100x - 116 &= 0 \\ 9\left(y^2 - 2y + 1\right) + 25\left(x^2 + 4x + 4\right) &= 0 + 116 + 9 + 100 \\ 9(y-1)^2 + 25(x+2)^2 &= 225 \\ \frac{(y-1)^2}{25} + \frac{(x+2)^2}{9} &= 1 \end{aligned}$$

The equation is now in standard form, where $k = 1$ and $h = -2$. So, the center is located at (–2, 1). The ellipse has a vertical orientation, so $a^2 = 25$, $a = 5$, and $b^2 = 9$.

Use the values of a and b to find c.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ c^2 &= 25 - 9 \\ c^2 &= 16 \\ c &= 4 \end{aligned}$$

The foci are c units from the center, so they are located at (–2, 5), (–2, –3). The vertices are a units from the center, so they are located at (–2, 6), (–2, –4).

7-2 Ellipses and Circles

47. $65x^2 + 16y^2 + 130x - 975 = 0$

SOLUTION:

First, write the equation in standard form.

$$65x^2 + 16y^2 + 130x - 975 = 0$$
$$65x^2 + 130x + 16y^2 - 975 = 0$$
$$65\left(x^2 + 2x + 1\right) + 16y^2 = 0 + 975 + 65$$
$$65(x+1)^2 + 16y^2 = 1040$$
$$\frac{(x+1)^2}{16} + \frac{y^2}{65} = 1$$

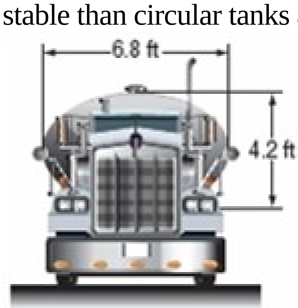
The equation is now in standard form, where $h = -1$ and $k = 0$. So, the center is located at (–1, 0). The ellipse has a vertical orientation, so $a^2 = 65$, $a = \sqrt{65}$, and $b^2 = 16$.

Use the values of a and b to find c.

$$c^2 = a^2 - b^2$$
$$c^2 = 65 - 16$$
$$c^2 = 49$$
$$c = 7$$

The foci are c units from the center, so they are located at (–1, ±7). The vertices are a units from the center, so they are located at $(-1, \pm\sqrt{65})$.

48. **TRUCKS** Elliptical tanker trucks like the one shown are often used to transport liquids because they are more stable than circular tanks and the movement of the fluid is minimized.

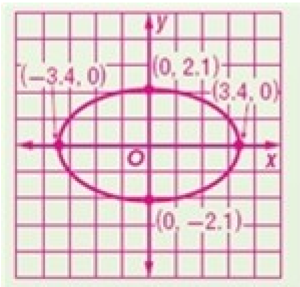


a. Draw and label the elliptical cross–section of the tank on a coordinate plane.
b. Write an equation to represent the elliptical shape of the tank.
c. Find the eccentricity of the ellipse.

SOLUTION:

a. The major axis of the ellipse is 6.8 ft, so $a = 3.4$. The minor axis is 4.2 ft, so $b = 2.1$. The ellipse is centered about the origin, so $h = k = 0$. The major axis is along the x–axis, so the ellipse is horizontal. The vertices are located at (0 ± 3.4, 0) and the co–vertices are located at (0, 0 ± 2.1).

Plot the vertices and co–vertices to sketch the ellipse.



b. The orientation of the ellipse is horizontal, so the standard form of the equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

Substitute the values of a and b into the equation.

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-0)^2}{3.4^2}+\frac{(y-0)^2}{2.1^2}=1$$

$$\frac{x^2}{11.56}+\frac{y^2}{4.41}=1$$

c. Use the values of a and b to find c.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ c^2 &= 11.56 - 4.41 \\ c^2 &= 7.15 \\ c &\approx 2.67 \end{aligned}$$

Substitute a and c into the eccentricity equation.

$$\begin{aligned} e &= \frac{c}{a} \\ &= \frac{2.67}{3.4} \\ &\approx 0.79 \end{aligned}$$

So, the eccentricity of the ellipse is about 0.79.

7-2 Ellipses and Circles

Write the standard form of the equation for each ellipse.

49. The vertices are at (–10, 0) and (10, 0) and the eccentricity e is $\frac{3}{5}$.

SOLUTION:

The vertices are located at (–10, 0) and (10, 0), so $2a = 20$, $a = 10$,and $a^2 = 100$.

The eccentricity of the ellipse is given, so you can use the eccentricity equation to find c.

$$\begin{aligned} e &= \frac{c}{a} \\ \frac{3}{5} &= \frac{c}{10} \\ 6 &= c \end{aligned}$$

Use the values of a and c to find b

$$\begin{aligned} c^2 &= a^2 - b^2 \\ 6^2 &= 10^2 - b^2 \\ 36 &= 100 - b^2 \\ 64 &= b^2 \end{aligned}$$

The y–term is the same for each vertex, so the ellipse is oriented horizontally. The center is the midpoint of the vertices or (0, 0). So, $h = 0$ and $k = 0$. Therefore, the equation of the ellipse is $\frac{x^2}{100} + \frac{y^2}{64} = 1$.

50. The co–vertices are at (0, 1) and (6, 1) and the eccentricity e is $\frac{4}{5}$.

SOLUTION:

The co–vertices are located at (0, 1) and (6, 1), so $2b = 6$, $b = 3$, and $b^2 = 9$. The center is located at the midpoint of the co–vertices, or (3, 1). So, $h = 3$ and $k = 1$. The y–term is the same for each co–vertex, so the ellipse is oriented vertically and the standard form of the equation is $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$.

Substitute $e = \frac{4}{5}$ into the eccentricity equation, and solve for c.

$$\begin{aligned} e &= \frac{c}{a} \\ \frac{4}{5} &= \frac{c}{a} \\ \frac{4}{5}a &= c \end{aligned}$$

Substitute b and c into $c^2 = a^2 - b^2$ to find a.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ \left(\frac{4}{5}a\right)^2 &= a^2 - 9 \\ \frac{16a^2}{25} &= a^2 - 9 \\ 9 &= \frac{9a^2}{25} \\ 25 &= a^2 \end{aligned}$$

Therefore, the equation of the ellipse is $\frac{(x-3)^2}{9} + \frac{(y-1)^2}{25} = 1$

51. The center is at (2, –4), one focus is at (2, $-4 + 2\sqrt{5}$), and the eccentricity e is $\frac{\sqrt{5}}{3}$.

SOLUTION:

The center is located at (2, –4), so $h = 2$ and $k = -4$. The distance from a focus to the center is equal to c. Since one focus is located at (2, $-4 + 2\sqrt{5}$), $c = 2\sqrt{5}$. Since the x–coordinates of the center and a focus are the same, the ellipse is oriented horizontally and the standard form of the equation is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

Find a.

$$\begin{aligned} e &= \frac{c}{a} \\ \frac{\sqrt{5}}{3} &= \frac{2\sqrt{5}}{a} \\ \sqrt{5}a &= 6\sqrt{5} \\ a &= 6 \end{aligned}$$

So, $a = 6$ and $a^2 = 36$.

Find b.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ (2\sqrt{5})^2 &= 6^2 - b^2 \\ 20 &= 36 - b^2 \\ b^2 &= 16 \end{aligned}$$

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(y+4)^2}{36} + \frac{(x-2)^2}{16} = 1$.

52. **ROLLER COASTERS** The shape of a roller coaster loop in an amusement park can be modeled by $\frac{y^2}{3306.25} + \frac{x^2}{2025} = 1$.

a. What is the width of the loop along the horizontal axis?

b. Determine the height of the roller coaster from the ground when it reaches the top of the loop, if the lower rail is 20 feet from ground level.

c. Find the eccentricity of the ellipse.

SOLUTION:

a. The equation is in standard form and 2025 is associated with the x^2–term and is less than 3306.25. So, $b^2 = 2025$ and $b = 45$. The width of the loop along the horizontal line is $2 \cdot 45$ or 90 ft.

b. Find a.

$$\begin{aligned} a^2 &= 3306.25 \\ a &= \sqrt{3306.25} \text{ or about } 57.5 \end{aligned}$$

So, $a = 57.5$ and $2a = 115$. Since the lower rail is 20 ft above ground level, the maximum height is 115 + 20 or 135 ft.

c. Find c.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ c^2 &= 3306.25 - 2025 \\ c &= \sqrt{3306.25 - 2025} \text{ or about } 35.79 \end{aligned}$$

Now you can find the eccentricity of the ellipse.

$$\begin{aligned} e &= \frac{c}{a} \\ &= \frac{35.79}{57.5} \\ &\approx 0.62 \end{aligned}$$

53. **FOREST FIRES** The radius of a forest fire is expanding at a rate of 4 miles per day. The current state of the fire is shown below, where a city is located 20 miles southeast of the fire.

a. Write the equation of the circle at the current time and the equation of the circle at the time the fire reaches the city.

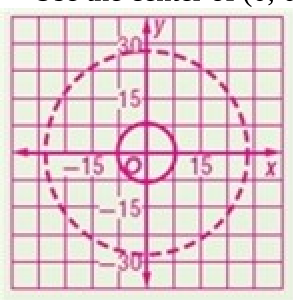
b. Graph both circles.

c. If the fire continues to spread at the same rate, how many days will it take to reach the city?

SOLUTION:

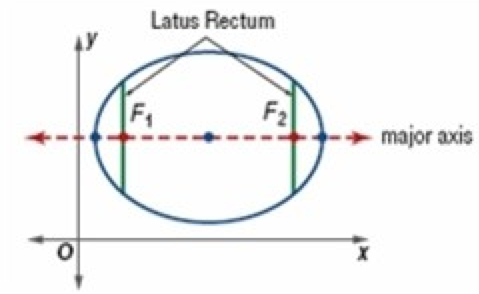
a. The radius of the small circle is 8. So, the equation is $x^2 + y^2 = 64$. When the fire reaches the city, the radius will be 8 + 20 or 28. The equation for the circle is $x^2 + y^2 = 784$.

b. Use the center of (0, 0), $r = 8$, and $r = 28$ to graph both circles.



c. The radius of the fire is growing 4 miles per day. It will reach the city if 20 ÷ 4 = 5 days.

54. The *latus rectum* of an ellipse is a line segment that passes through a focus, is perpendicular to the major axis of the ellipse, and has endpoints on the ellipse. The length of each latus rectum is $\frac{2b^2}{a}$ units, where a is half the length of the major axis and b is half the length of the minor axis.



Write the equation of a horizontal ellipse with center at (3, 2), major axis is 16 units long, and latus rectum 12 units long.

SOLUTION:

The length of the major axis is equal to 16. So, $2a = 16$ and $a = 8$, and $a^2 = 64$. The length of the latus rectum is 12, so $12 = \frac{2b^2}{a}$.

Find b^2.

$$\begin{aligned} 12 &= \frac{2b^2}{a} \\ 12 &= \frac{b^2}{4} \\ 48 &= b^2 \end{aligned}$$

The ellipse is oriented horizontally, so the standard form of the equation is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x-3)^2}{64} + \frac{(y-2)^2}{48} = 1$.

Find the coordinates of points where a line intersects a circle.

55. $y = x - 8$, $(x - 7)^2 + (y + 5)^2 = 16$

SOLUTION:

Solve the system of equations.

$$\begin{aligned} (x-7)^2 + (y+5)^2 &= 16 \\ (x-7)^2 + (x-8+5)^2 &= 16 \\ x^2 - 14x + 49 + x^2 - 6x + 9 &= 16 \\ 2x^2 - 20x + 42 &= 0 \\ x^2 - 10x + 21 &= 0 \\ (x-7)(x-3) &= 0 \\ x &= 7 \text{ or } 3 \end{aligned}$$

Substitute $x = 3$ and $x = 7$ back into the first equation to find the y–coordinate for each point of intersection.

$$\begin{aligned} y &= x - 8 \\ &= 7 - 8 \text{ or } -1 \end{aligned}$$

$$\begin{aligned} y &= x - 8 \\ &= 3 - 8 \text{ or } -5 \end{aligned}$$

So, the coordinates of the intersection are (7, –1) and (3, –5).

56. $y = x + 9$, $(x - 3)^2 + (y + 5)^2 = 169$

SOLUTION:

Solve the system of equations.

$$\begin{aligned} (x-3)^2 + (y+5)^2 &= 169 \\ (x-3)^2 + (x+9+5)^2 &= 169 \\ x^2 - 6x + 9 + x^2 + 28x + 196 &= 169 \\ 2x^2 + 22x + 36 &= 0 \\ x^2 + 11x + 18 &= 0 \\ (x+9)(x+2) &= 0 \\ x &= -9 \text{ or } -2 \end{aligned}$$

Substitute $x = -9$ and $x = -2$ back into the first equation to find the y–coordinate for each point of intersection.

$$\begin{aligned} y &= x + 9 \\ &= -9 + 9 \text{ or } 0 \end{aligned}$$

$$\begin{aligned} y &= x + 9 \\ &= -2 + 9 \text{ or } 7 \end{aligned}$$

So, the coordinates of the intersection are (–2, 7) and (–9, 0).

7-2 Ellipses and Circles

57. $y = -x + 1$, $(x - 5)^2 + (y - 2)^2 = 50$

SOLUTION:

Solve the system of equations.

$$\begin{aligned} (x-5)^2 + (y-2)^2 &= 50 \\ (x-5)^2 + (-x+1-2)^2 &= 50 \\ x^2 - 10x + 25 + x^2 + 2x + 1 &= 50 \\ 2x^2 - 8x - 24 &= 0 \\ x^2 - 4x - 12 &= 0 \\ (x-6)(x+2) &= 0 \\ x &= 6 \text{ or } -2 \end{aligned}$$

Substitute $x = 6$ and $x = -2$ back into the first equation to find the y–coordinate for each point of intersection.

$$\begin{aligned} y &= -x + 1 \\ &= -6 + 1 \text{ or } -5 \end{aligned}$$

$$\begin{aligned} y &= -x + 1 \\ &= -(-2) + 1 \text{ or } 3 \end{aligned}$$

So, the coordinates of the intersection are $(-2, 3)$ and $(6, -5)$.

58. $y = -\frac{1}{3}x - 3$, $(x + 3)^2 + (y - 3)^2 = 25$

SOLUTION:

Solve the system of equations.

$$
\begin{aligned}
(x + 3)^2 + (y - 3)^2 &= 25 \\
(x + 3)^2 + \left(-\frac{1}{3}x - 3 - 3\right)^2 &= 25 \\
x^2 + 6x + 9 + \frac{1}{9}x^2 + 4x + 36 &= 25 \\
\frac{10}{9}x^2 + 10x + 20 &= 0 \\
10x^2 + 90x + 180 &= 0 \\
x^2 + 9x + 18 &= 0 \\
(x + 6)(x + 3) &= 0 \\
x &= -6 \text{ or } -3
\end{aligned}
$$

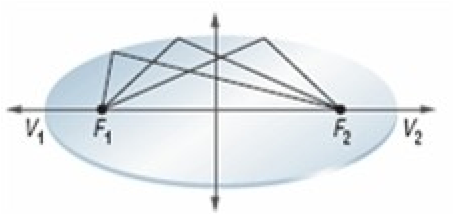
Substitute $x = -6$ and $x = -3$ back into the first equation to find the y–coordinate for each point of intersection.

$$
\begin{aligned}
y &= -\frac{1}{3}(-6) - 3 \\
&= 2 - 3 \text{ or } -1
\end{aligned}
$$

$$
\begin{aligned}
y &= -\frac{1}{3}(-3) - 3 \\
&= 1 - 3 \text{ or } -2
\end{aligned}
$$

So, the coordinates of the intersection are (–3, –2) and (–6, –1).

59. **REFLECTION** *Silvering* is the process of coating glass with a reflective substance. The interior of an ellipse can be silvered to produce a mirror with rays that originate at the ellipse's focus and then reflect to the other focus, as shown. If the segment V_1F_1 is 2 cm long and the eccentricity of the mirror is 0.5, find the equation of the ellipse in standard form.



SOLUTION:

If V_1F_1 is 2 cm, then the distance from the vertex to the center is 2 cm greater than the distance from the focus to the center, and $a = c + 2$. If the eccentricity is 0.5, then $\frac{c}{a} = 0.5$.

Solve the system of equations.

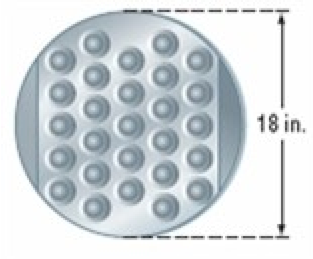
$$\begin{aligned} \frac{c}{c+2} &= 0.5 \\ c &= 0.5c + 1 \\ 0.5c &= 1 \\ c &= 2 \end{aligned}$$

By substituting $c = 2$ back into one of the original equations, you can find that $a = 4$. Use the values of a and c to find b.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ 2^2 &= 4^2 - b^2 \\ 4 &= 16 - b^2 \\ b^2 &= 12 \end{aligned}$$

So, the equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{12} = 1$.

60. **CHEMISTRY** Distillation columns are used to separate chemical substances based on the differences in their rates of evaporation. The columns may contain plates with bubble caps or small circular openings.
a. Write an equation for the plate shown, assuming that the center is at (−4, −1).
b. What is the surface area of the plate not covered by bubble caps if each cap is 2 inches in diameter?

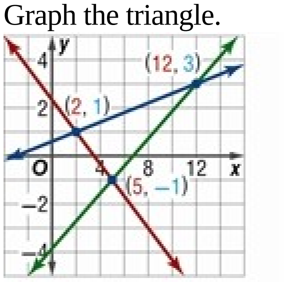


SOLUTION:

a. The plate is circular with center (–4, –1). The diameter is 18 inches, so the radius r is 9 and r^2 is 81. The equation of the circle is $(x + 4)^2 + (y + 1)^2 = 81$.
b. The area of the plate is $\pi \cdot 9^2$ or 81π. The area of a cap is $\pi \cdot 1^2$ or π. There are 27 caps, so the area not covered by caps is $81\pi - 27\pi$ or about 169.6 in^2.

61. **GEOMETRY** The graphs of $x - 5y = -3$, $2x + 3y = 7$, and $4x - 7y = 27$ contain the sides of a triangle. Write the equation of a circle that circumscribes the triangle.

SOLUTION:
Graph the triangle.



The equation of the circle that circumscribes the triangle is found by locating the circumcenter of the triangle. Locate the perpendicular bisectors of each side of the triangle.

Find the midpoints of each segment
(2, 1) and (12, 3)

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{2 + 12}{2}, \frac{1 + 3}{2}\right) = \left(\frac{14}{2}, \frac{4}{2}\right) = (7, 2)$$

(5, –1) and (12, 3)

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{5 + 12}{2}, \frac{-1 + 3}{2}\right) = \left(\frac{17}{2}, \frac{2}{2}\right) = \left(\frac{17}{2}, 1\right)$$

(2, 1) and (5, –1)

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$
$$= \left(\frac{5+2}{2}, \frac{-1+1}{2}\right) = \left(\frac{7}{2}, \frac{0}{2}\right) = (3.5, 0)$$

Thus, the midpoints of the segments are (8.5, 1), (3.5, 0), and (7, 2).

Find the equations for the perpendicular bisectors.
Find the slope of segment through points (2, 1) and (12, 3).
$m = \frac{1-3}{2-12} = \frac{-2}{-10} = \frac{1}{5}$
$m_{\perp}$ is –5.

Find equation of line perpendicular through (7, 2)
$$y - y_1 = m(x - x_1)$$
$$y - 2 = -5(x - 7)$$
$$y = -5x + 37$$

Find the slope of segment through points (5, –1) and (12, 3).
$m = \frac{-1-3}{5-12} = \frac{4}{7}$
$m_{\perp}$ is $-\frac{7}{4}$.

Find equation of line perpendicular through (8.5, 1).
$$y - y_1 = m(x - x_1)$$
$$y - 1 = -\frac{7}{4}(x - 8.5)$$
$$y = -\frac{7}{4}x + \frac{127}{8}$$

Find the slope of segment through points (2, 1) and (5, –1).
$m = \frac{-1-1}{5-2} = -\frac{2}{3}$
$m_{\perp}$ is $\frac{3}{2}$.

Find equation of line perpendicular through (3.5, 0).
$$y - y_1 = m(x - x_1)$$
$$y - 0 = \frac{3}{2}(x - 3.5)$$
$$y = -\frac{3}{2}x - \frac{21}{4}$$

Thus, the equations of the perpendicular bisectors are:
$y = -\frac{7}{4}x + \frac{127}{8}$
$y = \frac{3}{2}x - \frac{21}{4}$
$y = -5x + 37$

Find the intersection of the perpendicular bisectors to find the circumcenter.

$$y = -\frac{7}{4}x + \frac{127}{8} \ \& \ y = \frac{3}{2}x - \frac{21}{4}$$

$$\begin{aligned} -\frac{7}{4}x + \frac{127}{8} &= \frac{3}{2}x - \frac{21}{4} \\ \frac{127}{8} + \frac{21}{4} &= \frac{3}{2}x + \frac{7}{4}x \\ \frac{169}{8} &= \frac{13}{4}x \\ \frac{13}{2} &= x \end{aligned}$$

$$y = \frac{3}{2}\left(\frac{13}{2}\right) - \frac{21}{4} = \frac{9}{2}$$

$$y = \frac{3}{2}x - \frac{21}{4} \ \& \ y = -5x + 37$$

$$\begin{aligned} -5x + 37 &= \frac{3}{2}x - \frac{21}{4} \\ 37 + \frac{21}{4} &= \frac{3}{2}x + 5x \\ \frac{169}{4} &= \frac{13}{2}x \\ \frac{13}{2} &= x \end{aligned}$$

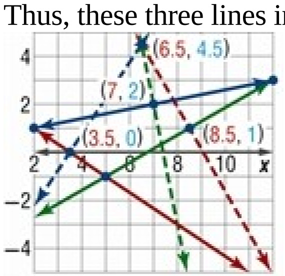
$$y = \frac{3}{2}\left(\frac{13}{2}\right) - \frac{21}{4} = \frac{9}{2}$$

$$y = -\frac{7}{4}x + \frac{127}{8} \text{ and } y = -5x + 37.$$

$$\begin{aligned} -5x + 37 &= -\frac{7}{4}x + \frac{127}{8} \\ 37 - \frac{127}{8} &= -\frac{7}{4}x + 5x \\ \frac{169}{8} &= \frac{13}{4}x \\ \frac{13}{2} &= x \end{aligned}$$

$$y = -5\left(\frac{13}{2}\right) + 37 = \frac{9}{2}$$

Thus, these three lines intersect at the circumcenter of the triangle, (6.5, 4.5).



(6.5, 4.5) is the center of the circle.

The radius of the circle is the distance between this point and any vertex
Use the distance formula to find the radius.
(6.5, 4.5) and (12, 3)

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$D = \sqrt{(12 - 6.5)^2 + (3 - 4.5)^2} = \sqrt{5.5^2 + 1.5^2} = \sqrt{32.5}$$

(6.5, 4.5) and (5, –1)

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

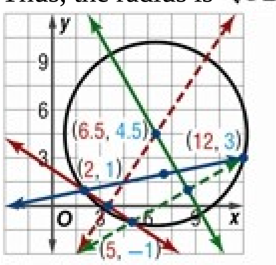
$$D = \sqrt{(5 - 6.5)^2 + (-1 - 4.5)^2} = \sqrt{(-1.5)^2 + (-5.5)^2} = \sqrt{32.5}$$

(6.5, 4.5) and (2, 1)

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$D = \sqrt{(2 - 6.5)^2 + (1 - 4.5)^2} = \sqrt{(-4.5)^2 + (-3.5)^2} = \sqrt{32.5}$$

Thus, the radius is $\sqrt{32.5}$ and $r^2 = 32.5$. Then equation of the circle is $(x - 6.5)^2 + (y - 4.5)^2 = 32.5$.



Write the standard form of the equation of a circle that passes through each set of points. Then identify the center and radius of the circle.

62. (2, 3), (8, 3), (5, 6)

SOLUTION:

The standard form of the equation for a circle is $(x - h)^2 + (y - k)^2 = r^2$. Substitute each point into the equation to obtain three equations.

$(2 - h)^2 + (3 - k)^2 = r^2$
$(8 - h)^2 + (3 - k)^2 = r^2$
$(5 - h)^2 + (6 - k)^2 = r^2$

Since $(2 - h)^2 + (3 - k)^2 = r^2$ and $(8 - h)^2 + (3 - k)^2 = r^2$ both have the same y–coordinate, you can solve that system of equations to find h.

$$\begin{aligned} (2-h)^2 + (3-k)^2 &= (8-h)^2 + (3-k)^2 \\ (2-h)^2 &= (8-h)^2 \\ 4 - 4h + h^2 &= 64 - 16h + h^2 \\ -4h + 4 &= -16h + 64 \\ 12h &= 60 \\ h &= 5 \end{aligned}$$

Next, substitute $h = 5$ into two of the equations to get a system of two equations with k and r.

$$\begin{aligned}(8-5)^2+(3-k)^2&=r^2\\ 9+k^2-6k+9&=r^2\\ k^2-6k+18&=r^2\end{aligned}$$

$$\begin{aligned}(5-5)^2+(6-k)^2&=r^2\\ (6-k)^2&=r^2\\ k^2-12k+36&=r^2\end{aligned}$$

Solve these two new equations for k.

$$\begin{aligned}k^2-12k+36&=k^2-6k+18\\ -12k+36&=-6k+18\\ 18&=6k\\ 3&=k\end{aligned}$$

Substitute h = 5 and k = 3 into any of the original equations to find r.

$$\begin{aligned}(2-h)^2+(3-k)^2&=r^2\\ (2-5)^2+(3-3)^2&=r^2\\ 9&=r^2\\ \pm 3&=r\end{aligned}$$

Since the radius must be positive, r = 3.

The equation of the circle is $(x - 5)^2 + (y - 3)^2 = 9$, with center (5, 3) and radius r = 3.

63. (1, –11), (–3, –7), (5, –7)

SOLUTION:

The standard form of the equation for a circle is $(x - h)^2 + (y - k)^2 = r^2$. Substitute each point into the equation to obtain three equations.

$(1 - h)^2 + (-11 - k)^2 = r^2$
$(-3 - h)^2 + (-7 - k)^2 = r^2$
$(5 - h)^2 + (-7 - k)^2 = r^2$

Since $(-3 - h)^2 + (-7 - k)^2 = r^2$ and $(5 - h)^2 + (-7 - k)^2 = r^2$ both have the same y–coordinate, you can solve that system of equations to find h.

$$\begin{aligned}(-3-h)^2+(-7-k)^2 &= (5-h)^2+(-7-k)^2\\ (-3-h)^2 &= (5-h)^2\\ 9+6h+h^2 &= 25-10h+h^2\\ 6h+9 &= -10h+25\\ 16h &= 16\\ h &= 1\end{aligned}$$

Next, Substitute h = 1 into tow equations to get a system of two equations with k and r.

$$\begin{aligned}(5-h)^2+(-7-k)^2 &= r^2\\ (5-1)^2+(-7-k)^2 &= r^2\\ 16+k^2+14k+49 &= r^2\\ k^2+14k+65 &= r^2\end{aligned}$$

$$\begin{aligned}(1-h)^2+(-11-k)^2 &= r^2\\ (1-1)^2+(-11-k)^2 &= r^2\\ k^2+22k+121 &= r^2\end{aligned}$$

Solve these two new equations for k.

$$\begin{aligned}k^2+22k+121 &= k^2+14k+65\\ 22k+121 &= 14k+65\\ 8k &= -56\\ k &= -7\end{aligned}$$

Substitute h = 1 and k = –7 into any of the equations to find r.

$$\begin{aligned}(5-h)^2+(-7-k)^2 &= r^2\\ (5-1)^2+(-7-(-7))^2 &= r^2\\ 16 &= r^2\\ \pm 4 &= r\end{aligned}$$

Since the radius must be positive, r = 4.

So, the equation of the circle is $(x-1)^2+(y+7)^2=16$, with center (1, –7) and radius r = 4.

64. (0, 9), (0, 3), (–3, 6)

SOLUTION:

The standard form of the equation for a circle is $(x - h)^2 + (y - k)^2 = r^2$. Substitute each point into the equation to obtain three equations.
$(0 - h)^2 + (9 - k)^2 = r^2$ or $h^2 + (9 - k)^2 = r^2$
$(0 - h)^2 + (3 - k)^2 = r^2$ or $h^2 + (3 - k)^2 = r^2$
$(-3 - h)^2 + (6 - k)^2 = r^2$

Since $h^2 + (9 - k)^2 = r^2$ and $h^2 + (9 - k)^2 = r^2$ both have 0 as a y–coordinate, you can solve that system of equations to find k.

$$\begin{aligned} h^2 + (9-k)^2 &= h^2 + (3-k)^2 \\ 81 - 18k + k^2 &= 9 - 6k + k^2 \\ 81 - 18k &= 9 - 6k \\ 72 &= 12k \\ 6 &= k \end{aligned}$$

Next, substitute $k = 6$ into two of the equations to get a system of two equations in h and r.

$$\begin{aligned} h^2 + (9-k)^2 &= r^2 \\ h^2 + (9-6)^2 &= r^2 \\ h^2 + 9 &= r^2 \end{aligned}$$

$$\begin{aligned} (-3-h)^2 + (6-k)^2 &= r^2 \\ (-3-h)^2 + (6-6)^2 &= r^2 \\ 9 + 6h + h^2 &= r^2 \end{aligned}$$

Solve these two new equations for h.

$$\begin{aligned} 9 + 6h + h^2 &= 9 + h^2 \\ 9 + 6h &= 9 \\ 6h &= 0 \\ h &= 0 \end{aligned}$$

Substitute $h = 0$ and $k = 6$ into any of the original equations to find r.

$$\begin{aligned} (-3-h)^2 + (6-k)^2 &= r^2 \\ (-3-0)^2 + (6-6)^2 &= r^2 \\ 9 &= r^2 \\ \pm 3 &= r \end{aligned}$$

Since the radius must be positive, $r = 3$.

So, the equation of the circle is $x^2 + (y - 6)^2 = 9$, with center (0, 6) and radius $r = 3$.

65. (7, 4), (–1, 12), (–9, 4)

SOLUTION:

The standard form of the equation for a circle is $(x - h)^2 + (y - k)^2 = r^2$. Substitute each point into the equation to obtain three equations.

$(7 - h)^2 + (4 - k)^2 = r^2$
$(-1 - h)^2 + (12 - k)^2 = r^2$
$(-9 - h)^2 + (4 - k)^2 = r^2$

Since $(7 - h)^2 + (4 - k)^2 = r^2$ and $(-9 - h)^2 + (4 - k)^2 = r^2$ both have the same y–coordinate, you can solve that system of equations to find h.

$$\begin{aligned}(7-h)^2+(4-k)^2 &= (-9-h)^2+(4-k)^2\\ (7-h)^2 &= (-9-h)^2\\ 49-14h+h^2 &= 81+18h+h^2\\ 49-14h &= 81+18h\\ -32 &= 32h\\ -1 &= h\end{aligned}$$

Next, substitute $h = -1$ into two equations to get a system of two equations with k and r.

$$\begin{aligned}(-9-h)^2+(4-k)^2 &= r^2\\ (-9-(-1))^2+(4-k)^2 &= r^2\\ 64+14-8k+k^2 &= r^2\\ 80-8k+k^2 &= r^2\end{aligned}$$

$$\begin{aligned}(-1-h)^2+(12-k)^2 &= r^2\\ (-1-(-1))^2+(12-k)^2 &= r^2\\ 144-24k+k^2 &= r^2\end{aligned}$$

Solve these two new equations for k.

$$\begin{aligned}k^2-8k+80 &= k^2-24k+144\\ -8k+80 &= -24k+144\\ 16k &= 64\\ k &= 4\end{aligned}$$

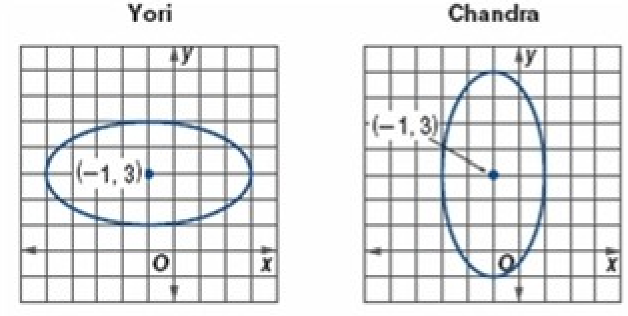
Substitute $h = -1$ and $k = 4$ into any of the original equations to find r.

$$\begin{aligned}(7-h)^2+(4-4)^2 &= r^2\\ (7-(-1))^2+(4-4)^2 &= r^2\\ 64 &= r^2\\ \pm 8 &= r\end{aligned}$$

Since the radius must be positive, $r = 8$.

So, the equation of the circle is $(x + 1)^2 + (y - 4)^2 = 64$, with center $(-1, 4)$ and radius $r = 8$.

66. **ERROR ANALYSIS** Yori and Chandra are graphing an ellipse that has a center at (–1, 3), a major axis of length 8, and a minor axis of length 4. Is either of them correct? Explain your reasoning.



SOLUTION:

Both Yori and Chandra are correct. In Yori's graph, the major axis is horizontal. In Chandra's graph, the major axis is vertical.

67. **REASONING** Determine whether an ellipse represented by $\frac{x^2}{p} + \frac{y^2}{p+r} = 1$, where $r > 0$, will have the same foci as the ellipse represented by $\frac{x^2}{p+r} + \frac{y^2}{p} = 1$. Explain your reasoning.

SOLUTION:

No, they will not have the same foci.
Sample answer:

For $\frac{x^2}{p} + \frac{y^2}{p+r} = 1$, $p > 0$ and $r > 0$, so $p + r > p$. Then $\frac{x^2}{p} + \frac{y^2}{p+r} = 1$ is of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ and is elongated vertically with $a^2 = p + r$ and $b^2 = p$. Use these values to find c and the foci.

$c^2 = a^2 - b^2$
$c^2 = p + r - p$
$c^2 = r$
$c = \sqrt{r}$

So, the foci are found $\sqrt{r}$ up and down from the center (0, 0). Therefore, the foci are $\left(0, \pm\sqrt{r}\right)$.

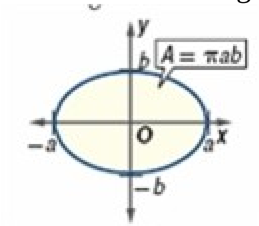
For $\frac{x^2}{p+r} + \frac{y^2}{p} = 1$, $p > 0$ and $r > 0$, so $p + r > p$. Then $\frac{x^2}{p+r} + \frac{y^2}{p} = 1$ is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and is elongated horizontally with $a^2 = p + r$ and $b^2 = p$. Use these values to find c and the foci.

$c^2 = a^2 - b^2$
$c^2 = p + r - p$
$c^2 = r$
$c = \sqrt{r}$

So, the foci are found $\sqrt{r}$ left and right from the center (0, 0). Therefore, the foci are $\left(\pm\sqrt{r}, 0\right)$.

7-2 Ellipses and Circles

CHALLENGE The area A of an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $A = \pi ab$. Write an equation of an ellipse with each of the following characteristics.



68. $b + a = 12, A = 35\pi$

SOLUTION:

The area of the ellipse is 35π, so $35 = ab$. Now, you have a system of two equations, $35 = ab$ and $b = 12 - a$.

Solve the system of equations.

$$\begin{aligned} 35 &= ab \\ 35 &= a(12 - a) \\ 35 &= 12a - a^2 \\ 0 &= -a^2 + 12a - 35 \\ 0 &= (-a + 7)(a - 5) \end{aligned}$$

Since a cannot be negative, $a = 5$ and $a^2 = 25$.

Substitute $a = 5$ into one of the equations to find b.

$$\begin{aligned} b + a &= 12 \\ b + 5 &= 12 \\ b &= 7 \end{aligned}$$

So, $b = 7$ and $b^2 = 49$.

Therefore, an equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{49} = 1$.

7-2 Ellipses and Circles

69. $a - b = -5, A = 24\pi$

SOLUTION:

The area of the ellipse is 24π, so $24 = ab$. Now, you have a system of two equations, $24 = ab$ and $a = 5 + b$.

Solve the system of equations.

$$\begin{aligned} 24 &= ab \\ 24 &= b(b + 5) \\ 24 &= b^2 + 5b \\ 0 &= b^2 + 5b - 24 \\ 0 &= (b + 8)(b - 3) \end{aligned}$$

Since b cannot be negative, $b = 3$ and $b^2 = 9$.

Substitute $b = 8$ into one of the equations to find a.

$$\begin{aligned} a - b &= 5 \\ a - 3 &= 5 \\ a &= 8 \end{aligned}$$

So, $a = 8$ and $a^2 = 64$.

Therefore, an equation of the ellipse is $\frac{x^2}{64} + \frac{y^2}{9} = 1$.

70. ***Writing in Math*** Explain how to find the foci and vertices of an ellipse if you are given the standard form of the equation.

SOLUTION:

Sample answer: First, use the equation to find the values of a, b, h, and k. Use $c^2 = a^2 - b^2$ to find the value of c. When the equation is in standard form, the square root of the larger denominator is a and the square root of the smaller denominator is b. h is associated with x and k is associated with y in the numerators.

Then determine whether the major axis is horizontal or vertical. If x corresponds with the larger denominator (a), then the orientation is horizontal. Otherwise, it is vertical.

If the major axis is horizontal, the foci are at $(h \pm c, k)$, the vertices are at $(h \pm a, k)$, and the co–vertices are at $(h, k \pm b)$. If the major axis is vertical, the foci are at $(h, k \pm c)$, the vertices are at $(h, k \pm a)$, and the co–vertices are at $(h \pm b, k)$.

7-2 Ellipses and Circles

71. **REASONING** Is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ symmetric with respect to the origin? Explain your reasoning.

SOLUTION:

Yes; sample answer: If (x, y) is a point on the ellipse, then $(–x, –y)$ must also be on the ellipse. This can be determined because the x- and y-terms are squared. When the center is not at the origin, then we will be squaring $x + h$ and/or $y + k$ instead of x and y. We know that $(x + h)^2 \neq x^2$ when $h \neq 0$, so the ellipse will not be symmetric with respect to the origin.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{(-x)^2}{a^2} + \frac{(-y)^2}{b^2} = 1$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Thus, $(–x, –y)$ is also a point on the ellipse and the ellipse is symmetric with respect to the origin.

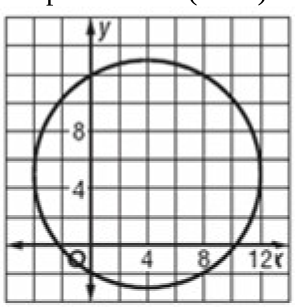
72. **OPEN ENDED** If the equation of a circle is $(x – h)^2 + (y – k)^2 = r^2$, where $h > 0$ and $k < 0$, what is the domain of the circle? Verify your answer with an example, both algebraically and graphically.

SOLUTION:

The value of k does not affect the domain. It only shifts the graph up and down.

The value of h does affect the domain as it shifts the graph left and right. The value of r determines the range of the domain.
D: $[h – r, h + r]$

Sample answer: $(x – 4)^2 + (y + 5)^2 = 8^2$, with domain $[4 – 8, 4 + 8]$ or $[–4, 12]$.



73. ***Writing in Math*** Explain why an ellipse becomes circular as the value of b approaches the value of a.

SOLUTION:

Sample answer: As the value of b approaches the value of a, $a^2 – b^2$ approaches 0. In the relationship $c^2 = a^2 – b^2$, if $a^2 – b^2$ approaches 0, so does c^2. This means that c approaches 0. If c approaches 0, then the eccentricity of an ellipse, defined as $e = \frac{c}{a}$, also approaches 0. As e approaches 0, an ellipse becomes less elongated and more circular. When $e = 0$, the resulting figure is a circle.

For each equation, identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola.

74. $y = 3x^2 - 24x + 50$

SOLUTION:

Write the standard form of the equation.

$$\begin{aligned} y &= 3x^2 - 24x + 50 \\ y &= 3(x^2 - 8x + 16) + 2 \\ y - 2 &= 3(x - 4)^2 \\ \frac{1}{3}(y - 2) &= (x - 4)^2 \end{aligned}$$

Because the x–term is squared and $p = \frac{1}{12}$, the graph opens up. Use the standard form to determine the characteristics of the parabola.

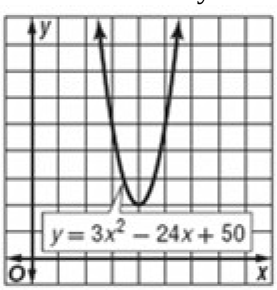
vertex: $(h, k) = (4, 2)$

focus: $(h + p, k) = \left(4, 2\frac{1}{12}\right)$

directrix: $y = k - p$ or $1\frac{11}{12}$

axis of symmetry: $x = h$ or 4

Graph the vertex, focus, axis, and directrix of the parabola. Then make a table of values to graph the curve. The curve should be symmetric about the axis of symmetry.



75. $y = -2x^2 + 5x - 10$

SOLUTION:

Write the standard form of the equation.

$$\begin{aligned} y &= -2x^2 + 5x - 10 \\ y &= -2\left(x^2 - \tfrac{5}{2}x + 5\right) \\ y &= -2\left(x^2 - \tfrac{5}{2}x + \tfrac{25}{16} - \tfrac{25}{16} + 5\right) \\ y &= -2\left(x^2 - \tfrac{5}{2}x + \tfrac{25}{16} + \tfrac{55}{16}\right) \\ y &= -2\left(x^2 - \tfrac{5}{2}x + \tfrac{25}{16}\right) - 2\left(\tfrac{55}{16}\right) \\ y &= -2\left(x - \tfrac{5}{4}\right)^2 - \tfrac{55}{8} \\ y + \tfrac{55}{8} &= -2\left(x - \tfrac{5}{4}\right)^2 \\ -\tfrac{1}{2}\left(y + \tfrac{55}{8}\right) &= \left(x - \tfrac{5}{4}\right)^2 \end{aligned}$$

Because the x–term is squared and $p = -\frac{1}{8}$, the graph opens down. Use the standard form to determine the characteristics of the parabola.

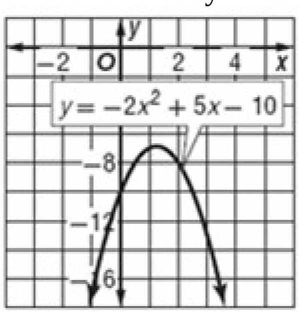
vertex: $(h, k) = \left(\frac{5}{4}, -\frac{55}{8}\right)$

focus: $(h, k + p) = \left(\frac{5}{4}, -7\right)$

directrix: $y = k - p$ or $-\frac{27}{4}$

axis of symmetry: $x = h$ or $\frac{5}{4}$

Graph the vertex, focus, axis, and directrix of the parabola. Then make a table of values to graph the curve. The curve should be symmetric about the axis of symmetry.



7-2 Ellipses and Circles

76. $x = 5y^2 - 10y + 9$

SOLUTION:

Write the standard form of the equation.

$x = 5y^2 - 10y + 9$

$x = 5(y^2 - 2y + 1) + 4$

$x - 4 = 5(y - 1)^2$

$\frac{1}{5}(x - 4) = (y - 1)^2$

Because the y–term is squared and $p = \frac{1}{20}$, the graph opens right. Use the standard form to determine the characteristics of the parabola.

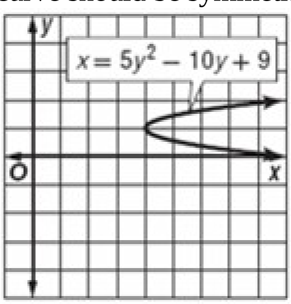
vertex: $(h, k) = (4, 1)$

focus: $(h + p, k) = \left(4\frac{1}{20}, 1\right)$

directrix: $x = h - p$ or $3\frac{19}{20}$

axis of symmetry: $y = k$ or 1

Graph the vertex, focus, axis, and directrix of the parabola. Then make a table of values to graph the curve. The curve should be symmetric about the axis of symmetry.



77. **MANUFACTURING** A toy company is introducing two new dolls to its customers: My First Baby, which talks, laughs, and cries; and My Real Baby, which uses a bottle and crawls. In one hour, the company can produce 8 First Babies or 20 Real Babies. Because of the demand, the company must produce at least twice as many First Babies as Real Babies. The company spends no more than 48 hours per week making these two dolls. Find the number and type of dolls that should be produced to maximize the profit.

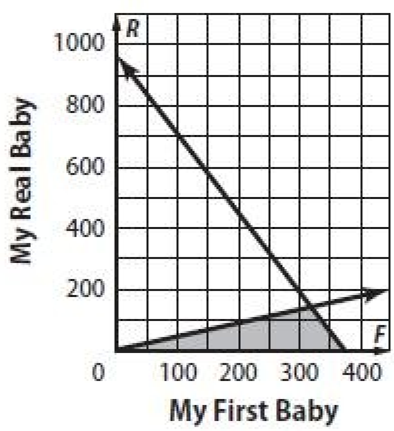
Profit per Doll ($)	
First Baby	Real Baby
3.00	7.50

SOLUTION:

Let *F* represnt the number of First Babies produced and *R* represent the number of Real Babies produced.

At least twice as many First Babies must be produced, so $F \geq 2R$. They can produce one First Baby in $\frac{1}{8}$ hour and one Real Baby in $\frac{1}{20}$ hour. They have at most 48 hours, so $\frac{1}{8}F + \frac{1}{20}R \leq 48$.

Graph both equations.



The vertices of the intersections are (160, 320), (0, 0), and (0, 384).
Because f is greatest at (160, 320), they need to produce 160 My Real Babies and 320 My First Babies.

Verify each identity.

78. $\sin(\theta + 30°) + \cos(\theta + 60°) = \cos\theta$

SOLUTION:

$$\begin{aligned}
&\sin(\theta + 30°) + \cos(\theta + 60°)\\
&= \sin\theta\cos 30° + \cos\theta\sin 30° + \cos\theta\cos 60° - \sin\theta\sin 60°\\
&= \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\\
&= \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta\\
&= \cos\theta
\end{aligned}$$

79. $\sin\left(\theta+\frac{\pi}{3}\right)-\cos\left(\theta+\frac{\pi}{6}\right)=\sin\theta$

SOLUTION:

$$\begin{aligned}
&\sin\left(\theta+\frac{\pi}{3}\right)-\cos\left(\theta+\frac{\pi}{6}\right)\\
&=\sin\theta\cos\frac{\pi}{3}+\cos\theta\sin\frac{\pi}{3}-\left(\cos\theta\cos\frac{\pi}{6}-\sin\theta\sin\frac{\pi}{6}\right)\\
&=\sin\theta\cos\frac{\pi}{3}+\cos\theta\sin\frac{\pi}{3}-\cos\theta\cos\frac{\pi}{6}+\sin\theta\sin\frac{\pi}{6}\\
&=\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta-\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta\\
&=\frac{1}{2}\sin\theta+\frac{1}{2}\sin\theta\\
&=\sin\theta
\end{aligned}$$

80. $\sin(3\pi-x)=\sin x$

SOLUTION:

$$\begin{aligned}
\sin(3\pi-x)&=\sin 3\pi\cos x-\cos 3\pi\sin x\\
&=0\cos x-(-1)\sin x\\
&=0+\sin x\\
&=\sin x
\end{aligned}$$

Find all solutions to each equation on the interval [0, 2π).

81. $\sin\theta=\cos\theta$

SOLUTION:

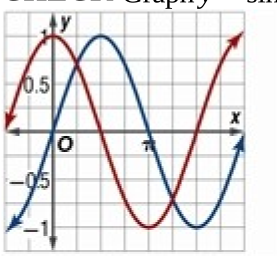
$$\begin{aligned}
\sin\theta&=\cos\theta\\
\sin^2\theta&=\cos^2\theta\\
\sin^2\theta-\cos^2\theta&=0\\
\sin^2\theta-(1-\sin^2\theta)&=0\\
\sin^2\theta-1+\sin^2\theta&=0\\
2\sin^2\theta-1&=0\\
2\sin^2\theta&=1\\
\sin^2\theta&=\frac{1}{2}\\
\sin\theta&=\pm\sqrt{\frac{1}{2}}\\
\sin\theta&=\pm\frac{\sqrt{2}}{2}\\
\theta&=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\text{ and }\frac{7\pi}{4}
\end{aligned}$$

Since squaring each side of a trigonometric equation can produce extraneous results, check each solution in the original equation.

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} \qquad \sin\frac{3\pi}{4} = \cos\frac{\pi}{4}$$
$$\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \qquad \frac{\sqrt{2}}{2} \neq -\frac{\sqrt{2}}{2}$$
$$\sin\frac{5\pi}{4} = \cos\frac{5\pi}{4}$$
$$-\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2} \qquad \sin\frac{7\pi}{4} = \cos\frac{7\pi}{4}$$
$$-\frac{\sqrt{2}}{2} \neq \frac{\sqrt{2}}{2}$$

Therefore, the only solutions to this equation on the interval $[0, 2\pi)$ are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

CHECK Graph $y = \sin\theta$ and $y = \cos\theta$ on the interval $[0, 2\pi)$.



The graphs intersect at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$, so the solutions are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

82. $\sin \theta = 1 + \cos \theta$

SOLUTION:

$$\begin{aligned}\sin\theta &= 1+\cos\theta\\ \sin^2\theta &= 1+2\cos\theta+\cos^2\theta\\ 1-\cos^2\theta &= 1+2\cos\theta+\cos^2\theta\\ 0 &= 2\cos\theta+2\cos^2\theta\end{aligned}$$

$$2\cos\theta+2\cos^2\theta=0$$
$$2\cos\theta(1+\cos\theta)=0$$

$$\begin{aligned}2\cos\theta &= 0 & 1+\cos\theta &= 0\\ \cos\theta &= 0 & \cos\theta &= -1\\ \theta &= \frac{\pi}{2}, \frac{3\pi}{2} & \theta &= \pi\end{aligned}$$

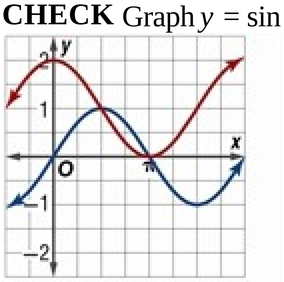
Since squaring each side of a trigonometric equation can produce extraneous results, check each solution in the original equation.

$$\begin{aligned}\sin\theta &= 1+\cos\theta & \sin\theta &= 1+\cos\theta\\ \sin\frac{\pi}{2} &= 1+\cos\frac{\pi}{2} & \sin\frac{3\pi}{2} &= 1+\cos\frac{3\pi}{2}\\ 1 &= 1+0 & -1 &= 1+0\\ 1 &= 1 & -1 &\neq 1\end{aligned}$$

$$\begin{aligned}\sin\theta &= 1+\cos\theta\\ \sin\pi &= 1+\cos\pi\\ 0 &= 1+(-1)\\ 0 &= 0\end{aligned}$$

Therefore, the only solutions to this equation on the interval $[0, 2\pi)$ are π and $\frac{\pi}{2}$.

CHECK Graph $y = \sin \theta$ and $y = 1 + \cos \theta$ on the interval $[0, 2\pi)$.



The graphs intersect at π and $\frac{\pi}{2}$, so the solutions are π and $\frac{\pi}{2}$.

83. $2\sin^2 x + 3\sin x + 1 = 0$

SOLUTION:

Let $u = \sin x$. Then factor the equation.

$2\sin^2 x + 3\sin x + 1 = 0$

$2u^2 + 3u + 1 = 0$

$(2u + 1)(u + 1) = 0$

Substitute sin x back into the equation for u.

$$\begin{aligned}(2u + 1)(u + 1) &= 0 \\ (2\sin x + 1)(\sin x + 1) &= 0\end{aligned}$$

Solve for x.

$$\begin{aligned}2\sin x + 1 &= 0 \\ \sin x &= -\frac{1}{2} \\ x &= \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}\end{aligned}$$

$$\begin{aligned}\sin x + 1 &= 0 \\ \sin x &= -1 \\ x &= \frac{3\pi}{2}\end{aligned}$$

So, the solutions are $\frac{3\pi}{2}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$.

Solve each inequality.

84. $x^2 - 5x - 24 > 0$

SOLUTION:

Let $f(x) = x^2 - 5x - 24$. Find the real zeros of this function.

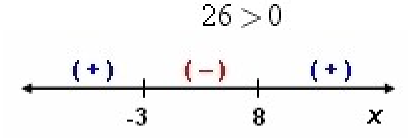
$x^2 - 5x - 24 = 0$
$(x - 8)(x + 3) = 0$
$x = 8$ or $x = -3$

$f(x)$ has real zeros at $x = -3$ and $x = 8$. Set up a sign chart. Substitute an x-value in each test interval into the polynomial to determine if $f(x)$ is positive or negative at that point.

$$\begin{aligned} for\,[-\infty, -3], x &= -4 \\ (x-8)(x+3) &\ ?\ 0 \\ (-4-8)(-4+3) &\ ?\ 0 \\ (-12)(-1) &\ ?\ 0 \\ 12 &> 0 \end{aligned}$$

$$\begin{aligned} for\,[-3, 8], x &= 0 \\ (x-8)(x+3) &\ ?\ 0 \\ (0-8)(0+3) &\ ?\ 0 \\ (-8)(3) &\ ?\ 0 \\ -24 &< 0 \end{aligned}$$

$$\begin{aligned} for\,[8, \infty], x &= 10 \\ (x-8)(x+3) &\ ?\ 0 \\ (10-8)(10+3) &\ ?\ 0 \\ (2)(13) &\ ?\ 0 \\ 26 &> 0 \end{aligned}$$



The solutions of $x^2 - 5x - 24 > 0$ are x-values such that $f(x)$ is positive. From the sign chart, you can see that the solution set is $(-\infty, -3) \cup (8, \infty)$.

7-2 Ellipses and Circles

85. $x^2 + 2x - 35 \le 0$

SOLUTION:

Let $f(x) = x^2 + 2x - 35$. Find the real zeros of this function.

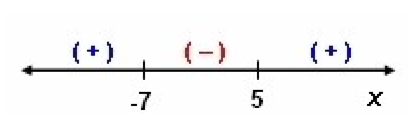
$$x^2 + 2x - 35 = 0$$
$$(x + 7)(x - 5) = 0$$
$$x = -7 \text{ or } x = 5$$

$f(x)$ has real zeros at $x = -7$ and $x = 5$. Set up a sign chart. Substitute an x-value in each test interval into the polynomial to determine if $f(x)$ is positive or negative at that point.

$$\begin{aligned} &for\,[-\infty, -7], x = -8 \\ &(x+7)(x-5) \ ?\ 0 \\ &(-8+7)(-8-5) \ ?\ 0 \\ &(-1)(-13) \ ?\ 0 \\ &13 \not\le 0 \end{aligned}$$

$$\begin{aligned} &for\,[-7, 5], x = 0 \\ &(x+7)(x-5) \ ?\ 0 \\ &(0+7)(0-5) \ ?\ 0 \\ &(7)(-5) \ ?\ 0 \\ &-35 \le 0 \end{aligned}$$

$$\begin{aligned} &for\,[5, \infty], x = 8 \\ &(x+7)(x-5) \ ?\ 0 \\ &(8+7)(8-5) \ ?\ 0 \\ &(15)(3) \ ?\ 0 \\ &45 \not\le 0 \end{aligned}$$



The solutions of $x^2 + 2x - 35 \le 0$ are x-values such that $f(x)$ is negative or zero. From the sign chart, you can see that the solution set is $[-7, 5]$.

86. $-2x^2 + 7x + 4 < 0$

SOLUTION:

Let $f(x) = -2x^2 + 7x + 4$. Find the real zeros of this function.

$$\begin{aligned} -2x^2 + 7x + 4 &= 0 \\ 2x^2 - 7x - 4 &= 0 \\ (2x + 1)(x - 4) &= 0 \end{aligned}$$

$2x + 1 = 0$ or $x - 4 = 0$

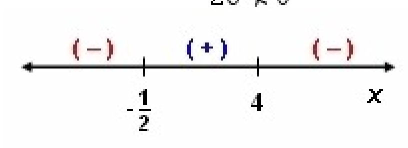
$x = -\frac{1}{2}$ $\quad x = 4$

$f(x)$ has real zeros at $x = -\frac{1}{2}$ and $x = 4$. Set up a sign chart. Substitute an x-value in each test interval into the polynomial to determine if $f(x)$ is positive or negative at that point.

$$for\left[-\infty, -\frac{1}{2}\right], x = -1$$
$$\begin{aligned} (2x+1)(x-4) &\ ?\ 0 \\ (2(-1)+1)(-1-4) &\ ?\ 0 \\ (-1)(-5) &\ ?\ 0 \\ 5 &\not< 0 \end{aligned}$$

$$for\left[-\frac{1}{2}, 4\right], x = 0$$
$$\begin{aligned} (2x+1)(x-4) &\ ?\ 0 \\ (2(0)+1)(0-4) &\ ?\ 0 \\ (1)(-4) &\ ?\ 0 \\ -4 &< 0 \end{aligned}$$

$$for\,[4, \infty], x = 6$$
$$\begin{aligned} (2x+1)(x-4) &\ ?\ 0 \\ (2(6)+1)(6-4) &\ ?\ 0 \\ (13)(2) &\ ?\ 0 \\ 26 &\not< 0 \end{aligned}$$



The solutions of $-2x^2 + 7x + 4 > 0$ are x-values such that $f(x)$ is positive. From the sign chart, you can see that the solution set is $\left(-\frac{1}{2}, 4\right)$.

State the number of possible real zeros and turning points of each function. Then determine all of the real zeros by factoring.

87. $f(x) = 3x^4 + 18x^3 + 24x^2$

SOLUTION:

The degree of the polynomial is 4, so there are possibly 4 real zeros and 3 turning points.

$$\begin{aligned} 3x^4 + 18x^3 + 24x^2 &= 0 \\ 3x^2(x^2 + 6x + 8) &= 0 \\ 3x^2(x + 4)(x + 2) &= 0 \end{aligned}$$

The zeros of the function are –4, –2, and 0.

88. $f(x) = 8x^6 + 48x^5 + 40x^4$

SOLUTION:

The degree of the polynomial is 6, so there are possibly 6 real zeros and 5 turning points.

$$\begin{aligned} 8x^6 + 48x^5 + 40x^4 &= 0 \\ 8x^4(x^2 + 6x + 5) &= 0 \\ 8x^4(x + 1)(x + 5) &= 0 \end{aligned}$$

The zeros of the function are –5, –1, and 0.

89. $f(x) = 5x^5 - 15x^4 - 50x^3$

SOLUTION:

The degree of the polynomial is 5, so there are possibly 5 real zeros and 4 turning points.

$$\begin{aligned} 5x^5 - 15x^4 - 50x^3 &= 0 \\ 5x^3(x^2 - 3x - 10) &= 0 \\ 5x^3(x - 5)(x + 2) &= 0 \end{aligned}$$

The zeros of the function are –2, 0, and 5.

Simplify.

90. $(2 + 4\boldsymbol{i}) + (-1 + 5\boldsymbol{i})$

SOLUTION:

$(2 + 4\boldsymbol{i}) + (-1 + 5\boldsymbol{i})$
$= 2 + (-1) + 4\boldsymbol{i} + 5\boldsymbol{i}$
$= 1 + 9\boldsymbol{i}$

91. $(-2 - \boldsymbol{i})^2$

SOLUTION:

$(-2 - \boldsymbol{i})^2$
$= 4 + 4\boldsymbol{i} + \boldsymbol{i}^2$
$= 4 + 4\boldsymbol{i} - 1$
$= 3 + 4\boldsymbol{i}$

92. $\frac{i}{1+2i}$

SOLUTION:

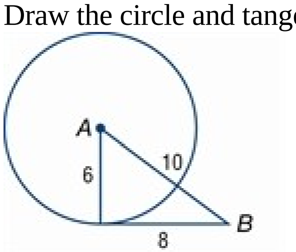
$$\begin{aligned}\frac{i}{1+2i}\cdot\frac{1-2i}{1-2i} &= \frac{i-2i^2}{1-4i^2}\\ &= \frac{i+2}{1+4}\\ &= \frac{i+2}{5}\\ &= \frac{2}{5}+\frac{1}{5}i\end{aligned}$$

93. **SAT/ACT** Point B lies 10 units from point A, which is the center of a circle of radius 6. If a tangent line is drawn from B to the circle, what is the distance from B to the point of tangency?

A 6
B 8
C 10
D $2\sqrt{34}$
E $2\sqrt{41}$

SOLUTION:

Draw the circle and tangent line.



The tangent line forms a right triangle.
Solving the right triangle shows the distance from point B to the point of tangency to be 8 units.

94. **REVIEW** What is the standard form of the equation of the conic given below?

$$2x^2 + 4y^2 - 8x + 24y + 32 = 0$$

F $\frac{(x-4)^2}{3} + \frac{(y+3)^2}{11} = 1$

G $\frac{(x-2)^2}{6} + \frac{(y+3)^2}{3} = 1$

H $\frac{(x+2)^2}{5} + \frac{(y+3)^2}{4} = 1$

J $\frac{(x-4)^2}{11} + \frac{(y+3)^2}{3} = 1$

SOLUTION:

Write the equation in standard form.

$$\begin{aligned}
2x^2 + 4y^2 - 8x + 24y + 32 &= 0 \\
2x^2 - 8x + 4y^2 + 24y + 32 &= 0 \\
x^2 - 4x + 2y^2 + 12y + 16 &= 0 \\
(x^2 - 4x) + 2(y^2 + 6y) &= -16 \\
(x^2 - 4x + 4) + 2(y^2 + 6y + 9) &= -16 + 4 + 18 \\
(x-2)^2 + 2(y+3)^2 &= 6 \\
\frac{(x-2)^2}{6} + \frac{2(y+3)^2}{6} &= 1 \\
\frac{(x-2)^2}{6} + \frac{(y+3)^2}{3} &= 1
\end{aligned}$$

95. Ruben is making an elliptical target for throwing darts. He wants the target to be 27 inches wide and 15 inches high. Which equation should Ruben use to draw the target?

A $\frac{x^2}{7.5} + \frac{y^2}{13.5} = 1$

B $\frac{x^2}{56.25} + \frac{y^2}{182.25} = 1$

C $\frac{x^2}{182.25} + \frac{y^2}{56.25} = 1$

D $\frac{x^2}{13.5} + \frac{y^2}{7.5} = 1$

SOLUTION:

The target is to be 27 inches wide, so $2a = 27$, $a = 13.5$, and $a^2 = 182.25$. The target is to be 15 inches wide, so $2b = 15$, $b = 7.5$, and $b^2 = 56.25$. The ellipse is horizontally oriented, so the a^2–term is with the x^2–term. So, the equation is $\frac{x^2}{182.25} + \frac{y^2}{56.25} = 1$.

96. **REVIEW** If $m = \frac{1}{x}$, $n = 7m, p = \frac{1}{n}$, $q = 14p$, and $r = \frac{1}{\frac{1}{2}q}$, find x.

F r
G q
H p
J $\frac{1}{r}$

SOLUTION:

First, find r in terms of p.

$$\begin{aligned} r &= \frac{1}{\frac{1}{2}q} \\ &= \frac{1}{\frac{1}{2}(14p)} \\ &= \frac{1}{7p} \end{aligned}$$

Find r in terms of n.

$$\begin{aligned} r &= \frac{1}{7p} \\ &= \frac{1}{7\left(\frac{1}{n}\right)} \\ &= \frac{1}{\frac{7}{n}} \\ &= \frac{n}{7} \end{aligned}$$

Find r in terms of m.

$$\begin{aligned} r &= \frac{n}{7} \\ &= \frac{7m}{7} \\ &= m \end{aligned}$$

Find x in terms of r.

$$\begin{aligned} m &= \frac{1}{x} \\ r &= \frac{1}{x} \\ x &= \frac{1}{r} \end{aligned}$$