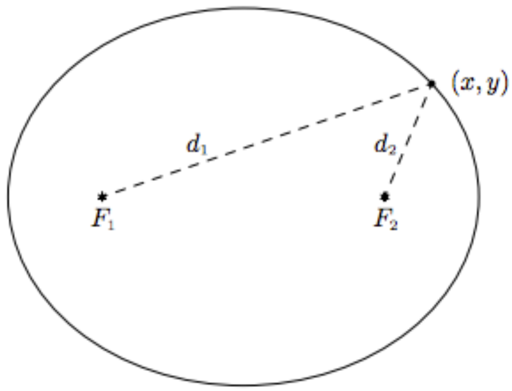
2.3 Conic Sections: Ellipse

Ellipse: (locus definition) set of all points (x, y) in the plane such that the sum of each of the distances from F_1 and F_2 is d.



$d_1 + d_2 = d$ for all (x, y) on the ellipse

Standard Form of an Ellipse:

Horizontal Ellipse

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Vertical Ellipse

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

center = (h,k)

2a = length of major axis

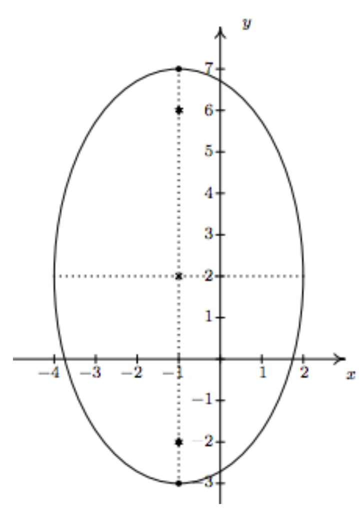
2b = length of minor axis

c = distance from center to focus

$c^2 = a^2 - b^2$

eccentricity $e = \frac{c}{a}$ ($0 < e < 1$ the closer to 0 the more circular)

Ex. Graph $\frac{(x+1)^2}{9}+\frac{(y-2)^2}{25}=1$



Center: $(-1,2)$

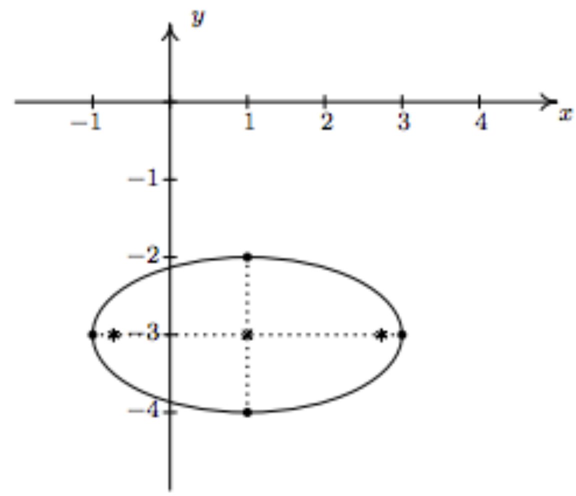
Endpoints of Major Axis: (-1, 7) & (-1, -3)

Endpoints of Minor Axis: (-4, -2) & (2, 2)

Foci: (-1, 6) & (-1, -2)

Eccentricity: 4/5

Ex. Graph $x^2+4y^2-2x+24y+33=0$



$$\begin{aligned}
&x^2+4y^2-2x+24y+33=0\\
&x^2-2x+4y^2+24y=-33\\
&x^2-2x+4(y^2+6y)=-33\\
&x^2-2x+\underline{1^2}+4(y^2+6y+\underline{3^2})=-33+1+4(9)\\
&(x-1)^2+4(y+3)^2=4\\
&\frac{(x-1)^2+4(y+3)^2}{4}=\frac{4}{4}\\
&\frac{(x-1)^2}{4}+\frac{(y+3)^2}{1}=1
\end{aligned}$$

Homework:

In Exercises 1-8, graph the ellipse. Find the center, the lines that contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci, and the eccentricity.

1. $\frac{x^2}{225}+\frac{y^2}{16}=1$
2. $\frac{x^2}{36}+\frac{y^2}{49}=1$
3. $\frac{(x-4)^2}{25}+\frac{(y+5)^2}{64}=1$
4. $\frac{(x+11)^2}{1}+\frac{(y+7)^2}{25}=1$
5. $\frac{(x-2)^2}{14}+\frac{(y-7)^2}{7}=1$
6. $\frac{(x+1)^2}{16}+\frac{(y+9)^2}{81}=1$
7. $\frac{(x+8)^2}{4}+\frac{(y-1)^2}{24}=1$
8. $\frac{(x-6)^2}{2}+\frac{(y-8)^2}{32}=1$

In Exercises 9-14, put the equation in standard form. Find the center, the lines that contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci, and the eccentricity.

9. $16x^2+4y^2-64x-28y-31=0$
10. $36x^2+4y^2+8y-104=0$
11. $2x^2+y^2-12x+14=0$
12. $x^2+4y^2-2x-24y+5=0$
13. $4x^2+24y^2-24y-42=0$
14. $8x^2+7y^2-32x+42y-17=0$

In Exercises 15-20, find the standard form of the equation of the ellipse given the following properties.

15. Center (2,10), Vertex (2,12), Focus (2,11)

16. Foci $(0,\pm 3)$, Vertices $(0,\pm 13)$

17. Foci, $(\pm 1,0)$ length of the Minor Axis 10

18. Vertices (8,4), (18,4), Endpoints of the Minor Axis (13,7), (13,1)

19. Center (6,1), Vertex (0,1), eccentricity $\frac{1}{4}$

20. All points on the ellipse are in Quadrant IV except (0,-6) and (10,0). (One might also say that the ellipse is "tangent to the axes" at those two points.)

21. Jamie and Jason want to exchange secrets (terrible secrets) from across a crowded whispering gallery. Recall that a whispering gallery is a room which, in cross section, is half of an ellipse. If the room is 15 feet high at the center and 60 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?

22. An elliptical arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch.

23. The Earth's orbit around the sun is an ellipse with the sun at one focus and eccentricity $e \approx 0.0167$. The length of the semimajor axis (half the major axis) is defined to be 1 astronomical unit (AU). The vertices of the elliptical orbit are given special names: 'aphelion' is the vertex farthest from the sun, and the 'perihelion' is the vertex closest to the sun.

a. Find the distance in AU between the perihelion and the sun.

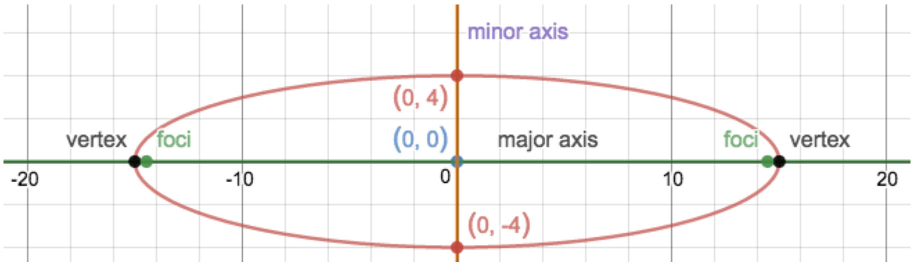
b. Find the distance in AU between the aphelion and the sun.

24. The graph of an ellipse clearly fails the Vertical Line Test, so the standard form of an ellipse does not define y as a function of x. However we can split an ellipse into a top half and a bottom half, each of which represents y as a function of x. Write an equation for an ellipse where y as a function of x.

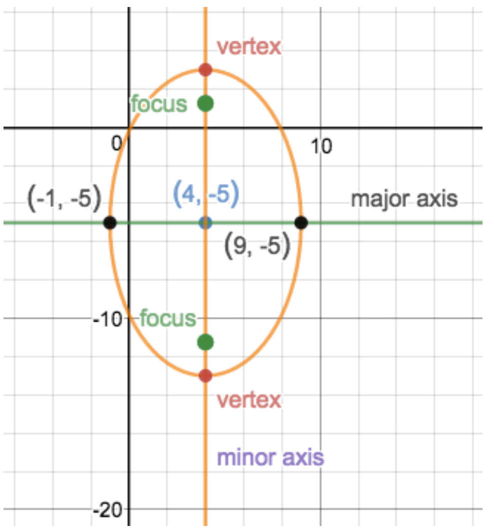
25. Can you write a circle in the standard form of an ellipse? When will an ellipse become a circle?

Selected Answers:

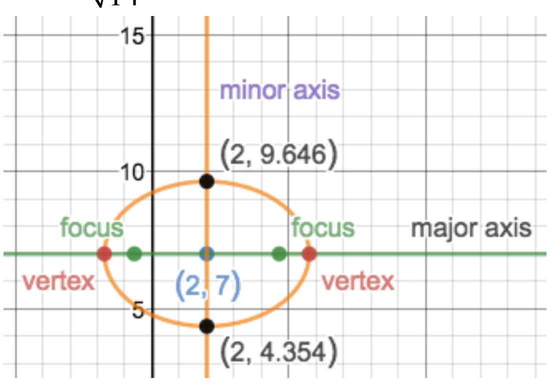
1. $e = \dfrac{c}{a} = \dfrac{\sqrt{209}}{15} \approx 0.96$



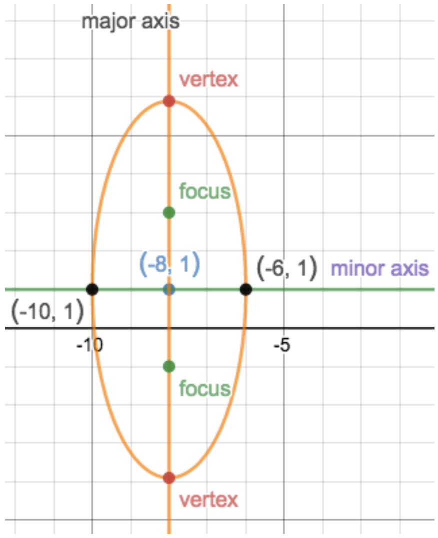
3. $e = \dfrac{\sqrt{39}}{8} \approx 0.78$



5. $e = \dfrac{\sqrt{7}}{\sqrt{14}} \approx 0.71$



7. $e=\frac{\sqrt{20}}{\sqrt{24}}=\frac{2\sqrt{5}}{2\sqrt{6}}\approx 0.91$



$$16x^2+4y^2-64x-28y-31=0$$
$$16x^2-64x+4y^2-28y=31$$
$$16(x^2-4x)+4(y^2-7y)=31$$

9. $16(x^2-4x+\underline{2^2})+4(y^2-7y+\underline{(\frac{7}{2})^2})=31+16(\underline{4})+4(\underline{\frac{49}{4}})$

$$16(x-2)^2+4(y-\frac{7}{2})^2=144$$

$$\frac{16(x-2)^2+4(y-\frac{7}{2})^2}{144}=\frac{144}{144}$$

$$\frac{(x-2)^2}{9}+\frac{(y-\frac{7}{2})^2}{36}=1$$

Center: $\left(2,\frac{7}{2}\right)$ Minor Axis: $y=\frac{7}{2}$ Major Axis: $x=2$ Vertices: $\left(2,\frac{7}{2}\pm 6\right)$

Minor Axis Endpoints: $\left(2\pm 3,\frac{7}{2}\right)$ Foci: $\left(2,\frac{7}{2}\pm 3\right)$ $e=\frac{\sqrt{27}}{6}=\frac{3\sqrt{3}}{6}=\frac{\sqrt{3}}{2}\approx 0.87$

$$2x^2 + y^2 - 12x + 14 = 0$$
$$2x^2 - 12x + y^2 = -14$$
$$2(x^2 - 6x) + y^2 = -14$$
11. $2(x^2 - 6x + \underline{3^2}) + y^2 = -14 + 2(\underline{9})$
$$2(x-3)^2 + y^2 = 4$$
$$\frac{2(x-3)^2 + y^2}{4} = \frac{4}{4}$$
$$\frac{(x-3)^2}{2} + \frac{y^2}{4} = 1$$

Center: $(3,0)$ Minor Axis: $y = 0$ Major Axis: $x = 3$ Vertices: $(3, 0 \pm 2)$

Minor Axis Endpoints: $(3 \pm \sqrt{2}, 0)$ Foci: $(3, 0 \pm \sqrt{2})$ $e = \frac{\sqrt{2}}{2} \approx 0.71$

$$4x^2 + 24y^2 - 24y - 42 = 0$$
$$4x^2 + 24(y^2 - y) = 42$$
$$4x^2 + 24(y^2 - y + \underline{(\tfrac{1}{2})^2} = 42 + 24(\underline{\tfrac{1}{4}})$$
13. $4x^2 + 24(y - \frac{1}{2})^2 = 48$
$$\frac{4x^2 + 24(y - \frac{1}{2})^2}{48} = \frac{48}{48}$$
$$\frac{4x^2}{12} + \frac{24(y - \frac{1}{2})^2}{2} = 1$$

Center: $\left(0, \frac{1}{2}\right)$ Minor Axis: $x = 0$ Major Axis: $y = \frac{1}{2}$ Vertices: $\left(0 \pm \sqrt{12}, \frac{1}{2}\right)$

Minor Axis Endpoints: $\left(0, \frac{1}{2} \pm \sqrt{2}\right)$ Foci: $\left(0 \pm \sqrt{2}, \frac{1}{2}\right)$ $e = \frac{\sqrt{10}}{\sqrt{12}} = \sqrt{\frac{5}{6}} \approx 0.91$

15. Vertex: (2,12) = (2,10+2) → a = 2 → $a^2 = 4$

Focus: (2,11) = (2,10+1) → c = 1 → $c^2 = 1$ → $b^2 = 3$

$$\frac{(x-2)^2}{3} + \frac{(y-10)^2}{4} = 1$$

17. Minor Axis length: 10 → Minor Axis endpoints: $(0, 0 \pm 5)$ → $b^2 = 25$

Focus: $\left(0 \pm 1, 0\right)$ $\rightarrow$ c = 1 and Center: (0,0)

$$b^2 = 25$$
$$c = 1$$
$$c^2 = a^2 - b^2$$
$$1 = a^2 - 25$$
$$a^2 = 26$$

$$\frac{x^2}{26} + \frac{y^2}{25} = 1$$

19. Center (6,1), Vertex (0,1), thus *a=6*.

Eccentricity $\frac{1}{4} = \frac{c}{a} = \frac{c}{6} \rightarrow c = 1.5$

$$c^2 = a^2 - b^2$$

$$\left(\frac{3}{2}\right)^2 = 6^2 - b^2$$

$$b^2 = 36 - \frac{9}{4} = \frac{135}{4}$$

$$\frac{(x-6)^2}{36} + \frac{(y-1)^2}{\frac{135}{4}} = 1$$

21. *a = 30, b= 15* …c = 25.9808. So each person should stand 4.0192 feet from the edge of the wall.

23. 0.9833 AU (perihelion to sun) 1.0167 AU (aphelion to sun)

25.