Calculus

Solutions to Practice Problems

Paul Dawkins

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Preface

First, here's a little bit of history on how this material was created (there's a reason for this, I promise). A long time ago (2002 or so) when I decided I wanted to put some mathematics stuff on the web I wanted a format for the source documents that could produce both a pdf version as well as a web version of the material. After some investigation I decided to use MS Word and MathType as the easiest/quickest method for doing that. The result was a pretty ugly HTML (*i.e* web page code) and had the drawback of the mathematics were images which made editing the mathematics painful. However, it was the quickest way or dealing with this stuff.

Fast forward a few years (don't recall how many at this point) and the web had matured enough that it was now much easier to write mathematics in LaTeX (https://en.wikipedia.org/wiki/LaTeX) and have it display on the web (LaTeX was my first choice for writing the source documents). So, I found a tool that could convert the MS Word mathematics in the source documents to LaTeX. It wasn't perfect and I had to write some custom rules to help with the conversion but it was able to do it without "messing" with the mathematics and so I didn't need to worry about any math errors being introduced in the conversion process. The only problem with the tool is that all it could do was convert the mathematics and not the rest of the source document into LaTeX. That meant I just converted the math into LaTeX for the website but didn't convert the source documents.

Now, here's the reason for this history lesson. Fast forward even more years and I decided that I really needed to convert the source documents into LaTeX as that would just make my life easier and I'd be able to enable working links in the pdf as well as a simple way of producing an index for the material. The only issue is that the original tool I'd use to convert the MS Word mathematics had become, shall we say, unreliable and so that was no longer an option and it still has the problem on not converting anything else into proper LaTeX code.

So, the best option that I had available to me is to take the web pages, which already had the mathematics in proper LaTeX format, and convert the rest of the HTML into LaTeX code. I wrote a set of tools to do his and, for the most part, did a pretty decent job. The only problem is that the tools weren't perfect. So, if you run into some "odd" stuff here (things like `<sup>`, `<span>`, `</span>`, `<div>`, *etc.*) please let me know the section with the code that I missed. I did my best to find all the "orphaned" HTML code but I'm certain I missed some on occasion as I did find my eyes glazing over every once in a while as I went over the converted document.

Now, with that out of the way, here are the solutions to the practice problems for the Calculus notes.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Please understand as well that this document was written up with the intent of eventual display on the web. That means that some of the formatting may seem a little out of place. For example, some of the problems/parts have hints for them. Online the solution that the hint refers to will be hidden until the reader clicks on a “show” link but here that is not possible and so the hint may not be as useful here as it would be if reading this online.

Outline

Here is a listing of sections for which practice problems, as well as solutions, have been written as well as a brief description of the material covered in the notes for that particular section.

Review - In this chapter we give a brief review of selected topics from Algebra and Trig that are vital to surviving a Calculus course. Included are Functions, Trig Functions, Solving Trig Equations, Exponential/Logarithm Functions and Solving Exponential/Logarithm Equations.

Functions - In this section we will cover function notation/evaluation, determining the domain and range of a function and function composition.

Inverse Functions - In this section we will define an inverse function and the notation used for inverse functions. We will also discuss the process for finding an inverse function.

Trig Functions - In this section we will give a quick review of trig functions. We will cover the basic notation, relationship between the trig functions, the right triangle definition of the trig functions. We will also cover evaluation of trig functions as well as the unit circle (one of the most important ideas from a trig class!) and how it can be used to evaluate trig functions.

Solving Trig Equations - In this section we will discuss how to solve trig equations. The answers to the equations in this section will all be one of the “standard” angles that most students have memorized after a trig class. However, the process used here can be used for any answer regardless of it being one of the standard angles or not.

Solving Trig Equations with Calculators, Part I - In this section we will discuss solving trig equations when the answer will (generally) require the use of a calculator (*i.e.* they aren’t one of the standard angles). Note however, the process used here is identical to that for when the answer is one of the standard angles. The only difference is that the answers in here can be a little messy due to the need of a calculator. Included is a brief discussion of inverse trig functions.

Solving Trig Equations with Calculators, Part II - In this section we will continue our discussion of solving trig equations when a calculator is needed to get the answer. The equations in this section tend to be a little trickier than the ”normal” trig equation and are not always covered in a trig class.

Exponential Functions - In this section we will discuss exponential functions. We will cover the basic definition of an exponential function, the natural exponential function, i.e. $\mathbf{e}^x$, as well as the properties and graphs of exponential functions

Logarithm Functions - In this section we will discuss logarithm functions, evaluation of logarithms and their properties. We will discuss many of the basic manipulations of logarithms that commonly occur in Calculus (and higher) classes. Included is a discussion of the natural (ln(x)) and common logarithm (log(x)) as well as the change of base formula.

Exponential and Logarithm Equations - In this section we will discuss various methods for solving equations that involve exponential functions or logarithm functions.

Common Graphs - In this section we will do a very quick review of many of the most common functions and their graphs that typically show up in a Calculus class.

Limits - In this chapter we introduce the concept of limits. We will discuss the interpretation/meaning of a limit, how to evaluate limits, the definition and evaluation of one-sided limits, evaluation of infinite limits, evaluation of limits at infinity, continuity and the Intermediate Value Theorem. We will also give a brief introduction to a precise definition of the limit and how to use it to evaluate limits.

Tangent Lines and Rates of Change -In this section we will introduce two problems that we will see time and again in this course : Rate of Change of a function and Tangent Lines to functions. Both of these problems will be used to introduce the concept of limits, although we won't formally give the definition or notation until the next section.

The Limit - In this section we will introduce the notation of the limit. We will also take a conceptual look at limits and try to get a grasp on just what they are and what they can tell us. We will be estimating the value of limits in this section to help us understand what they tell us. We will actually start computing limits in a couple of sections.

One-Sided Limits - In this section we will introduce the concept of one-sided limits. We will discuss the differences between one-sided limits and limits as well as how they are related to each other.

Limit Properties - In this section we will discuss the properties of limits that we'll need to use in computing limits (as opposed to estimating them as we've done to this point). We will also compute a couple of basic limits in this section.

Computing Limits - In this section we will looks at several types of limits that require some work before we can use the limit properties to compute them. We will also look at computing limits of piecewise functions and use of the Squeeze Theorem to compute some limits.

Infinite Limits - In this section we will look at limits that have a value of infinity or negative infinity. We'll also take a brief look at vertical asymptotes.

Limits At Infinity, Part I - In this section we will start looking at limits at infinity, *i.e.* limits in which the variable gets very large in either the positive or negative sense. We will concentrate on polynomials and rational expressions in this section. We'll also take a brief look at horizontal asymptotes.

Limits At Infinity, Part II - In this section we will continue covering limits at infinity. We'll be looking at exponentials, logarithms and inverse tangents in this section.

Continuity - In this section we will introduce the concept of continuity and how it relates to limits. We will also see the Intermediate Value Theorem in this section and how it can be used to determine if functions have solutions in a given interval.

The Definition of the Limit - In this section we will give a precise definition of several of the limits covered in this section. We will work several basic examples illustrating how to use this precise definition to compute a limit. We'll also give a precise definition of continuity.

Derivatives - In this chapter we will start looking at the next major topic in a calculus class, derivatives. This chapter is devoted almost exclusively to finding derivatives. We will be looking at one application of them in this chapter. We will be leaving most of the applications of derivatives to the next chapter.

The Definition of the Derivative - In this section we define the derivative, give various notations for the derivative and work a few problems illustrating how to use the definition of the derivative to actually compute the derivative of a function.

Interpretation of the Derivative - In this section we give several of the more important interpretations of the derivative. We discuss the rate of change of a function, the velocity of a moving object and the slope of the tangent line to a graph of a function.

Differentiation Formulas - In this section we give most of the general derivative formulas and properties used when taking the derivative of a function. Examples in this section concentrate mostly on polynomials, roots and more generally variables raised to powers.

Product and Quotient Rule - In this section we will give two of the more important formulas for differentiating functions. We will discuss the Product Rule and the Quotient Rule allowing us to differentiate functions that, up to this point, we were unable to differentiate.

Derivatives of Trig Functions - In this section we will discuss differentiating trig functions. Derivatives of all six trig functions are given and we show the derivation of the derivative of $\sin(x)$ and $\tan(x)$.

Derivatives of Exponential and Logarithm Functions - In this section we derive the formulas for the derivatives of the exponential and logarithm functions.

Derivatives of Inverse Trig Functions - In this section we give the derivatives of all six inverse trig functions. We show the derivation of the formulas for inverse sine, inverse cosine and inverse tangent.

Derivatives of Hyperbolic Functions - In this section we define the hyperbolic functions, give the relationships between them and some of the basic facts involving hyperbolic functions. We also give the derivatives of each of the six hyperbolic functions and show the derivation of the formula for hyperbolic sine.

Chain Rule - In this section we discuss one of the more useful and important differentiation formulas, The Chain Rule. With the chain rule in hand we will be able to differentiate a much wider variety of functions. As you will see throughout the rest of your Calculus courses a great many of derivatives you take will involve the chain rule!

Implicit Differentiation - In this section we will discuss implicit differentiation. Not every function can be explicitly written in terms of the independent variable, e.g. y = f(x) and yet we will still need to know what f'(x) is. Implicit differentiation will allow us to find the derivative in these cases. Knowing implicit differentiation will allow us to do one of the more important applications of derivatives, Related Rates (the next section).

Related Rates - In this section we will discuss the only application of derivatives in this section, Related Rates. In related rates problems we are give the rate of change of one quantity in a problem and asked to determine the rate of one (or more) quantities in the problem. This is often one of the more difficult sections for students. We work quite a few problems in this section so hopefully by the end of this section you will get a decent understanding on how these problems work.

Higher Order Derivatives - In this section we define the concept of higher order derivatives and give a quick application of the second order derivative and show how implicit differentiation works for higher order derivatives.

Logarithmic Differentiation - In this section we will discuss logarithmic differentiation. Logarithmic differentiation gives an alternative method for differentiating products and quotients (sometimes easier than using product and quotient rule). More importantly, however, is the fact that logarithm differentiation allows us to differentiate functions that are in the form of one function raised to another function, *i.e.* there are variables in both the base and exponent of the function.

Derivative Applications - In the previous chapter we focused almost exclusively on the computation of derivatives. In this chapter will focus on applications of derivatives. It is important to always remember that we didn't spend a whole chapter talking about computing derivatives just to be talking about them. There are many very important applications to derivatives.

The two main applications that we'll be looking at in this chapter are using derivatives to determine information about graphs of functions and optimization problems. These will not be the only applications however. We will be revisiting limits and taking a look at an application of derivatives that will allow us to compute limits that we haven't been able to compute previously. We will also see how derivatives can be used to estimate solutions to equations.

Rates of Change - In this section we review the main application/interpretation of derivatives from the previous chapter (i.e. rates of change) that we will be using in many of the applications in this chapter.

Critical Points - In this section we give the definition of critical points. Critical points will show up in most of the sections in this chapter, so it will be important to understand them and how to find them. We will work a number of examples illustrating how to find them for a wide variety of functions.

Minimum and Maximum Values - In this section we define absolute (or global) minimum and maximum values of a function and relative (or local) minimum and maximum values of a function. It is important to understand the difference between the two types of minimum/-maximum (collectively called extrema) values for many of the applications in this chapter and so we use a variety of examples to help with this. We also give the Extreme Value Theorem and Fermat's Theorem, both of which are very important in the many of the applications we'll see in this chapter.

Finding Absolute Extrema - In this section we discuss how to find the absolute (or global) minimum and maximum values of a function. In other words, we will be finding the largest and smallest values that a function will have.

The Shape of a Graph, Part I - In this section we will discuss what the first derivative of a function can tell us about the graph of a function. The first derivative will allow us to identify the relative (or local) minimum and maximum values of a function and where a function will be increasing and decreasing. We will also give the First Derivative test which will allow us to classify critical points as relative minimums, relative maximums or neither a minimum or a maximum.

The Shape of a Graph, Part II - In this section we will discuss what the second derivative of a function can tell us about the graph of a function. The second derivative will allow us to determine where the graph of a function is concave up and concave down. The second derivative will also allow us to identify any inflection points (i.e. where concavity changes) that a function may have. We will also give the Second Derivative Test that will give an alternative method for identifying some critical points (but not all) as relative minimums or relative maximums.

The Mean Value Theorem - In this section we will give Rolle's Theorem and the Mean Value Theorem. With the Mean Value Theorem we will prove a couple of very nice facts, one of which will be very useful in the next chapter.

Optimization Problems - In this section we will be determining the absolute minimum and/or maximum of a function that depends on two variables given some constraint, or relationship, that the two variables must always satisfy. We will discuss several methods for determining the absolute minimum or maximum of the function. Examples in this section tend to center around geometric objects such as squares, boxes, cylinders, etc.

More Optimization Problems - In this section we will continue working optimization problems. The examples in this section tend to be a little more involved and will often involve situations that will be more easily described with a sketch as opposed to the 'simple' geometric objects we looked at in the previous section.

L'Hospital's Rule and Indeterminate Forms - In this section we will revisit indeterminate forms and limits and take a look at L'Hospital's Rule. L'Hospital's Rule will allow us to evaluate some limits we were not able to previously.

Linear Approximations - In this section we discuss using the derivative to compute a linear approximation to a function. We can use the linear approximation to a function to approximate values of the function at certain points. While it might not seem like a useful thing to do with when we have the function there really are reasons that one might want to do this. We give two ways this can be useful in the examples.

Differentials - In this section we will compute the differential for a function. We will give an application of differentials in this section. However, one of the more important uses of differentials will come in the next chapter and unfortunately we will not be able to discuss it until then.

Newton's Method - In this section we will discuss Newton's Method. Newton's Method is an application of derivatives that will allow us to approximate solutions to an equation. There are many equations that cannot be solved directly and with this method we can get approximations to the solutions to many of those equations.

Business Applications - In this section we will give a cursory discussion of some basic applications of derivatives to the business field. We will revisit finding the maximum and/or minimum function value and we will define the marginal cost function, the average cost, the revenue function, the marginal revenue function and the marginal profit function. Note that this section is only intended to introduce these concepts and not teach you everything about them.

Integrals In this chapter we will be looking at integrals. Integrals are the third and final major topic that will be covered in this class. As with derivatives this chapter will be devoted almost exclusively to finding and computing integrals. Applications will be given in the following chapter. There are really two types of integrals that we'll be looking at in this chapter : Indefinite Integrals and Definite Integrals. The first half of this chapter is devoted to indefinite integrals and the last half is devoted to definite integrals. As we will see in the last half of the chapter if we don't know indefinite integrals we will not be able to do definite integrals.

Indefinite Integrals - In this section we will start off the chapter with the definition and properties of indefinite integrals. We will not be computing many indefinite integrals in this section. This section is devoted to simply defining what an indefinite integral is and to give many of the properties of the indefinite integral. Actually computing indefinite integrals will start in the next section.

Computing Indefinite Integrals - In this section we will compute some indefinite integrals. The integrals in this section will tend to be those that do not require a lot of manipulation of the function we are integrating in order to actually compute the integral. As we will see starting in the next section many integrals do require some manipulation of the function before we can actually do the integral. We will also take a quick look at an application of indefinite integrals.

Substitution Rule for Indefinite Integrals - In this section we will start using one of the more common and useful integration techniques - The Substitution Rule. With the substitution rule we will be able integrate a wider variety of functions. The integrals in this section will all require some manipulation of the function prior to integrating unlike most of the integrals from the previous section where all we really needed were the basic integration formulas.

More Substitution Rule - In this section we will continue to look at the substitution rule. The problems in this section will tend to be a little more involved than those in the previous section.

Area Problem - In this section we start off with the motivation for definite integrals and give one of the interpretations of definite integrals. We will be approximating the amount of area that lies between a function and the x-axis. As we will see in the next section this problem will lead us to the definition of the definite integral and will be one of the main interpretations of the definite integral that we'll be looking at in this material.

Definition of the Definite Integral - In this section we will formally define the definite integral, give many of its properties and discuss a couple of interpretations of the definite integral. We will also look at the first part of the Fundamental Theorem of Calculus which shows the very close relationship between derivatives and integrals

Computing Definite Integrals - In this section we will take a look at the second part of the Fundamental Theorem of Calculus. This will show us how we compute definite integrals without using (the often very unpleasant) definition. The examples in this section can all be done with a basic knowledge of indefinite integrals and will not require the use of the substitution rule. Included in the examples in this section are computing definite integrals of piecewise and absolute value functions.

Substitution Rule for Definite Integrals - In this section we will revisit the substitution rule as it applies to definite integrals. The only real requirements to being able to do the examples in this section are being able to do the substitution rule for indefinite integrals and understanding how to compute definite integrals in general.

Applications of Integrals In this last chapter of this course we will be taking a look at a couple of Applications of Integrals. There are many other applications, however many of them require integration techniques that are typically taught in Calculus II. We will therefore be focusing on applications that can be done only with knowledge taught in this course.

Because this chapter is focused on the applications of integrals it is assumed in all the examples that you are capable of doing the integrals. There will not be as much detail in the integration process in the examples in this chapter as there was in the examples in the previous chapter.

Average Function Value - In this section we will look at using definite integrals to determine the average value of a function on an interval. We will also give the Mean Value Theorem for Integrals.

Area Between Curves - In this section we'll take a look at one of the main applications of definite integrals in this chapter. We will determine the area of the region bounded by two curves.

Volumes of Solids of Revolution / Method of Rings - In this section, the first of two sections devoted to finding the volume of a solid of revolution, we will look at the method of rings/disks to find the volume of the object we get by rotating a region bounded by two curves (one of which may be the x or y-axis) around a vertical or horizontal axis of rotation.

Volumes of Solids of Revolution / Method of Cylinders - In this section, the second of two sections devoted to finding the volume of a solid of revolution, we will look at the method of cylinders/shells to find the volume of the object we get by rotating a region bounded by two curves (one of which may be the x or y-axis) around a vertical or horizontal axis of rotation.

More Volume Problems - In the previous two sections we looked at solids that could be found by treating them as a solid of revolution. Not all solids can be thought of as solids of revolution and, in fact, not all solids of revolution can be easily dealt with using the methods from the previous two sections. So, in this section we'll take a look at finding the volume of some solids that are either not solids of revolutions or are not easy to do as a solid of revolution.

Work - In this section we will look at is determining the amount of work required to move an object subject to a force over a given distance.

Integration Techniques In this chapter we are going to be looking at various integration techniques. There are a fair number of them and some will be easier than others. The point of the chapter is to teach you these new techniques and so this chapter assumes that you've got a fairly good working knowledge of basic integration as well as substitutions with integrals. In fact, most integrals involving "simple" substitutions will not have any of the substitution work shown. It is going to be assumed that you can verify the substitution portion of the integration yourself.

Also, most of the integrals done in this chapter will be indefinite integrals. It is also assumed that once you can do the indefinite integrals you can also do the definite integrals and so to conserve space we concentrate mostly on indefinite integrals. There is one exception to this and that is the Trig Substitution section and in this case there are some subtleties involved with definite integrals that we're going to have to watch out for. Outside of that however, most sections will have at most one definite integral example and some sections will not have any definite integral examples.

Integration by Parts - In this section we will be looking at Integration by Parts. Of all the techniques we'll be looking at in this class this is the technique that students are most likely to run into down the road in other classes. We also give a derivation of the integration by parts formula.

Integrals Involving Trig Functions - In this section we look at integrals that involve trig functions. In particular we concentrate integrating products of sines and cosines as well as products of secants and tangents. We will also briefly look at how to modify the work for products of these trig functions for some quotients of trig functions.

Trig Substitutions - In this section we will look at integrals (both indefinite and definite) that require the use of a substitutions involving trig functions and how they can be used to simplify certain integrals.

Partial Fractions - In this section we will use partial fractions to rewrite integrands into a form that will allow us to do integrals involving some rational functions.

Integrals Involving Roots - In this section we will take a look at a substitution that can, on occasion, be used with integrals involving roots.

Integrals Involving Quadratics - In this section we are going to look at some integrals that involve quadratics for which the previous techniques won't work right away. In some cases, manipulation of the quadratic needs to be done before we can do the integral. We will see several cases where this is needed in this section.

Integration Strategy - In this section we give a general set of guidelines for determining how to evaluate an integral. The guidelines give here involve a mix of both Calculus I and Calculus II techniques to be as general as possible. Also note that there really isn't one set of guidelines that will always work and so you always need to be flexible in following this set of guidelines.

Improper Integrals - In this section we will look at integrals with infinite intervals of integration and integrals with discontinuous integrands in this section. Collectively, they are called improper integrals and as we will see they may or may not have a finite (i.e. not infinite) value. Determining if they have finite values will, in fact, be one of the major topics of this section.

Comparison Test for Improper Integrals - It will not always be possible to evaluate improper integrals and yet we still need to determine if they converge or diverge (i.e. if they have a finite value or not). So, in this section we will use the Comparison Test to determine if improper integrals converge or diverge.

Approximating Definite Integrals - In this section we will look at several fairly simple methods of approximating the value of a definite integral. It is not possible to evaluate every definite integral (i.e. because it is not possible to do the indefinite integral) and yet we may need to know the value of the definite integral anyway. These methods allow us to at least get an approximate value which may be enough in a lot of cases.

More Applications of Integrals In this section we're going to take a look at some of the Applications of Integrals. It should be noted as well that these applications are presented here, as opposed to Calculus I, simply because many of the integrals that arise from these applications tend to require techniques that we discussed in the previous chapter.

Arc Length - In this section we'll determine the length of a curve over a given interval.

Surface Area - In this section we'll determine the surface area of a solid of revolution, *i.e.* a solid obtained by rotating a region bounded by two curves about a vertical or horizontal axis.

Center of Mass - In this section we will determine the center of mass or centroid of a thin plate where the plate can be described as a region bounded by two curves (one of which may the x or y-axis).

Hydrostatic Pressure and Force - In this section we'll determine the hydrostatic pressure and force on a vertical plate submerged in water. The plates used in the examples can all be described as regions bounded by one or more curves/lines.

Probability - Many quantities can be described with probability density functions. For example, the length of time a person waits in line at a checkout counter or the life span of a light bulb. None of these quantities are fixed values and will depend on a variety of factors. In this section we will look at probability density functions and computing the mean (think average wait in line or average life span of a light blub) of a probability density function.

Parametric Equations and Polar Coordinates In this section we will be looking at parametric equations and polar coordinates. While the two subjects don't appear to have that much in common on the surface we will see that several of the topics in polar coordinates can be done in terms of parametric equations and so in that sense they make a good match in this chapter

We will also be looking at how to do many of the standard calculus topics such as tangents and area in terms of parametric equations and polar coordinates.

Parametric Equations and Curves - In this section we will introduce parametric equations and parametric curves (i.e. graphs of parametric equations). We will graph several sets of parametric equations and discuss how to eliminate the parameter to get an algebraic equation which will often help with the graphing process.

Tangents with Parametric Equations - In this section we will discuss how to find the derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for parametric curves. We will also discuss using these derivative formulas to find the tangent line for parametric curves as well as determining where a parametric curve in increasing/decreasing and concave up/concave down.

Area with Parametric Equations - In this section we will discuss how to find the area between a parametric curve and the x-axis using only the parametric equations (rather than eliminating the parameter and using standard Calculus I techniques on the resulting algebraic equation).

Arc Length with Parametric Equations - In this section we will discuss how to find the arc length of a parametric curve using only the parametric equations (rather than eliminating the parameter and using standard Calculus techniques on the resulting algebraic equation).

Surface Area with Parametric Equations - In this section we will discuss how to find the surface area of a solid obtained by rotating a parametric curve about the x or y-axis using only the parametric equations (rather than eliminating the parameter and using standard Calculus techniques on the resulting algebraic equation).

Polar Coordinates - In this section we will introduce polar coordinates an alternative coordinate system to the 'normal' Cartesian/Rectangular coordinate system. We will derive formulas to convert between polar and Cartesian coordinate systems. We will also look at many of the standard polar graphs as well as circles and some equations of lines in terms of polar coordinates.

Tangents with Polar Coordinates - In this section we will discuss how to find the derivative $\frac{dy}{dx}$ for polar curves. We will also discuss using this derivative formula to find the tangent line for polar curves using only polar coordinates (rather than converting to Cartesian coordinates and using standard Calculus techniques).

Area with Polar Coordinates - In this section we will discuss how to the area enclosed by a polar curve. The regions we look at in this section tend (although not always) to be shaped vaguely like a piece of pie or pizza and we are looking for the area of the region from the outer boundary (defined by the polar equation) and the origin/pole. We will also discuss finding the area between two polar curves.

Arc Length with Polar Coordinates - In this section we will discuss how to find the arc length of a polar curve using only polar coordinates (rather than converting to Cartesian coordinates and using standard Calculus techniques).

Surface Area with Polar Coordinates - In this section we will discuss how to find the surface area of a solid obtained by rotating a polar curve about the x or y-axis using only polar coordinates (rather than converting to Cartesian coordinates and using standard Calculus techniques).

Arc Length and Surface Area Revisited - In this section we will summarize all the arc length and surface area formulas we developed over the course of the last two chapters.

Series and Sequences In this chapter we'll be taking a look at sequences and (infinite) series. In fact, this chapter will deal almost exclusively with series. However, we also need to understand some of the basics of sequences in order to properly deal with series. We will therefore, spend a little time on sequences as well.

Series is one of those topics that many students don't find all that useful. To be honest, many students will never see series outside of their calculus class. However, series do play an important role in the field of ordinary differential equations and without series large portions of the field of partial differential equations would not be possible.

In other words, series is an important topic even if you won't ever see any of the applications. Most of the applications are beyond the scope of most Calculus courses and tend to occur in classes that many students don't take. So, as you go through this material keep in mind that these do have applications even if we won't really be covering many of them in this class.

Sequences - In this section we define just what we mean by sequence in a math class and give the basic notation we will use with them. We will focus on the basic terminology, limits of sequences and convergence of sequences in this section. We will also give many of the basic facts and properties we'll need as we work with sequences.

More on Sequences - In this section we will continue examining sequences. We will determine if a sequence in an increasing sequence or a decreasing sequence and hence if it is a monotonic sequence. We will also determine a sequence is bounded below, bounded above and/or bounded.

Series - The Basics - In this section we will formally define an infinite series. We will also give many of the basic facts, properties and ways we can use to manipulate a series. We will also briefly discuss how to determine if an infinite series will converge or diverge (a more in depth discussion of this topic will occur in the next section).

Convergence/Divergence of Series - In this section we will discuss in greater detail the convergence and divergence of infinite series. We will illustrate how partial sums are used to determine if an infinite series converges or diverges. We will also give the Divergence Test for series in this section.

Special Series - In this section we will look at three series that either show up regularly or have some nice properties that we wish to discuss. We will examine Geometric Series, Telescoping Series, and Harmonic Series.

Integral Test - In this section we will discuss using the Integral Test to determine if an infinite series converges or diverges. The Integral Test can be used on a infinite series provided the terms of the series are positive and decreasing. A proof of the Integral Test is also given.

Comparison Test/Limit Comparison Test - In this section we will discuss using the Comparison Test and Limit Comparison Tests to determine if an infinite series converges or diverges. In order to use either test the terms of the infinite series must be positive. Proofs for both tests are also given.

Alternating Series Test - In this section we will discuss using the Alternating Series Test to determine if an infinite series converges or diverges. The Alternating Series Test can be used only if the terms of the series alternate in sign. A proof of the Alternating Series Test is also given.

Absolute Convergence - In this section we will have a brief discussion on absolute convergence and conditionally convergent and how they relate to convergence of infinite series.

Ratio Test - In this section we will discuss using the Ratio Test to determine if an infinite series converges absolutely or diverges. The Ratio Test can be used on any series, but unfortunately will not always yield a conclusive answer as to whether a series will converge absolutely or diverge. A proof of the Ratio Test is also given.

Root Test - In this section we will discuss using the Root Test to determine if an infinite series converges absolutely or diverges. The Root Test can be used on any series, but unfortunately will not always yield a conclusive answer as to whether a series will converge absolutely or diverge. A proof of the Root Test is also given.

Strategy for Series - In this section we give a general set of guidelines for determining which test to use in determining if an infinite series will converge or diverge. Note as well that there really isn't one set of guidelines that will always work and so you always need to be flexible in following this set of guidelines. A summary of all the various tests, as well as conditions that must be met to use them, we discussed in this chapter are also given in this section.

Estimating the Value of a Series - In this section we will discuss how the Integral Test, Comparison Test, Alternating Series Test and the Ratio Test can, on occasion, be used to estimating the value of an infinite series.

Power Series - In this section we will give the definition of the power series as well as the definition of the radius of convergence and interval of convergence for a power series. We will also illustrate how the Ratio Test and Root Test can be used to determine the radius and interval of convergence for a power series.

Power Series and Functions - In this section we discuss how the formula for a convergent Geometric Series can be used to represent some functions as power series. To use the Geometric Series formula, the function must be able to be put into a specific form, which is often impossible. However, use of this formula does quickly illustrate how functions can be represented as a power series. We also discuss differentiation and integration of power series.

Taylor Series - In this section we will discuss how to find the Taylor/Maclaurin Series for a function. This will work for a much wider variety of function than the method discussed in the previous section at the expense of some often unpleasant work. We also derive some well known formulas for Taylor series of $\mathbf{e}^{x}$, $\cos(x)$ and $\sin(x)$ around $x = 0$.

Applications of Series - In this section we will take a quick look at a couple of applications of series. We will illustrate how we can find a series representation for indefinite integrals that cannot be evaluated by any other method. We will also see how we can use the first few terms of a power series to approximate a function.

Binomial Series - In this section we will give the Binomial Theorem and illustrate how it can be used to quickly expand terms in the form $(a+b)^n$ when n is an integer. In addition, when n is not an integer an extension to the Binomial Theorem can be used to give a power series representation of the term.

Vectors This is a fairly short chapter. We will be taking a brief look at vectors and some of their properties. We will need some of this material in the next chapter and those of you heading on towards Calculus III will use a fair amount of this there as well.

Basic Concepts - In this section we will introduce some common notation for vectors as well as some of the basic concepts about vectors such as the magnitude of a vector and unit vectors. We also illustrate how to find a vector from its starting and end points.

Vector Arithmetic - In this section we will discuss the mathematical and geometric interpretation of the sum and difference of two vectors. We also define and give a geometric interpretation for scalar multiplication. We also give some of the basic properties of vector arithmetic and introduce the common i, j, k notation for vectors.

Dot Product - In this section we will define the dot product of two vectors. We give some of the basic properties of dot products and define orthogonal vectors and show how to use the dot product to determine if two vectors are orthogonal. We also discuss finding vector projections and direction cosines in this section.

Cross Product - In this section we define the cross product of two vectors and give some of the basic facts and properties of cross products.

Three Dimensional Space In this chapter we will start taking a more detailed look at three dimensional space (3-D space or $\mathbb{R}^3$). This is a very important topic for Calculus III since a good portion of Calculus III is done in three (or higher) dimensional space.

We will be looking at the equations of graphs in 3-D space as well as vector valued functions and how we do calculus with them. We will also be taking a look at a couple of new coordinate systems for 3-D space.

This is the only chapter that exists in two places in the notes. When we originally wrote these notes all of these topics were covered in Calculus II however, we have since moved several of them into Calculus III. So, rather than split the chapter up we kept it in the Calculus II notes and also put a copy in the Calculus III notes. Many of the sections not covered in Calculus III will be used on occasion there anyway and so they serve as a quick reference for when we need them. In addition this allows those that teach the topic in either place to have the notes quickly available to them.

The 3-D Coordinate System - In this section we will introduce the standard three dimensional coordinate system as well as some common notation and concepts needed to work in three dimensions.

Equations of Lines - In this section we will derive the vector form and parametric form for the equation of lines in three dimensional space. We will also give the symmetric equations of lines in three dimensional space. Note as well that while these forms can also be useful for lines in two dimensional space.

Equations of Planes - In this section we will derive the vector and scalar equation of a plane. We also show how to write the equation of a plane from three points that lie in the plane.

Quadric Surfaces - In this section we will be looking at some examples of quadric surfaces. Some examples of quadric surfaces are cones, cylinders, ellipsoids, and elliptic paraboloids.

Functions of Several Variables - In this section we will give a quick review of some important topics about functions of several variables. In particular we will discuss finding the domain of a function of several variables as well as level curves, level surfaces and traces.

Vector Functions - In this section we introduce the concept of vector functions concentrating primarily on curves in three dimensional space. We will however, touch briefly on surfaces as well. We will illustrate how to find the domain of a vector function and how to graph a vector function. We will also show a simple relationship between vector functions and parametric equations that will be very useful at times.

Calculus with Vector Functions - In this section here we discuss how to do basic calculus, i.e. limits, derivatives and integrals, with vector functions.

Tangent, Normal and Binormal Vectors - In this section we will define the tangent, normal and binormal vectors.

Arc Length with Vector Functions - In this section we will extend the arc length formula we used early in the material to include finding the arc length of a vector function. As we will see the new formula really is just an almost natural extension of one we've already seen.

Curvature - In this section we give two formulas for computing the curvature (*i.e.* how fast the function is changing at a given point) of a vector function.

Velocity and Acceleration - In this section we will revisit a standard application of derivatives, the velocity and acceleration of an object whose position function is given by a vector function. For the acceleration we give formulas for both the normal acceleration and the tangential acceleration.

Cylindrical Coordinates - In this section we will define the cylindrical coordinate system, an alternate coordinate system for the three dimensional coordinate system. As we will see cylindrical coordinates are really nothing more than a very natural extension of polar coordinates into a three dimensional setting.

Spherical Coordinates - In this section we will define the spherical coordinate system, yet another alternate coordinate system for the three dimensional coordinate system. This coordinates system is very useful for dealing with spherical objects. We will derive formulas to convert between cylindrical coordinates and spherical coordinates as well as between Cartesian and spherical coordinates (the more useful of the two).

Partial Derivatives In Calculus I and in most of Calculus II we concentrated on functions of one variable. In Calculus III we will extend our knowledge of calculus into functions of two or more variables. Despite the fact that this chapter is about derivatives we will start out the chapter with a section on limits of functions of more than one variable. In the remainder of this chapter we will be looking at differentiating functions of more than one variable. As we will see, while there are differences with derivatives of functions of one variable, if you can do derivatives of functions of one variable you shouldn't have any problems differentiating functions of more than one variable. You'll just need to keep one subtlety in mind as we do the work.

Limits - In the section we'll take a quick look at evaluating limits of functions of several variables. We will also see a fairly quick method that can be used, on occasion, for showing that some limits do not exist.

Partial Derivatives - In this section we will look at the idea of partial derivatives. We will give the formal definition of the partial derivative as well as the standard notations and how to compute them in practice (i.e. without the use of the definition). As you will see if you can do derivatives of functions of one variable you won't have much of an issue with partial derivatives. There is only one (very important) subtlety that you need to always keep in mind while computing partial derivatives.

Interpretations of Partial Derivatives - In the section we will take a look at a couple of important interpretations of partial derivatives. First, the always important, rate of change of the function. Although we now have multiple 'directions' in which the function can change (unlike in Calculus I). We will also see that partial derivatives give the slope of tangent lines to the traces of the function.

Higher Order Partial Derivatives - In the section we will take a look at higher order partial derivatives. Unlike Calculus I however, we will have multiple second order derivatives, multiple third order derivatives, etc. because we are now working with functions of multiple variables. We will also discuss Clairaut's Theorem to help with some of the work in finding higher order derivatives.

Differentials - In this section we extend the idea of differentials we first saw in Calculus I to functions of several variables.

Chain Rule - In the section we extend the idea of the chain rule to functions of several variables. In particular, we will see that there are multiple variants to the chain rule here all depending on how many variables our function is dependent on and how each of those variables can, in turn, be written in terms of different variables. We will also give a nice method for writing down the chain rule for pretty much any situation you might run into when dealing with functions of multiple variables. In addition, we will derive a very quick way of doing implicit differentiation so we no longer need to go through the process we first did back in Calculus I.

Directional Derivatives - In the section we introduce the concept of directional derivatives. With directional derivatives we can now ask how a function is changing if we allow all the independent variables to change rather than holding all but one constant as we had to do with partial derivatives. In addition, we will define the gradient vector to help with some of the notation and work here. The gradient vector will be very useful in some later sections as well. We will also give a nice fact that will allow us to determine the direction in which a given function is changing the fastest.

Line Integrals In this section we are going to start looking at Calculus with vector fields (which we'll define in the first section). In particular we will be looking at a new type of integral, the line integral and some of the interpretations of the line integral. We will also take a look at one of the more important theorems involving line integrals, Green's Theorem.

Vector Fields - In this section we introduce the concept of a vector field and give several examples of graphing them. We also revisit the gradient that we first saw a few chapters ago.

Line Integrals - Part I - In this section we will start off with a quick review of parameterizing curves. This is a skill that will be required in a great many of the line integrals we evaluate and so needs to be understood. We will then formally define the first kind of line integral we will be looking at : line integrals with respect to arc length..

Line Integrals - Part II - In this section we will continue looking at line integrals and define the second kind of line integral we'll be looking at : line integrals with respect to x, y, and/or z. We also introduce an alternate form of notation for this kind of line integral that will be useful on occasion.

Line Integrals of Vector Fields - In this section we will define the third type of line integrals we'll be looking at : line integrals of vector fields. We will also see that this particular kind of line integral is related to special cases of the line integrals with respect to x, y and z.

Fundamental Theorem for Line Integrals - In this section we will give the fundamental theorem of calculus for line integrals of vector fields. This will illustrate that certain kinds of line integrals can be very quickly computed. We will also give quite a few definitions and facts that will be useful.

Conservative Vector Fields - In this section we will take a more detailed look at conservative vector fields than we've done in previous sections. We will also discuss how to find potential functions for conservative vector fields.

Green's Theorem - In this section we will discuss Green's Theorem as well as an interesting application of Green's Theorem that we can use to find the area of a two dimensional region.

Surface Integrals In the previous chapter we looked at evaluating integrals of functions or vector fields where the points came from a curve in two- or three-dimensional space. We now want to extend this idea and integrate functions and vector fields where the points come from a surface in three-dimensional space. These integrals are called surface integrals.

Curl and Divergence - In this section we will introduce the concepts of the curl and the divergence of a vector field. We will also give two vector forms of Green's Theorem and show how the curl can be used to identify if a three dimensional vector field is conservative field or not.

Parametric Surfaces - In this section we will take a look at the basics of representing a surface with parametric equations. We will also see how the parameterization of a surface can be used to find a normal vector for the surface (which will be very useful in a couple of sections) and how the parameterization can be used to find the surface area of a surface.

Surface Integrals - In this section we introduce the idea of a surface integral. With surface integrals we will be integrating over the surface of a solid. In other words, the variables will always be on the surface of the solid and will never come from inside the solid itself. Also, in this section we will be working with the first kind of surface integrals we'll be looking at in this chapter : surface integrals of functions.

Surface Integrals of Vector Fields - In this section we will introduce the concept of an oriented surface and look at the second kind of surface integral we'll be looking at : surface integrals of vector fields.

Stokes' Theorem - In this section we will discuss Stokes' Theorem.

Divergence Theorem - In this section we will discuss the Divergence Theorem.

1 Review

Technically a student coming into a Calculus class is supposed to know both Algebra and Trigonometry. Unfortunately, the reality is often much different. Most students enter a Calculus class woefully unprepared for both the algebra and the trig that is in a Calculus class. This is very unfortunate since good algebra skills are absolutely vital to successfully completing any Calculus course and if your Calculus course includes trig (as this one does) good trig skills are also important in many sections.

The above statement is not meant to denigrate your favorite Algebra or Trig instructor. It is simply an acknowledgment of the fact that many of these courses, especially Algebra courses, are aimed at a more general audience and so do not always put the time into topics that are vital to a Calculus course and/or the level of difficulty is kept lower than might be best for students heading on towards Calculus.

Far too often the biggest impediment to students being successful in a Calculus course is they do not have sufficient skills in the underlying algebra and trig that will be in many of the calculus problems we'll be looking at. These students end up struggling with the algebra and trig in the problems rather than working to understand the calculus topics which in turn negatively impacts their grade in a Calculus course. The intent of this chapter, therefore, is to do a very cursory review of some algebra and trig skills that are vital to a calculus course that many students just didn't learn as well as they should have from their Algebra and Trig courses.

This chapter does not include all the algebra and trig skills that are needed to be successful in a Calculus course. It only includes those topics that most students are particularly deficient in. For instance, factoring is also vital to completing a standard calculus class but is not included here as it is assumed that if you are taking a Calculus course then you do know how to factor. Likewise, it is assumed that if you are taking a Calculus course then you know how to solve linear and quadratic equations so those topics are not covered here either. For a more in depth review of Algebra topics you should check out the full set of Algebra notes at http://tutorial.math.lamar.edu.

Note that even though these topics are very important to a Calculus class we rarely cover all of them in the actual class itself. We simply don't have the time to do that. We will cover certain portions of this chapter in class, but for the most part we leave it to the students to read this chapter on their own to make sure they are ready for these topics as they arise in class.

The following sections are the practice problems, with solutions, for this material.

1.1 Functions

1. Perform the indicated function evaluations for $f(x) = 3 - 5x - 2x^2$.

(a) $f(4)$ **(b)** $f(0)$ **(c)** $f(-3)$

(d) $f(6-t)$ **(e)** $f(7-4x)$ **(f)** $f(x+h)$

Solutions

(a) $f(4)$

Solution

$$f(4) = 3 - 5(4) - 2(4)^2 = -49$$

(b) $f(0)$

Solution

$$f(0) = 3 - 5(0) - 2(0)^2 = 3$$

(c) $f(-3)$

Solution

$$f(-3) = 3 - 5(-3) - 2(-3)^2 = 0$$

Hint

Don't let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(d) $f(6-t)$

Solution

$$\begin{aligned} f(6-t) &= 3 - 5(6-t) - 2(6-t)^2 \\ &= 3 - 5(6-t) - 2\left(36 - 12t + t^2\right) \\ &= 3 - 30 + 5t - 72 + 24t - 2t^2 \\ &= -99 + 29t - 2t^2 \end{aligned}$$

Hint

Don't let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(e) $f(7-4x)$

Solution

$$\begin{aligned} f(7-4x) &= 3 - 5(7-4x) - 2(7-4x)^2 \\ &= 3 - 5(7-4x) - 2\left(49 - 56x + 16x^2\right) \\ &= 3 - 35 + 20x - 98 + 112x - 32x^2 \\ &= -130 + 132x - 32x^2 \end{aligned}$$

Hint

Don't let the fact that there are now variables here instead of numbers get you confused. Also, don't get excited about the fact that there is both an x and an h here. This works exactly the same way as the first three it will just have a little more algebra involved.

(f) $f(x+h)$

Solution

$$\begin{aligned} f(x+h) &= 3 - 5(x+h) - 2(x+h)^2 \\ &= 3 - 5(x+h) - 2\left(x^2 + 2xh + h^2\right) \\ &= 3 - 5x - 5h - 2x^2 - 4xh - 2h^2 \end{aligned}$$

---

2. Perform the indicated function evaluations for $g(t) = \dfrac{t}{2t+6}$.

(a) $g(0)$ **(b)** $g(-3)$ **(c)** $g(10)$

(d) $g\left(x^2\right)$ **(e)** $g(t+h)$ **(f)** $g\left(t^2 - 3t + 1\right)$

Solutions

(a) $g(0)$

Solution

$$g(0) = \frac{0}{2(0)+6} = \frac{0}{6} = 0$$

(b) $g(-3)$

Solution

$$g(-3) = \frac{-3}{2(-3)+6} = \frac{-3}{0} \quad \times$$

The minute we see the division by zero we know that $g(-3)$ **does not exist**.

(c) $g(10)$

Solution

$$g(10) = \frac{10}{2(10)+6} = \frac{10}{26} = \frac{5}{13}$$

Hint

Don't let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(d) $g\left(x^2\right)$

Solution

$$g\left(x^2\right) = \frac{x^2}{2x^2+6}$$

Hint

Don't let the fact that there are now variables here instead of numbers get you confused. Also, don't get excited about the fact that there is both a t and an h here. This works exactly the same way as the first three it will just have a little more algebra involved.

(e) $g(t+h)$

Solution

$$g(t+h) = \frac{t+h}{2(t+h)+6} = \frac{t+h}{2t+2h+6}$$

Hint

Don't let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(f) $g\left(t^2-3t+1\right)$

Solution

$$g\left(t^2-3t+1\right) = \frac{t^2-3t+1}{2\left(t^2-3t+1\right)+6} = \frac{t^2-3t+1}{2t^2-6t+8}$$

3. Perform the indicated function evaluations for $h(z) = \sqrt{1 - z^2}$.

(a) $h(0)$ **(b)** $h\left(-\frac{1}{2}\right)$ **(c)** $h\left(\frac{1}{2}\right)$

(d) $h(9z)$ **(e)** $h\left(z^2 - 2z\right)$ **(f)** $h(z + k)$

Solutions

(a) $h(0)$

Solution

$$h(0) = \sqrt{1 - 0^2} = \sqrt{1} = 1$$

(b) $h\left(-\frac{1}{2}\right)$

Solution

$$h\left(-\frac{1}{2}\right) = \sqrt{1 - \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

(c) $h\left(\frac{1}{2}\right)$

Solution

$$h\left(\frac{1}{2}\right) = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Hint

Don't let the fact that there are new variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(d) $h(9z)$

Solution

$$h(9z) = \sqrt{1 - (9z)^2} = \sqrt{1 - 81z^2}$$

Hint

Don't let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(e) $h\left(z^2 - 2z\right)$

Solution

$$h\left(z^2 - 2z\right) = \sqrt{1 - \left(z^2 - 2z\right)^2} = \sqrt{1 - \left(z^4 - 4z^3 + 4z^2\right)} = \sqrt{1 - 4z^2 + 4z^3 - z^4}$$

Hint

Don't let the fact that there are now variables here instead of numbers get you confused. Also, don't get excited about the fact that there is both a z and a k here. This works exactly the same way as the first three it will just have a little more algebra involved.

(f) $h(z + k)$

Solution

$$h(z + k) = \sqrt{1 - (z + k)^2} = \sqrt{1 - \left(z^2 + 2zk + k^2\right)} = \sqrt{1 - z^2 - 2zk - k^2}$$

---

4. Perform the indicated function evaluations for $R(x) = \sqrt{3+x} - \dfrac{4}{x+1}$.

(a) $R(0)$ **(b)** $R(6)$ **(c)** $R(-9)$

(d) $R(x+1)$ **(e)** $R(x^4-3)$ **(f)** $R\left(\frac{1}{x}-1\right)$

Solutions

(a) $R(0)$

Solution

$$R(0) = \sqrt{3+0} - \frac{4}{0+1} = \sqrt{3} - 4$$

(b) $R(6)$

Solution

$$R(6) = \sqrt{3+6} - \frac{4}{6+1} = \sqrt{9} - \frac{4}{7} = 3 - \frac{4}{7} = \frac{17}{7}$$

(c) $R(-9)$

Solution

$$R(-9) = \sqrt{3+(-9)} - \frac{4}{-9+1} = \sqrt{-6} - \frac{4}{-8}$$

In this class we only deal with functions that give real values as answers. Therefore, because we have the square root of a negative number in the first term this function is **not defined**.

Note that the fact that the second term is perfectly acceptable has no bearing on the fact that the function will not be defined here. If any portion of the function is not defined upon evaluation, then the whole function is not defined at that point. Also note that if we allow complex numbers this function will be defined.

Hint

Don't let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(d) $R(x+1)$

Solution

$$R(x+1) = \sqrt{3+(x+1)} - \frac{4}{(x+1)+1} = \sqrt{4+x} - \frac{4}{x+2}$$

Hint

Don't let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(e) $R(x^4-3)$

Solution

$$R(x^4-3) = \sqrt{3+(x^4-3)} - \frac{4}{(x^4-3)+1} = \sqrt{x^4} - \frac{4}{x^4-2} = x^2 - \frac{4}{x^4-2}$$

Hint

Don't let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(f) $R\left(\frac{1}{x}-1\right)$

Solution

$$R\left(\frac{1}{x}-1\right) = \sqrt{3+\left(\frac{1}{x}-1\right)} - \frac{4}{\left(\frac{1}{x}-1\right)+1} = \sqrt{2+\frac{1}{x}} - \frac{4}{\frac{1}{x}} = \sqrt{2+\frac{1}{x}} - 4x$$

---

5. Compute the difference quotient fo $f(x) = 4x - 9$.

Difference Quotient

The **difference quotient** of a function $f(x)$ is defined to be,

$$\frac{f(x+h) - f(x)}{h}$$

Hint

Compute $f(x+h)$, then compute the numerator and finally compute the difference quotient.

Step 1

$$f(x+h) = 4(x+h) - 9 = 4x + 4h - 9$$

Step 2

$$f(x+h) - f(x) = 4x + 4h - 9 - (4x - 9) = 4h$$

Step 3

$$\frac{f(x+h) - f(x)}{h} = \frac{4h}{h} = 4$$

---

6. Compute the difference quotient fo $g(x) = 6 - x^2$.

Difference Quotient

The **difference quotient** of a function $f(x)$ is defined to be,

$$\frac{f(x+h) - f(x)}{h}$$

Hint

Don't get excited about the fact that the function is now named $g(x)$, the difference quotient still works in the same manner it just has g's instead of f's now. So, compute $g(x+h)$, then compute the numerator and finally compute the difference quotient.

Step 1

$$g(x+h) = 6 - (x+h)^2 = 6 - x^2 - 2xh - h^2$$

Step 2

$$g(x+h) - g(x) = 6 - x^2 - 2xh - h^2 - \left(6 - x^2\right) = -2xh - h^2$$

Step 3

$$\frac{g(x+h) - g(x)}{h} = \frac{-2xh - h^2}{h} = -2x - h$$

---

7. Compute the difference quotient fo $f(t) = 2t^2 - 3t + 9$.

Difference Quotient

The **difference quotient** of a function $f(x)$ is defined to be,

$$\frac{f(x+h) - f(x)}{h}$$

Hint

Don't get excited about the fact that the function is now $f(t)$, the difference quotient still works in the same manner it just has t's instead of x's now. So, compute $f(t+h)$, then compute the numerator and finally compute the difference quotient.

Step 1

$$\begin{aligned} f(t+h) &= 2(t+h)^2 - 3(t+h) + 9 \\ &= 2\left(t^2 + 2th + h^2\right) - 3t - 3h + 9 \\ &= 2t^2 + 4th + 2h^2 - 3t - 3h + 9 \end{aligned}$$

Step 2

$$f(t+h) - f(t) = 2t^2 + 4th + 2h^2 - 3t - 3h + 9 - \left(2t^2 - 3t + 9\right) = 4th + 2h^2 - 3h$$

Step 3

$$\frac{f(t+h) - f(t)}{h} = \frac{4th + 2h^2 - 3h}{h} = 4t + 2h - 3$$

---

8. Compute the difference quotient fo $y(z) = \dfrac{1}{z+2}$.

Difference Quotient

The **difference quotient** of a function $f(x)$ is defined to be,

$$\frac{f(x+h) - f(x)}{h}$$

Hint

Don't get excited about the fact that the function is now named $y(z)$, the difference quotient still works in the same manner it just has y's and z's instead of f's and x's now. So, compute $y(z+h)$, then compute the numerator and finally compute the difference quotient.

Step 1

$$y(z+h) = \frac{1}{z+h+2}$$

Step 2

$$y(z+h) - y(z) = \frac{1}{z+h+2} - \frac{1}{z+2} = \frac{z+2-(z+h+2)}{(z+h+2)(z+2)} = \frac{-h}{(z+h+2)(z+2)}$$

Note that, when dealing with difference quotients, it will almost always be advisable to combine rational expressions into a single term in preparation of the next step.

Step 3

$$\begin{aligned} \frac{y(z+h)-y(z)}{h} = \frac{1}{h}\left(y(z+h)-y(z)\right) &= \frac{1}{h}\left(\frac{-h}{(z+h+2)(z+2)}\right) \\ &= \frac{-1}{(z+h+2)(z+2)} \end{aligned}$$

In this step we rewrote the difference quotient a little to make the numerator a little easier to deal with. All that we're doing here is using the fact that,

$$\frac{a}{b} = (a)\left(\frac{1}{b}\right) = \left(\frac{1}{b}\right)(a)$$

---

9. Compute the difference quotient fo $A(t) = \dfrac{2t}{3-t}$.

Difference Quotient

The **difference quotient** of a function $f(x)$ is defined to be,

$$\frac{f(x+h)-f(x)}{h}$$

Hint

Don't get excited about the fact that the function is now named $A(t)$, the difference quotient still works in the same manner it just has A's and t's instead of f's and x's now. So, compute $A(t+h)$, then compute the numerator and finally compute the difference quotient.

Step 1

$$A(t+h) = \frac{2(t+h)}{3-(t+h)} = \frac{2t+2h}{3-t-h}$$

Step 2

$$\begin{aligned} A(t+h) - A(t) &= \frac{2t+2h}{3-t-h} - \frac{2t}{3-t} \\ &= \frac{(2t+2h)(3-t) - 2t(3-t-h)}{(3-t-h)(3-t)} \\ &= \frac{6t - 2t^2 + 6h - 2ht - \left(6t - 2t^2 - 2th\right)}{(3-t-h)(3-t)} \\ &= \frac{6h}{(3-t-h)(3-t)} \end{aligned}$$

Note that, when dealing with difference quotients, it will almost always be advisable to combine rational expressions into a single term in preparation of the next step. Also, when doing this don't forget to simplify the numerator as much as possible. With most difference quotients you'll see a lot of cancelation as we did here.

Step 3

$$\begin{aligned} \frac{A(t+h) - A(t)}{h} = \frac{1}{h}\left(A(t+h) - A(t)\right) &= \frac{1}{h}\left(\frac{6h}{(3-t-h)(3-t)}\right) \\ &= \frac{6}{(3-t-h)(3-t)} \end{aligned}$$

In this step we rewrote the difference quotient a little to make the numerator a little easier to deal with. All that we're doing here is using the fact that,

$$\frac{a}{b} = (a)\left(\frac{1}{b}\right) = \left(\frac{1}{b}\right)(a)$$

---

10. Determine all the roots of $f(x) = x^5 - 4x^4 - 32x^3$.

Solution

Set the function equal to zero and factor the left side.

$$x^5 - 4x^4 - 32x^3 = x^3\left(x^2 - 4x - 32\right) = x^3(x-8)(x+4) = 0$$

After factoring we can see that the three roots of this function are,

$$x = -4, \quad x = 0, \quad x = 8$$

11. Determine all the roots of $R(y) = 12y^2 + 11y - 5$.

Solution

Set the function equal to zero and factor the left side.

$$12y^2 + 11y - 5 = (4y+5)(3y-1) = 0$$

After factoring we see that the two roots of this function are,

$$y = -\frac{5}{4}, \quad y = \frac{1}{3}$$

12. Determine all the roots of $h(t) = 18 - 3t - 2t^2$.

Solution

Set the function equal to zero and because the left side will not factor we'll need to use the quadratic formula to find the roots of the function.

$$18 - 3t - 2t^2 = 0$$

$$\begin{aligned} t = \frac{3 \pm \sqrt{(-3)^2 - 4(-2)(18)}}{2(-2)} &= \frac{3 \pm \sqrt{153}}{-4} \\ &= \frac{3 \pm \sqrt{(9)(17)}}{-4} = \frac{3 \pm 3\sqrt{17}}{-4} = -\frac{3}{4}\left(1 \pm \sqrt{17}\right) \end{aligned}$$

So, the quadratic formula gives the following two roots of the function,

$$-\frac{3}{4}\left(1 + \sqrt{17}\right) = -3.842329 \qquad -\frac{3}{4}\left(1 - \sqrt{17}\right) = 2.342329$$

13. Determine all the roots of $g(x) = x^3 + 7x^2 - x$.

Solution

Set the equation equal to zero and factor the left side as much as possible.

$$x^3 + 7x^2 - x = x\left(x^2 + 7x - 1\right) = 0$$

So, we can see that one root is $x = 0$ and because the quadratic doesn't factor we'll need to use the quadratic formula on that to get the remaining two roots.

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(1)(-1)}}{2(1)} = \frac{-7 \pm \sqrt{53}}{2}$$

We then have the following three roots of the function,

$$x = 0, \qquad \frac{-7 + \sqrt{53}}{2} = 0.140055, \qquad \frac{-7 - \sqrt{53}}{2} = -7.140055$$

14. Determine all the roots of $W(x) = x^4 + 6x^2 - 27$.

Solution

Set the function equal to zero and factor the left side as much as possible.

$$x^4 + 6x^2 - 27 = \left(x^2 - 3\right)\left(x^2 + 9\right) = 0$$

Don't so locked into quadratic equations that the minute you see an equation that is not quadratic you decide you can't deal with it. While this function was not a quadratic it still factored in an obvious manner.

Now, the second term will never be zero (for any real value of x anyway and in this class those tend to be the only ones we are interested in) and so we can ignore that term. The first will be zero if,

$$x^2 - 3 = 0 \qquad \Rightarrow \qquad x^2 = 3 \qquad \Rightarrow \qquad x = \pm\sqrt{3}$$

So, we have two real roots of this function. Note that if we allowed complex roots (which again, we aren't really interested in for this course) there would also be two complex roots from the second term as well.

15. Determine all the roots of $f(t) = t^{\frac{5}{3}} - 7t^{\frac{4}{3}} - 8t$.

Solution

Set the function equal to zero and factor the left side as much as possible.

$$t^{\frac{5}{3}} - 7t^{\frac{4}{3}} - 8t = t\left(t^{\frac{2}{3}} - 7t^{\frac{1}{3}} - 8\right) = t\left(t^{\frac{1}{3}} - 8\right)\left(t^{\frac{1}{3}} + 1\right) = 0$$

Don't so locked into quadratic equations that the minute you see an equation that is not quadratic you decide you can't deal with it. While this function was not a quadratic it still factored, it just wasn't as obvious that it did in this case. You could have clearly seen that if factored if it had been,

$$t\left(t^2 - 7t - 8\right)$$

but notice that the only real difference is that the exponents are fractions now, but it still has the same basic form and so can be factored.

Okay, back to the problem. From the factored form we get,

$$\begin{array}{lllll} t = 0 & & & & \\ t^{\frac{1}{3}} - 8 = 0 & \Rightarrow & t^{\frac{1}{3}} = 8 & \Rightarrow & t = 8^3 = 512 \\ t^{\frac{1}{3}} + 1 = 0 & \Rightarrow & t^{\frac{1}{3}} = -1 & \Rightarrow & t = (-1)^3 = -1 \end{array}$$

So, the function has three roots,

$$t = -1, \quad t = 0, \quad t = 512$$

16. Determine all the roots of $h(z) = \frac{z}{z-5} - \frac{4}{z-8}$.

Solution

Set the function equal to zero and clear the denominator by multiplying by the least common denominator, $(z-5)(z-8)$, and then solve the resulting equation.

$$\begin{aligned} (z-5)(z-8)\left(\frac{z}{z-5} - \frac{4}{z-8}\right) &= 0 \\ z(z-8) - 4(z-5) &= 0 \\ z^2 - 12z + 20 &= 0 \\ (z-10)(z-2) &= 0 \end{aligned}$$

So, it looks like the function has two roots, $z = 2$ and $z = 10$ however recall that because we started off with a function that contained rational expressions we need to go back to the original function and make sure that neither of these will create a division by zero problem in the original function. In this case neither do and so the two roots are,

$$z = 2 \qquad z = 10$$

17. Determine all the roots of $g(w) = \dfrac{2w}{w+1} + \dfrac{w-4}{2w-3}$.

Solution

Set the function equal to zero and clear the denominator by multiplying by the least common denominator, $(w+1)(2w-3)$, and then solve the resulting equation.

$$\begin{aligned} (w+1)(2w-3)\left(\frac{2w}{w+1} + \frac{w-4}{2w-3}\right) &= 0 \\ 2w(2w-3) + (w-4)(w+1) &= 0 \\ 5w^2 - 9w - 4 &= 0 \end{aligned}$$

This quadratic doesn't factor so we'll need to use the quadratic formula to get the solution.

$$w = \frac{9 \pm \sqrt{(-9)^2 - 4(5)(-4)}}{2(5)} = \frac{9 \pm \sqrt{161}}{10}$$

So, it looks like this function has the following two roots,

$$\frac{9+\sqrt{161}}{10} = 2.168858 \qquad \frac{9-\sqrt{161}}{10} = -0.368858$$

Recall that because we started off with a function that contained rational expressions we need to go back to the original function and make sure that neither of these will create a division by zero problem in the original function. Neither of these do and so they are the two roots of this function.

18. Find the domain and range of $Y(t) = 3t^2 - 2t + 1$.

Solution

This is a polynomial (a 2^{nd} degree polynomial in fact) and so we know that we can plug any value of t into the function and so the domain is all real numbers or,

$$\text{Domain : } -\infty < t < \infty \quad \text{or} \quad (-\infty, \infty)$$

The graph of this 2^{nd} degree polynomial (or quadratic) is a parabola that opens upwards (because the coefficient of the t^2 is positive) and so we know that the vertex will be the lowest point on the graph. This also means that the function will take on all values greater than or equal to the y-coordinate of the vertex which will in turn give us the range.

So, we need the vertex of the parabola. The t-coordinate is,

$$t = -\frac{-2}{2(3)} = \frac{1}{3}$$

and the y coordinate is then, $Y\left(\frac{1}{3}\right) = \frac{2}{3}$.

The range is then,

$$\text{Range : } \left[\frac{2}{3}, \infty\right)$$

19. Find the domain and range of $g(z) = -z^2 - 4z + 7$.

Solution

This is a polynomial (a 2^{nd} degree polynomial in fact) and so we know that we can plug any value of z into the function and so the domain is all real numbers or,

$$\text{Domain : } -\infty < z < \infty \quad \text{or} \quad (-\infty, \infty)$$

The graph of this 2^{nd} degree polynomial (or quadratic) is a parabola that opens downwards (because the coefficient of the z^2 is negative) and so we know that the vertex will be the highest point on the graph. This also means that the function will take on all values less than or equal to the y-coordinate of the vertex which will in turn give us the range.

So, we need the vertex of the parabola. The z-coordinate is,

$$z = -\frac{-4}{2(-1)} = -2$$

and the y coordinate is then, $g(-2) = 11$.

The range is then,

$$\text{Range : } (-\infty, 11]$$

20. Find the domain and range of $f(z) = 2 + \sqrt{z^2 + 1}$.

Solution

We know that when we have square roots that we can't take the square root of a negative number. However, because,

$$z^2 + 1 \geq 1$$

we will never be taking the square root of a negative number in this case and so the domain is all real numbers or,

$$\text{Domain : } -\infty < z < \infty \text{ or } (-\infty, \infty)$$

For the range we need to recall that square roots will only return values that are positive or zero and in fact the only way we can get zero out of a square root will be if we take the square root of zero. For our function, as we've already noted, the quantity that is under the root is always at least 1 and so this root will never be zero. Also recall that we have the following fact about square roots,

$$\text{If } x \geq 1 \text{ then } \sqrt{x} \geq 1$$

So, we now know that,

$$\sqrt{z^2 + 1} \geq 1$$

Finally, we are adding 2 onto the root and so we know that the function must always be greater than or equal to 3 and so the range is,

$$\text{Range : } [3, \infty)$$

---

21. Find the domain and range of $h(y) = -3\sqrt{14 + 3y}$.

Solution

In this case we need to require that,

$$14 + 3y \geq 0 \qquad \Rightarrow \qquad y \geq -\frac{14}{3}$$

in order to make sure that we don't take the square root of negative numbers. The domain is then,

$$\text{Domain : } -\frac{14}{3} \leq y < \infty \text{ or } \left[-\frac{14}{3}, \infty\right)$$

For the range for this function we can notice that the quantity under the root can be zero (if $y = -\frac{14}{3}$). Also note that because the quantity under the root is a line it will take on all positive values and so the square root will in turn take on all positive value and zero. The square root is then multiplied by -3. This won't change the fact that the root can be zero, but the minus sign will change the sign of the non-zero values from positive to negative. The 3 will only affect the general size of the square root but it won't change the fact that the square root will still take on all positive (or negative after we add in the minus sign) values.

The range is then,

$$\text{Range}: \ (-\infty, 0]$$

---

22. Find the domain and range of $M(x) = 5 - |x + 8|$.

Solution

We're dealing with an absolute value here and the quantity inside is a line, which we can plug all values of x into, and so the domain is all real numbers or,

$$\text{Domain}: \ -\infty < x < \infty \ \text{ or } \ (-\infty, \infty)$$

For the range let's again note that the quantity inside the absolute value is a linear function that will take on all real values. We also know that absolute value functions will never be negative and will only be zero if we take the absolute value of zero. So, we now know that,

$$|x + 8| \geq 0$$

However, we are subtracting this from 5 and so we'll be subtracting a positive or zero number from 5 and so the range is,

$$\text{Range}: \ (-\infty, 5]$$

---

23. Find the domain of $f(w) = \dfrac{w^3 - 3w + 1}{12w - 7}$.

Solution

In this case we need to avoid division by zero issues so we'll need to determine where the denominator is zero. To do this we will solve,

$$12w - 7 = 0 \qquad \Rightarrow \qquad w = \frac{7}{12}$$

We can plug all other values of w into the function without any problems and so the domain is,

$$\text{Domain : All real numbers except } w = \frac{7}{12}$$

24. Find the domain of $R(z) = \dfrac{5}{z^3 + 10z^2 + 9z}$.

Solution

In this case we need to avoid division by zero issues so we'll need to determine where the denominator is zero. To do this we will solve,

$$z^3 + 10z^2 + 9z = z\left(z^2 + 10z + 9\right) = z(z+1)(z+9) = 0 \qquad \Rightarrow \qquad z = 0,\ z = -1,\ z = -9$$

The three values above are the only values of z that we can't plug into the function. All other values of z can be plugged into the function and will return real values. The domain is then,

$$\text{Domain : All real numbers except } z = 0,\ \ z = -1,\ \ z = -9$$

25. Find the domain of $g(t) = \dfrac{6t - t^3}{7 - t - 4t^2}$.

Solution

In this case we need to avoid division by zero issues so we'll need to determine where the denominator is zero. To do this we will solve,

$$7 - t - 4t^2 = 0 \qquad \Rightarrow \qquad t = \frac{1 \pm \sqrt{(-1)^2 - 4(-4)(7)}}{2(-4)} = -\frac{1}{8}\left(1 \pm \sqrt{113}\right)$$

The two values above are the only values of t that we can't plug into the function. All other values of t can be plugged into the function and will return real values. The domain is then,

$$\text{Domain : All real numbers except } t = -\frac{1}{8}\left(1 \pm \sqrt{113}\right)$$

26. Find the domain of $g(x) = \sqrt{25 - x^2}$.

Solution

In this case we need to avoid square roots of negative numbers so we need to require,

$$25 - x^2 \geq 0$$

Note that once we have the original inequality written down we can do a little rewriting of things as follows to make things a little easier to see.

$$x^2 \leq 25 \qquad \Rightarrow \qquad -5 \leq x \leq 5$$

At this point it should be pretty easy to find the values of x that will keep the quantity under the radical positive or zero so we won't need to do a number line or sign table to determine the range.

The domain is then,

$$\text{Domain : } -5 \leq x \leq 5$$

---

27. Find the domain of $h(x) = \sqrt{x^4 - x^3 - 20x^2}$.

Hint

We need to avoid negative numbers under the square root and because the quantity under the root is a polynomial we know that it can only change sign if it goes through zero and so we first need to determine where it is zero.

Step 1

In this case we need to avoid square roots of negative numbers so we need to require,

$$x^4 - x^3 - 20x^2 = x^2\left(x^2 - x - 20\right) = x^2(x-5)(x+4) \geq 0$$

Once we have the polynomial in factored form we can see that the left side will be zero at $x = 0$, $x = -4$ and $x = 5$. Because the quantity under the radical is a polynomial we know that it can only change sign if it goes through zero and so these are the only points the only places where the polynomial on the left can change sign.

Hint

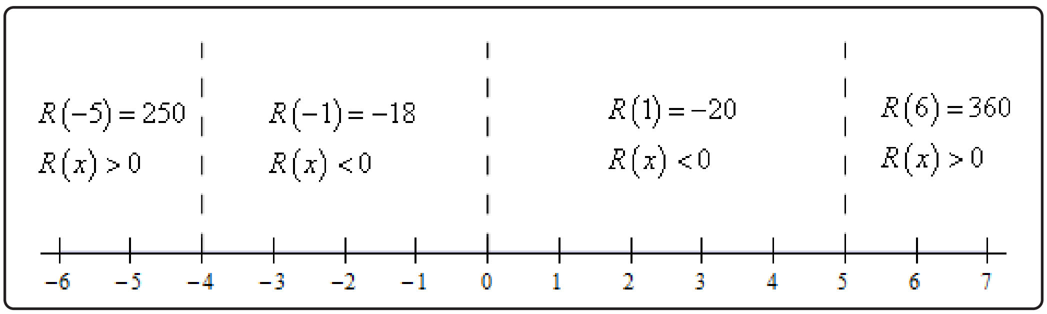
Because the polynomial can only change sign at these points we know that it will be the same sign in each region defined by these points and so all we need to know is the value of the polynomial as a single point in each region.

Step 2

Here is a number line giving the value/sign of the polynomial at a *test point* in each of the region defined by these three points. To make it a little easier to read the number line let's define the polynomial under the radical to be,

$$R(x) = x^4 - x^3 - 20x^2 = x^2(x-5)(x+4)$$

Now, here is the number line,



Hint

Now all we need to do is write down the values of x where the polynomial under the root will be positive or zero and we'll have the domain. Be careful with the points where the polynomial is zero.

Step 3

The domain will then be all the points where the polynomial under the root is positive or zero and so the domain is,

$$\text{Domain} : -\infty < x \leq -4, \quad x = 0, \quad 5 \leq x < \infty$$

In this case we need to be very careful and not miss $x = 0$. This is the point separating two regions which give negative values of the polynomial, but it will give zero and so it also part of the domain. This point is often very is very easy to miss.

28. Find the domain of $P(t) = \dfrac{5t+1}{\sqrt{t^3 - t^2 - 8t}}$.

Hint

We need to avoid negative numbers under the square root and because the quantity under the root is a polynomial we know that it can only change sign if it goes through zero and so we first need to determine where it is zero.

Step 1

In this case we need to avoid square roots of negative numbers and because the square root is in the denominator we'll also need to avoid division by zero issues. We can satisfy both needs by requiring,

$$t^3 - t^2 - 8t = t\left(t^2 - t - 8\right) > 0$$

Note that there is nothing wrong with the square root of zero, but we know that the square root of zero is zero and so if we require that the polynomial under the root is strictly positive we'll know that we won't have square roots of negative numbers and we'll avoid division by zero.

Now, despite the fact that we need to avoid where the polynomial is zero we know that it will only change signs if it goes through zero and so we'll next need to determine where the polynomial is zero.

Clearly one value is $t = 0$ and because the quadratic does not factor we can use the quadratic formula on it to get the following two additional points.

$$t = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-8)}}{2} = \frac{1 \pm \sqrt{33}}{2} \qquad \begin{aligned} t &= \frac{1 + \sqrt{33}}{2} = 3.372281 \\ t &= \frac{1 - \sqrt{33}}{2} = -2.372281 \end{aligned}$$

So, these three points ($t = 0$, $t = -2.372281$ and $t = 3.372281$ are the only places that the polynomial under the root can change sign.

Hint

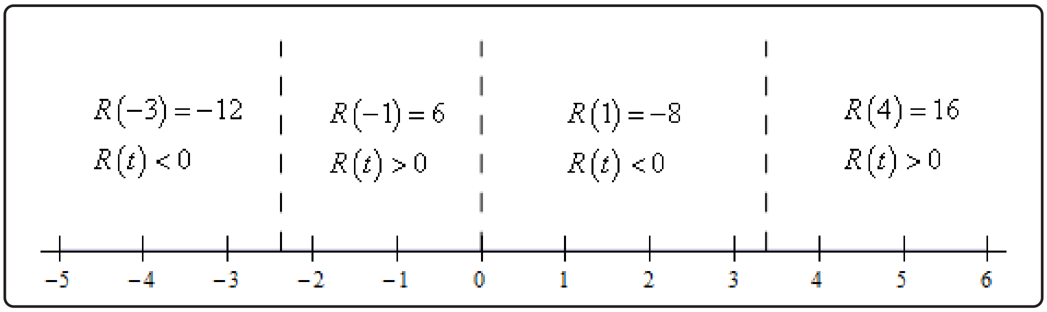
Because the polynomial can only change sign at these points we know that it will be the same sign in each region defined by these points and so all we need to know is the value of the polynomial as a single point in each region.

Step 2

Here is a number line giving the value/sign of the polynomial at a *test point* in each of the region defined by these three points. To make it a little easier to read the number line let's define the polynomial under the radical to be,

$$R(t) = t^3 - t^2 - 8t = t\left(t^2 - t - 8\right) > 0$$

Now, here is the number line,



Hint

Now all we need to do is write down the values of x where the polynomial under the root will be positive (recall we need to avoid division by zero) and we'll have the domain.

Step 3

The domain will then be all the points where the polynomial under the root is positive, but not zero as we also need to avoid division by zero, and so the domain is,

$$\text{Domain} : \frac{1-\sqrt{33}}{2} < t < 0, \quad \frac{1+\sqrt{33}}{2} < t < \infty$$

---

29. Find the domain of $f(z) = \sqrt{z-1} + \sqrt{z+6}$.

Hint

The domain of this function will be the set of all values of z that will work in both terms of this function.

Step 1

The domain of this function will be the set of all z's that we can plug into both terms in this function and get a real number back as a value. This means that we first need to determine the domain of each of the two terms.

For the first term we need to require,

$$z - 1 \geq 0 \qquad \Rightarrow \qquad z \geq 1$$

For the second term we need to require,

$$z + 6 \geq 0 \qquad \Rightarrow \qquad z \geq -6$$

Hint

What values of z are in both of these?

Step 2

Now, we just need the set of z's that are in both conditions above. In this case notice that all the z that satisfy $z \geq 1$ will also satisfy $z \geq -6$. The reverse is not true however. Any z that is in the range $-6 \leq z < 1$ will satisfy $z \geq 6$ but will not satisfy $z \geq 1$.

So, in this case, the domain is in fact just the first condition above or,

$$\text{Domain : } z \geq 1$$

---

30. Find the domain of $h(y) = \sqrt{2y + 9} - \dfrac{1}{\sqrt{2 - y}}$.

Hint

The domain of this function will be the set of all values of y that will work in both terms of this function.

Step 1

The domain of this function will be the set of all y's that we can plug into both terms in this function and get a real number back as a value. This means that we first need to determine the domain of each of the two terms.

For the first term we need to require,

$$2y+9\geq 0 \qquad \Rightarrow \qquad y\geq -\frac{9}{2}$$

For the second term we need to require,

$$2-y>0 \qquad \Rightarrow \qquad y<2$$

Note that we need the second condition to be strictly positive to avoid division by zero as well.

Hint

What values of y are in both of these?

Step 2

Now, we just need the set of y's that are in both conditions above. In this case we need all the y's that will be greater than or equal to $-\frac{9}{2}$ AND less than 2. The domain is then,

$$\text{Domain : } -\frac{9}{2}\leq y<2$$

---

31. Find the domain of $A(x)=\dfrac{4}{x-9}-\sqrt{x^2-36}$.

Hint

The domain of this function will be the set of all values of x that will work in both terms of this function.

Step 1

The domain of this function will be the set of all x's that we can plug into both terms in this function and get a real number back as a value. This means that we first need to determine the domain of each of the two terms.

For the first term we need to require,

$$x-9\neq 0 \qquad \Rightarrow \qquad x\neq 9$$

For the second term we need to require,

$$x^2 - 36 \geq 0 \qquad \rightarrow \qquad x^2 \geq 36 \qquad \Rightarrow \qquad x \leq -6 \;\;\&\;\; x \geq 6$$

Hint

What values of x are in both of these?

Step 2

Now, we just need the set of x's that are in both conditions above. In this case the second condition gives us most of the domain as it is the most restrictive. The first term is okay as long as we avoid $x = 9$ and because this point will in fact satisfy the second condition we'll need to make sure and exclude it. The domain is then,

$$\text{Domain : } x \leq -6 \;\;\&\;\; x \geq 6, \;\; x \neq 9$$

---

32. Find the domain of $Q(y) = \sqrt{y^2 + 1} - \sqrt[3]{1 - y}$.

Solution

The domain of this function will be the set of y's that will work in both terms of this function. So, we need the domain of each of the terms.

For the first term let's note that,

$$y^2 + 1 \geq 1$$

and so will always be positive. The domain of the first term is then all real numbers.

For the second term we need to notice that we're dealing with the cube root in this case and we can plug all real numbers into a cube root and so the domain of this term is again all real numbers.

So, the domain of both terms is all real numbers and so the domain of the function as a whole must also be all real numbers or,

$$\text{Domain : } -\infty < y < \infty$$

---

33. Compute $(f \circ g)(x)$ and $(g \circ f)(x)$ for $f(x) = 4x - 1$, $g(x) = \sqrt{6 + 7x}$.

Solution

Not much to do here other than to compute each of these.

$$(f \circ g)(x) = f[g(x)] = f\left[\sqrt{6+7x}\right] = 4\sqrt{6+7x} - 1$$

$$(g \circ f)(x) = g[f(x)] = g[4x-1] = \sqrt{6+7(4x-1)} = \sqrt{28x-1}$$

---

34. Compute $(f \circ g)(x)$ and $(g \circ f)(x)$ for $f(x) = 5x + 2$, $g(x) = x^2 - 14x$.

Solution

Not much to do here other than to compute each of these.

$$(f \circ g)(x) = f[g(x)] = f\left[x^2 - 14x\right] = 5\left(x^2 - 14x\right) + 2 = 5x^2 - 70x + 2$$

$$(g \circ f)(x) = g[f(x)] = g[5x+2] = (5x+2)^2 - 14(5x+2) = 25x^2 - 50x - 24$$

---

35. Compute $(f \circ g)(x)$ and $(g \circ f)(x)$ for $f(x) = x^2 - 2x + 1$, $g(x) = 8 - 3x^2$.

Solution

Not much to do here other than to compute each of these.

$$\begin{aligned}(f \circ g)(x) = f[g(x)] = f\left[8 - 3x^2\right] &= \left(8 - 3x^2\right)^2 - 2\left(8 - 3x^2\right) + 1 \\ &= 9x^4 - 42x^2 + 49\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) = g[f(x)] &= g\left[x^2 - 2x + 1\right] \\ &= 8 - 3\left(x^2 - 2x + 1\right)^2 = -3x^4 + 12x^3 - 18x^2 + 12x + 5\end{aligned}$$

---

36. Compute $(f \circ g)(x)$ and $(g \circ f)(x)$ for $f(x) = x^2 + 3$, $g(x) = \sqrt{5 + x^2}$.

Solution

Not much to do here other than to compute each of these.

$$(f \circ g)(x) = f[g(x)] = f\left[\sqrt{5 + x^2}\right] = \left(\sqrt{5 + x^2}\right)^2 + 3 = 8 + x^2$$

$$(g \circ f)(x) = g[f(x)] = g\left[x^2 + 3\right] = \sqrt{5 + (x^2 + 3)^2} = \sqrt{x^4 + 6x^2 + 14}$$

1.2 Inverse Functions

1. Find the inverse for $f(x) = 6x + 15$. Verify your inverse by computing one or both of the composition as discussed in this section.

Hint

Remember the process described in this section. Replace the $f(x)$, interchange the x's and y's, solve for y and the finally replace the y with $f^{-1}(x)$.

Step 1

$$y = 6x + 15$$

Step 2

$$x = 6y + 15$$

Step 3

$$x - 15 = 6y$$

$$y = \frac{1}{6}(x - 15) \qquad \rightarrow \qquad \boxed{f^{-1}(x) = \frac{1}{6}(x - 15)}$$

Finally, compute either $\left(f \circ f^{-1}\right)(x)$ or $\left(f^{-1} \circ f\right)(x)$ to verify our work.

Step 4

Either composition can be done so let's do $\left(f \circ f^{-1}\right)(x)$ in this case.

$$\begin{aligned}\left(f \circ f^{-1}\right)(x) &= f\left[f^{-1}(x)\right] \\ &= 6\left[\frac{1}{6}(x-15)\right]+15 \\ &= x-15+15 \\ &= x\end{aligned}$$

So, we got x out of the composition and so we know we've done our work correctly.

---

2. Find the inverse for $h(x) = 3 - 29x$. Verify your inverse by computing one or both of the composition as discussed in this section.

Hint

Remember the process described in this section. Replace the $h(x)$, interchange the x's and y's, solve for y and the finally replace the y with $h^{-1}(x)$.

Step 1

$$y = 3 - 29x$$

Step 2

$$x = 3 - 29y$$

Step 3

$$x - 3 = -29y$$

$$y = -\frac{1}{29}(x-3) \qquad \rightarrow \qquad \boxed{h^{-1}(x) = \frac{1}{29}(3-x)}$$

Notice that we multiplied the minus sign into the parenthesis. We did this in order to avoid potentially losing the minus sign if it had stayed out in front. This does not need to be done in order to get the inverse.

Finally, compute either $\left(h \circ h^{-1}\right)(x)$ or $\left(h^{-1} \circ h\right)(x)$ to verify our work.

Step 4

Either composition can be done so let's do $\left(h \circ h^{-1}\right)(x)$ in this case.

$$\begin{aligned}\left(h \circ h^{-1}\right)(x) &= h\left[h^{-1}(x)\right] \\ &= 3 - 29\left[\frac{1}{29}(3-x)\right] \\ &= 3 - (3-x) \\ &= x\end{aligned}$$

So, we got x out of the composition and so we know we've done our work correctly.

---

3. Find the inverse for $R(x) = x^3 + 6$. Verify your inverse by computing one or both of the composition as discussed in this section.

Hint

Remember the process described in this section. Replace the $R(x)$, interchange the x's and y's, solve for y and the finally replace the y with $R^{-1}(x)$.

Step 1

$$y = x^3 + 6$$

Step 2

$$x = y^3 + 6$$

Step 3

$$x - 6 = y^3$$

$$y = \sqrt[3]{x-6} \qquad \to \qquad \boxed{R^{-1}(x) = \sqrt[3]{x-6}}$$

Finally, compute either $\left(R \circ R^{-1}\right)(x)$ or $\left(R^{-1} \circ R\right)(x)$ to verify our work.

Step 4

Either composition can be done so let's do $\left(R^{-1} \circ R\right)(x)$ in this case.

$$\begin{aligned} \left(R^{-1} \circ R\right)(x) &= R^{-1}\left[R(x)\right] \\ &= \sqrt[3]{(x^3+6)-6} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned}$$

So, we got x out of the composition and so we know we've done our work correctly.

---

4. Find the inverse for $g(x) = 4(x-3)^5 + 21$. Verify your inverse by computing one or both of the composition as discussed in this section.

Hint

Remember the process described in this section. Replace the $g(x)$, interchange the x's and y's, solve for y and the finally replace the y with $g^{-1}(x)$.

Step 1

$$y = 4(x-3)^5 + 21$$

Step 2

$$x = 4(y-3)^5 + 21$$

Step 3

$$
\begin{aligned}
x - 21 &= 4(y-3)^5 \\
\frac{1}{4}(x-21) &= (y-3)^5 \\
\sqrt[5]{\frac{1}{4}(x-21)} &= y-3
\end{aligned}
$$

$$
y = 3 + \sqrt[5]{\frac{1}{4}(x-21)} \qquad \to \qquad \boxed{g^{-1}(x) = 3 + \sqrt[5]{\frac{1}{4}(x-21)}}
$$

Finally, compute either $\left(g \circ g^{-1}\right)(x)$ or $\left(g^{-1} \circ g\right)(x)$ to verify our work.

Step 4

Either composition can be done so let's do $\left(g \circ g^{-1}\right)(x)$ in this case.

$$
\begin{aligned}
\left(g \circ g^{-1}\right)(x) &= g\left[g^{-1}(x)\right] \\
&= 4\left(\left[3 + \sqrt[5]{\frac{1}{4}(x-21)}\right] - 3\right)^5 + 21 \\
&= 4\left(\sqrt[5]{\frac{1}{4}(x-21)}\right)^5 + 21 \\
&= 4\left(\frac{1}{4}(x-21)\right) + 21 \\
&= (x-21) + 21 \\
&= x
\end{aligned}
$$

So, we got x out of the composition and so we know we've done our work correctly.

5. Find the inverse for $W(x) = \sqrt[5]{9-11x}$. Verify your inverse by computing one or both of the composition as discussed in this section.

Hint

Remember the process described in this section. Replace the $W(x)$, interchange the x's and y's, solve for y and the finally replace the y with $W^{-1}(x)$.

Step 1

$$y = \sqrt[5]{9-11x}$$

Step 2

$$x = \sqrt[5]{9-11y}$$

Step 3

$$\begin{aligned} x &= \sqrt[5]{9-11y} \\ x^5 &= 9-11y \\ x^5 - 9 &= -11y \\ y &= -\frac{1}{11}\left(x^5-9\right) \qquad \to \qquad \boxed{W^{-1}(x) = \frac{1}{11}\left(9-x^5\right)} \end{aligned}$$

Notice that we multiplied the minus sign into the parenthesis. We did this in order to avoid potentially losing the minus sign if it had stayed out in front. This does not need to be done in order to get the inverse.

Finally, compute either $\left(W \circ W^{-1}\right)(x)$ or $\left(W^{-1} \circ W\right)(x)$ to verify our work.

Step 4

Either composition can be done so let's do $\left(W^{-1} \circ W\right)(x)$ in this case.

$$\begin{aligned}\left(W^{-1}\circ W\right)(x) &= W^{-1}\left[W(x)\right]\\ &= \frac{1}{11}\left(9-\left[\sqrt[5]{9-11x}\right]^{5}\right)\\ &= \frac{1}{11}\left(9-\left[9-11x\right]\right)\\ &= \frac{1}{11}\left(11x\right)\\ &= x\end{aligned}$$

So, we got x out of the composition and so we know we've done our work correctly.

6. Find the inverse for $f(x) = \sqrt[7]{5x+8}$. Verify your inverse by computing one or both of the composition as discussed in this section.

Hint

Remember the process described in this section. Replace the $f(x)$, interchange the x's and y's, solve for y and the finally replace the y with $f^{-1}(x)$.

Step 1

$$y = \sqrt[7]{5x+8}$$

Step 2

$$x = \sqrt[7]{5y+8}$$

Step 3

$$\begin{aligned} x &= \sqrt[7]{5y+8} \\ x^7 &= 5y+8 \\ x^7 - 8 &= 5y \\ y &= \frac{1}{5}\left(x^7-8\right) \qquad \rightarrow \qquad \boxed{f^{-1}(x) = \frac{1}{5}\left(x^7-8\right)} \end{aligned}$$

Finally, compute either $\left(f \circ f^{-1}\right)(x)$ or $\left(f^{-1} \circ f\right)(x)$ to verify our work.

Step 4

Either composition can be done so let's do $\left(f \circ f^{-1}\right)(x)$ in this case.

$$\begin{aligned} \left(f \circ f^{-1}\right)(x) &= f\left[f^{-1}(x)\right] \\ &= \sqrt[7]{5\left[\frac{1}{5}\left(x^7-8\right)\right]+8} \\ &= \sqrt[7]{\left[x^7-8\right]+8} \\ &= \sqrt[7]{x^7} \\ &= x \end{aligned}$$

So, we got x out of the composition and so we know we've done our work correctly.

---

7. Find the inverse for $h(x) = \dfrac{1+9x}{4-x}$. Verify your inverse by computing one or both of the composition as discussed in this section.

Hint

Remember the process described in this section. Replace the $h(x)$, interchange the x's and y's, solve for y and the finally replace the y with $h^{-1}(x)$.

Step 1

$$y = \frac{1+9x}{4-x}$$

Step 2

$$x = \frac{1+9y}{4-y}$$

Step 3

$$\begin{aligned} x &= \frac{1+9y}{4-y} \\ x\left(4-y\right) &= 1+9y \\ 4x - xy &= 1+9y \\ 4x-1 &= 9y+xy \\ 4x-1 &= \left(9+x\right)y \\ y &= \frac{4x-1}{9+x} \qquad \to \qquad \boxed{h^{-1}\left(x\right) = \frac{4x-1}{9+x}} \end{aligned}$$

Note that the Algebra in these kinds of problems can often be fairly messy, but don't let that make you decide that you can't do these problems. Messy Algebra will be a fairly common occurrence in a Calculus class so you'll need to get used to it!

Finally, compute either $\left(h \circ h^{-1}\right)(x)$ or $\left(h^{-1} \circ h\right)(x)$ to verify our work.

Step 4

Either composition can be done so let's do $\left(h^{-1} \circ h\right)(x)$ in this case. As with the previous step, the Algebra here is going to be messy and in fact will probably be messier.

$$\begin{aligned}\left(h^{-1}\circ h\right)(x) &= h^{-1}\left[h(x)\right]\\ &= \frac{4\left[\frac{1+9x}{4-x}\right]-1}{9+\left[\frac{1+9x}{4-x}\right]}\;\frac{4-x}{4-x}\\ &= \frac{4(1+9x)-(4-x)}{9(4-x)+1+9x}\\ &= \frac{4+36x-4+x}{36-9x+1+9x}\\ &= \frac{37x}{37}\\ &= x\end{aligned}$$

In order to do the simplification we multiplied the numerator and denominator of the initial fraction by $4-x$ in order to clear out some of the denominators. This in turn allowed a fair amount of simplification.

So, we got x out of the composition and so we know we've done our work correctly.

---

8. Find the inverse for $f(x) = \dfrac{6-10x}{8x+7}$. Verify your inverse by computing one or both of the composition as discussed in this section.

Hint

Remember the process described in this section. Replace the $f(x)$, interchange the x's and y's, solve for y and the finally replace the y with $f^{-1}(x)$.

Step 1

$$y = \frac{6-10x}{8x+7}$$

Step 2

$$x = \frac{6-10y}{8y+7}$$

Step 3

$$\begin{aligned}
x &= \frac{6-10y}{8y+7} \\
x\left(8y+7\right) &= 6-10y \\
8xy+7x &= 6-10y \\
8xy+10y &= 6-7x \\
\left(8x+10\right)y &= 6-7x \\
y &= \frac{6-7x}{8x+10} \qquad \rightarrow \qquad \boxed{f^{-1}\left(x\right) = \frac{6-7x}{8x+10}}
\end{aligned}$$

Note that the Algebra in these kinds of problems can often be fairly messy, but don't let that make you decide that you can't do these problems. Messy Algebra will be a fairly common occurrence in a Calculus class so you'll need to get used to it!

Finally, compute either $\left(f \circ f^{-1}\right)(x)$ or $\left(f^{-1} \circ f\right)(x)$ to verify our work.

Step 4

Either composition can be done so let's do $\left(f \circ f^{-1}\right)(x)$ in this case. As with the previous step, the Algebra here is going to be messy and in fact will probably be messier.

$$\begin{aligned}
\left(f \circ f^{-1}\right)(x) &= f\left[f^{-1}\left(x\right)\right] \\
&= \frac{6-10\left[\frac{6-7x}{8x+10}\right]}{8\left[\frac{6-7x}{8x+10}\right]+7}\;\frac{8x+10}{8x+10} \\
&= \frac{6\left(8x+10\right)-10\left(6-7x\right)}{8\left(6-7x\right)+7\left(8x+10\right)} \\
&= \frac{48x+60-60+70x}{48-56x+56x+70} \\
&= \frac{118x}{118} \\
&= x
\end{aligned}$$

So, we got x out of the composition and so we know we've done our work correctly.

1.3 Trig Functions

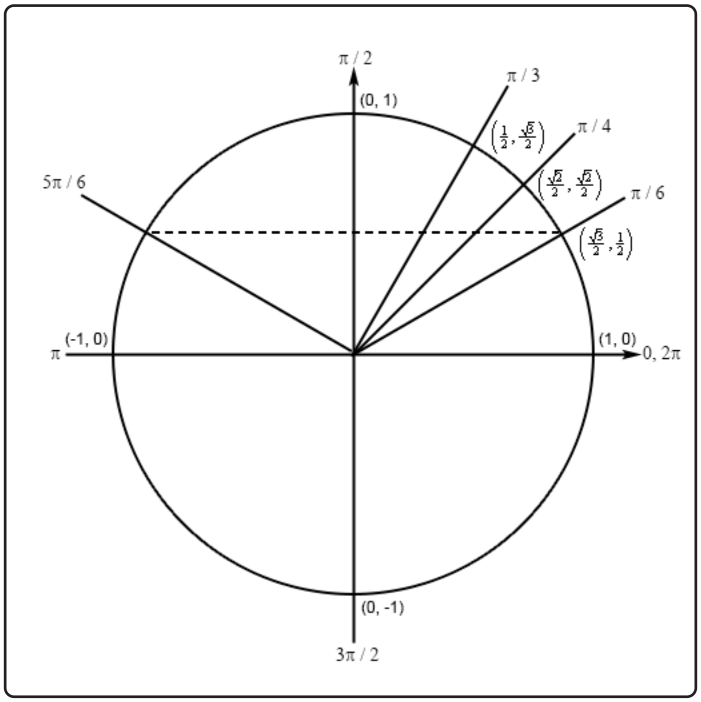
1. Determine the exact value of cos $\left(\frac{5\pi}{6}\right)$ without using a calculator.

Hint

Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First, we can notice that $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ and so the terminal line for $\frac{5\pi}{6}$ will form an angle of $\frac{\pi}{6}$ with the negative x-axis in the second quadrant and we'll have the following unit circle for this problem.



Hint

Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{5\pi}{6}$ to the coordinates of the line representing $\frac{\pi}{6}$ and use those to answer the question.

Step 2

The coordinates of the line representing $\frac{5\pi}{6}$ will be the same as the coordinates of the line representing $\frac{\pi}{6}$ except that the x coordinate will now be negative. So, our new coordinates will then be $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and so the answer is,

$$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

---

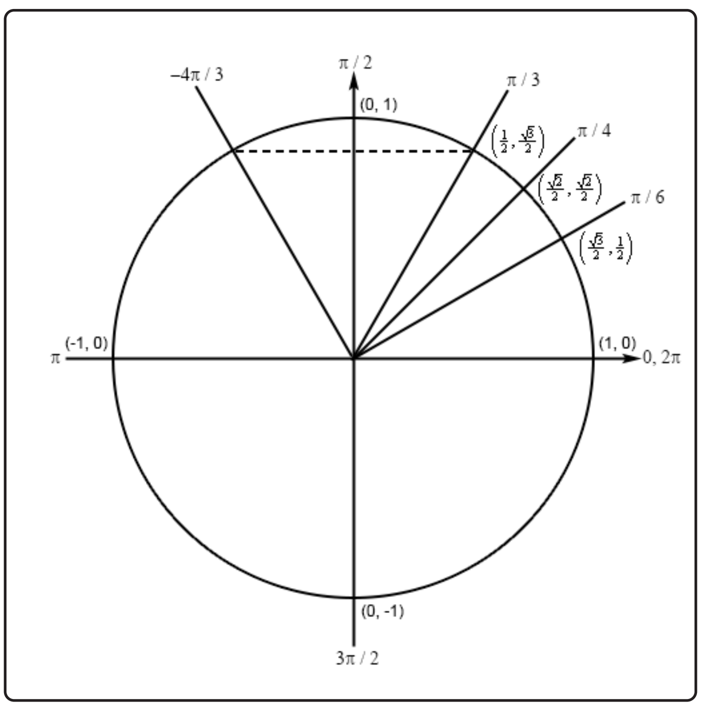
2. Determine the exact value of $\sin\left(-\frac{4\pi}{3}\right)$ without using a calculator.

Hint

Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $-\pi - \frac{\pi}{3} = -\frac{4\pi}{3}$ and so (remembering that negative angles are rotated clockwise) we can see that the terminal line for $-\frac{4\pi}{3}$ will form an angle of $\frac{\pi}{3}$ with the negative x-axis in the second quadrant and we'll have the following unit circle for this problem.



Hint

Given the obvious symmetry in the unit circle relate the coordinates of the line representing $-\frac{4\pi}{3}$ to the coordinates of the line representing $\frac{\pi}{3}$ and use those to answer the question.

Step 2

The coordinates of the line representing $-\frac{4\pi}{3}$ will be the same as the coordinates of the line representing $\frac{\pi}{3}$ except that the x coordinate will now be negative. So, our new coordinates will then be $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and so the answer is,

$$\sin\left(-\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

---

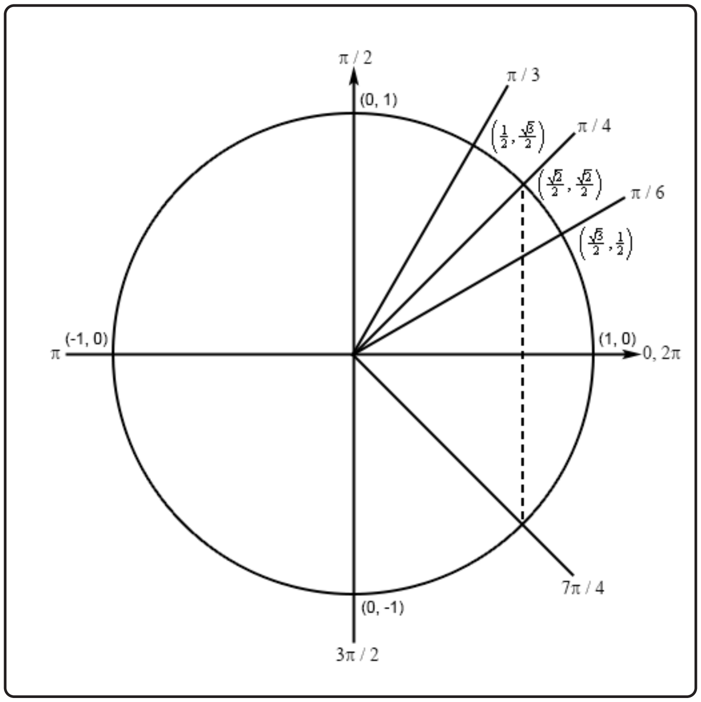
3. Determine the exact value of $\sin\left(\frac{7\pi}{4}\right)$ without using a calculator.

Hint

Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ and so the terminal line for $\frac{7\pi}{4}$ will form an angle of $\frac{\pi}{4}$ with the positive x-axis in the fourth quadrant and we'll have the following unit circle for this problem.



Hint

Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{7\pi}{4}$ to the coordinates of the line representing $\frac{\pi}{4}$ and use those to answer the question.

Step 2

The coordinates of the line representing $\frac{7\pi}{4}$ will be the same as the coordinates of the line representing $\frac{\pi}{4}$ except that the y coordinate will now be negative. So, our new coordinates will then be $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and so the answer is,

$$\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

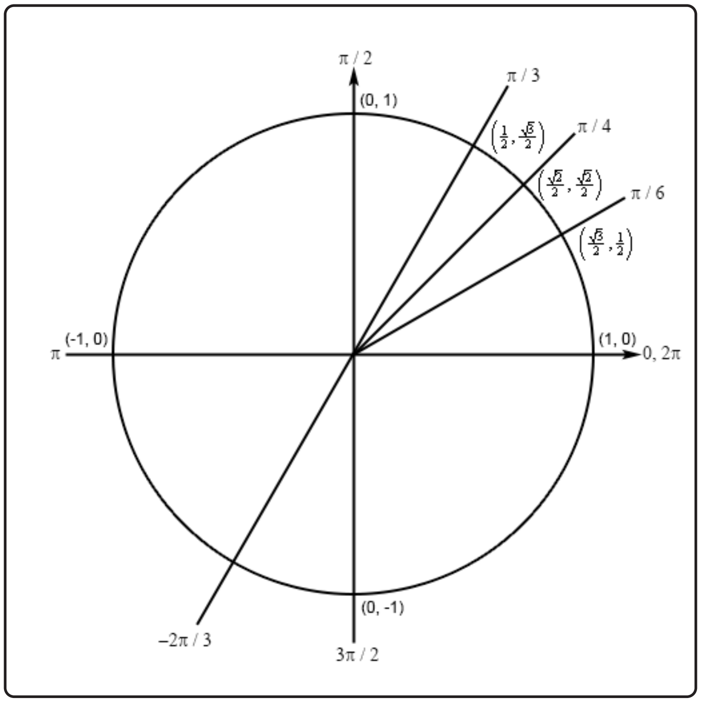
4. Determine the exact value of $\cos\left(-\frac{2\pi}{3}\right)$ without using a calculator.

Hint

Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $-\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$ so (recalling that negative angles rotate clockwise and positive angles rotation counter clockwise) the terminal line for $-\frac{2\pi}{3}$ will form an angle of $\frac{\pi}{3}$ with the negative x-axis in the third quadrant and we'll have the following unit circle for this problem.



Hint

Given the obvious symmetry in the unit circle relate the coordinates of the line representing $-\frac{2\pi}{3}$ to the coordinates of the line representing $\frac{\pi}{3}$ and use those to answer the question.

Step 2

The line representing $-\frac{2\pi}{3}$ is a mirror image of the line representing $\frac{\pi}{3}$ and so the coordinates for $-\frac{2\pi}{3}$ will be the same as the coordinates for $\frac{\pi}{3}$ except that both coordinates will now be negative. So, our new coordinates will then be $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and so the answer is,

$$\cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2}$$

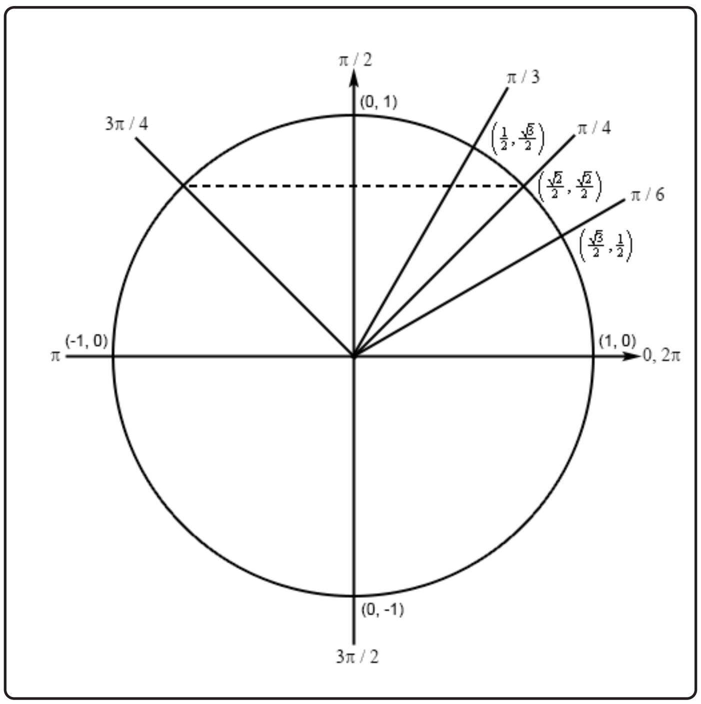
5. Determine the exact value of $\tan\left(\frac{3\pi}{4}\right)$ without using a calculator.

Hint

Even though a unit circle only tells us information about sine and cosine it is still useful for tangents so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$ and so (remembering that negative angles are rotated clockwise) we can see that the terminal line for $\frac{3\pi}{4}$ will form an angle of $\frac{\pi}{4}$ with the negative x-axis in the second quadrant and we'll have the following unit circle for this problem.



Hint

Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{3\pi}{4}$ to the coordinates of the line representing $\frac{\pi}{4}$ and and then recall how tangent is defined in terms of sine and cosine to answer the question.

Step 2

The coordinates of the line representing $\frac{3\pi}{4}$ will be the same as the coordinates of the line representing $\frac{\pi}{4}$ except that the x coordinate will now be negative. So, our new coordinates will then be $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and so the answer is,

$$\tan\left(\frac{3\pi}{4}\right) = \frac{\sin\left(\frac{3\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right)} = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = -1$$

---

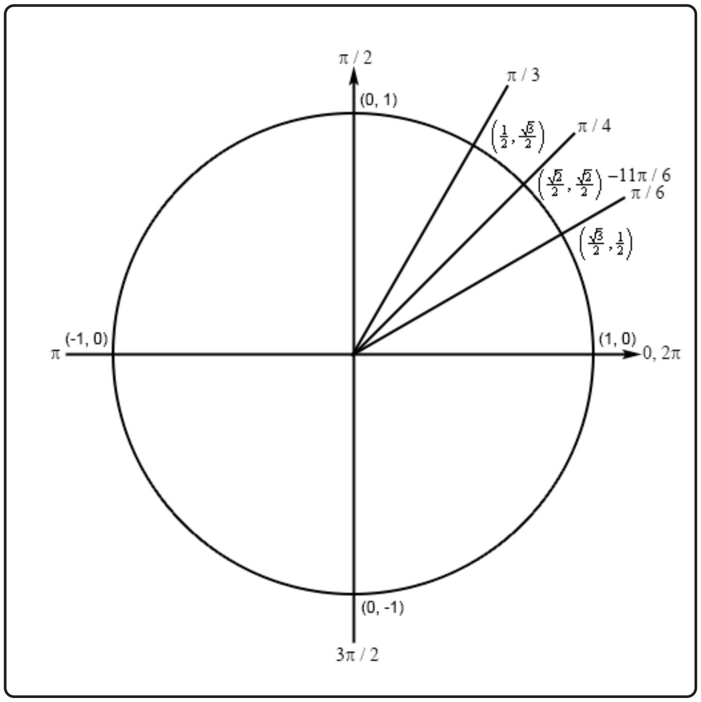
6. Determine the exact value of $\sec\left(-\frac{11\pi}{6}\right)$ without using a calculator.

Hint

Even though a unit circle only tells us information about sine and cosine it is still useful for secant so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First, we can notice that $\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ and so (remembering that negative angles are rotated clockwise) we can see that the terminal line for $-\frac{11\pi}{6}$ will form an angle of $\frac{\pi}{6}$ with the positive x-axis in the first quadrant. In other words, $-\frac{11\pi}{6}$ and $\frac{\pi}{6}$ represent the same angle. So, we'll have the following unit circle for this problem.



Hint

Given the obvious symmetry here use the definition of secant in terms of cosine to write down the solution.

Step 2

Because the two angles $-\frac{11\pi}{6}$ and $\frac{\pi}{6}$ have the same coordinates the answer is,

$$\sec\left(-\frac{11\pi}{6}\right) = \frac{1}{\cos\left(-\frac{11\pi}{6}\right)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$$

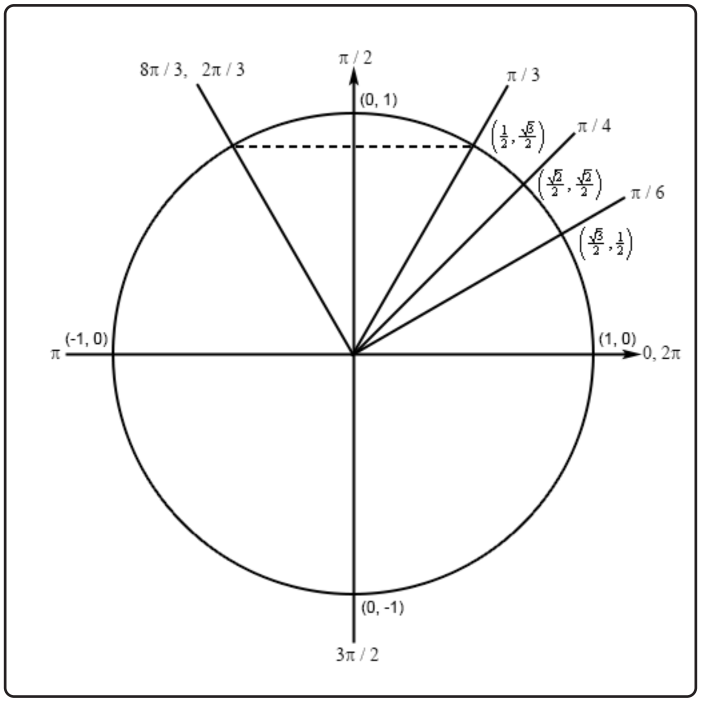
7. Determine the exact value of $\cos\left(\frac{8\pi}{3}\right)$ without using a calculator.

Hint

Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First, we can notice that $2\pi + \frac{2\pi}{3} = \frac{8\pi}{3}$ and because 2π is one complete revolution the angles $\frac{8\pi}{3}$ and $\frac{2\pi}{3}$ are the same angle. Also, note that $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and so the terminal line for $\frac{8\pi}{3}$ will form an angle of $\frac{\pi}{3}$ with the negative x-axis in the second quadrant and we'll have the following unit circle for this problem.



Hint

Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{8\pi}{3}$ to the coordinates of the line representing $\frac{2\pi}{3}$ and use those to answer the question.

Step 2

The coordinates of the line representing $\frac{8\pi}{3}$ will be the same as the coordinates of the line representing $\frac{\pi}{3}$ except that the x coordinate will now be negative. So, our new coordinates will then be $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and so the answer is,

$$\cos\left(\frac{8\pi}{3}\right) = -\frac{1}{2}$$

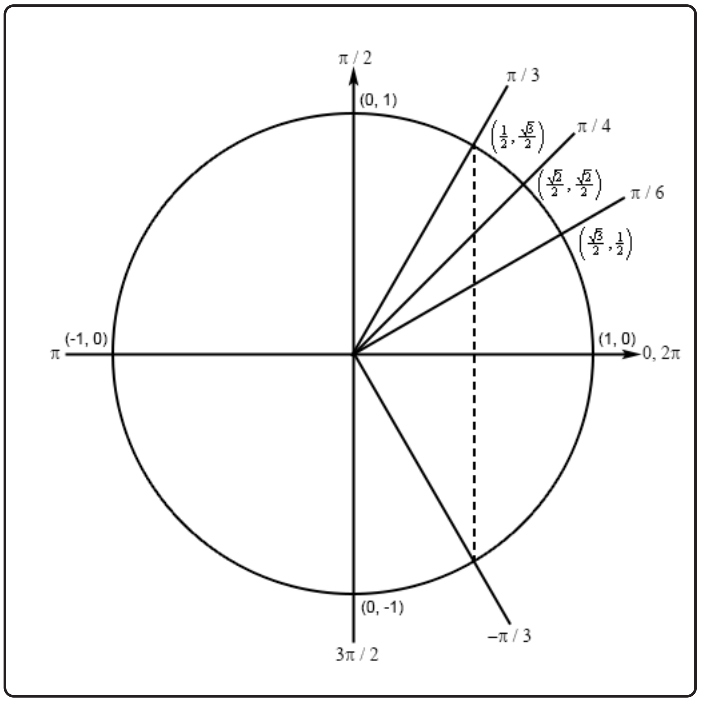
8. Determine the exact value of tan $\left(-\frac{\pi}{3}\right)$ without using a calculator.

Hint

Even though a unit circle only tells us information about sine and cosine it is still useful for tangents so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

To do this problem all we need to notice is that $-\frac{\pi}{3}$ will form an angle of $\frac{\pi}{3}$ with the positive x-axis in the fourth quadrant and we'll have the following unit circle for this problem.



Hint

Given the obvious symmetry in the unit circle relate the coordinates of the line representing $-\frac{\pi}{3}$ to the coordinates of the line representing $\frac{\pi}{3}$ and use the definition of tangent in terms of sine and cosine to answer the question.

Step 2

The coordinates of the line representing $-\frac{\pi}{3}$ will be the same as the coordinates of the line representing $\frac{\pi}{3}$ except that the y coordinate will now be negative. So, our new coordinates will then be $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and so the answer is,

$$\tan\left(-\frac{\pi}{3}\right) = \frac{\sin\left(-\frac{\pi}{3}\right)}{\cos\left(-\frac{\pi}{3}\right)} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

---

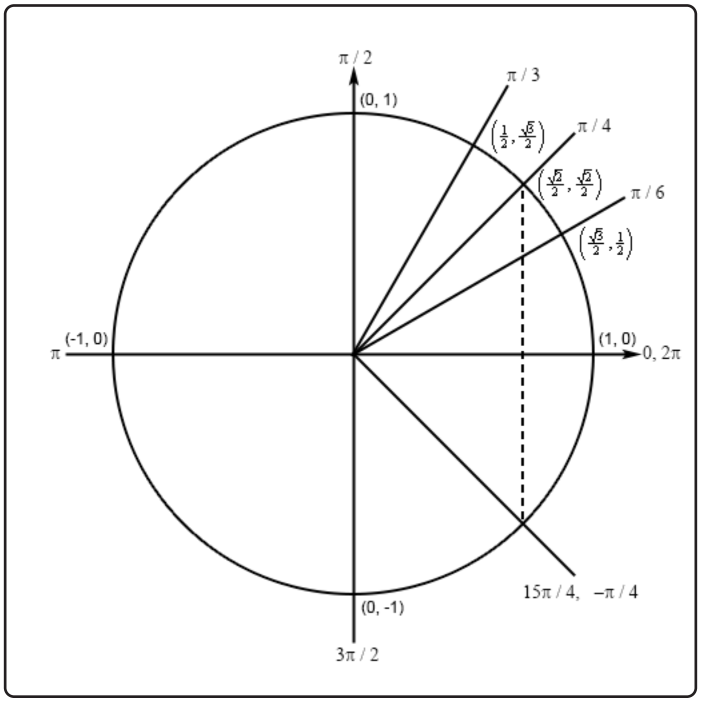
9. Determine the exact value of $\tan\left(\dfrac{15\pi}{4}\right)$ without using a calculator.

Hint

Even though a unit circle only tells us information about sine and cosine it is still useful for tangents so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $4\pi - \frac{\pi}{4} = \frac{15\pi}{4}$ and also note that 4π is two complete revolutions so the terminal line for $\frac{15\pi}{4}$ and $-\frac{\pi}{4}$ represent the same angle. Also note that $-\frac{\pi}{4}$ will form an angle of $\frac{\pi}{4}$ with the positive x-axis in the fourth quadrant and we'll have the following unit circle for this problem.



Hint

Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{15\pi}{4}$ to the coordinates of the line representing $\frac{\pi}{4}$ and the definition of tangent in terms of sine and cosine to answer the question.

Step 2

The coordinates of the line representing $\frac{15\pi}{4}$ will be the same as the coordinates of the line representing $\frac{\pi}{4}$ except that the y coordinate will now be negative. So, our new coordinates will then be $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and so the answer is,

$$\tan\left(\frac{15\pi}{4}\right) = \frac{\sin\left(\frac{15\pi}{4}\right)}{\cos\left(\frac{15\pi}{4}\right)} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1$$

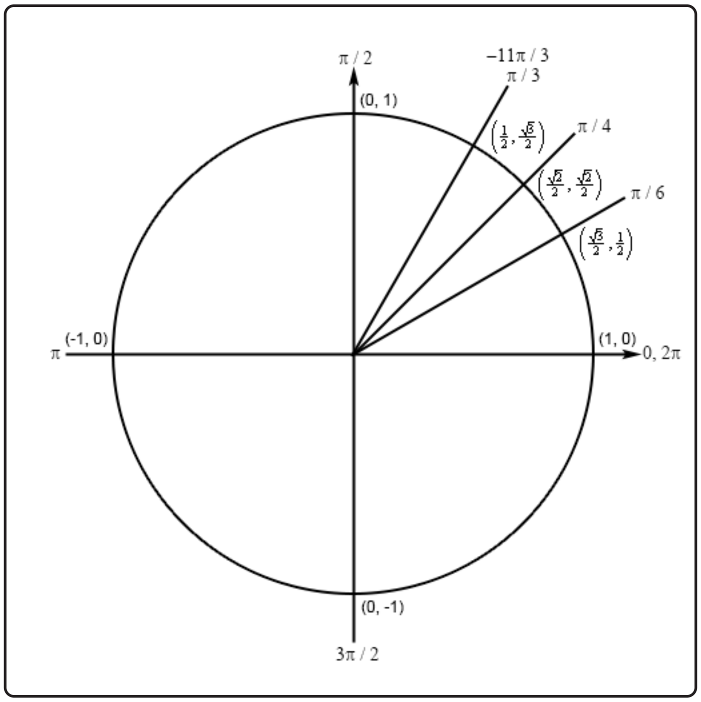
10. Determine the exact value of $\sin\left(-\frac{11\pi}{3}\right)$ without using a calculator.

Hint

Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $\frac{\pi}{3} - 4\pi = -\frac{11\pi}{3}$ and note that 4π is two complete revolutions (also, remembering that negative angles are rotated clockwise) we can see that the terminal line for $-\frac{11\pi}{3}$ and $\frac{\pi}{3}$ are the same angle and so we'll have the following unit circle for this problem.



Hint

Given the very obvious symmetry here write down the answer to the question.

Step 2

Because $-\frac{11\pi}{3}$ and $\frac{\pi}{3}$ are the same angle the answer is,

$$\sin\left(-\frac{11\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

---

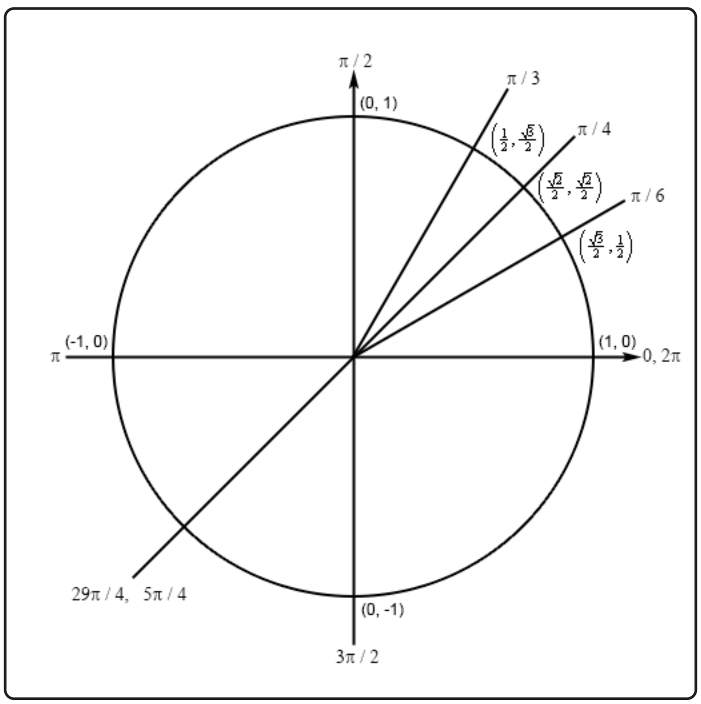
11. Determine the exact value of $\sec\left(\frac{29\pi}{4}\right)$ without using a calculator.

Hint

Even though a unit circle only tells us information about sine and cosine it is still useful for secant so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $\frac{5\pi}{4} + 6\pi = \frac{29\pi}{4}$ and recalling that 6π is three complete revolutions we can see that $\frac{29\pi}{4}$ and $\frac{5\pi}{4}$ represent the same angle. Next, note that $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$ and so the line representing $\frac{5\pi}{4}$ will form an angle of $\frac{\pi}{4}$ with the negative x-axis in the third quadrant and we'll have the following unit circle for this problem.



Hint

Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{29\pi}{4}$ to the coordinates of the line representing $\frac{\pi}{4}$ and the recall how secant is defined in terms of cosine to answer the question.

Step 2

The line representing $\frac{95\pi}{4}$ is a mirror image of the line representing $\frac{\pi}{4}$ and so the coordinates for $\frac{29\pi}{4}$ will be the same as the coordinates for $\frac{\pi}{4}$ except that both coordinates will now be negative. So, our new coordinates will then be $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and so the answer is,

$$\sec\left(\frac{29\pi}{4}\right) = \frac{1}{\cos\left(\frac{29\pi}{4}\right)} = \frac{1}{-\sqrt{2}/2} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

1.4 Solving Trig Equations

1. Without using a calculator find all the solutions to $4\sin(3t) = 2$.

Hint

Isolate the sine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the sine (with a coefficient of one) on one side of the equation gives,

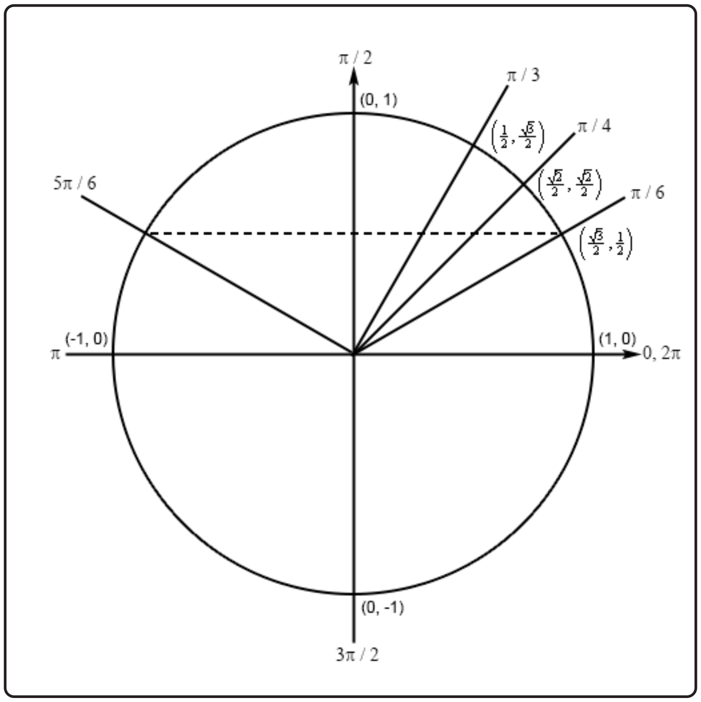
$$\sin(3t) = \frac{1}{2}$$

Hint

Use your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which sine will have this value.

Step 2

Because we're dealing with sine in this problem and we know that the y-axis represents sine on a unit circle we're looking for angles that will have a y coordinate of $\frac{1}{2}$. This means we'll have an angle in the first quadrant and an angle in the second quadrant (that we can use the angle in the first quadrant to find). Here is a unit circle for this situation.



Clearly the angle in the first quadrant is $\frac{\pi}{6}$ and by some basic symmetry we can see that the terminal line for the second angle must form an angle of $\frac{\pi}{6}$ with the negative x-axis as shown above and so it will be : $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Hint

Using the two angles above write down all the angles for which sine will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+ 2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$3t = \frac{\pi}{6} + 2\pi n \qquad \text{OR} \qquad 3t = \frac{5\pi}{6} + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is divide both sides by 3.

$$\boxed{t = \frac{\pi}{18} + \frac{2\pi n}{3} \qquad \text{OR} \qquad t = \frac{5\pi}{18} + \frac{2\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \ldots}$$

---

2. Without using a calculator find the solution(s) to $4\sin(3t) = 2$ that are in $\left[0, \frac{4\pi}{3}\right]$.

Hint

First, find all the solutions to the equation without regard to the given interval.

Step 1

Because we found all the solutions to this equation in Problem 1 of this section we'll just list the result here. For full details on how these solutions were obtained please see the solution to Problem 1.

All solutions to the equation are,

$$t = \frac{\pi}{18} + \frac{2\pi n}{3} \qquad \text{OR} \qquad t = \frac{5\pi}{18} + \frac{2\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in this interval.

Step 2

Note that because at least some of the solutions will have a denominator of 18 it will probably be convenient to also have the interval written in terms of fractions with denominators of 18. Doing this will make it much easier to identify solutions that fall inside the interval so,

$$\left[0, \frac{4\pi}{3}\right] = \left[0, \frac{24\pi}{18}\right]$$

With the interval written in this form, if our potential solutions have a denominator of 18, all we need to do is compare numerators. As long as the numerators are positive and less than 24π we'll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of n it will be much easier to have both fractions in the solutions have denominators of 18. So, the solutions, written in this form, are.

$$t = \frac{\pi}{18} + \frac{12\pi n}{18} \qquad \text{OR} \qquad t = \frac{5\pi}{18} + \frac{12\pi n}{18} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now let's find all the solutions. First notice that, in this case, if we plug in negative values of n we will get negative solutions and these will not be in the interval and so there is no reason to even try these. So, let's start at $n = 0$ and see what we get.

$$\begin{array}{llll} n=0: & t = \frac{\pi}{18} & \text{OR} & t = \frac{5\pi}{18} \\ n=1: & t = \frac{13\pi}{18} & \text{OR} & t = \frac{17\pi}{18} \\ n=2: & t = \cancel{\frac{25\pi}{18}} > \frac{24\pi}{18} & \text{OR} & t = \cancel{\frac{29\pi}{18}} > \frac{24\pi}{18} \end{array}$$

Note that we didn't really need to plug in $n = 2$ above to see that they would not work. With each increase in n we were really just adding another $\frac{12\pi}{18}$ onto the previous results and by a quick inspection we could see that adding 12π to the numerator of either solution from the $n = 1$ step would result in a numerator that is larger than 24π and so would result in a solution that is outside of the interval. This is not something that must be noticed in order to work the problem, but noticing this would definitely help reduce the amount of actual work.

So, it looks like we have the four solutions to this equation in the given interval.

$$\boxed{t = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}}$$

---

3. Without using a calculator find all the solutions to $2\cos\left(\frac{x}{3}\right) + \sqrt{2} = 0$.

Hint

Isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the cosine (with a coefficient of one) on one side of the equation gives,

$$\cos\left(\frac{x}{3}\right) = -\frac{\sqrt{2}}{2}$$

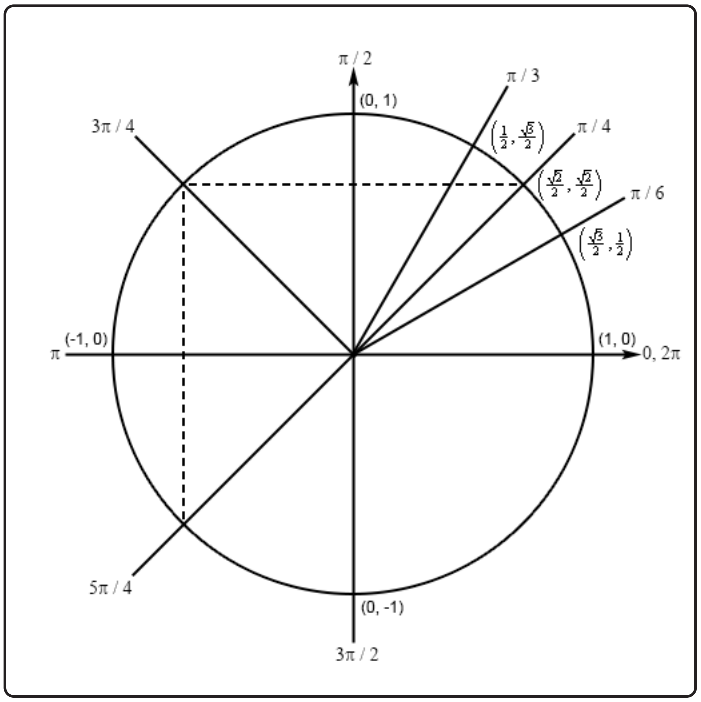
Hint

Use your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosine will have this value.

Step 2

Because we're dealing with cosine in this problem and we know that the x-axis represents cosine on a unit circle we're looking for angles that will have a x coordinate of $-\frac{\sqrt{2}}{2}$. This means that we'll have angles in the second and third quadrant.

Because of the negative value we can't just find the corresponding angle in the first quadrant and use that to find the second angle. However, we can still use the angles in the first quadrant to find the two angles that we need. Here is a unit circle for this situation.



If we didn't have the negative value then the angle would be $\frac{\pi}{4}$. Now, based on the symmetry in the unit circle, the terminal line for both of the angles will form an angle of $\frac{\pi}{4}$ with the negative x-axis. The angle in the second quadrant will then be $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$ and the angle in the third quadrant will be $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$.

Hint

Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these. This then means that we must have,

$$\frac{x}{3} = \frac{3\pi}{4} + 2\pi n \qquad \text{OR} \qquad \frac{x}{3} = \frac{5\pi}{4} + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 3.

$$\boxed{x = \frac{9\pi}{4} + 6\pi n \qquad \text{OR} \qquad x = \frac{15\pi}{4} + 6\pi n \qquad n = 0, \pm 1, \pm 2, \ldots}$$

---

4. Without using a calculator find the solution(s) to $2\cos\left(\frac{x}{3}\right) + \sqrt{2} = 0$ that are in $[-7\pi, 7\pi]$.

Hint

First, find all the solutions to the equation without regard to the given interval.

Step 1

Because we found all the solutions to this equation in Problem 3 of this section we'll just list the result here. For full details on how these solutions were obtained please see the solution to Problem 3.

All solutions to the equation are,

$$x = \frac{9\pi}{4} + 6\pi n \qquad \text{OR} \qquad x = \frac{15\pi}{4} + 6\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in this interval.

Step 2

Note that because at least some of the solutions will have a denominator of 4 it will probably be convenient to also have the interval written in terms of fractions with denominators of 4. Doing this will make it much easier to identify solutions that fall inside the interval so,

$$[-7\pi, 7\pi] = \left[-\frac{28\pi}{4}, \frac{28\pi}{4}\right]$$

With the interval written in this form, if our potential solutions have a denominator of 4, all we need to do is compare numerators. As long as the numerators are between -28π and 28π we'll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of n it will be much easier to have both fractions in the solutions have denominators of 4. So, the solutions, written in this form, are.

$$x = \frac{9\pi}{4} + \frac{24\pi n}{4} \qquad \text{OR} \qquad x = \frac{15\pi}{4} + \frac{24\pi n}{4} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now let's find all the solutions.

$$\begin{array}{llll} n = -2: & x = \cancel{-\tfrac{39\pi}{4}} < -\tfrac{28\pi}{4} & \text{OR} & x = \cancel{-\tfrac{33\pi}{4}} < -\tfrac{28\pi}{4} \\ n = -1: & x = -\tfrac{15\pi}{4} & \text{OR} & x = -\tfrac{9\pi}{4} \\ n = 0: & x = \tfrac{9\pi}{4} & \text{OR} & x = \tfrac{15\pi}{4} \\ n = 1: & x = \cancel{\tfrac{33\pi}{4}} > \tfrac{28\pi}{4} & \text{OR} & x = \cancel{\tfrac{39\pi}{4}} > \tfrac{28\pi}{4} \end{array}$$

Note that we didn't really need to plug in $n = 1$ or $n = -2$ above to see that they would not work. With each increase in n we were really just adding (for positive n) or subtracting (for negative n) another $\frac{24\pi}{4}$ from the previous results. By a quick inspection we could see that adding 24π to the numerator of either solution from the $n = 1$ step would result in a numerator that is larger than 28π and so would result in a solution that is outside of the interval. Likewise, for the $n = -2$ case, subtracting 24π from each of the numerators will result in numerators that will be less than -28π and so will not be in the interval. This is not something that must be noticed in order to work the problem, but noticing this would definitely help reduce the amount of actual work.

So, it looks like we have the four solutions to this equation in the given interval.

$$\boxed{x = -\frac{15\pi}{4}, -\frac{9\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}}$$

5. Without using a calculator find the solution(s) to $4\cos(6z) = \sqrt{12}$ that are in $\left[0, \frac{\pi}{2}\right]$.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the cosine (with a coefficient of one) on one side of the equation gives,

$$\cos(6z) = \frac{\sqrt{12}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

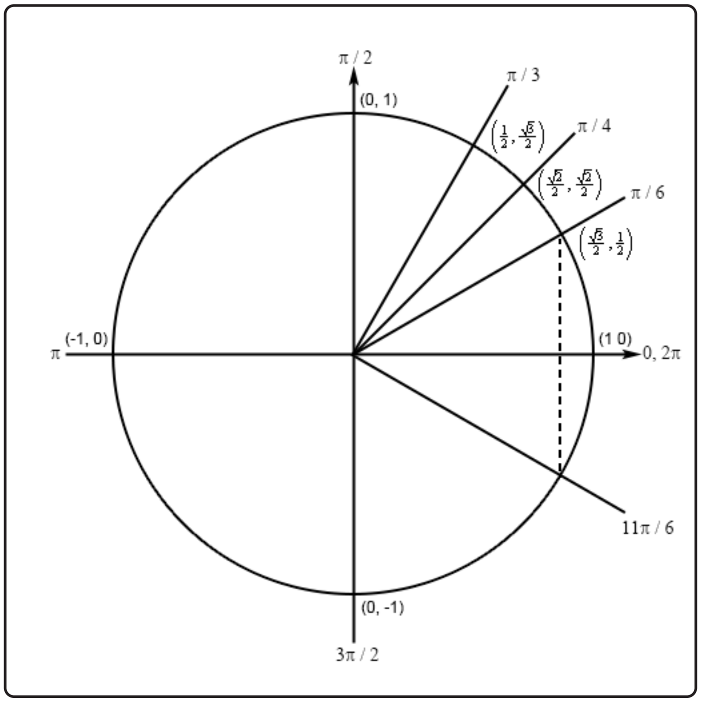
Notice that we needed to do a little simplification of the root to get the value into a more recognizable form. This kind of simplification is usually a good thing to do.

Hint

Use your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosine will have this value.

Step 2

Because we're dealing with cosine in this problem and we know that the x-axis represents cosine on a unit circle we're looking for angles that will have a x coordinate of $\frac{\sqrt{3}}{2}$. This means we'll have an angle in the first quadrant and an angle in the fourth quadrant (that we can use the angle in the first quadrant to find). Here is a unit circle for this situation.



Clearly the angle in the first quadrant is $\frac{\pi}{6}$ and by some basic symmetry we can see that the terminal line for the second angle must form an angle of $\frac{\pi}{6}$ with the positive x-axis as shown above and so it will be : $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Note that you don't really need a positive angle for the second one. If you wanted to you could just have easily used $-\frac{\pi}{6}$ for the second angle. There is nothing wrong with this and you'll get the same solutions in the end. The reason we chose to go with the positive angle is simply to avoid inadvertently losing the minus sign on the second solution at some point in the future. That kind of mistake is easy to make on occasion and by using positive angles here we will not need to worry about making it.

Hint

Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these. This then means that we must have,

$$6z = \frac{\pi}{6} + 2\pi n \qquad \text{OR} \qquad 6z = \frac{11\pi}{6} + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is divide both sides by 6.

$$z = \frac{\pi}{36} + \frac{\pi n}{3} \qquad \text{OR} \qquad z = \frac{11\pi}{36} + \frac{\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Note that because at least some of the solutions will have a denominator of 36 it will probably be convenient to also have the interval written in terms of fractions with denominators of 36. Doing this will make it much easier to identify solutions that fall inside the interval so,

$$\left[0, \frac{\pi}{2}\right] = \left[0, \frac{18\pi}{36}\right]$$

With the interval written in this form, if our potential solutions have a denominator of 36, all we need to do is compare numerators. As long as the numerators are positive and less than 18π we'll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of n it will be much easier to have both fractions in the solutions have denominators of 36. So the solutions, written in this form, are.

$$z = \frac{\pi}{36} + \frac{12\pi n}{36} \qquad \text{OR} \qquad z = \frac{11\pi}{36} + \frac{12\pi n}{36} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now let's find all the solutions. First notice that, in this case, if we plug in negative values of n we will get negative solutions and these will not be in the interval and so there is no reason to even try these. So, let's start at $n = 0$ and see what we get.

$$\begin{aligned} n = 0 : \quad & z = \tfrac{\pi}{36} \qquad \text{OR} \qquad z = \tfrac{11\pi}{36} \\ n = 1 : \quad & z = \tfrac{13\pi}{36} \qquad \text{OR} \qquad z = \cancel{\tfrac{23\pi}{36}} > \tfrac{18\pi}{36} \end{aligned}$$

There are a couple of things we should note before proceeding. First, it is important to understand both solutions from a given value of n will not necessarily be in the given interval. It is completely possible, as this problem shows, that we will only get one or the other solution from a given value of n to fall in the given interval.

Next notice that with each increase in n we were really just adding another $\frac{12\pi}{36}$ onto the previous results and by a quick inspection we could see that adding 12π to the numerator of the first solution from the $n = 1$ step would result in a numerator that is larger than 18π and so would result in a solution that is outside of the interval. Therefore, there was no reason to plug in $n = 2$ into the first set of solutions. Of course, we also didn't plug $n = 2$ into the second set because once we've gotten out of the interval adding anything else on will remain out of the interval.

So, it looks like we have the three solutions to this equation in the given interval.

$$\boxed{z = \frac{\pi}{36}, \frac{11\pi}{36}, \frac{13\pi}{36}}$$

---

6. Without using a calculator find the solution(s) to $2\sin\left(\frac{3y}{2}\right) + \sqrt{3} = 0$ that are in $\left[-\frac{7\pi}{3}, 0\right]$.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the sine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the sine (with a coefficient of one) on one side of the equation gives,

$$\sin\left(\frac{3y}{2}\right) = -\frac{\sqrt{3}}{2}$$

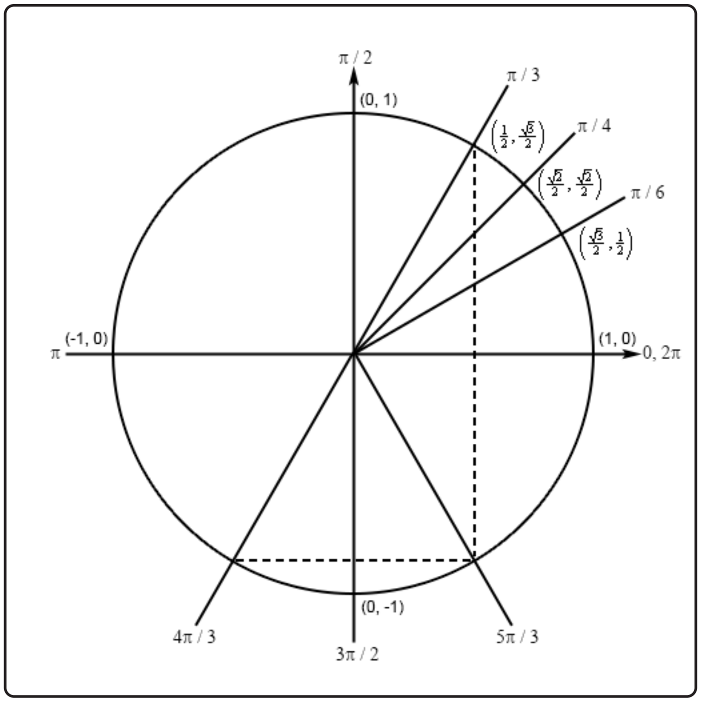
Hint

Use your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosine will have this value.

Step 2

Because we're dealing with sine in this problem and we know that the y-axis represents sine on a unit circle we're looking for angles that will have a y coordinate of $-\frac{\sqrt{3}}{2}$. This means that we'll have angles in the third and fourth quadrant.

Because of the negative value we can't just find the corresponding angle in the first quadrant and use that to find the second angle. However, we can still use the angles in the first quadrant to find the two angles that we need. Here is a unit circle for this situation.



If we didn't have the negative value then the angle would be $\frac{\pi}{3}$. Now, based on the symmetry in the unit circle, the terminal line for the first angle will form an angle of $\frac{\pi}{3}$ with the negative x-axis and the terminal line for the second angle will form an angle of $\frac{\pi}{3}$ with the positive x-axis. The angle in the third quadrant will then be $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$ and the angle in the fourth quadrant will be $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

Note that you don't really need a positive angle for the second one. If you wanted to you could just have easily used $-\frac{\pi}{3}$ for the second angle. There is nothing wrong with this and you'll get the same solutions in the end. The reason we chose to go with the positive angle is simply to avoid inadvertently losing the minus sign on the second solution at some point in the future. That kind of mistake is easy to make on occasion and by using positive angles here we will not need to worry about making it.

Hint

Using the two angles above write down all the angles for which sine will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$\frac{3y}{2} = \frac{4\pi}{3} + 2\pi n \qquad \text{OR} \qquad \frac{3y}{2} = \frac{5\pi}{3} + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by $\frac{2}{3}$.

$$y = \frac{8\pi}{9} + \frac{4\pi n}{3} \qquad \text{OR} \qquad y = \frac{10\pi}{9} + \frac{4\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Note that because at least some of the solutions will have a denominator of 9 it will probably be convenient to also have the interval written in terms of fractions with denominators of 9. Doing this will make it much easier to identify solutions that fall inside the interval so,

$$\left[-\frac{7\pi}{3}, 0\right] = \left[-\frac{21\pi}{9}, 0\right]$$

With the interval written in this form, if our potential solutions have a denominator of 9, all we need to do is compare numerators. As long as the numerators are negative and greater than -21π we'll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of n it will be much easier to have both fractions in the solutions have denominators of 9. So the solutions, written in this form, are.

$$y = \frac{8\pi}{9} + \frac{12\pi n}{9} \qquad \text{OR} \qquad y = \frac{10\pi}{9} + \frac{12\pi n}{9} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now let's find all the solutions. First notice that, in this case, if we plug in positive values of n or zero we will get positive solutions and these will not be in the interval and so there is no reason to even try these. So, let's start at $n = -1$ and see what we get.

$$n = -1: \quad y = -\tfrac{4\pi}{9} \quad \text{OR} \quad y = -\tfrac{2\pi}{9}$$
$$n = -2: \quad y = -\tfrac{16\pi}{9} \quad \text{OR} \quad y = -\tfrac{14\pi}{9}$$

Notice that with each increase (in the negative sense anyway) in n we were really just subtracting another $\frac{12\pi}{9}$ from the previous results and by a quick inspection we could see that subtracting 12π from either of the numerators from the $n = -2$ solutions the numerators will be less than -21π and so will be out of the interval. There is no reason to write down the $n = -3$ solutions since we know that they will not be in the given interval.

So, it looks like we have the four solutions to this equation in the given interval.

$$\boxed{y = -\frac{16\pi}{9},\ -\frac{14\pi}{9},\ -\frac{4\pi}{9},\ -\frac{2\pi}{9}}$$

7. Without using a calculator find the solution(s) to $8\tan(2x) - 5 = 3$ that are in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the tangent (with a coefficient of one) on one side of the equation.

Step 1

Isolating the tangent (with a coefficient of one) on one side of the equation gives,

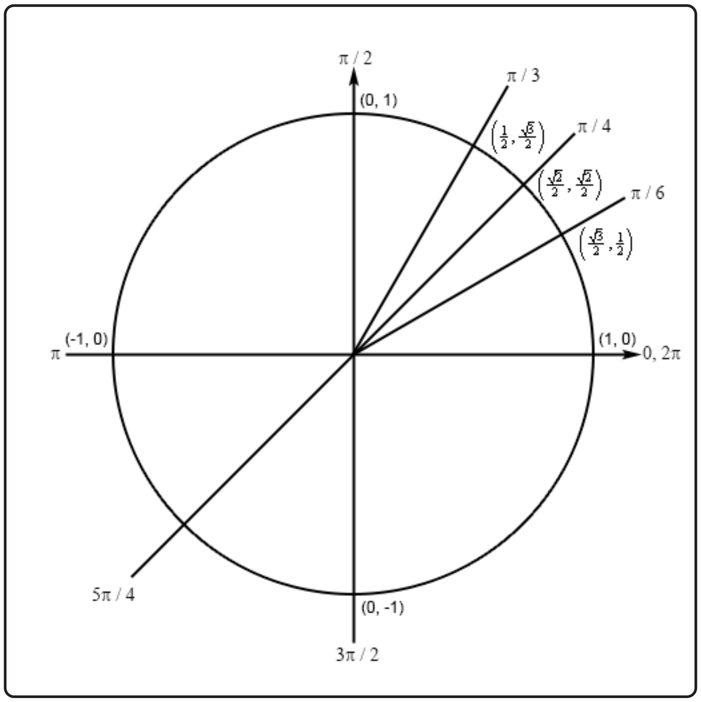
$$\tan(2x) = 1$$

Hint

Determine all the angles in the range $[0, 2\pi]$ for which tangent will have this value.

Step 2

If tangent has a value of 1 then we know that sine and cosine must be the same. This means that, in the first quadrant, the solution is $\frac{\pi}{4}$. We also know that sine and cosine will be the same in the third quadrant and we can use the basic symmetry on our unit circle to determine this value. Here is a unit circle for this situation.



By basic symmetry we can see that the line terminal line for the second angle must form an angle of $\frac{\pi}{4}$ with the negative x-axis as shown above and so it will be : $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$.

Hint

Hint :Using the two angles above write down all the angles for which tangent will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+\,2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$2x = \frac{\pi}{4} + 2\pi n \qquad \text{OR} \qquad 2x = \frac{5\pi}{4} + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is divide both sides by 2.

$$x = \frac{\pi}{8} + \pi n \qquad \text{OR} \qquad x = \frac{5\pi}{8} + \pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Note that because at least some of the solutions will have a denominator of 8 it will probably be convenient to also have the interval written in terms of fractions with denominators of 8. Doing this will make it much easier to identify solutions that fall inside the interval so,

$$\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] = \left[-\frac{4\pi}{8}, \frac{12\pi}{8}\right]$$

With the interval written in this form, if our potential solutions have a denominator of 8, all we need to do is compare numerators. As long as the numerators are between -4π and 12π we'll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of n it will be much easier to have both fractions in the solutions have denominators of 8. So, the solutions, written in this form, are.

$$x = \frac{\pi}{8} + \frac{8\pi n}{8} \qquad \text{OR} \qquad x = \frac{5\pi}{8} + \frac{8\pi n}{8} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now let's find all the solutions.

$$\begin{array}{lll} n = -1: & x = \cancel{-\frac{7\pi}{8}} < -\frac{4\pi}{8} \quad \text{OR} & x = -\frac{3\pi}{8} \\ n = 0: & x = \frac{\pi}{8} \quad \text{OR} & x = \frac{5\pi}{8} \\ n = 1: & x = \frac{9\pi}{8} \quad \text{OR} & x = \cancel{\frac{13\pi}{8}} > \frac{12\pi}{8} \end{array}$$

There are a couple of things we should note before proceeding. First, it is important to understand both solutions from a given value of n will not necessarily be in the given interval. It is completely possible, as this problem shows, that we will only get one or the other solution from a given value of n to fall in the given interval.

Next notice that with each increase in n we were really just adding/subtracting (depending upon the sign of n) another $\frac{8\pi}{8}$ from the previous results and by a quick inspection we could see that adding 8π to the numerator of the $n = 1$ solutions would result in numerators that are larger than 12π and so would result in solutions that are outside of the interval. Likewise, subtracting 8π from the $n = -1$ solutions would result in numerators that are smaller than -4π and so would result in solutions that are outside the interval. Therefore, there is no reason to even go past the values of n listed here.

So, it looks like we have the four solutions to this equation in the given interval.

$$\boxed{x = -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}}$$

---

8. Without using a calculator find the solution(s) to $16 = -9\sin(7x) - 4$ that are in $\left[-2\pi, \frac{9\pi}{4}\right]$.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the sine (with a coefficient of one) on one side of the equation.

Solution

Isolating the sine (with a coefficient of one) on one side of the equation gives,

$$\sin(7x) = -\frac{20}{9} < -1$$

Okay, at this point we can stop all work. We know that $-1 \leq \sin\theta \leq 1$ for any argument and so in this case there is **no solution**. This will happen on occasion and we shouldn't get to excited about it when it happens.

---

9. Without using a calculator find the solution(s) to $\sqrt{3}\tan\left(\frac{t}{4}\right) + 5 = 4$ that are in $[0, 4\pi]$.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the tangent (with a coefficient of one) on one side of the equation.

Step 1

Isolating the tangent (with a coefficient of one) on one side of the equation gives,

$$\tan\left(\frac{t}{4}\right) = -\frac{1}{\sqrt{3}}$$

Hint

Determine all the angles in the range $[0, 2\pi]$ for which tangent will have this value.

Step 2

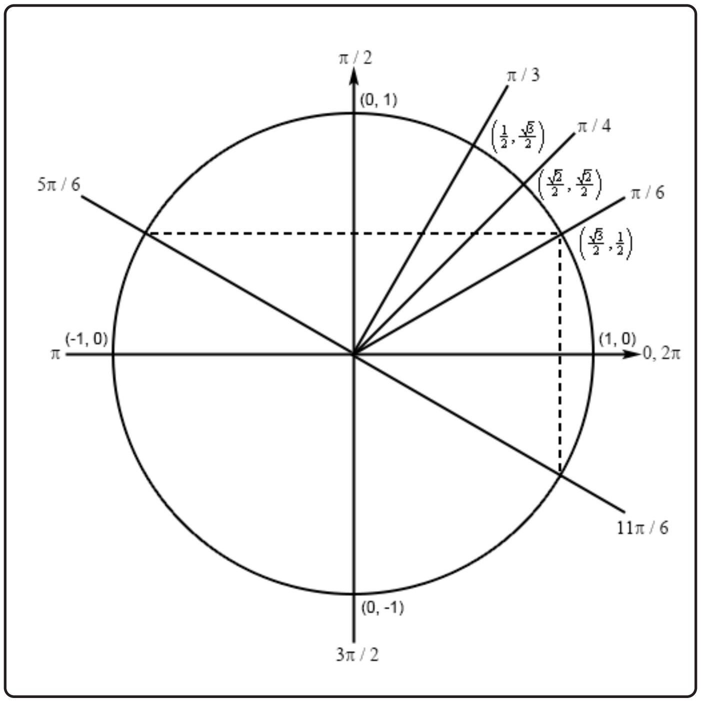
To get the first angle here let's recall the definition of tangent in terms of sine and cosine.

$$\tan\left(\frac{t}{4}\right) = \frac{\sin\left(\frac{t}{4}\right)}{\cos\left(\frac{t}{4}\right)} = -\frac{1}{\sqrt{3}}$$

Now, because of the section we're in, we know that the angle must be one of the "standard" angles and from a quick look at a unit circle (shown below) we know that for $\frac{\pi}{6}$ we will have,

$$\frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} = \frac{{}^{1}/_{2}}{{}^{\sqrt{3}}/_{2}} = \frac{1}{\sqrt{3}}$$

So, if we had a positive value on the tangent we'd have the first angle. We do have a negative value however, but this work will allow us to get the two angles we're after. Because the value is negative this simply means that the sine and cosine must have the same values that they have for $\frac{\pi}{6}$ except that one must be positive and the other must be negative. This means that the angles that we're after must be in the second and fourth quadrants. Here is a unit circle for this situation.



By basic symmetry we can see that the terminal line for the angle in the second quadrant must form an angle of $\frac{\pi}{6}$ with the negative x-axis and the terminal line in the fourth quadrant must form an angle of $\frac{\pi}{6}$ with the positive x-axis as shown above. The angle in the second quadrant will then be : $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ while the angle in the fourth quadrant will be $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Note that you don't really need a positive angle for the second one. If you wanted to you could just have easily used $-\frac{\pi}{6}$ for the second angle. There is nothing wrong with this and you'll get the same solutions in the end. The reason we chose to go with the positive angle is simply to avoid inadvertently losing the minus sign on the second solution at some point in the future. That kind of mistake is easy to make on occasion and by using positive angles here we will not need to worry about making it.

Hint

Using the two angles above write down all the angles for which tangent will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$\frac{t}{4} = \frac{5\pi}{6} + 2\pi n \qquad \text{OR} \qquad \frac{t}{4} = \frac{11\pi}{6} + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 4.

$$t = \frac{10\pi}{3} + 8\pi n \qquad \text{OR} \qquad t = \frac{22\pi}{3} + 8\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Note that because at least some of the solutions will have a denominator of 3 it will probably be convenient to also have the interval written in terms of fractions with denominators of 3. Doing this will make it much easier to identify solutions that fall inside the interval so,

$$[0, 4\pi] = \left[0, \frac{12\pi}{3}\right]$$

With the interval written in this form, if our potential solutions have a denominator of 3, all we need to do is compare numerators. As long as the numerators are positive and less than 12π we'll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of n it will be much easier to have both fractions in the solutions have denominators of 3. So the solutions, written in this form, are.

$$t = \frac{10\pi}{3} + \frac{24\pi n}{3} \qquad \text{OR} \qquad t = \frac{22\pi}{3} + \frac{24\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now let's find all the solutions. First notice that, in this case, if we plug in negative values of n we will get negative solutions and these will not be in the interval and so there is no reason to even try these. Next, notice that for any positive n we will be adding $\frac{24\pi}{3}$ onto a positive quantity and so are guaranteed to be greater than $\frac{12\pi}{3}$ and so will out of the given interval. This leaves $n = 0$ and for this one we can notice that the only solution that will fall in the given interval is then,

$$\boxed{\frac{10\pi}{3}}$$

Before leaving this problem let's note that on occasion we will only get a single solution out of all the possible solutions that will fall in the given interval. So, don't get excited about it if this should happen.

---

10. Without using a calculator find the solution(s) to $\sqrt{3}\csc(9z) - 7 = -5$ that are in $\left[-\frac{\pi}{3}, \frac{4\pi}{9}\right]$.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosecant (with a coefficient of one) on one side of the equation.

Step 1

Isolating the cosecant (with a coefficient of one) on one side of the equation gives,

$$\csc(9z) = \frac{2}{\sqrt{3}}$$

Hint

We need to determine all the angles in the range $[0, 2\pi]$ for which cosecant will have this value. The best way to do this is to rewrite this equation into one in terms of a different trig function that we can more easily deal with.

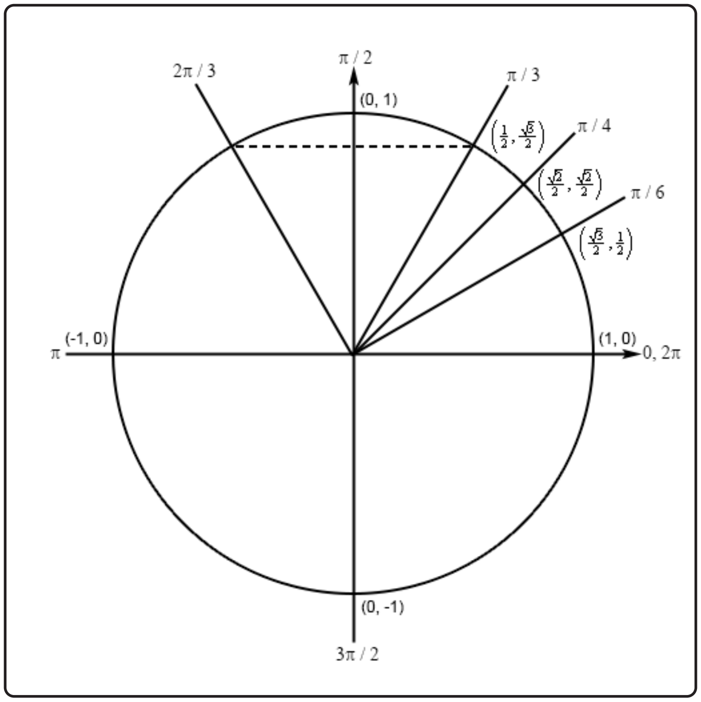
Step 2

The best way to do this is to recall the definition of cosecant in terms of sine and rewrite the equation in terms sine instead as that will be easier to deal with. Doing this gives,

$$\csc(9z) = \frac{1}{\sin(9z)} = \frac{2}{\sqrt{3}} \qquad \Rightarrow \qquad \sin(9z) = \frac{\sqrt{3}}{2}$$

The solution(s) to the equation with sine in it are the same as the solution(s) to the equation with cosecant in it and so let's work with that instead.

At this point we are now dealing with sine and we know that the y-axis represents sine on a unit circle. So, we're looking for angles that will have a y coordinate of $\frac{\sqrt{3}}{2}$. This means we'll have an angle in the first quadrant and an angle in the second quadrant (that we can use the angle in the first quadrant to find). Here is a unit circle for this situation.



Clearly the angle in the first quadrant is $\frac{\pi}{3}$ and by some basic symmetry we can see that the terminal line for the second angle must form an angle of $\frac{\pi}{3}$ with the negative x-axis as shown above and so it will be : $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

Hint

Using the two angles above write down all the angles for which sine/cosecant will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+\,2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$9z = \frac{\pi}{3} + 2\pi n \qquad \text{OR} \qquad 9z = \frac{2\pi}{3} + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is divide both sides by 9.

$$z = \frac{\pi}{27} + \frac{2\pi n}{9} \qquad \text{OR} \qquad z = \frac{2\pi}{27} + \frac{2\pi n}{9} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Note that because at least some of the solutions will have a denominator of 27 it will probably be convenient to also have the interval written in terms of fractions with denominators of 27. Doing this will make it much easier to identify solutions that fall inside the interval so,

$$\left[-\frac{\pi}{3}, \frac{4\pi}{9}\right] = \left[-\frac{9\pi}{27}, \frac{12\pi}{27}\right]$$

With the interval written in this form, if our potential solutions have a denominator of 27, all we need to do is compare numerators. As long as the numerators are between -9π and 12π we'll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of n it will be much easier to have both fractions in the solutions have denominators of 27. So the solutions, written in this form, are.

$$z = \frac{\pi}{27} + \frac{6\pi n}{27} \qquad \text{OR} \qquad z = \frac{2\pi}{27} + \frac{6\pi n}{27} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now let's find all the solutions.

$$\begin{array}{llll} n = -1: & z = -\frac{5\pi}{27} & \text{OR} & z = -\frac{4\pi}{27} \\ n = 0: & z = \frac{\pi}{27} & \text{OR} & z = \frac{2\pi}{27} \\ n = 1: & z = \frac{7\pi}{27} & \text{OR} & z = \frac{8\pi}{27} \end{array}$$

Notice that with each increase in n we were really just adding/subtracting (depending upon the sign of n) another $\frac{6\pi}{27}$ from the previous results and by a quick inspection we could see that adding 6π to the numerator of the $n = 1$ solutions would result in numerators that are larger than 12π and so would result in solutions that are outside of the interval. Likewise, subtracting 6π from the $n = -1$ solutions would result in numerators that are smaller than -9π and so would result in solutions that are outside the interval. Therefore, there is no reason to even go past the values of n listed here.

So, it looks like we have the six solutions to this equation in the given interval.

$$\boxed{x = -\frac{5\pi}{27}, \ -\frac{4\pi}{27}, \ \frac{\pi}{27}, \ \frac{2\pi}{27}, \ \frac{7\pi}{27}, \ \frac{8\pi}{27}}$$

---

11. Without using a calculator find the solution(s) to $1 - 14\cos\left(\frac{2x}{5}\right) = -6$ that are in $\left[5\pi, \frac{40\pi}{3}\right]$.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the cosine (with a coefficient of one) on one side of the equation gives,

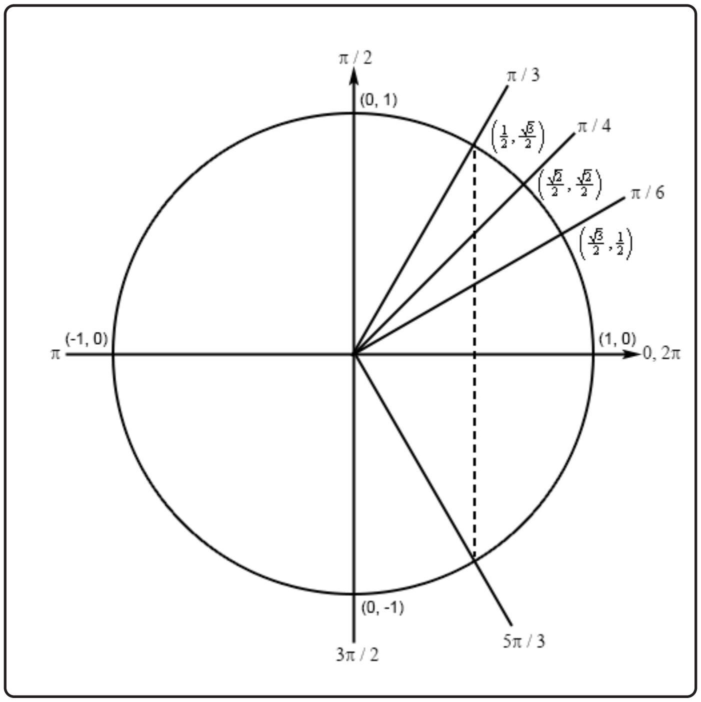
$$\cos\left(\frac{2x}{5}\right) = \frac{1}{2}$$

Hint

Use your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosine will have this value.

Step 2

Because we're dealing with cosine in this problem and we know that the x-axis represents cosine on a unit circle we're looking for angles that will have a x coordinate of $\frac{1}{2}$. This means we'll have an angle in the first quadrant and an angle in the fourth quadrant (that we can use the angle in the first quadrant to find). Here is a unit circle for this situation.



Clearly the angle in the first quadrant is $\frac{\pi}{3}$ and by some basic symmetry we can see that the terminal line for the second angle must form an angle of $\frac{\pi}{3}$ with the positive x-axis as shown above and so it will be : $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

Note that you don't really need a positive angle for the second one. If you wanted to you could just have easily used $-\frac{\pi}{3}$ for the second angle. There is nothing wrong with this and you'll get the same solutions in the end. The reason we chose to go with the positive angle is simply to avoid inadvertently losing the minus sign on the second solution at some point in the future. That kind of mistake is easy to make on occasion and by using positive angles here we will not need to worry about making it.

Hint

Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$\frac{2x}{5} = \frac{\pi}{3} + 2\pi n \qquad \text{OR} \qquad \frac{2x}{5} = \frac{5\pi}{3} + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by $\frac{5}{2}$.

$$x = \frac{5\pi}{6} + 5\pi n \qquad \text{OR} \qquad x = \frac{25\pi}{6} + 5\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Note that because at least some of the solutions will have a denominator of 6 it will probably be convenient to also have the interval written in terms of fractions with denominators of 6. Doing this will make it much easier to identify solutions that fall inside the interval so,

$$\left[5\pi, \frac{40\pi}{3}\right] = \left[\frac{30\pi}{6}, \frac{80\pi}{6}\right]$$

With the interval written in this form, if our potential solutions have a denominator of 6, all we need to do is compare numerators. As long as the numerators are between 30π and 80π we'll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of n it will be much easier to have both fractions in the solutions have denominators of 6. So the solutions, written in this form, are.

$$x = \frac{5\pi}{6} + \frac{30\pi n}{6} \qquad \text{OR} \qquad x = \frac{25\pi}{6} + \frac{30\pi n}{6} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now let's find all the solutions. First notice that, in this case, if we plug in negative values of n we will get negative solutions and these will not be in the interval and so there is no reason to even try these. We can also see from a quick inspection that $n = 0$ will result in solutions that are not in the interval and so let's start at $n = 1$ and see what we get.

$$\begin{array}{llll} n = 1: & x = \frac{35\pi}{6} & \text{OR} & x = \frac{55\pi}{6} \\ n = 2: & x = \frac{65\pi}{6} & \text{OR} & x = \xcancel{\frac{85\pi}{36}} > \frac{80\pi}{6} \end{array}$$

There are a couple of things we should note before proceeding. First, it is important to understand both solutions from a given value of n will not necessarily be in the given interval. It is completely possible, as this problem shows, that we will only get one or the other solution from a given value of n to fall in the given interval.

Next notice that with each increase in n we were really just adding another $\frac{30\pi}{6}$ onto the previous results and by a quick inspection we could see that adding 30π to the numerator of the first solution from the $n = 2$ step would result in a numerator that is larger than 80π and so would result in a solution that is outside of the interval. Therefore, there was no reason to plug in $n = 3$ into the first set of solutions. Of course, we also didn't plug $n = 3$ into the second set because once we've gotten out of the interval adding anything else on will remain out of the interval.

Finally, unlike most of the problems in this section $n = 0$ did not produce any solutions that were in the given interval. This will happen on occasion so don't get excited about it when it happens.

So, it looks like we have the three solutions to this equation in the given interval.

$$\boxed{z = \frac{35\pi}{6}, \frac{55\pi}{6}, \frac{65\pi}{6}}$$

---

12. Without using a calculator find the solution(s) to $15 = 17 + 4\cos\left(\frac{y}{7}\right)$ that are in $[10\pi, 15\pi]$.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the cosine (with a coefficient of one) on one side of the equation gives,

$$\cos\left(\frac{y}{7}\right) = -\frac{1}{2}$$

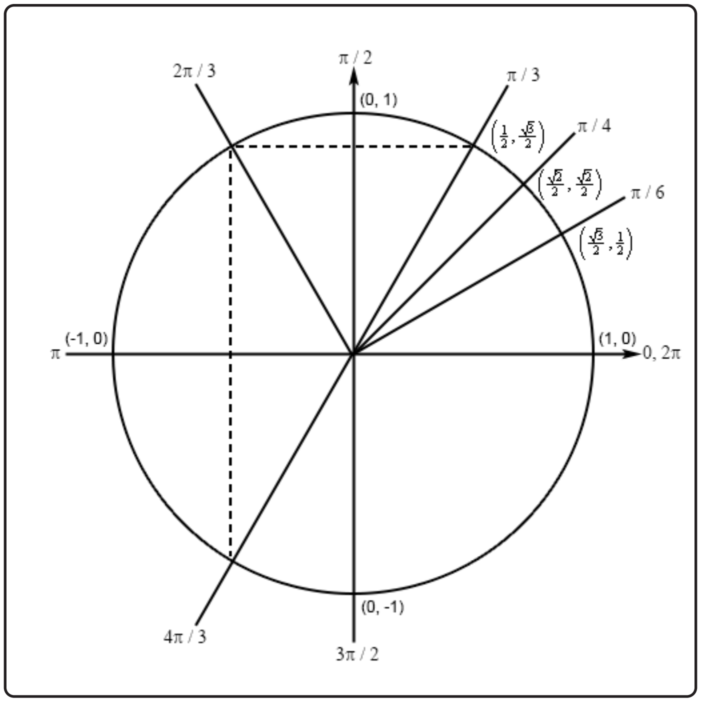
Hint

Use your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosine will have this value.

Step 2

Because we're dealing with cosine in this problem and we know that the x-axis represents cosine on a unit circle we're looking for angles that will have a x coordinate of $-\frac{1}{2}$. This means that we'll have angles in the second and third quadrant.

Because of the negative value we can't just find the corresponding angle in the first quadrant and use that to find the second angle. However, we can still use the angles in the first quadrant to find the two angles that we need. Here is a unit circle for this situation.



If we didn't have the negative value then the angle would be $\frac{\pi}{3}$. Now, based on the symmetry in the unit circle, the terminal line for both of the angles will form an angle of $\frac{\pi}{3}$ with the negative x-axis. The angle in the second quadrant will then be $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and the angle in the third quadrant will be $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

Hint

Hint :Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$\frac{y}{7} = \frac{2\pi}{3} + 2\pi n \qquad \text{OR} \qquad \frac{y}{7} = \frac{4\pi}{3} + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 7.

$$y = \frac{14\pi}{3} + 14\pi n \qquad \text{OR} \qquad y = \frac{28\pi}{3} + 14\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Note that because at least some of the solutions will have a denominator of 3 it will probably be convenient to also have the interval written in terms of fractions with denominators of 3. Doing this will make it much easier to identify solutions that fall inside the interval so,

$$\left[\frac{30\pi}{3}, \frac{45\pi}{3}\right]$$

With the interval written in this form, if our potential solutions have a denominator of 3, all we need to do is compare numerators. As long as the numerators are between 30π and 45π we'll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of n it will be much easier to have both fractions in the solutions have denominators of 3. So, the solutions, written in this form, are.

$$y = \frac{14\pi}{3} + \frac{42\pi n}{3} \qquad \text{OR} \qquad y = \frac{28\pi}{3} + \frac{42\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now let's find all the solutions. First notice that, in this case, if we plug in negative values of n we will get negative solutions and these will not be in the interval and so there is no reason to even try these. We can also see from a quick inspection that $n = 0$ will result in solutions that are not in the interval and so let's start at $n = 1$ and see what we get.

$$n = 1: \quad x = \cancel{\frac{56\pi}{3}} > \frac{45\pi}{3} \qquad \text{OR} \qquad x = \cancel{\frac{70\pi}{3}} > \frac{45\pi}{3}$$

So, by plugging in $n = 1$ we get solutions that are already outside of the interval and increasing n will simply mean adding another $\frac{42\pi}{3}$ onto these and so will remain outside of the given interval. We also noticed earlier than all other value of n will result in solutions outside of the given interval.

What all this means is that while there are solutions to the equation none fall inside the given interval and so the official answer would then be **no solutions in the given interval**.

1.5 Solving Trig Equations with Calculators, Part I

1. Find all the solutions to $7\cos(4x) + 11 = 10$. Use at least 4 decimal places in your work.

Hint

Isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the cosine (with a coefficient of one) on one side of the equation gives,

$$\cos(4x) = -\frac{1}{7}$$

Hint

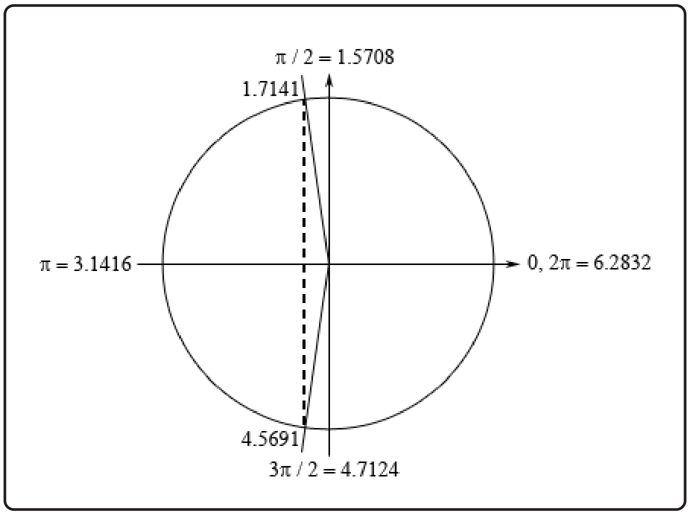
Using a calculator and your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosine will have this value.

Step 2

First, using our calculator we can see that,

$$4x = \cos^{-1}\left(-\frac{1}{7}\right) = 1.7141$$

Now we're dealing with cosine in this problem and we know that the x-axis represents cosine on a unit circle and so we're looking for angles that will have a x coordinate of $-\frac{1}{7}$. This means that we'll have angles in the second (this is the one our calculator gave us) and third quadrant. Here is a unit circle for this situation.



From the symmetry of the unit circle we can see that we can either use -1.7141 or $2\pi - 1.7141 = 4.5691$ for the second angle. Each will give the same set of solutions. However, because it is easy to lose track of minus signs we will use the positive angle for our second solution.

Hint

Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$4x = 1.7141 + 2\pi n \qquad \text{OR} \qquad 4x = 4.5691 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is divide both sides by 4.

$$\boxed{x = 0.4285 + \frac{\pi n}{2} \qquad \text{OR} \qquad x = 1.1423 + \frac{\pi n}{2} \qquad n = 0, \pm 1, \pm 2, \ldots}$$

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

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2. Find the solution(s) to $6 + 5\cos\left(\frac{x}{3}\right) = 10$ that are in $[0, 38]$. Use at least 4 decimal places in your work.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the cosine (with a coefficient of one) on one side of the equation gives,

$$\cos\left(\frac{x}{3}\right) = \frac{4}{5}$$

Hint

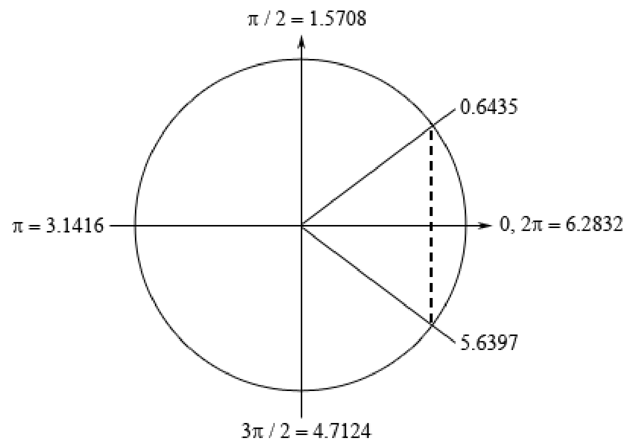
Using a calculator and your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosine will have this value.

Step 2

First, using our calculator we can see that,

$$\frac{x}{3} = \cos^{-1}\left(\frac{4}{5}\right) = 0.6435$$

Now we're dealing with cosine in this problem and we know that the x-axis represents cosine on a unit circle and so we're looking for angles that will have a x coordinate of $\frac{4}{5}$. This means that we'll have angles in the first (this is the one our calculator gave us) and fourth quadrant. Here is a unit circle for this situation.



From the symmetry of the unit circle we can see that we can either use -0.6435 or $2\pi - 0.6435 = 5.6397$ for the second angle. Each will give the same set of solutions. However, because it is easy to lose track of minus signs we will use the positive angle for our second solution.

Hint

Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+\,2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$\frac{x}{3} = 0.6435 + 2\pi n \qquad \text{OR} \qquad \frac{x}{3} = 5.6397 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 3 and we'll convert everything to decimals to help with the final step.

$$\begin{aligned} x &= 1.9305 + 6\pi n & \text{OR} \qquad x &= 16.9191 + 6\pi n & n &= 0, \pm 1, \pm 2, \ldots \\ &= 1.9305 + 18.8496n & \text{OR} \qquad &= 16.9191 + 18.8496n & n &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Now let's find all the solutions. First notice that, in this case, if we plug in negative values of n we will get negative solutions and these will not be in the interval and so there is no reason to even try these. So, let's start at $n = 0$ and see what we get.

$$\begin{array}{llll} n = 0: & x = 1.9305 & \text{OR} & x = 16.9191 \\ n = 1: & x = 20.7801 & \text{OR} & x = 35.7687 \end{array}$$

Notice that with each increase in n we were really just adding another 18.8496 onto the previous results and by doing this to the results from the $n = 1$ step we will get solutions that are outside of the interval and so there is no reason to even plug in $n = 2$.

So, it looks like we have the four solutions to this equation in the given interval.

$$\boxed{x = 1.9305,\ 16.9191,\ 20.7801,\ 35.7687}$$

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

---

3. Find all the solutions to $3 = 6 - 11\sin\left(\frac{t}{8}\right)$. Use at least 4 decimal places in your work.

Hint

Isolate the sine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the sine (with a coefficient of one) on one side of the equation gives,

$$\sin\left(\frac{t}{8}\right) = \frac{3}{11}$$

Hint

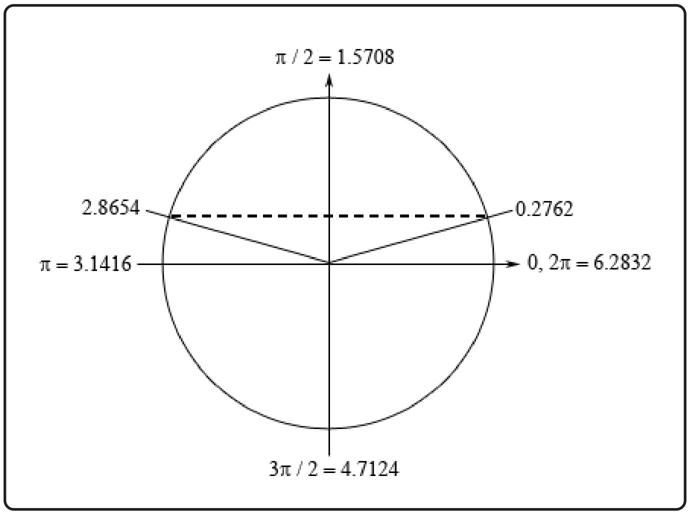
Using a calculator and your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which sine will have this value.

Step 2

First, using our calculator we can see that,

$$\frac{t}{8} = \sin^{-1}\left(\frac{3}{11}\right) = 0.2762$$

Now we're dealing with sine in this problem and we know that the y-axis represents sine on a unit circle and so we're looking for angles that will have a y coordinate of $\frac{3}{11}$. This means that we'll have angles in the first (this is the one our calculator gave us) and second quadrant. Here is a unit circle for this situation.



From the symmetry of the unit circle we can see that $\pi - 0.2762 = 2.8654$ is the second angle.

Hint

Using the two angles above write down all the angles for which sine will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$\frac{t}{8} = 0.2762 + 2\pi n \qquad \text{OR} \qquad \frac{t}{8} = 2.8654 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 8.

$$\boxed{t = 2.2096 + 16\pi n \qquad \text{OR} \qquad t = 22.9232 + 16\pi n \qquad n = 0, \pm 1, \pm 2, \ldots}$$

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

---

4. Find the solution(s) to $4\sin(6z) + \frac{13}{10} = -\frac{3}{10}$ that are in $[0, 2]$. Use at least 4 decimal places in your work.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the sine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the sine (with a coefficient of one) on one side of the equation gives,

$$\sin(6z) = -\frac{2}{5}$$

Hint

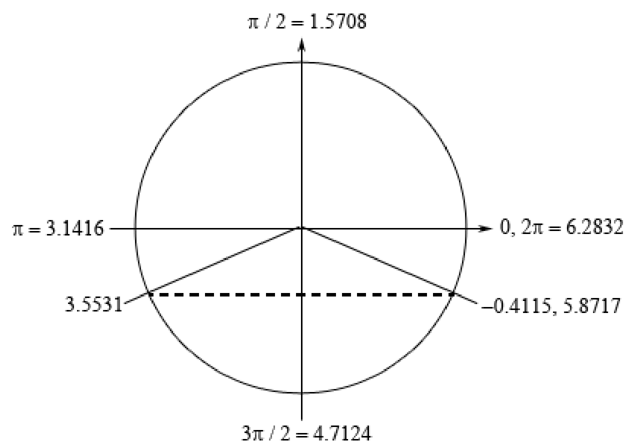
Using a calculator and your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which sine will have this value.

Step 2

First, using our calculator we can see that,

$$6z = \sin^{-1}\left(-\frac{2}{5}\right) = -0.4115$$

Now we're dealing with sine in this problem and we know that the y-axis represents sine on a unit circle and so we're looking for angles that will have a y coordinate of $-\frac{2}{5}$. This means that we'll have angles in the fourth (this is the one our calculator gave us) and third quadrant. Here is a unit circle for this situation.



From the symmetry of the unit circle we can see that the second angle will make an angle of 0.4115 with the negative x-axis and so the second angle will be $\pi + 0.4115 = 3.5531$. Also, as noted on the unit circle above a positive angle that represents the first angle (*i.e.* the angle in the fourth quadrant) is $2\pi - 0.4115 = 5.8717$. We can use either the positive or the negative angle here and we'll get the same solutions. However, because it is often easy to lose track of minus signs we will be using the positive angle in the fourth quadrant for our work here.

Hint

Using the two angles above write down all the angles for which sine will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$6z = 3.5531 + 2\pi n \qquad \text{OR} \qquad 6z = 5.8717 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is divide both sides by 6 and we'll convert everything to decimals to help with the final step.

$$\begin{aligned} z &= 0.5922 + \frac{\pi n}{3} & \text{OR} \qquad z &= 0.9786 + \frac{\pi n}{3} & n &= 0, \pm 1, \pm 2, \ldots \\ &= 0.5922 + 1.0472n & \text{OR} \qquad &= 0.9786 + 1.0472n & n &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Now let's find all the solutions. First notice that, in this case, if we plug in negative values of n we will get negative solutions and these will not be in the interval and so there is no reason to even try these. So, let's start at $n = 0$ and see what we get.

$$\begin{aligned} n &= 0: \quad z = 0.5922 & \text{OR} \qquad & z = 0.9786 \\ n &= 1: \quad z = 1.6394 & \text{OR} \qquad & \cancel{z = 2.0258} > 2 \end{aligned}$$

Notice that with each increase in n we were really just adding another 1.0472 onto the previous results and by doing this to the results from the $n = 1$ step we will get solutions that are outside of the interval and so there is no reason to even plug in $n = 2$. Also, as we've seen in this problem it is completely possible for only one of the solutions from a given interval to be in the given interval so don't worry about that when it happens.

So, it looks like we have the three solutions to this equation in the given interval.

$$\boxed{z = 0.5922, \; 0.9786, \; 1.6394}$$

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

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5. Find the solution(s) to $9\cos\left(\frac{4z}{9}\right)+21\sin\left(\frac{4z}{9}\right)=0$ that are in $[-10,10]$. Use at least 4 decimal places in your work.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to reduce the equation down to a single trig function (with a coefficient of one) on one side of the equation.

Step 1

Because we've got both a sine and a cosine here it makes some sense to reduce this down to tangent. So, reducing to a tangent (with a coefficient of one) on one side of the equation gives,

$$\tan\left(\frac{4z}{9}\right)=-\frac{3}{7}$$

Hint

Using a calculator and your knowledge of solving trig equations involving tangents to determine all the angles in the range $[0,2\pi]$ for which tangent will have this value.

Step 2

First, using our calculator we can see that,

$$\frac{4z}{9}=\tan^{-1}\left(-\frac{3}{7}\right)=-0.4049$$

As we discussed in Example 5 of this section the second angle for equations involving tangent will always be the π plus the first angle. Therefore, $\pi+(-0.4049)=2.7367$ will be the second angle.

Also, because it is very easy to lose track of minus signs we'll use the fact that we know that any angle plus 2π will give another angle whose terminal line is identical to the original angle to eliminate the minus sign on the first angle. So, another angle that will work for the first angle is $2\pi + (-0.4049) = 5.8783$. Note that there is nothing wrong with using the negative angle and if you chose to work with that you will get the same solutions. We are using the positive angle only to make sure we don't accidentally lose the minus sign on the angle we received from our calculator.

Hint

Using the two angles above write down all the angles for which tangent will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$\frac{4z}{9} = 2.7367 + 2\pi n \quad \text{OR} \quad \frac{4z}{9} = 5.8783 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by $\frac{9}{4}$ and we'll convert everything to decimals to help with the final step.

$$\begin{aligned} z &= 6.1576 + \frac{9\pi n}{2} & \text{OR} \qquad z &= 13.2262 + \frac{9\pi n}{2} & n = 0, \pm 1, \pm 2, \ldots \\ &= 6.1576 + 14.1372n & \text{OR} \qquad &= 13.2262 + 14.1372n & n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Now let's find all the solutions.

$$\begin{aligned} n = -1: \quad & z = -7.9796 & \text{OR} \quad & z = -0.9110 \\ n = 0: \quad & z = 6.1576 & \text{OR} \quad & \cancel{z = 13.2262} > 10 \end{aligned}$$

Notice that with each increase in n we were really just adding/subtracting (depending on the sign of n) another 14.1372 onto the previous results. A quick inspection of the results above will quickly show us that we don't need to go any farther and we won't bother with any other values of n. Also, as we've seen in this problem it is completely possible for only one of the solutions from a given interval to be in the given interval so don't worry about that when it happens.

So, it looks like we have the three solutions to this equation in the given interval.

$$\boxed{z = -7.9796, \ -0.9110, \ 6.1576}$$

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

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6. Find the solution(s) to $3\tan\left(\frac{w}{4}\right) - 1 = 11 - 2\tan\left(\frac{w}{4}\right)$ that are in $[-50, 0]$. Use at least 4 decimal places in your work.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the tangent (with a coefficient of one) on one side of the equation.

Step 1

Isolating the tangent (with a coefficient of one) on one side of the equation gives,

$$\tan\left(\frac{w}{4}\right) = \frac{12}{5}$$

Hint

Using a calculator and your knowledge of solving trig equations involving tangents to determine all the angles in the range $[0, 2\pi]$ for which tangent will have this value.

Step 2

First, using our calculator we can see that,

$$\frac{w}{4} = \tan^{-1}\left(\frac{12}{5}\right) = 1.1760$$

As we discussed in Example 5 of this section the second angle for equations involving tangent will always be the π plus the first angle. Therefore, $\pi + 1.1760 = 4.3176$ will be the second angle.

Hint

Using the two angles above write down all the angles for which tangent will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+ 2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$\frac{w}{4} = 1.1760 + 2\pi n \qquad \text{OR} \qquad \frac{w}{4} = 4.3176 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 4 and we'll convert everything to decimals to help with the final step.

$$\begin{aligned} w &= 4.7040 + 8\pi n && \text{OR} && w = 17.2704 + 8\pi n && n = 0, \pm 1, \pm 2, \ldots \\ &= 4.7040 + 25.1327n && \text{OR} && \quad = 17.2704 + 25.1327n && n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Now let's find all the solutions. First, notice that if we plug in positive n or $n = 0$ we will have positive solutions and these solutions will be out of the interval. Therefore, we'll start with $n = -1$.

$$n = -1: \quad w = -20.4287 \quad \text{OR} \quad w = -7.8623$$
$$n = -2: \quad w = -45.5614 \quad \text{OR} \quad w = -32.9950$$

Notice that with each increase in n we were really just subtracting another 25.1327 from the previous results. A quick inspection of the results above will quickly show us that we don't need to go any farther and we won't bother with any other values of n.

So, it looks like we have the four solutions to this equation in the given interval.

$$\boxed{w = -45.5614, \ -32.9950, \ -20.4287, \ -7.8623}$$

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

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7. Find the solution(s) to $17 - 3\sec\left(\frac{z}{2}\right) = 2$ that are in $[20, 45]$. Use at least 4 decimal places in your work.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the secant (with a coefficient of one) on one side of the equation.

Step 1

Isolating the secant (with a coefficient of one) on one side of the equation gives,

$$\sec\left(\frac{z}{2}\right) = 5$$

Hint

Using a calculator and your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which secant will have this value. The best way to do this is to rewrite the equation into one in terms of a different trig function that we can more easily deal with.

Step 2

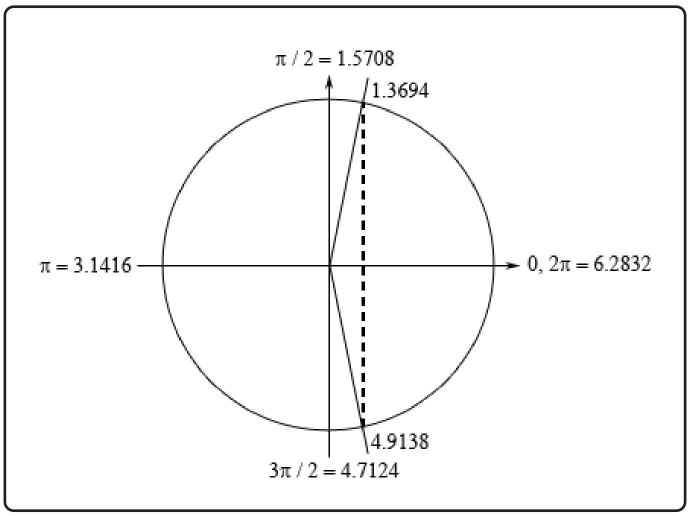
In order to get the solutions it will be much easier to recall the definition of secant in terms of cosine and rewrite the equation into one involving cosine. Doing this gives,

$$\sec\left(\frac{z}{2}\right) = \frac{1}{\cos\left(\frac{z}{2}\right)} = 5 \qquad\Rightarrow\qquad \cos\left(\frac{z}{2}\right) = \frac{1}{5}$$

The solution(s) to the equation involving the cosine are the same as the solution(s) to the equation involving the secant and so working with that will be easier. Using our calculator we can see that,

$$\frac{z}{2} = \cos^{-1}\left(\frac{1}{5}\right) = 1.3694$$

Now we're dealing with cosine in this problem and we know that the x-axis represents cosine on a unit circle and so we're looking for angles that will have a x coordinate of $\frac{1}{5}$. This means that we'll have angles in the first (this is the one our calculator gave us) and fourth quadrant. Here is a unit circle for this situation.



From the symmetry of the unit circle we can see that we can either use -1.3694 or $2\pi - 1.3694 = 4.9138$ for the second angle. Each will give the same set of solutions. However, because it is easy to lose track of minus signs we will use the positive angle for our second solution.

Hint

Using the two angles above write down all the angles for which cosine/secant will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$\frac{z}{2} = 1.3694 + 2\pi n \qquad \text{OR} \qquad \frac{z}{2} = 4.9138 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 2 and we'll convert everything to decimals to help with the final step.

$$\begin{aligned} z &= 2.7388 + 4\pi n & \text{OR} \qquad z &= 9.8276 + 4\pi n & n &= 0, \pm 1, \pm 2, \ldots \\ &= 2.7388 + 12.5664n & \text{OR} \qquad &= 9.8276 + 12.5664n & n &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Now let's find all the solutions. First notice that, in this case, if we plug in negative values of n we will get negative solutions and these will not be in the interval and so there is no reason to even try these. Also note that if we use $n = 0$ we will still not be in the interval and so let's start things off at $n = 1$.

$$\begin{aligned} n = 1: &\quad \cancel{z = 15.3052} < 20 & \text{OR} &\quad z = 22.3940 \\ n = 2: &\quad z = 27.8716 & \text{OR} &\quad z = 34.9604 \\ n = 3: &\quad z = 40.4380 & \text{OR} &\quad \cancel{z = 47.5268} > 45 \end{aligned}$$

Notice that with each increase in n we were really just adding another 12.5664 onto the previous results and by a quick inspection of the results above we can see that we don't need to go any farther. Also, as we've seen in this problem it is completely possible for only one of the solutions from a given interval to be in the given interval so don't worry about that when it happens.

So, it looks like we have the four solutions to this equation in the given interval.

$$\boxed{z = 22.3940,\ 27.8716,\ 34.9604,\ 40.4380}$$

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

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8. Find the solution(s) to $12\sin(7y) + 11 = 3 + 4\sin(7y)$ that are in $\left[-2, -\frac{1}{2}\right]$. Use at least 4 decimal places in your work.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the sine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the sine (with a coefficient of one) on one side of the equation gives,

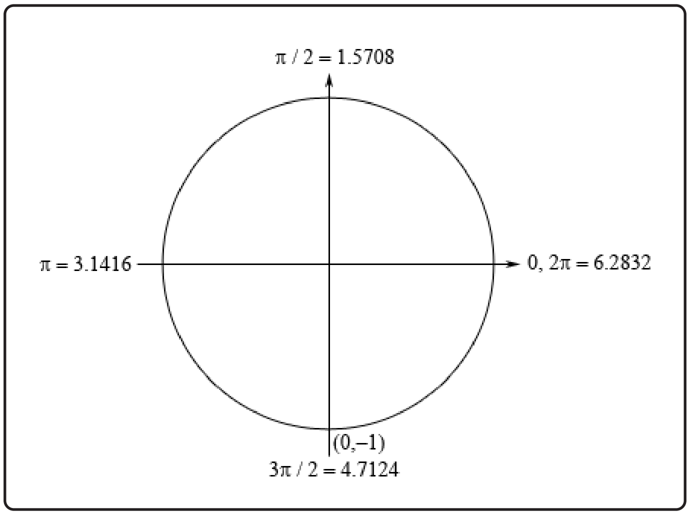
$$\sin(7y) = -1$$

Hint

Use your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which sine will have this value.

Step 2

If you need to use a calculator to get the solution for this that is fine, but this is also one of the *standard* angles as we can see from the unit circle below.



Because we're dealing with sine in this problem and we know that the y-axis represents sine on a unit circle we're looking for angle(s) that will have a y coordinate of -1. The only angle that will have this y coordinate will be $\frac{3\pi}{2} = 4.7124$.

Note that unlike all the other problems that we've worked to this point this will be the only angle. There is simply not another angle in the range $[0, 2\pi]$ for which sine will have this value. Don't get so locked into the *usual* case where we get two possible angles in the $[0, 2\pi]$ that when these single solution cases roll around you decide you must have done something wrong. They happen on occasion and we need to be able to deal with them when they occur.

Hint

Using the angle above write down all the angles for which sine will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have the angle above we can get all possible angles by simply adding "$+\,2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto the angle.

This then means that we must have,

$$7y = 4.7124 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is divide both sides by 7

and we'll convert everything to decimals to help with the final step.

$$\begin{aligned} y &= 0.6732 + \frac{2\pi n}{7} & n &= 0, \pm 1, \pm 2, \ldots \\ &= 0.6732 + 0.8976n & n &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Now let's find all the solutions. First notice that, in this case, if we plug in positive values of n or $n = 0$ we will get positive solutions and these will not be in the interval and so there is no reason to even try these. So, let's start at $n = -1$ and see what we get.

$$\begin{aligned} n &= -1: & &\cancel{y = -0.2244} > -0.5 \\ n &= -2: & &y = -1.122 \\ n &= -3: & &\cancel{y = -2.0196} < -2 \end{aligned}$$

So, it looks like we have only a single solution to this equation in the given interval.

$$\boxed{y = -1.122}$$

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

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9. Find the solution(s) to $5 - 14\tan(8x) = 30$ that are in $[-1, 1]$. Use at least 4 decimal places in your work.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the tangent (with a coefficient of one) on one side of the equation.

Step 1

Isolating the tangent (with a coefficient of one) on one side of the equation gives,

$$\tan(8x) = -\frac{25}{14}$$

Hint

Using a calculator and your knowledge of solving trig equations involving tangents to determine all the angles in the range $[0, 2\pi]$ for which tangent will have this value.

Step 2

First, using our calculator we can see that,

$$8x = \tan^{-1}\left(-\frac{25}{14}\right) = -1.0603$$

As we discussed in Example 5 of this section the second angle for equations involving tangent will always be the π plus the first angle. Therefore, $\pi + (-1.0603) = 2.0813$ will be the second angle.

Hint

Using the two angles above write down all the angles for which tangent will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$8x = -1.0603 + 2\pi n \qquad \text{OR} \qquad 8x = 2.0813 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is divide both sides by 8

and we'll convert everything to decimals to help with the final step.

$$\begin{aligned} x &= -0.1325 + \frac{\pi n}{4} \quad &\text{OR}\quad x &= 0.2602 + \frac{\pi n}{4} \quad & n = 0, \pm 1, \pm 2, \ldots \\ &= -0.1325 + 0.7854n \quad &\text{OR}\quad &= 0.2602 + 0.7854n \quad & n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Now let's find all the solutions.

$$\begin{array}{lll} n = -1: & x = -0.9179 & \text{OR} \quad x = -0.5252 \\ n = 0: & x = -0.1325 & \text{OR} \quad x = 0.2602 \\ n = 1: & x = 0.6529 & \text{OR} \quad x = \cancel{1.0456} > 1 \end{array}$$

Notice that with each increase in n we were really just adding/subtracting another 0.7854 from the previous results. A quick inspection of the results above will quickly show us that we don't need to go any farther and we won't bother with any other values of n. Also, as we've seen in this problem it is completely possible for only one of the solutions from a given interval to be in the given interval so don't worry about that when it happens.

So, it looks like we have the five solutions to this equation in the given interval.

$$\boxed{x = -0.9179,\ -0.5252,\ -0.1325,\ 0.2602,\ 0.6529}$$

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

---

10. Find the solution(s) to $0 = 18 + 2\csc\left(\frac{t}{3}\right)$ that are in $[0, 5]$. Use at least 4 decimal places in your work.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosecant (with a coefficient of one) on one side of the equation.

Step 1

Isolating the cosecant (with a coefficient of one) on one side of the equation gives,

$$\csc\left(\frac{t}{3}\right) = -9$$

Hint

Using a calculator and your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosecant will have this value. The best way to do this is to rewrite the equation into one in terms of a different trig function that we can more easily deal with.

Step 2

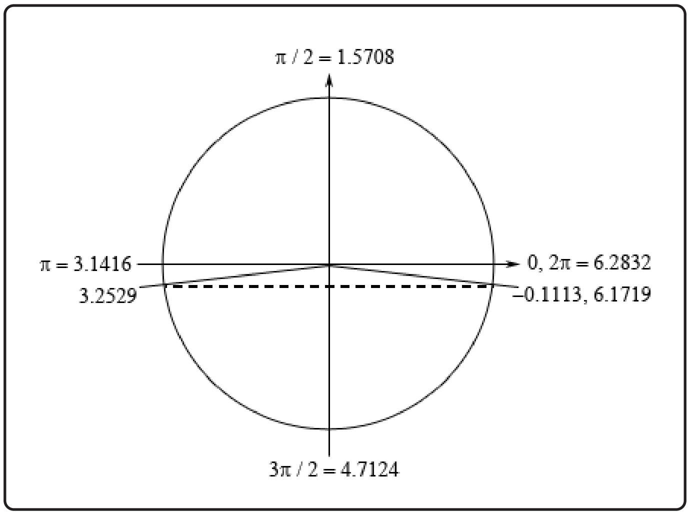
In order to get the solutions it will be much easier to recall the definition of cosecant in terms of sine and rewrite the equation into one involving sine. Doing this gives,

$$\csc\left(\frac{t}{3}\right) = \frac{1}{\sin\left(\frac{t}{3}\right)} = -9 \qquad \Rightarrow \qquad \sin\left(\frac{t}{3}\right) = -\frac{1}{9}$$

The solution(s) to the equation involving the sine are the same as the solution(s) to the equation involving the cosecant and so working with that will be easier. Using our calculator we can see that,

$$\frac{t}{3} = \sin^{-1}\left(-\frac{1}{9}\right) = -0.1113$$

Now we're dealing with sine in this problem and we know that the y-axis represents sine on a unit circle and so we're looking for angles that will have a y coordinate of $-\frac{1}{9}$. This means that we'll have angles in the fourth (this is the one our calculator gave us) and third quadrant. Here is a unit circle for this situation.



From the symmetry of the unit circle we can see that the second angle will make an angle of 0.1113 with the negative x-axis and so the second angle will be $\pi + 0.1113 = 3.2529$. Also, as noted on the unit circle above a positive angle that represents the first angle (*i.e.* the angle in the fourth quadrant) is $2\pi - 0.1113 = 6.1719$. We can use either the positive or the negative angle here and we'll get the same solutions. However, because it is often easy to lose track of minus signs we will be using the positive angle in the fourth quadrant for our work here.

Hint

Using the two angles above write down all the angles for which sine/cosecant will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+\,2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$\frac{t}{3} = 3.2529 + 2\pi n \qquad \text{OR} \qquad \frac{t}{3} = 6.1719 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 3

and we'll convert everything to decimals to help with the final step.

$$\begin{aligned} t &= 9.7587 + 6\pi n & \text{OR} \quad t &= 18.5157 + 6\pi n & n &= 0, \pm 1, \pm 2, \ldots \\ &= 9.7587 + 18.8496n & \text{OR} \quad &= 18.5157 + 18.8496n & n &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Now let's find all the solutions. First notice that, in this case, if we plug in negative values of n we will get negative solutions and these will not be in the interval and so there is no reason to even try these. Also note that even if we start off with $n = 0$ we will get solutions that are already out of the given interval.

So, despite the fact that there are solutions to this equation none of them fall in the given interval and so there are **no solutions** to this equation. Do not get excited about the answer here. This kind of situation will happen on occasion and so we need to be aware of it and able to deal with it.

---

11. Find the solution(s) to $\frac{1}{2}\cos\left(\frac{x}{8}\right) + \frac{1}{4} = \frac{2}{3}$ that are in $[0, 100]$. Use at least 4 decimal places in your work.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the cosine (with a coefficient of one) on one side of the equation gives,

$$\cos\left(\frac{x}{8}\right) = \frac{5}{6}$$

Hint

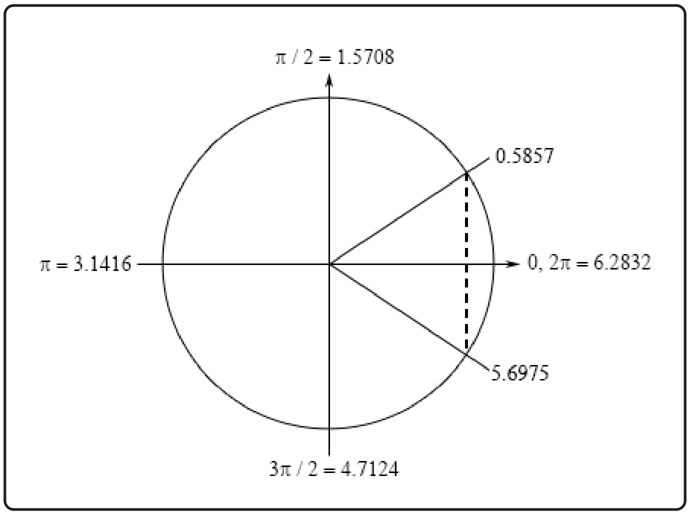
Using a calculator and your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosine will have this value.

Step 2

First, using our calculator we can see that,

$$\frac{x}{8} = \cos^{-1}\left(\frac{5}{6}\right) = 0.5857$$

Now we're dealing with cosine in this problem and we know that the x-axis represents cosine on a unit circle and so we're looking for angles that will have a x coordinate of $\frac{5}{6}$. This means that we'll have angles in the first (this is the one our calculator gave us) and fourth quadrant. Here is a unit circle for this situation.



From the symmetry of the unit circle we can see that we can either use -0.5857 or $2\pi - 0.5857 = 5.6975$ for the second angle. Each will give the same set of solutions. However, because it is easy to lose track of minus signs we will use the positive angle for our second solution.

Hint

Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3

From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding "$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$" onto each of these.

This then means that we must have,

$$\frac{x}{8} = 0.5857 + 2\pi n \qquad \text{OR} \qquad \frac{x}{8} = 5.6975 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 8 and we'll convert everything to decimals to help with the final step.

$$\begin{aligned} x &= 4.6856 + 16\pi n & \text{OR} \qquad x &= 45.5800 + 16\pi n & n &= 0, \pm 1, \pm 2, \ldots \\ &= 4.6856 + 50.2655n & \text{OR} \qquad &= 45.58 + 50.2655n & n &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Hint

Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4

Now let's find all the solutions. First notice that, in this case, if we plug in negative values of n we will get negative solutions and these will not be in the interval and so there is no reason to even try these. So, let's start at $n = 0$ and see what we get.

$$\begin{aligned} n = 0: &\quad x = 4.6856 & \text{OR} \quad & x = 45.58 \\ n = 1: &\quad x = 54.9511 & \text{OR} \quad & x = 95.8455 \end{aligned}$$

Notice that with each increase in n we were really just adding another 50.2655 onto the previous results and by doing this to the results from the $n = 1$ step we will get solutions that are outside of the interval and so there is no reason to even plug in $n = 2$.

So, it looks like we have the four solutions to this equation in the given interval.

$$\boxed{x = 4.6856,\ 45.58,\ 54.9511,\ 95.8455}$$

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

12. Find the solution(s) to $\frac{4}{3} = 1 + 3\sec(2t)$ that are in $[-4, 6]$. Use at least 4 decimal places in your work.

Hint

Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the secant (with a coefficient of one) on one side of the equation.

Solution

Isolating the secant (with a coefficient of one) on one side of the equation gives,

$$\sec(2t) = \frac{1}{9}$$

At this point we can stop. We know that

$$\sec\theta \le -1 \qquad \text{or} \qquad \sec\theta \ge 1$$

This means that it is impossible for secant to ever be $\frac{1}{9}$ and so there will be **no solution** to this equation.

Note that if you didn't recall the restrictions on secant the next step would have been to convert this to cosine so let's do that.

$$\sec(2t) = \frac{1}{\cos(2t)} = \frac{1}{9} \qquad \Rightarrow \qquad \cos(2t) = 9$$

At this point we can note that $-1 \le \cos\theta \le 1$ and so again there is no way for cosine to be 9 and again we get that there will be **no solution** to this equation.

1.6 Solving Trig Equations with Calculators, Part II

1. Find all the solutions to $3 - 14\sin(12t + 7) = 13$. Use at least 4 decimal places in your work.

Hint

With the exception of the argument, which is a little more complex, this is identical to the equations that we solved in the previous section.

Solution

The argument of the sine is a little more complex in this equation than those we saw in the previous section, but the solution process is identical. Therefore, we will be assuming that you recall the process from the previous section and do not need all the hints or quite as many details as we put into the solutions there. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the section.

First, isolating the sine on one side of the equation gives,

$$\sin(12t + 7) = -\frac{5}{7}$$

Using a calculator we get,

$$12t + 7 = \sin^{-1}\left(-\frac{5}{7}\right) = -0.7956$$

From our knowledge of the unit circle we can see that a positive angle that corresponds to this angle is $2\pi - 0.7956 = 5.4876$. Either these angles can be used here but we'll use the positive angle to avoid the possibility of losing the minus sign. Also, from a quick look at a unit circle we can see that a second angle in the range $[0, 2\pi]$ will be $\pi + 0.7965 = 3.9372$.

Now, all possible angles for which sine will have this value are,

$$12t + 7 = 3.9372 + 2\pi n \qquad \text{OR} \qquad 12t + 7 = 5.4876 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

At this point all we need to do is solve each of these for t and we'll have all the solutions to the equation. Doing this gives,

$$\boxed{t = -0.2552 + \frac{\pi n}{6} \qquad \text{OR} \qquad t = -0.1260 + \frac{\pi n}{6} \qquad n = 0, \pm 1, \pm 2, \ldots}$$

If an interval had been given we would next proceed with plugging in values of n to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

---

2. Find all the solutions to $3\sec(4-9z)-24=0$. Use at least 4 decimal places in your work.

Hint

With the exception of the argument, which is a little more complex, this is identical to the equations that we solved in the previous section.

Solution

The argument of the secant is a little more complex in this equation than those we saw in the previous section, but the solution process is identical. Therefore, we will be assuming that you recall the process from the previous section and do not need all the hints or quite as many details as we put into the solutions there. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the section.

First, isolating the secant on one side of the equation gives and converting the equation into one involving cosine (to make the work a little easier) gives,

$$\sec(4-9z)=8 \qquad \Rightarrow \qquad \cos(4-9z)=\frac{1}{8}$$

Using a calculator we get,

$$4-9z=\cos^{-1}\left(\frac{1}{8}\right)=1.4455$$

From a quick look at a unit circle we can see that a second angle in the range $[0,2\pi]$ will be $2\pi-1.4455=4.8377$. Now, all possible angles for which secant will have this value are,

$$4-9z=1.4455+2\pi n \qquad \text{OR} \qquad 4-9z=4.8377+2\pi n \qquad n=0,\pm1,\pm2,\ldots$$

At this point all we need to do is solve each of these for z and we'll have all the solutions to the equation. Doing this gives,

$$\boxed{z=0.2838-\frac{2\pi n}{9} \qquad \text{OR} \qquad z=-0.09308-\frac{2\pi n}{9} \qquad n=0,\pm1,\pm2,\ldots}$$

If an interval had been given we would next proceed with plugging in values of n to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

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3. Find all the solutions to $4\sin(x+2) - 15\sin(x+2)\tan(4x) = 0$. Use at least 4 decimal places in your work.

Hint

Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques.

Step 1

Notice that each term has a sine in it and so we can factor this out of each term to get,

$$\sin(x+2)(4 - 15\tan(4x)) = 0$$

Now, we have a product of two factors that equals zero and so by basic algebraic properties we know that we must have,

$$\sin(x+2) = 0 \qquad \text{OR} \qquad 4 - 15\tan(4x) = 0$$

Hint

Solve each of these two equations to attain all the solutions to the original equation.

Step 2

Each of these equations are similar to equations solved in the previous section or in the earlier problems of this section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

We'll start with,

$$\sin(x+2)=0$$

From a unit circle we can see that we must have,

$$x+2=0+2\pi n \qquad \text{OR} \qquad x+2=\pi+2\pi n \qquad n=0,\pm1,\pm2,\ldots$$

Notice that we can further reduce this down to,

$$x+2=\pi n \qquad n=0,\pm1,\pm2,\ldots$$

Finally, the solutions from this equation are,

$$x=\pi n-2 \qquad n=0,\pm1,\pm2,\ldots$$

The second equation will take a little more (but not much more) work. First, isolating the tangent gives,

$$\tan(4x)=\frac{4}{15}$$

Using our calculator we get,

$$4x=\tan^{-1}\left(\frac{4}{15}\right)=0.2606$$

From our knowledge on solving equations involving tangents we know that the second angle in the range $[0,2\pi]$ will be $\pi+0.2606=3.4022$.

Finally, the solutions to this equation are,

$$\begin{array}{rclcrcl}
4x=0.2606+2\pi n & & \text{OR} & & 4x=3.4022+2\pi n & & n=0,\pm1,\pm2,\ldots \\
x=0.06515+\dfrac{\pi n}{2} & & \text{OR} & & x=0.8506+\dfrac{\pi n}{2} & & n=0,\pm1,\pm2,\ldots
\end{array}$$

Putting all of this together gives the following set of solutions.

$$\boxed{x=\pi n-2,\quad x=0.06515+\frac{\pi n}{2}, \quad \text{OR} \quad x=0.8506+\frac{\pi n}{2} \quad n=0,\pm1,\pm2,\ldots}$$

If an interval had been given we would next proceed with plugging in values of n to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

4. Find all the solutions to $3\cos\left(\frac{3y}{7}\right)\sin\left(\frac{y}{2}\right)+14\cos\left(\frac{3y}{7}\right)=0$. Use at least 4 decimal places in your work.

Hint

Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques.

Step 1

Notice that each term has a cosine in it and so we can factor this out of each term to get,

$$\cos\left(\frac{3y}{7}\right)\left(3\sin\left(\frac{y}{2}\right)+14\right)=0$$

Now, we have a product of two factors that equals zero and so by basic algebraic properties we know that we must have,

$$\cos\left(\frac{3y}{7}\right)=0 \qquad \text{OR} \qquad 3\sin\left(\frac{y}{2}\right)+14=0$$

Hint

Solve each of these two equations to attain all the solutions to the original equation.

Step 2

Each of these equations are similar to equations solved in the previous section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

We'll start with,

$$\cos\left(\frac{3y}{7}\right)=0$$

From a unit circle we can see that we must have,

$$\frac{3y}{7}=\frac{\pi}{2}+2\pi n \qquad \text{OR} \qquad \frac{3y}{7}=\frac{3\pi}{2}+2\pi n \qquad n=0,\pm 1,\pm 2,\ldots$$

Notice that we can further reduce this down to,

$$\frac{3y}{7} = \frac{\pi}{2} + \pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, the solutions from this equation are,

$$y = \frac{7\pi}{6} + \frac{7\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \ldots$$

The second equation will take a little more (but not much more) work. First, isolating the sine gives,

$$\sin\left(\frac{y}{2}\right) = -\frac{14}{3} < -1$$

At this point recall that we know $-1 \le \sin\theta \le 1$ and so this equation will have no solutions.

Therefore, the only solutions to this equation are,

$$\boxed{y = \frac{7\pi}{6} + \frac{7\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \ldots}$$

Do get too excited about the fact that we only got solutions from one of the two equations we got after factoring. This will happen on occasion and so we need to be ready for these cases when they happen.

If an interval had been given we would next proceed with plugging in values of n to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

5. Find all the solutions to $7\cos^2(3x) - \cos(3x) = 0$. Use at least 4 decimal places in your work.

Hint

Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques.

Step 1

Notice that we can factor a cosine out of each term to get,

$$\cos(3x)(7\cos(3x) - 1) = 0$$

Now, we have a product of two factors that equals zero and so by basic algebraic properties we know that we must have,

$$\cos(3x) = 0 \qquad \text{OR} \qquad 7\cos(3x) - 1 = 0$$

Hint

Solve each of these two equations to attain all the solutions to the original equation.

Step 2

Each of these equations are similar to equations solved in the previous section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

We'll start with,

$$\cos(3x) = 0$$

From a unit circle we can see that we must have,

$$3x = \frac{\pi}{2} + 2\pi n \qquad \text{OR} \qquad 3x = \frac{3\pi}{2} + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Notice that we can further reduce this down to,

$$3x = \frac{\pi}{2} + \pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, the solutions from this equation are,

$$x = \frac{\pi}{6} + \frac{\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \ldots$$

The second equation will take a little more (but not much more) work. First, isolating the cosine gives,

$$\cos(3x) = \frac{1}{7}$$

Using our calculator we get,

$$3x = \cos^{-1}\left(\frac{1}{7}\right) = 1.4274$$

From a quick look at a unit circle we know that the second angle in the range $[0, 2\pi]$ will be $2\pi - 1.4274 = 4.8558$.

Finally, the solutions to this equation are,

$$\begin{aligned} 3x &= 1.4274 + 2\pi n & \text{OR} \quad 3x &= 4.8558 + 2\pi n & n &= 0, \pm1, \pm2, \ldots \\ x &= 0.4758 + \frac{2\pi n}{3} & \text{OR} \quad x &= 1.6186 + \frac{2\pi n}{3} & n &= 0, \pm1, \pm2, \ldots \end{aligned}$$

Putting all of this together gives the following set of solutions.

$$\boxed{x = \frac{\pi}{6} + \frac{\pi n}{3}, \quad x = 0.4758 + \frac{2\pi n}{3}, \quad \text{OR} \quad x = 1.6186 + \frac{2\pi n}{3} \quad n = 0, \pm1, \pm2, \ldots}$$

If an interval had been given we would next proceed with plugging in values of n to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

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6. Find all the solutions to $\tan^2\left(\frac{w}{4}\right) = \tan\left(\frac{w}{4}\right) + 12$. Use at least 4 decimal places in your work.

Hint

Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques. If you're not sure how to factor this think about how you would factor $x^2 - x - 12 = 0$.

Step 1

This equation may look very different from anything that we've ever been asked to factor, however it is something that we can factor. First think about factoring the following,

$$x^2 = x + 12 \qquad \to \qquad x^2 - x - 12 = (x-4)(x+3) = 0$$

If we can factor this algebraic equation then we can factor the given equation in exactly the same manner.

$$\begin{aligned}\tan^2\left(\frac{w}{4}\right) &= \tan\left(\frac{w}{4}\right) + 12\\ \tan^2\left(\frac{w}{4}\right) - \tan\left(\frac{w}{4}\right) - 12 &= 0\\ \left(\tan\left(\frac{w}{4}\right) - 4\right)\left(\tan\left(\frac{w}{4}\right) + 3\right) &= 0\end{aligned}$$

Now, we have a product of two factors that equals zero and so by basic algebraic properties we know that we must have,

$$\tan\left(\frac{w}{4}\right) - 4 = 0 \qquad \text{OR} \qquad \tan\left(\frac{w}{4}\right) + 3 = 0$$

Hint

Solve each of these two equations to attain all the solutions to the original equation.

Step 2

Each of these equations are similar to equations solved in the previous section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

We'll start with the first equation and isolate the tangent to get,

$$\tan\left(\frac{w}{4}\right) = 4$$

Using our calculator we get,

$$\frac{w}{4} = \tan^{-1}(4) = 1.3258$$

From our knowledge on solving equations involving tangents we know that the second angle in the range $[0, 2\pi]$ will be $\pi + 1.3258 = 4.4674$.

All the solutions to the first equation are then,

$$\begin{array}{lcll}\dfrac{w}{4} = 1.3258 + 2\pi n & \text{OR} & \dfrac{w}{4} = 4.4674 + 2\pi n & n = 0, \pm 1, \pm 2, \ldots\\ w = 5.3032 + 8\pi n & \text{OR} & w = 17.8696 + 8\pi n & n = 0, \pm 1, \pm 2, \ldots\end{array}$$

Now, let's solve the second equation.

$$\tan\left(\frac{w}{4}\right) = -3 \qquad \rightarrow \qquad \frac{w}{4} = \tan^{-1}(-3) = -1.2490$$

From our knowledge of the unit circle we can see that a positive angle that corresponds to this angle is $2\pi - 1.2490 = 5.0342$. Either these angles can be used here but we'll use the positive angle to avoid the possibility of losing the minus sign. Also, the second angle in the range $[0, 2\pi]$ is $\pi + (-1.2490) = 1.8926$.

All the solutions to the second equation are then,

$$\frac{w}{4} = 1.8926 + 2\pi n \qquad \text{OR} \qquad \frac{w}{4} = 5.0342 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$w = 7.5704 + 8\pi n \qquad \text{OR} \qquad w = 20.1368 + 8\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Putting all of this together gives the following set of solutions.

$$\boxed{\begin{array}{lll} w = 5.3032 + 8\pi n, & w = 7.5704 + 8\pi n & \\ & & n = 0, \pm 1, \pm 2, \ldots \\ w = 17.8696 + 8\pi n, & w = 20.1368 + 8\pi n & \end{array}}$$

If an interval had been given we would next proceed with plugging in values of n to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

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7. Find all the solutions to $4\csc^2(1-t) + 6 = 25\csc(1-t)$. Use at least 4 decimal places in your work.

Hint

Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques. If you're not sure how to factor this think about how you would factor $4x^2 - 25x + 6 = 0$.

Step 1

This equation may look very different from anything that we've ever been asked to factor, however it is something that we can factor. First think about factoring the following,

$$4x^2 + 6 = 25x \qquad \to \qquad 4x^2 - 25x + 6 = (4x - 1)(x - 6) = 0$$

If we can factor this algebraic equation then we can factor the given equation in exactly the same manner.

$$\begin{aligned}4\csc^2(1-t)+6&=25\csc(1-t)\\4\csc^2(1-t)-25\csc(1-t)+6&=0\\(4\csc(1-t)-1)(\csc(1-t)-6)&=0\end{aligned}$$

Now, we have a product of two factors that equals zero and so by basic algebraic properties we know that we must have,

$$4\csc(1-t)-1=0 \qquad \text{OR} \qquad \csc(1-t)-6=0$$

Hint

Solve each of these two equations to attain all the solutions to the original equation.

Step 2

Each of these equations are similar to equations solved in the previous section and earlier in this section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

We'll start with the first equation, isolate the cosecant and convert to an equation in terms of sine for easier solving. Doing this gives,

$$\csc(1-t)=\frac{1}{4} \qquad \to \qquad \sin(1-t)=4>1$$

We now know that there are now solutions to the first equation because we know $-1\le\sin\theta\le1$.

Now, let's solve the second equation.

$$\csc(1-t)=6 \qquad \to \qquad \sin(1-t)=\frac{1}{6}$$

Using our calculator we get,

$$1-t=\sin^{-1}\left(\frac{1}{6}\right)=0.1674$$

A quick glance at a unit circle shows us that the second angle in the range $[0,2\pi]$ is $\pi-0.1674=2.9742$.

All the solutions to the second equation are then,

$$\begin{array}{rclcrcll} 1-t &=& 0.1674+2\pi n & \text{OR} & 1-t &=& 2.9742+2\pi n & n=0,\pm1,\pm2,\ldots \\ t &=& 0.8326-2\pi n & \text{OR} & t &=& -1.9742-2\pi n & n=0,\pm1,\pm2,\ldots \end{array}$$

Because we had not solutions to the first equation all the solutions to the original equation are then,

$$\boxed{t = 0.8326-2\pi n \quad \text{OR} \quad t=-1.9742-2\pi n \quad n=0,\pm1,\pm2,\ldots}$$

Do get too excited about the fact that we only got solutions from one of the two equations we got after factoring. This will happen on occasion and so we need to be ready for these cases when they happen.

If an interval had been given we would next proceed with plugging in values of n to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

---

8. Find all the solutions to $4y\sec(7y) = -21y$. Use at least 4 decimal places in your work.

Hint

Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques.

Step 1

Notice that if we move all the terms to one side we can then factor a y out of the equation. Doing this gives,

$$\begin{aligned} 4y\sec(7y)+21y &= 0 \\ y\left(4\sec(7y)+21\right) &= 0 \end{aligned}$$

Now, we have a product of two factors that equals zero and so by basic algebraic properties we know that we must have,

$$y=0 \qquad \text{OR} \qquad 4\sec(7y)+21=0$$

Be careful with this type of equation to not make the mistake of just canceling the y from both sides in the initial step. Had you done that you would have missed the $y = 0$ solution.

When solving equations it is important to remember that you can't cancel anything from both sides unless you know for a fact that what you are canceling will never be zero.

Hint

Solve each of these two equations to attain all the solutions to the original equation.

Step 2

There really isn't anything that we need to do with the first equation and so we can move right on to the second equation. Note that this equation is similar to equations solved in the previous section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

First, isolating the secant and converting to cosines (to make the solving a little easier) gives,

$$\sec(7y) = -\frac{21}{4} \qquad \rightarrow \qquad \cos(7y) = -\frac{4}{21}$$

Using our calculator we get,

$$7y = \cos^{-1}\left(-\frac{4}{21}\right) = 1.7624$$

From a quick look at a unit circle we know that the second angle in the range $[0, 2\pi]$ will be $2\pi - 1.7624 = 4.5208$.

Finally, the solutions to this equation are,

$$\begin{array}{lll} 7y = 1.7624 + 2\pi n & \text{OR} \quad 7y = 4.5208 + 2\pi n & n = 0, \pm 1, \pm 2, \ldots \\ y = 0.2518 + \dfrac{2\pi n}{7} & \text{OR} \quad y = 0.6458 + \dfrac{2\pi n}{7} & n = 0, \pm 1, \pm 2, \ldots \end{array}$$

Putting all of this together gives the following set of solutions.

$$\boxed{y = 0, \quad y = 0.2518 + \frac{2\pi n}{7}, \quad \text{OR} \quad y = 0.6458 + \frac{2\pi n}{7} \quad n = 0, \pm 1, \pm 2, \ldots}$$

If an interval had been given we would next proceed with plugging in values of n to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

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9. Find all the solutions to $10x^2 \sin(3x+2) = 7x \sin(3x+2)$. Use at least 4 decimal places in your work.

Hint

Factor the equation and using basic algebraic properties get some equations that can be dealt with using known techniques.

Step 1

Notice that if we move all the terms to one side we can then factor an x and a sine out of the equation. Doing this gives,

$$\begin{aligned} 10x^2 \sin(3x+2) - 7x \sin(3x+2) &= 0 \\ x(10x-7)\sin(3x+2) &= 0 \end{aligned}$$

Now, we have a product of three factors that equals zero and so by basic algebraic properties we know that we must have,

$$x = 0, \qquad 10x - 7 = 0, \qquad \text{OR} \qquad \sin(3x+2) = 0$$

Be careful with this type of equation to not make the mistake of just canceling the x or the sine from both sides in the initial step. Had you done that you would have missed the $x = 0$ solution and the solutions we will get from solving $\sin(3x+2) = 0$.

When solving equations it is important to remember that you can't cancel anything from both sides unless you know for a fact that what you are canceling will never be zero.

Hint

Solve each of these three equations to attain all the solutions to the original equation.

Step 2

There really isn't anything that we need to do with the first equation and so we can move right on to the second equation (which also doesn't really present any problems). Solving the second equation gives,

$$x = \frac{7}{10}$$

Now let's take a look at the third equation. This equation is similar to equations solved earlier in this section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

From a unit circle we can see that we must have,

$$3x + 2 = 0 + 2\pi n \qquad \text{OR} \qquad 3x + 2 = \pi + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Notice that we can further reduce this down to,

$$3x + 2 = \pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, the solutions from this equation are,

$$x = \frac{\pi n - 2}{3} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Putting all of this together gives the following set of solutions.

$$\boxed{x = 0, \qquad x = \frac{7}{10}, \qquad \text{OR} \qquad x = \frac{\pi n - 2}{3} \qquad n = 0, \pm 1, \pm 2, \ldots}$$

If an interval had been given we would next proceed with plugging in values of n to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

10. Find all the solutions to $(2t-3)\tan\left(\frac{6t}{11}\right) = 15-10t$. Use at least 4 decimal places in your work.

Hint

Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques.

Step 1

This one may be a little trickier to factor than the others in this section, but it can be factored. First get everything on one side of the equation and then notice that we can factor out a $2t-3$ from the equation as follows,

$$\begin{aligned}(2t-3)\tan\left(\frac{6t}{11}\right) + 10t - 15 &= 0\\(2t-3)\tan\left(\frac{6t}{11}\right) + 5(2t-3) &= 0\\(2t-3)\left(\tan\left(\frac{6t}{11}\right) + 5\right) &= 0\end{aligned}$$

Now, we have a product of two factors that equals zero and so by basic algebraic properties we know that we must have,

$$2t-3=0 \qquad \text{OR} \qquad \tan\left(\frac{6t}{11}\right)+5=0$$

Be careful with this type of equation to not make the mistake of just canceling the $2t-3$ from both sides. Had you done that you would have missed the solution from the first equation.

When solving equations it is important to remember that you can't cancel anything from both sides unless you know for a fact that what you are canceling will never be zero.

Hint

Solve each of these two equations to attain all the solutions to the original equation.

Step 2

Solving the first equation gives,

$$t = \frac{3}{2}$$

Now we can move onto the second equation and note that this equation is similar to equations solved in the previous section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

First, isolating the tangent gives,

$$\tan\left(\frac{6t}{11}\right) = -5$$

Using our calculator we get,

$$\frac{6t}{11} = \tan^{-1}(-5) = -1.3734$$

From our knowledge of the unit circle we can see that a positive angle that corresponds to this angle is $2\pi - 1.3734 = 4.9098$. Either these angles can be used here but we'll use the positive angle to avoid the possibility of losing the minus sign. Also, the second angle in the range $[0, 2\pi]$ is $\pi + (-1.3734) = 1.7682$.

Finally, the solutions to this equation are,

$$\frac{6t}{11} = 1.7682 + 2\pi n \qquad \text{OR} \qquad \frac{6t}{11} = 4.9098 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$t = 3.2417 + \frac{11\pi n}{3} \qquad \text{OR} \qquad t = 9.0013 + \frac{11\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Putting all of this together gives the following set of solutions.

$$\boxed{t = \frac{3}{2}, \quad t = 3.2417 + \frac{11\pi n}{3}, \quad \text{OR} \quad t = 9.0013 + \frac{11\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots}$$

If an interval had been given we would next proceed with plugging in values of n to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4^{th} decimal place or so however.

1.7 Exponential Functions

1. Sketch the graph of $f(x) = 3^{1+2x}$.

Solution

There are several methods that can be used for getting the graph of this function. One way would be to use some of the various algebraic transformations. The point of the problems in this section however are more to force you to do some evaluation of these kinds of functions to make sure you can do them. So, while you could use transformations, we'll be doing these the "old fashioned" way of plotting points. If you'd like some practice of the transformations you can check out the practice problems for the Common Graphs section of this chapter.

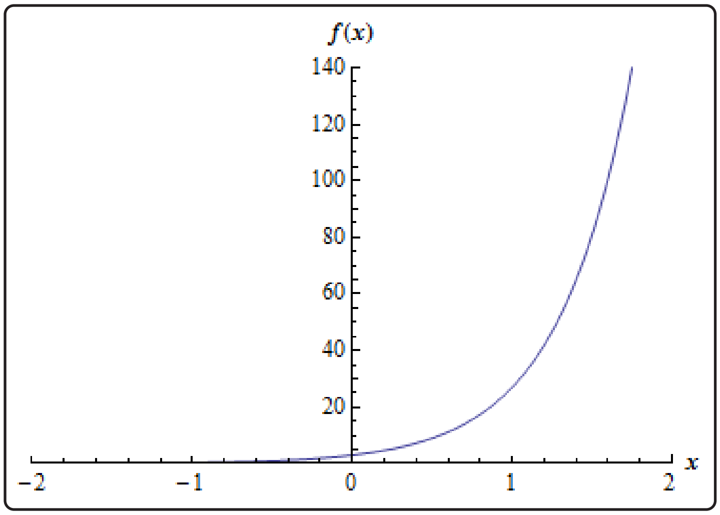
So, with that out of the way here is a table of values for this function.

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{27}$	$\frac{1}{3}$	3	27	243

A natural question at this point is "how did we know to use these values of x"? That is a good question and not always an easy one to answer. For exponential functions the key is to recall that when the exponent is positive the function will grow very quickly and when the exponent is negative the function will quickly get close to zero. This means that *often* (but not always) we'll want to keep the exponent in the range of about $[-4, 4]$ and by exponent we mean the value of $1 + 2x$ after we plug in the x.

Note that we often won't need the whole range given above to see what the curve looks like. As we plug in values of x we can look at our answers and if they aren't changing much then we'll know that the exponent has gone far enough in the negative direction so that the exponential is essentially zero. Likewise, once the value really starts changing fast we'll know that the exponent has gone far enough in the positive direction as well. The given above is just a way to give us some starting values of x and nothing more.

Here is the sketch of the graph of this function.



2. Sketch the graph of $h\left(x\right) = 2^{3-\frac{x}{4}} - 7$.

Solution

There are several methods that can be used for getting the graph of this function. One way would be to use some of the various algebraic transformations. The point of the problems in this section however are more to force you to do some evaluation of these kinds of functions to make sure you can do them. So, while you could use transformations, we'll be doing these the "old fashioned" way of plotting points. If you'd like some practice of the transformations you can check out the practice problems for the Common Graphs section of this chapter.

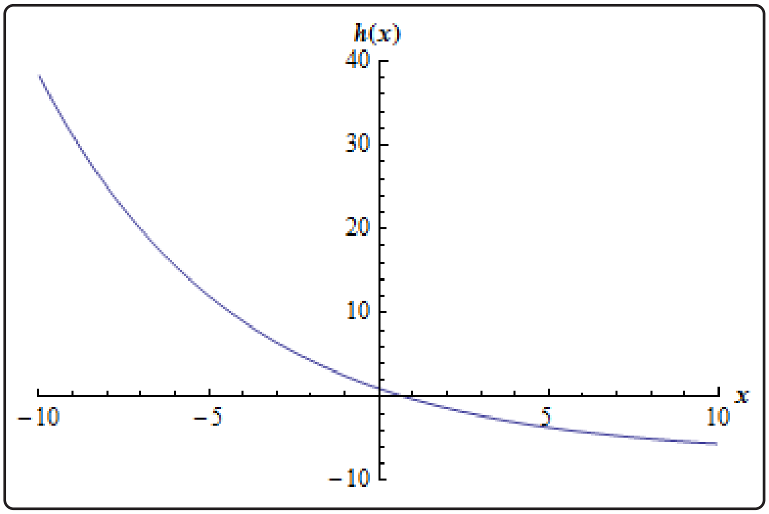
So, with that out of the way here is a table of values for this function.

x	-10	-6	-2	0	2	6	10
$h\left(x\right)$	38.2548	15.6274	4.3137	1	-1.3431	-4.1716	-5.5858

A natural question at this point is "how did we know to use these values of x"? That is a good question and not always an easy one to answer. For exponential functions the key is to recall that when the exponent is positive the function will grow very quickly and when the exponent is negative the function will quickly get close to zero. This means that *often* (but not always) we'll want to keep the exponent in the range of about $[-4, 4]$ and by exponent we mean the value of $3 - \frac{x}{4}$ after we plug in the x.

Note that we often won't need the whole range given above to see what the curve looks like. As we plug in values of x we can look at our answers and if they aren't changing much then we'll know that the exponent has gone far enough in the negative direction so that the exponential is essentially zero. Likewise, once the value really starts changing fast we'll know that the exponent has gone far enough in the positive direction as well. The given above is just a way to give us some starting values of x and nothing more.

Here is the sketch of the graph of this function.



3. Sketch the graph of $h(t) = 8 + 3\mathbf{e}^{2t-4}$.

Solution

There are several methods that can be used for getting the graph of this function. One way would be to use some of the various algebraic transformations. The point of the problems in this section however are more to force you to do some evaluation of these kinds of functions to make sure you can do them. So, while you could use transformations, we'll be doing these the "old fashioned" way of plotting points. If you'd like some practice of the transformations you can check out the practice problems for the Common Graphs section of this chapter.

So, with that out of the way here is a table of values for this function.

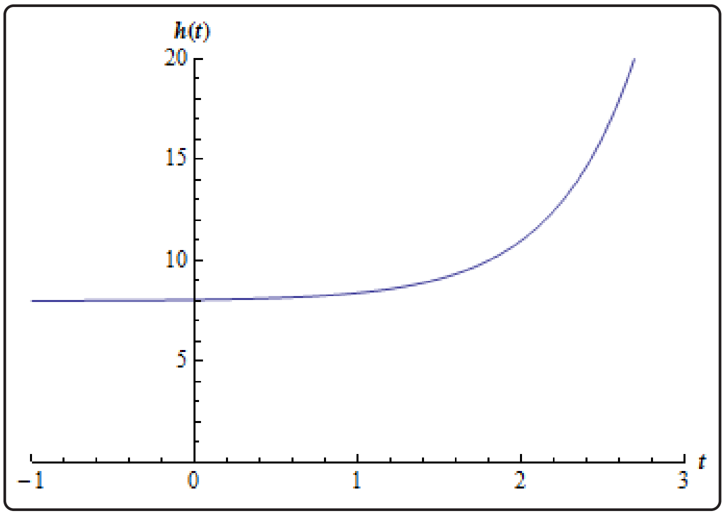
t	-1	0	1	2	3
$h(t)$	8.0074	8.0549	8.4060	11	30.1672

A natural question at this point is "how did we know to use these values of t"? That is a good question and not always an easy one to answer. For exponential functions the key is to recall that when the exponent is positive the function will grow very quickly and when the exponent is negative the function will quickly get close to zero. This means that *often* (but not always) we'll want to keep the exponent in the range of about $[-4, 4]$

and by exponent we mean the value of $2t - 4$ after we plug in the t.

Note that we often won't need the whole range given above to see what the curve looks like. As we plug in values of t we can look at our answers and if they aren't changing much then we'll know that the exponent has gone far enough in the negative direction so that the exponential is essentially zero. Likewise, once the value really starts changing fast we'll know that the exponent has gone far enough in the positive direction as well. The given above is just a way to give us some starting values of t and nothing more.

Here is the sketch of the graph of this function.



4. Sketch the graph of $g(z) = 10 - \frac{1}{4}\mathbf{e}^{-2-3z}$.

Solution

There are several methods that can be used for getting the graph of this function. One way would be to use some of the various algebraic transformations. The point of the problems in this section however are more to force you to do some evaluation of these kinds of functions to make sure you can do them. So, while you could use transformations, we'll be doing these the "old fashioned" way of plotting points. If you'd like some practice of the transformations you can check out the practice problems for the Common Graphs section of this chapter.

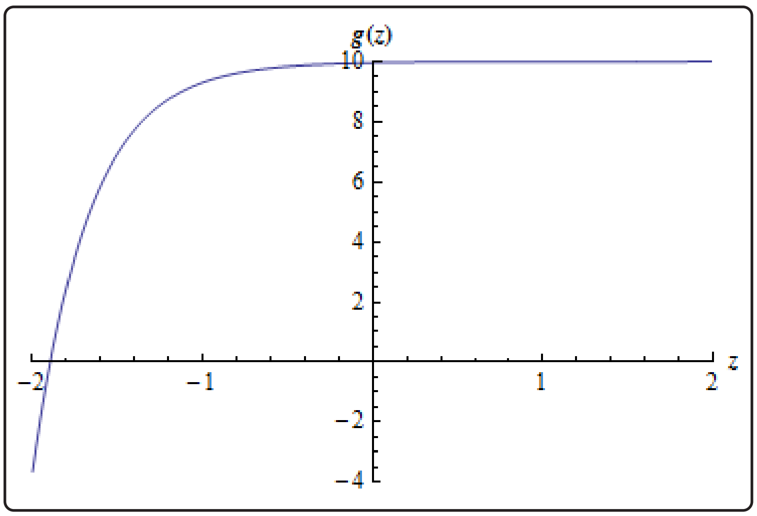
So, with that out of the way here is a table of values for this function.

z	-2	-1	0	1	2
$g(z)$	-3.6495	9.3204	9.9662	9.9983	9.9999

A natural question at this point is "how did we know to use these values of z"? That is a good question and not always an easy one to answer. For exponential functions the key is to recall that when the exponent is positive the function will grow very quickly and when the exponent is negative the function will quickly get close to zero. This means that *often* (but not always) we'll want to keep the exponent in the range of about $[-4, 4]$ and by exponent we mean the value of $-2 - 3z$ after we plug in the z.

Note that we often won't need the whole range given above to see what the curve looks like. As we plug in values of z we can look at our answers and if they aren't changing much then we'll know that the exponent has gone far enough in the negative direction so that the exponential is essentially zero. Likewise, once the value really starts changing fast we'll know that the exponent has gone far enough in the positive direction as well. The given above is just a way to give us some starting values of z and nothing more.

Here is the sketch of the graph of this function.



1.8 Logarithm Functions

1. Without using a calculator determine the exact value of $\log_3 81$.

Hint

Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Solution

Converting the logarithm to exponential form gives,

$$\log_3 81 = ? \qquad \Rightarrow \qquad 3^? = 81$$

From this we can quickly see that $3^4 = 81$ and so we must have,

$$\boxed{\log_3 81 = 4}$$

2. Without using a calculator determine the exact value of $\log_5 125$.

Hint

Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Solution

Converting the logarithm to exponential form gives,

$$\log_5 125 = ? \qquad \Rightarrow \qquad 5^? = 125$$

From this we can quickly see that $5^3 = 125$ and so we must have,

$$\boxed{\log_5 125 = 3}$$

3. Without using a calculator determine the exact value of $\log_2 \frac{1}{8}$.

Hint

Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Solution

Converting the logarithm to exponential form gives,

$$\log_2 \frac{1}{8} = ? \qquad \Rightarrow \qquad 2^{?} = \frac{1}{8}$$

Now, we know that if we raise an integer to a negative exponent we'll get a fraction and so we must have a negative exponent and then we know that $2^3 = 8$. Therefore we can see that $2^{-3} = \frac{1}{8}$ and so we must have,

$$\boxed{\log_2 \frac{1}{8} = -3}$$

4. Without using a calculator determine the exact value of $\log_{\frac{1}{4}} 16$.

Hint

Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Solution

Converting the logarithm to exponential form gives,

$$\log_{\frac{1}{4}} 16 = ? \qquad \Rightarrow \qquad \left(\frac{1}{4}\right)^{?} = 16$$

Now, we know that if we raise an fraction to a power and get an integer out we must have had a negative exponent. Now, we also know that $4^2 = 16$. Therefore we can see that $\left(\frac{1}{4}\right)^{-2} = \left(\frac{4}{1}\right)^{2} = 16$ and so we must have,

$$\boxed{\log_{\frac{1}{4}} 16 = -2}$$

5. Without using a calculator determine the exact value of $\ln \mathbf{e}^4$.

Hint

Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms. Also recall what the base is for a natural logarithm.

Solution

Recalling that the base for a natural logarithm is **e** and converting the logarithm to exponential form gives,

$$\ln \mathbf{e}^4 = \log_{\mathbf{e}} \mathbf{e}^4 = ? \qquad \Rightarrow \qquad \mathbf{e}^? = \mathbf{e}^4$$

From this we can quickly see that $\mathbf{e}^4 = \mathbf{e}^4$ and so we must have,

$$\boxed{\ln \mathbf{e}^4 = 4}$$

Note that an easier method of determining the value of this logarithm would have been to recall the properties of logarithm. In particular the property that states,

$$\log_b b^x = x$$

Using this we can also very quickly see what the value of the logarithm is.

6. Without using a calculator determine the exact value of $\log \frac{1}{100}$.

Hint

Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms. Also recall what the base is for a common logarithm.

Solution

Recalling that the base for a common logarithm is 10 and converting the logarithm to exponential form gives,

$$\log \frac{1}{100} = \log_{10} \frac{1}{100} = ? \qquad \Rightarrow \qquad 10^? = \frac{1}{100}$$

Now, we know that if we raise an integer to a negative exponent we'll get a fraction and so we must have a negative exponent and then we know that $10^2 = 100$. Therefore we can see that $10^{-2} = \frac{1}{100}$ and so we must have,

$$\boxed{\log \frac{1}{100} = -2}$$

---

7. Write $\log\left(3x^4y^{-7}\right)$ in terms of simpler logarithms.

Solution

So, we're being asked here to use as many of the properties as we can to reduce this down into simpler logarithms. So, here is the work for this problem.

$$\begin{aligned}\log\left(3x^4y^{-7}\right) &= \log(3) + \log\left(x^4\right) + \log\left(y^{-7}\right) \\ &= \boxed{\log(3) + 4\log(x) - 7\log(y)}\end{aligned}$$

Remember that we can only bring an exponent out of a logarithm if is on the whole argument of the logarithm. In other words, we couldn't bring any of the exponents out of the logarithms until we had dealt with the product.

---

8. Write $\ln\left(x\sqrt{y^2+z^2}\right)$ in terms of simpler logarithms.

Solution

So, we're being asked here to use as many of the properties as we can to reduce this down into simpler logarithms. So, here is the work for this problem.

$$\begin{aligned}\ln\left(x\sqrt{y^2+z^2}\right) &= \ln(x) + \ln\left(\left(y^2+z^2\right)^{\frac{1}{2}}\right) \\ &= \boxed{\ln(x) + \frac{1}{2}\ln\left(y^2+z^2\right)}\end{aligned}$$

Remember that we can only bring an exponent out of a logarithm if is on the whole argument of the logarithm. In other words, we couldn't bring any of the exponents out of the logarithms until we had dealt with the product. Also, in the second logarithm while each term is squared the whole argument is not squared, *i.e.* it's not $(x+y)^2$ and so we can't bring those 2's out of the logarithm.

9. Write $\log_4\left(\dfrac{x-4}{y^2\sqrt[5]{z}}\right)$ in terms of simpler logarithms.

Solution

So, we're being asked here to use as many of the properties as we can to reduce this down into simpler logarithms. So, here is the work for this problem.

$$\begin{aligned}\log_4\left(\frac{x-4}{y^2\sqrt[5]{z}}\right) &= \log_4(x-4) - \log_4\left(y^2 z^{\frac{1}{5}}\right)\\ &= \log_4(x-4) - \left(\log_4\left(y^2\right) + \log_4\left(z^{\frac{1}{5}}\right)\right)\\ &= \boxed{\log_4(x-4) - 2\log_4(y) - \frac{1}{5}\log_4(z)}\end{aligned}$$

Remember that we can only bring an exponent out of a logarithm if is on the whole argument of the logarithm. In other words, we couldn't bring any of the exponents out of the logarithms until we had dealt with the quotient and product. Recall as well that we can't split up an sum/difference in a logarithm. Finally, make sure that you are careful in dealing with the minus sign we get from breaking up the quotient when dealing with the product in the denominator.

10. Combine $2\log_4(x) + 5\log_4(y) - \dfrac{1}{2}\log_4(z)$ into a single logarithm with a coefficient of one.

Hint

The properties that we use to break up logarithms can be used in reverse as well.

Solution

To convert this into a single logarithm we'll be using the properties that we used to break up logarithms in reverse. The first step in this process is to use the property,

$$\log_b\left(x^r\right) = r\log_b(x)$$

to make sure that all the logarithms have coefficients of one. This needs to be done first because all the properties that allow us to combine sums/differences of logarithms require coefficients of one on individual logarithms. So, using this property gives,

$$\log_4\left(x^2\right) + \log_4\left(y^5\right) - \log_4\left(z^{\frac{1}{2}}\right)$$

Now, there are several ways to proceed from this point. We can use either of the two properties,

$$\log_b(xy) = \log_b(x) + \log_b(y) \qquad \log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

and in fact we'll need to use both in the end. Which we use first does not matter as we'll end up with the same result in the end. Here is the rest of the work for this problem.

$$2\log_4(x) + 5\log_4(y) - \frac{1}{2}\log_4(z) = \log_4\left(x^2y^5\right) - \log_4\left(\sqrt{z}\right)$$
$$= \boxed{\log_4\left(\frac{x^2y^5}{\sqrt{z}}\right)}$$

Note that the only reason we converted the fractional exponent to a root was to make the final answer a little nicer.

11. Combine $3\ln(t+5) - 4\ln(t) - 2\ln(s-1)$ into a single logarithm with a coefficient of one.

Hint

The properties that we use to break up logarithms can be used in reverse as well.

Solution

To convert this into a single logarithm we'll be using the properties that we used to break up logarithms in reverse. The first step in this process is to use the property,

$$\log_b(x^r) = r\log_b(x)$$

to make sure that all the logarithms have coefficients of one. This needs to be done first because all the properties that allow us to combine sums/differences of logarithms require coefficients of one on individual logarithms. So, using this property gives,

$$\ln(t+5)^3 - \ln\left(t^4\right) - \ln(s-1)^2$$

Now, there are several ways to proceed from this point. We can use either of the two properties,

$$\log_b(xy) = \log_b(x) + \log_b(y) \qquad \log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

and in fact we'll need to use both in the end.

We should also be careful with the fact that there are two minus signs in here as that sometimes adds confusion to the problem. They are easy to deal with however if we just factor a minus sign out of the last two terms and then proceed from there as follows.

$$3\ln(t+5) - 4\ln(t) - 2\ln(s-1) = \ln(t+5)^3 - \left(\ln\left(t^4\right) + \ln(s-1)^2\right)$$

$$= \ln(t+5)^3 - \ln\left(t^4(s-1)^2\right) = \boxed{\ln\frac{(t+5)^3}{t^4(s-1)^2}}$$

---

12. Combine $\frac{1}{3}\log(a) - 6\log(b) + 2$ into a single logarithm with a coefficient of one.

Hint

The properties that we use to break up logarithms can be used in reverse as well. For the constant see if you figure out a way to write that as a logarithm.

Solution

To convert this into a single logarithm we'll be using the properties that we used to break up logarithms in reverse. The first step in this process is to use the property,

$$\log_b\left(x^r\right) = r\log_b(x)$$

to make sure that all the logarithms have coefficients of one. This needs to be done first because all the properties that allow us to combine sums/differences of logarithms require coefficients of one on individual logarithms. So, using this property gives,

$$\log\left(a^{\frac{1}{3}}\right) - \log\left(b^6\right) + 2$$

Now, for the 2 let's notice that we can write this in terms of a logarithm as,

$$2 = \log 10^2 = \log 100$$

Note that this is really just using the property,

$$\log_b b^x = x$$

So, we now have,

$$\log\left(a^{\frac{1}{3}}\right) - \log\left(b^6\right) + \log 100$$

Now, there are several ways to proceed from this point. We can use either of the two properties,

$$\log_b(xy) = \log_b(x) + \log_b(y) \qquad \log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

and in fact we'll need to use both in the end. Which we use first does not matter as we'll end up with the same result in the end. Here is the rest of the work for this problem.

$$\log\left(a^{\frac{1}{3}}\right) - \log\left(b^6\right) + \log 10^2 = \log\left(100\sqrt[3]{a}\right) - \log\left(b^6\right)$$
$$= \boxed{\log\left(\frac{100\sqrt[3]{a}}{b^6}\right)}$$

Note that the only reason we converted the fractional exponent to a root was to make the final answer a little nicer.

---

13. Use the change of base formula and a calculator to find the value of $\log_{12}(35)$.

Solution

We can use either the natural logarithm or the common logarithm to do this so we'll do both.

$$\log_{12}(35) = \frac{\ln(35)}{\ln(12)} = \frac{3.55534806}{2.48490665} = \boxed{1.43077731}$$

$$\log_{12}(35) = \frac{\log(35)}{\log(12)} = \frac{1.54406804}{1.07918125} = \boxed{1.43077731}$$

So, as we noted at the start it doesn't matter which logarithm we use we'll get the same answer in the end.

---

14. Use the change of base formula and a calculator to find the value of $\log_{\frac{2}{3}}(53)$.

Solution

We can use either the natural logarithm or the common logarithm to do this so we'll do both.

$$\log_{\frac{2}{3}}(53) = \frac{\ln(53)}{\ln\left(\frac{2}{3}\right)} = \frac{3.97029191}{-0.40546511} = \boxed{-9.79194469}$$

$$\log_{\frac{2}{3}}(53) = \frac{\log(53)}{\log\left(\frac{2}{3}\right)} = \frac{1.72427587}{-0.17609126} = \boxed{-9.79194469}$$

So, as we noted at the start it doesn't matter which logarithm we use we'll get the same answer in the end.

1.9 Exponential And Logarithm Equations

1. Find all the solutions to $12 - 4\mathbf{e}^{7+3x} = 7$. If there are no solutions clearly explain why.

Step 1

There isn't all that much to do here for this equation. First, we need to isolate the exponential on one side by itself with a coefficient of one.

$$-4\mathbf{e}^{7+3x} = -5 \qquad \Rightarrow \qquad \mathbf{e}^{7+3x} = \frac{5}{4}$$

Step 2

Now all we need to do is take the natural logarithm of both sides and then solve for x.

$$\begin{aligned} \ln\left(\mathbf{e}^{7+3x}\right) &= \ln\left(\frac{5}{4}\right) \\ 7+3x &= \ln\left(\frac{5}{4}\right) \\ x &= \boxed{\frac{1}{3}\left(\ln\left(\frac{5}{4}\right) - 7\right) = -2.25895} \end{aligned}$$

Depending upon your preferences either the exact or decimal solution can be used.

2. Find all the solutions to $1 = 10 - 3\mathbf{e}^{z^2-2z}$. If there are no solutions clearly explain why.

Step 1

There isn't all that much to do here for this equation. First, we need to isolate the exponential on one side by itself with a coefficient of one.

$$-9 = -3\mathbf{e}^{z^2-2z} \qquad \Rightarrow \qquad \mathbf{e}^{z^2-2z} = 3$$

Step 2

Now all we need to do is take the natural logarithm of both sides and then solve for z.

$$\begin{aligned} \ln\left(\mathbf{e}^{z^2-2z}\right) &= \ln(3) \\ z^2-2z &= \ln(3) \\ z^2-2z-\ln(3) &= 0 \end{aligned}$$

Now, before proceeding with the solution here let's pause and make sure that we don't get too excited about the "strangeness" of the quadratic above. If we'd had the quadratic,

$$z^2-2z-5=0$$

for instance, we'd know that all we would need to do is use the quadratic formula to get the solutions.

That's all we need to as well for the quadratic that we have from our work. Of course we don't have a 5 we have a $ln(3)$, but $ln(3)$ is just a number and so we can use the quadratic formula to find the solutions here as well. Here is the work for that.

$$\begin{aligned} z &= \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-\ln(3))}}{2(1)} = \frac{2\pm\sqrt{4+4\ln(3)}}{2} \\ &= \frac{2\pm\sqrt{4(1+\ln(3))}}{2} = \frac{2\pm2\sqrt{1+\ln(3)}}{2} \\ &= \boxed{1\pm\sqrt{1+\ln(3)} = -0.4487, \;\; 2.4487} \end{aligned}$$

Notice that we did a little simplification on the root. This doesn't need to be done, but can make the exact solution a little easier to deal with. Also, depending upon your preferences either the exact or decimal solution can be used.

Before leaving this solution we should again make a point that not all quadratics will be the "simple" type of quadratics that you may be used to solving from an Algebra class. They can, and often will be, messier that those. That doesn't mean that you can't solve them. They are, for all intents and purposes, identical to the types of problems you are used to working. The only real difference is that they numbers are a little messier.

So, don't get too excited about this kind of problem. They will happen on occasion and you are capable of solving them!

3. Find all the solutions to $2t - t\mathbf{e}^{6t-1} = 0$. If there are no solutions clearly explain why.

Hint

Be careful to not cancel terms that shouldn't be canceled. Remember that you can't cancel something unless you know for a fact that it won't ever be zero. Also, note that if you can cancel something then it can be factored out of the equation.

Step 1

First notice that we can factor a t out of both terms to get,

$$t\left(2 - \mathbf{e}^{6t-1}\right) = 0$$

Be careful to not cancel the t from both terms. When solving equations you can only cancel something if you know for a fact that it won't be zero. If the term can be zero and you cancel it you will miss solutions and that will the case here.

Step 2

We now have a product of terms that is equal to zero so we know,

$$t = 0 \qquad \text{OR} \qquad 2 - \mathbf{e}^{6t-1} = 0$$

So, we have one solution already, $t = 0$, and again note that if we had canceled the t at the beginning we would have missed this solution. Now all we need to do is solve the equation involving the exponential.

Step 3

We can now solve the exponential equation in the same manner as the first couple of problems in this section.

$$\begin{aligned} \mathbf{e}^{6t-1} &= 2 \\ \ln\left(\mathbf{e}^{6t-1}\right) &= \ln(2) \\ 6t - 1 &= \ln(2) \\ t &= \frac{1}{6}\left(1 + \ln(2)\right) = 0.2822 \end{aligned}$$

Depending upon your preferences either the exact or decimal solution can be used.

Step 4

So, we have the following solutions to this equation.

$$\boxed{t = 0 \qquad \text{OR} \qquad t = \frac{1}{6}\left(1 + \ln(2)\right) = 0.2822}$$

4. Find all the solutions to $4x + 1 = (12x + 3)\,\mathbf{e}^{x^2-2}$. If there are no solutions clearly explain why.

Hint

Be careful to not cancel terms that shouldn't be canceled. Remember that you can't cancel something unless you know for a fact that it won't ever be zero. Also, note that if you can cancel something then it can be factored out of the equation.

Step 1

It may not be apparent at first glance, but with some work we can do a little factoring on this equation. To do that first move everything to one side and then the factoring might become a little more apparent.

$$\begin{aligned} 4x + 1 - (12x + 3)\,\mathbf{e}^{x^2-2} &= 0 \\ (4x + 1) - 3(4x + 1)\,\mathbf{e}^{x^2-2} &= 0 \\ (4x + 1)\left(1 - 3\mathbf{e}^{x^2-2}\right) &= 0 \end{aligned}$$

Note that in the second step we put parenthesis around the first couple of terms solely to make the factoring in the next step a little more apparent. It does not need to be done in practice.

Be careful to not cancel the $4x + 1$ from both terms. When solving equations you can only cancel something if you know for a fact that it won't be zero. If the term can be zero and you cancel it you will miss solutions, and that will be the case here.

Step 2

We now have a product of terms that is equal to zero so we know,

$$4x + 1 = 0 \qquad \text{OR} \qquad 1 - 3\mathbf{e}^{x^2-2} = 0$$

From the first equation we can quickly arrive at one solution, $x = -\frac{1}{4}$, and again note that if we had canceled the $4x + 1$ at the beginning we would have missed this solution. Now all we need to do is solve the equation involving the exponential.

Step 3

We can now solve the exponential equation in the same manner as the first couple of problems in this section.

$$\begin{aligned} \mathbf{e}^{x^2-2} &= \frac{1}{3} \\ \ln\left(\mathbf{e}^{x^2-2}\right) &= \ln\left(\frac{1}{3}\right) \\ x^2 - 2 &= \ln\left(\frac{1}{3}\right) \\ x^2 &= 2 + \ln\left(\frac{1}{3}\right) \\ x &= \pm\sqrt{2 + \ln\left(\frac{1}{3}\right)} = \pm 0.9494 \end{aligned}$$

Depending upon your preferences either the exact or decimal solution can be used.

Step 4

So, we have the following solutions to this equation.

$$\boxed{x = -\frac{1}{4} \qquad \text{OR} \qquad x = \pm\sqrt{2 + \ln\left(\frac{1}{3}\right)} = \pm 0.9494}$$

5. Find all the solutions to $2\mathbf{e}^{3y+8} - 11\mathbf{e}^{5-10y} = 0$. If there are no solutions clearly explain why.

Hint

The best way to proceed here is to reduce the equation down to a single exponential.

Step 1

With both exponentials in the equation this may be a little difficult to solve, so let's do some work to reduce this down to an equation with a single exponential.

$$\begin{aligned} 2\mathbf{e}^{3y+8} &= 11\mathbf{e}^{5-10y} \\ \frac{\mathbf{e}^{3y+8}}{\mathbf{e}^{5-10y}} &= \frac{11}{2} \\ \mathbf{e}^{13y+3} &= \frac{11}{2} \end{aligned}$$

Note that we could have divided by either exponential but by dividing by the one that we did we avoid a negative exponent on the y, which is sometimes easy to lose track of.

Step 2

Now all we need to do is take the logarithm of both sides and solve for y.

$$\begin{aligned} \ln\left(\mathbf{e}^{13y+3}\right) &= \ln\left(\frac{11}{2}\right) \\ 13y + 3 &= \ln\left(\frac{11}{2}\right) \\ y &= \boxed{\frac{1}{13}\left(\ln\left(\frac{11}{2}\right) - 3\right) = -0.09963} \end{aligned}$$

Depending upon your preferences either the exact or decimal solution can be used.

6. Find all the solutions to $14\mathbf{e}^{6-x} + \mathbf{e}^{12x-7} = 0$. If there are no solutions clearly explain why.

Hint

The best way to proceed here is to reduce the equation down to a single exponential.

Solution

With both exponentials in the equation this may be a little difficult to solve, so let's do some work to reduce this down to an equation with a single exponential.

$$\begin{aligned}14\mathbf{e}^{6-x} &= -\mathbf{e}^{12x-7}\\ \frac{\mathbf{e}^{12x-7}}{\mathbf{e}^{6-x}} &= -14\\ \mathbf{e}^{13x-13} &= -14\end{aligned}$$

At this point we can stop. We know that exponential functions are always positive and there is no way for this to be negative and therefore there is **no solution** to this equation.

Note that if we hadn't caught the exponent being negative our next step would have been to take the logarithm of both side and we also know that we can only take the logarithm of positive numbers and so again we'd see that there is no solution to this equation.

---

7. Find all the solutions to $1 - 8\ln\left(\frac{2x-1}{7}\right) = 14$. If there are no solutions clearly explain why.

Step 1

There isn't all that much to do here for this equation. First, we need to isolate the logarithm on one side by itself with a coefficient of one.

$$1 - 8\ln\left(\frac{2x-1}{7}\right) = 14 \qquad \Rightarrow \qquad \ln\left(\frac{2x-1}{7}\right) = -\frac{13}{8}$$

Step 2

Now all we need to do is exponentiate both sides using **e** (because we're working with the natural logarithm) and then solve for x.

$$\begin{aligned}\mathbf{e}^{\ln\left(\frac{2x-1}{7}\right)} &= \mathbf{e}^{-\frac{13}{8}}\\ \frac{2x-1}{7} &= \mathbf{e}^{-\frac{13}{8}}\\ x &= \frac{1}{2}\left(1 + 7\mathbf{e}^{-\frac{13}{8}}\right) = 1.1892\end{aligned}$$

Step 3

We're dealing with logarithms so we need to make sure that we won't have any problems with any of our potential solutions. In other words, we need to make sure that if we plug in the potential solution into the original equation we won't end up taking the logarithm of a negative number or zero.

Plugging in we can see that we won't be taking the logarithm of a negative number and so the solution is,

$$\boxed{x = \frac{1}{2}\left(1 + 7\mathbf{e}^{-\frac{13}{8}}\right) = 1.1892}$$

Depending upon your preferences either the exact or decimal solution can be used.

---

8. Find all the solutions to $\ln\left(y-1\right) = 1 + \ln\left(3y+2\right)$. If there are no solutions clearly explain why.

Hint

Don't forget about the basic logarithm properties and how they can be used to combine multiple logarithms into a single logarithm.

Step 1

We need to reduce this down to an equation with a single logarithm and to do that we first should rewrite it a little. Upon doing that we can use the basic logarithm properties to combine the two logarithms into a single logarithm as follows,

$$\begin{aligned} \ln\left(y-1\right) - \ln\left(3y+2\right) &= 1 \\ \ln\left(\frac{y-1}{3y+2}\right) &= 1 \end{aligned}$$

Step 2

Now all we need to do is exponentiate both sides using $\mathbf{e}$ (because we're working with the natural logarithm) and then solve for y.

$$\begin{aligned} \mathbf{e}^{\ln\left(\frac{y-1}{3y+2}\right)} &= \mathbf{e}^1 \\ \frac{y-1}{3y+2} &= \mathbf{e} \\ y-1 &= \mathbf{e}\left(3y+2\right) = 3\mathbf{e}y+2\mathbf{e} \\ \left(1-3\mathbf{e}\right)y &= 1+2\mathbf{e} \\ y &= \frac{1+2\mathbf{e}}{1-3\mathbf{e}} = -0.8996 \end{aligned}$$

Step 3

We're dealing with logarithms so we need to make sure that we won't have any problems with any of our potential solutions. In other words, we need to make sure that if we plug in the potential solutions into the original equation we won't end up taking the logarithm of a negative number or zero.

Upon inspection we can quickly see that if we plug in our potential solution into the first logarithm we'll be taking the logarithm of a negative number. The same will be true for the second logarithm and so $y = -0.8996$ can't be a solution.

Because this was our only potential solution we know now that there will be **no solutions** to this equation.

---

9. Find all the solutions to $\log\left(w\right)+\log\left(w-21\right)=2$. If there are no solutions clearly explain why.

Hint

Don't forget about the basic logarithm properties and how they can be used to combine multiple logarithms into a single logarithm.

Step 1

We need to reduce this down to an equation with a single logarithm and to do that we first should rewrite it a little. Upon doing that we can use the basic logarithm properties to combine the two logarithms into a single logarithm as follows,

$$\begin{aligned} \log\left(w\left(w-21\right)\right) &= 2 \\ \log\left(w^2-21w\right) &= 2 \end{aligned}$$

Step 2

Now all we need to do is exponentiate both sides using 10 (because we're working with the common logarithm) and then solve for y.

$$\begin{aligned}
\log\left(w^2 - 21w\right) &= 2\\
10^{\log\left(w^2-21w\right)} &= 10^2\\
w^2 - 21w &= 100\\
w^2 - 21w - 100 &= 0\\
(w-25)(w+4) &= 0 \qquad \Rightarrow \qquad w = -4, \quad w = 25
\end{aligned}$$

Step 3

We're dealing with logarithms so we need to make sure that we won't have any problems with any of our potential solutions. In other words, we need to make sure that if we plug either of the two potential solutions into the original equation we won't end up taking the logarithm of a negative number or zero.

Upon inspection we can quickly see that if we plug in $w = -4$ we will be taking a logarithm of a negative number (in both of the logarithms in this case) and so $w = -4$ can't be a solution. On the other hand, if we plug in $w = 25$ we won't be taking logarithms of negative numbers and so $w = 25$ is a solution.

In summary then, the only solution to the equation is : $\boxed{w = 25}$.

10. Find all the solutions to $2\log(z) - \log(7z-1) = 0$. If there are no solutions clearly explain why.

Hint

This problem can be worked in the same manner as the previous two or because each term is a logarithm an easier solution would be to use the fact that,

$$\text{If } \log_b(x) = \log_b(y) \text{ then } x = y$$

Step 1

While we could use the same method we used in the previous couple of examples to solve this equation there is an easier method. Because each of the terms is a logarithm and it's all equal to zero we can use the fact that,

$$\text{If } \log_b(x) = \log_b(y) \text{ then } x = y$$

So, a quick rewrite of the equation gives,

$$\begin{aligned} 2\log(z) &= \log(7z-1) \\ \log(z^2) &= \log(7z-1) \end{aligned}$$

Note that in order to use the fact above we need both logarithms to have coefficients of one so we also had to make quick use of one of the logarithm properties to make sure we had a coefficient of one.

Step 2

Now all we need to do use the fact and solve for z.

$$\begin{aligned} z^2 &= 7z-1 \\ z^2 - 7z + 1 &= 0 \end{aligned}$$

In this case we'll need to use the quadratic formula to finish this out.

$$z = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(1)}}{2(1)} = \frac{7 \pm \sqrt{45}}{2} = 0.1459, \quad 6.8541$$

Step 3

We're dealing with logarithms so we need to make sure that we won't have any problems with any of our potential solutions. In other words, we need to make sure that if we plug either of the two potential solutions into the original equation we won't end up taking the logarithm of a negative number or zero.

In this case it is pretty easy to plug them in and see that neither of the two potential solutions will result in taking logarithms of negative numbers and so both are solutions to the equation.

In summary then, the solutions to the equation are,

$$\boxed{z = \frac{7 \pm \sqrt{45}}{2} = 0.1459, \quad 6.8541}$$

Depending upon your preferences either the exact or decimal solution can be used.

Before leaving this solution we should again make a point that not all quadratics will be the "simple" type of quadratics that you may be used to solving from an Algebra class. They can, and often will be, messier than those. That doesn't mean that you can't solve them. They are, for all intents and purposes, identical to the types of problems you are used to working. The only real difference is that the numbers are a little messier.

So, don't get too excited about this kind of problem. They will happen on occasion and you are capable of solving them!

---

11. Find all the solutions to $16 = 17^{t-2} + 11$. If there are no solutions clearly explain why.

Hint

These look a little different from the first few problems in this section, but they work in essentially the same manner. The main difference is that we're not dealing with **e**$^{\text{power}}$ or 10^{power} and so there is no obvious logarithm to use and so can use any logarithm.

Step 1

First we need to isolate the term with the exponent in it on one side by itself.

$$17^{t-2} = 5$$

Step 2

At this point we need to take the logarithm of both sides so we can use logarithm properties to get the t out of the exponent. It doesn't matter which logarithm we use, but if we want a decimal value for the answer it will need to be one that we can work with. For this solution we'll use the natural logarithm.

Upon taking the logarithm we then need to use logarithm properties to get the t's out of the exponent at which point we can solve for t. Here is the rest of the work for this problem,

$$\begin{aligned}\ln\left(17^{t-2}\right) &= \ln(5)\\ (t-2)\ln(17) &= \ln(5)\\ t-2 &= \frac{\ln(5)}{\ln(17)}\\ x &= \boxed{2+\frac{\ln(5)}{\ln(17)} = 2.5681}\end{aligned}$$

Depending upon your preferences either the exact or decimal solution can be used. Also note that if you had used, say the common logarithm, you would get exactly the same answer.

---

12. Find all the solutions to $2^{3-8w} - 7 = 11$. If there are no solutions clearly explain why.

Hint

These look a little different from the first few problems in this section, but they work in essentially the same manner. The main difference is that we're not dealing with **e**$^{\text{power}}$ or 10^{power} and so there is no obvious logarithm to use and so can use any logarithm.

Step 1

First we need to isolate the term with the exponent in it on one side by itself.

$$2^{3-8w} = 18$$

Step 2

At this point we need to take the logarithm of both sides so we can use logarithm properties to get the w out of the exponent. It doesn't matter which logarithm we use, but if we want a decimal value for the answer it will need to be one that we can work with. For this solution we'll use the natural logarithm.

Upon taking the logarithm we then need to use logarithm properties to get the w's out of the exponent at which point we can solve for w. Here is the rest of the work for this problem,

$$\begin{aligned} \ln\left(2^{3-8w}\right) &= \ln(18) \\ (3-8w)\ln(2) &= \ln(18) \\ 3-8w &= \frac{\ln(18)}{\ln(2)} \\ w &= \boxed{\frac{1}{8}\left(3-\frac{\ln(18)}{\ln(2)}\right) = -0.1462} \end{aligned}$$

Depending upon your preferences either the exact or decimal solution can be used. Also note that if you had used, say the common logarithm, you would get exactly the same answer.

---

13. We have \$10,000 to invest for 44 months. How much money will we have if we put the money into an account that has an annual interest rate of 5.5% and interest is compounded

(a) quarterly **(b)** monthly **(c)** continuously

Compound Interest.

If we put P dollars into an account that earns interest at a rate of r (written as a decimal as opposed to the standard percent) for t years then,

(a) if interest is compounded m times per year we will have,

$$A = P\left(1+\frac{r}{m}\right)^{tm}$$

dollars after t years.

(b) if interest is compounded continuously we will have,

$$A = P\mathbf{e}^{rt}$$

dollars after t years.

Hint

There really isn't a whole lot to these other than to identify the quantities and then plug into the appropriate equation and compute the amount. Also note that you'll need to make sure that you don't do too much in the way of rounding with the numbers here. A little rounding can lead to very large errors in these kinds of computations.

Solutions

(a) quarterly

Solution

From the problem statement we can see that,

$$P = 10000 \qquad r = \frac{5.5}{100} = 0.055 \qquad t = \frac{44}{12} = \frac{11}{3}$$

Remember that the value of r must be given as a decimal, *i.e.* the percentage divided by 100. Also remember that t must be in years so we'll need to convert to years.

For this part we are compounding interest rate quarterly and that means it will compound 4 times per year so we also then know that,

$$m = 4$$

At this point all that we need to do is plug into the equation and run the numbers through a calculator to compute the amount of money that we'll have.

$$\begin{aligned} A &= 10000\left(1 + \frac{0.055}{4}\right)^{\frac{11}{3}(4)} = 10000(1.01375)^{\frac{44}{3}} \\ &= 10000\,(1.221760422) = 12217.60 \end{aligned}$$

So, we'll have \$12,217.60 in the account after 44 months.

(b) monthly

Solution

From the problem statement we can see that,

$$P = 10000 \qquad r = \frac{5.5}{100} = 0.055 \qquad t = \frac{44}{12} = \frac{11}{3}$$

Remember that the value of r must be given as a decimal, *i.e.* the percentage divided by 100. Also remember that t must be in years so we'll need to convert to years.

For this part we are compounding interest rate monthly and that means it will compound 12 times per year and so we also then know that,

$$m = 12$$

At this point all that we need to do is plug into the equation and run the numbers through a calculator to compute the amount of money that we'll have.

$$\begin{aligned} A &= 10000\left(1+\frac{0.055}{12}\right)^{\frac{11}{3}(12)} = 10000(1.00453333)^{44} \\ &= 10000\,(1.222876562) = 12228.77 \end{aligned}$$

So, we'll have \$12,228.77 in the account after 44 months.

(c) continuously

Solution

From the problem statement we can see that,

$$P = 10000 \qquad r = \frac{5.5}{100} = 0.055 \qquad t = \frac{44}{12} = \frac{11}{3}$$

Remember that the value of r must be given as a decimal, *i.e.* the percentage divided by 100. Also remember that t must be in years so we'll need to convert to years.

For this part we are compounding continuously so we won't have an m and will be using the other equation and all we have all we need to do the computation so,

$$A = 10000\mathbf{e}^{(0.055)\left(\frac{11}{3}\right)} = 10000\mathbf{e}^{0.2016666667} = 10000\,(1.223440127) = 12234.40$$

So, we'll have \$12,234.40 in the account after 44 months.

---

14. We are starting with \$5000 and we're going to put it into an account that earns an annual interest rate of 12%. How long should we leave the money in the account in order to double our money if interest is compounded

(a) quarterly **(b)** monthly **(c)** continuously

Compound Interest.

If we put P dollars into an account that earns interest at a rate of r (written as a decimal as opposed to the standard percent) for t years then,

(a) if interest is compounded m times per year we will have,

$$A = P\left(1 + \frac{r}{m}\right)^{t\,m}$$

dollars after t years.

(b) if interest is compounded continuously we will have,

$$A = P\mathbf{e}^{r\,t}$$

dollars after t years.

Hint

Identify the given quantities, plug into the appropriate equation and use the techniques from earlier problem to solve for t.

Solutions

(a) quarterly

Solution

From the problem statement we can see that,

$$A = 10000 \qquad P = 5000 \qquad r = \frac{12}{100} = 0.12$$

Remember that the value of r must be given as a decimal, *i.e.* the percentage divided by 100. Also, for this part we are compounding interest rate quarterly and that means it will compound 4 times per year so we also then know that,

$$m = 4$$

Plugging into the equation gives us,

$$10000 = 5000\left(1 + \frac{0.12}{4}\right)^{4t} = 5000(1.03)^{4t}$$

Using the techniques from this section we can solve for t.

$$\begin{aligned} 2 &= 1.03^{4t} \\ \ln(2) &= \ln\left(1.03^{4t}\right) \\ \ln(2) &= 4t\ln(1.03) \\ t &= \boxed{\frac{\ln(2)}{4\ln(1.03)} = 5.8624} \end{aligned}$$

So, we'll double our money in approximately 5.8624 years.

(b) monthly

Solution

From the problem statement we can see that,

$$A = 10000 \qquad P = 5000 \qquad r = \frac{12}{100} = 0.12$$

Remember that the value of r must be given as a decimal, *i.e.* the percentage divided by 100. Also, for this part we are compounding interest rate monthly and that means it will compound 12 times per year so we also then know that,

$$m = 12$$

Plugging into the equation gives us,

$$10000 = 5000\left(1 + \frac{0.12}{12}\right)^{12t} = 5000(1.01)^{12t}$$

Using the techniques from this section we can solve for t.

$$\begin{aligned} 2 &= 1.01^{12t} \\ \ln(2) &= \ln\left(1.01^{12t}\right) \\ \ln(2) &= 12t\ln(1.01) \\ t &= \boxed{\frac{\ln(2)}{12\ln(1.01)} = 5.8051} \end{aligned}$$

So, we'll double our money in approximately 5.8051 years.

(c) continuously

Solution

From the problem statement we can see that,

$$A = 10000 \qquad P = 5000 \qquad r = \frac{12}{100} = 0.12$$

Remember that the value of r must be given as a decimal, *i.e.* the percentage divided by 100. For this part we are compounding continuously so we won't have an m and will be using the other equation.

Plugging into the continuously compounding interest equation gives,

$$10000 = 5000\mathbf{e}^{0.12t}$$

Now, solving this gives,

$$\begin{aligned} 2 &= \mathbf{e}^{0.12t} \\ \ln(2) &= \ln\left(\mathbf{e}^{0.12t}\right) \\ \ln(2) &= 0.12t \\ t &= \boxed{\frac{\ln(2)}{0.12} = 5.7762} \end{aligned}$$

So, we'll double our money in approximately 5.7762 years.

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15. A population of bacteria initially has 250 present and in 5 days there will be 1600 bacteria present.

(a) Determine the exponential growth equation for this population.

(b) How long will it take for the population to grow from its initial population of 250 to a population of 2000?

Exponential Growth/Decay.

Many quantities in the world can be modeled (at least for a short time) by the exponential growth/decay equation.

$$Q = Q_0 \mathbf{e}^{k\,t}$$

If k is positive we will get exponential growth and if k is negative we will get exponential decay.

Solutions

(a) Determine the exponential growth equation for this population.

Hint

We have an equation with two unknowns and two values of the population at two times so use these values to find the two unknowns.

Solution

We can start off here by acknowledging that we know,

$$Q(0) = 250 \qquad \text{and} \qquad Q(5) = 1600$$

If we use the first condition in the equation we get,

$$250 = Q(0) = Q_0 \mathbf{e}^{k(0)} = Q_0 \qquad \rightarrow \qquad Q_0 = 250$$

We now know the first unknown in the equation. Plugging this as well as the second condition into the equation gives us,

$$1600 = Q(5) = 250\mathbf{e}^{5k}$$

We can use techniques from earlier problems in this section to determine the value of k.

$$\begin{aligned}1600 &= 250\mathbf{e}^{5k}\\ \frac{1600}{250} &= \mathbf{e}^{5k}\\ \ln\left(\frac{32}{5}\right) &= 5k\\ k &= \frac{1}{5}\ln\left(\frac{32}{5}\right) = 0.3712596\end{aligned}$$

Depending upon your preferences we can use either the exact value or the decimal value. Note however that because k is in the exponent of an exponential function we'll need to use quite a few decimal places to avoid potentially large differences in the value that we'd get if we rounded off too much.

Putting all of this together the exponential growth equation for this population is,

$$\boxed{Q = 250\mathbf{e}^{\frac{1}{5}\ln\left(\frac{32}{5}\right)t}}$$

(b) How long will it take for the population to grow from its initial population of 250 to a population of 2000?

Solution

What we're really being asked to do here is to solve the equation,

$$2000 = Q(t) = 250\mathbf{e}^{\frac{1}{5}\ln\left(\frac{32}{5}\right)t}$$

and we know from earlier problems in this section how to do that. Here is the solution work for this part.

$$\begin{aligned}\frac{2000}{250} &= \mathbf{e}^{\frac{1}{5}\ln\left(\frac{32}{5}\right)t}\\ \ln(8) &= \frac{1}{5}\ln\left(\frac{32}{5}\right)t\\ t &= \boxed{\frac{5\ln(8)}{\ln\left(\frac{32}{5}\right)} = 5.6010}\end{aligned}$$

It will take 5.601 days for the population to reach 2000.

16. We initially have 100 grams of a radioactive element and in 1250 years there will be 80 grams left.

    (a) Determine the exponential decay equation for this element.

    (b) How long will it take for half of the element to decay?

    (c) How long will it take until there is only 1 gram of the element left?

Exponential Growth/Decay.

Many quantities in the world can be modeled (at least for a short time) by the exponential growth/decay equation.

$$Q = Q_0 \mathbf{e}^{k\,t}$$

If k is positive we will get exponential growth and if k is negative we will get exponential decay.

Solutions

(a) Determine the exponential decay equation for this element.

Hint

We have an equation with two unknowns and two values of the amount of the element left at two times so use these values to find the two unknowns.

Solution

We can start off here by acknowledging that we know,

$$Q\left(0\right) = 100 \qquad \text{and} \qquad Q\left(1250\right) = 80$$

If we use the first condition in the equation we get,

$$100 = Q\left(0\right) = Q_0 \mathbf{e}^{k(0)} = Q_0 \qquad \rightarrow \qquad Q_0 = 100$$

We now know the first unknown in the equation. Plugging this as well as the second condition into the equation gives us,

$$80 = Q\left(1250\right) = 100\mathbf{e}^{1250k}$$

We can use techniques from earlier problems in this section to determine the value of k.

$$\begin{aligned} 80 &= 100\mathbf{e}^{1250k} \\ \frac{80}{100} &= \mathbf{e}^{1250k} \\ \ln\left(\frac{4}{5}\right) &= 1250k \\ k &= \frac{1}{1250}\ln\left(\frac{4}{5}\right) = -0.000178515 \end{aligned}$$

Depending upon your preferences we can use either the exact value or the decimal value. Note however that because k is in the exponent of an exponential function we'll need to use quite a few decimal places to avoid potentially large differences in the value that we'd get if we rounded off too much.

Putting all of this together the exponential decay equation for this population is,

$$\boxed{Q = 100\mathbf{e}^{\frac{1}{1250}\ln\left(\frac{4}{5}\right)t}}$$

(b) How long will it take for half of the element to decay?

Solution

What we're really being asked to do here is to solve the equation,

$$50 = Q(t) = 100\mathbf{e}^{\frac{1}{1250}\ln\left(\frac{4}{5}\right)t}$$

and we know from earlier problems in this section how to do that. Here is the solution work for this part.

$$\begin{aligned} \frac{50}{100} &= \mathbf{e}^{\frac{1}{1250}\ln\left(\frac{4}{5}\right)t} \\ \ln\left(\frac{1}{2}\right) &= \frac{1}{1250}\ln\left(\frac{4}{5}\right)t \\ t &= \boxed{\frac{1250\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{4}{5}\right)} = 3882.8546} \end{aligned}$$

It will take 3882.8546 years for half of the element to decay. On a side note this time is called the **half-life** of the element.

(c) How long will it take until there is only 1 gram of the element left?

Solution

In this part we're being asked to solve the equation,

$$1 = Q(t) = 100\mathbf{e}^{\frac{1}{1250}\ln\left(\frac{4}{5}\right)t}$$

and we know from earlier problems in this section how to do that. Here is the solution work for this part.

$$\begin{aligned} \frac{1}{100} &= \mathbf{e}^{\frac{1}{1250}\ln\left(\frac{4}{5}\right)t} \\ \ln\left(\frac{1}{100}\right) &= \frac{1}{1250}\ln\left(\frac{4}{5}\right)t \\ t &= \boxed{\frac{1250\ln\left(\frac{1}{100}\right)}{\ln\left(\frac{4}{5}\right)} = 25797.1279} \end{aligned}$$

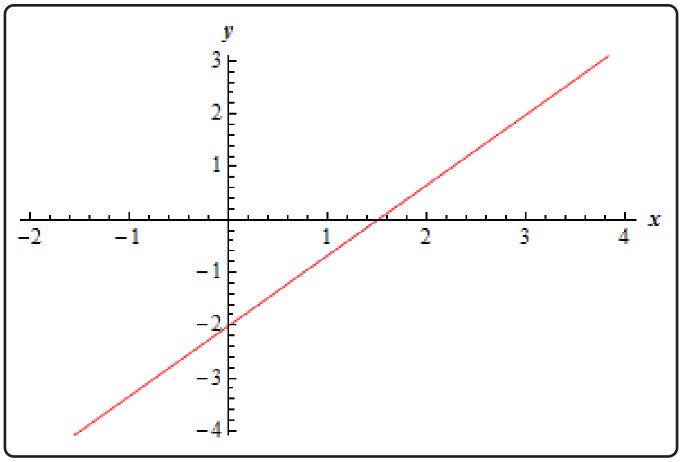
There will only be 1 gram of the element left after 25,797.1279 years.

1.10 Common Graphs

1. Without using a graphing calculator sketch the graph of $y = \frac{4}{3}x - 2$.

Solution

This is just a line with slope $\frac{4}{3}$ and y-intercept $(0, -2)$ so here is the graph.



2. Without using a graphing calculator sketch the graph of $f(x) = |x - 3|$.

Hint

Recall that the graph of $g(x + c)$ is simply the graph of $g(x)$ shifted right by c units if $c < 0$ or shifted left by c units if $c > 0$.

Solution

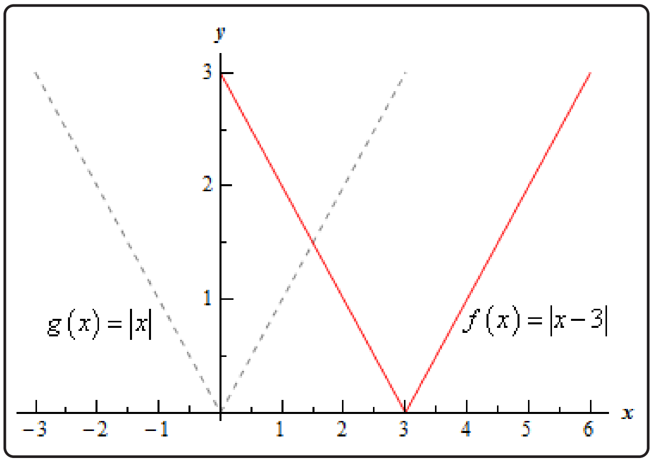
Recall the basic Algebraic transformations. If we know the graph of $g(x)$ then the graph of $g(x + c)$ is simply the graph of $g(x)$ shifted right by c units if $c < 0$ or shifted left by c units if $c > 0$.

So, in our case if $g(x) = |x|$ we can see that,

$$f(x) = |x - 3| = g(x - 3)$$

and so the graph we're being asked to sketch is the graph of the absolute value function shifted right by 3 units.

Here is the graph of $f(x) = |x - 3|$ and note that to help see the transformation we have also sketched in the graph of $g(x) = |x|$.



3. Without using a graphing calculator sketch the graph of $g(x) = \sin(x) + 6$.

Hint

Recall that the graph of $f(x) + c$ is simply the graph of $f(x)$ shifted down by c units if $c < 0$ or shifted up by c units if $c > 0$.

Solution

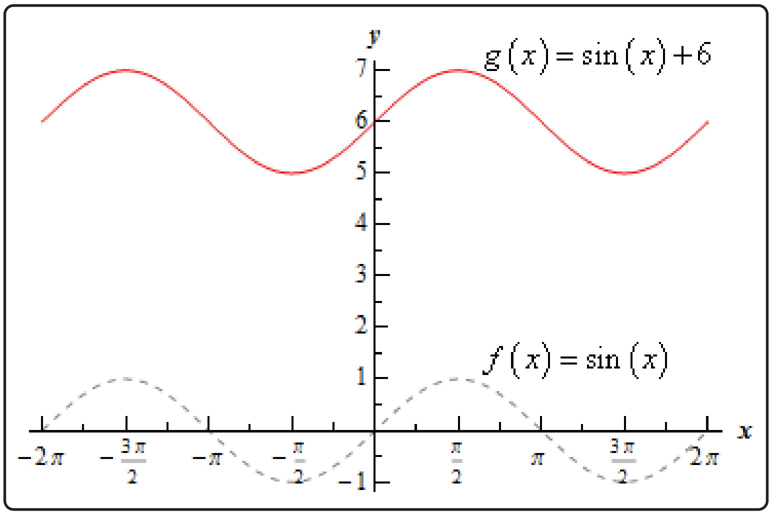
Recall the basic Algebraic transformations. If we know the graph of $f(x)$ then the graph of $f(x) + c$ is simply the graph of $f(x)$ shifted down by c units if $c < 0$ or shifted up by c units if $c > 0$.

So, in our case if $f(x) = \sin(x)$ we can see that,

$$g(x) = \sin(x) + 6 = f(x) + 6$$

and so the graph we're being asked to sketch is the graph of the sine function shifted up by 6 units.

Here is the graph of $g(x) = \sin(x) + 6$ and note that to help see the transformation we have also sketched in the graph of $f(x) = \sin(x)$.



4. Without using a graphing calculator sketch the graph of $f(x) = \ln(x) - 5$.

Hint

Recall that the graph of $g(x) + c$ is simply the graph of $g(x)$ shifted down by c units if $c < 0$ or shifted up by c units if $c > 0$.

Solution

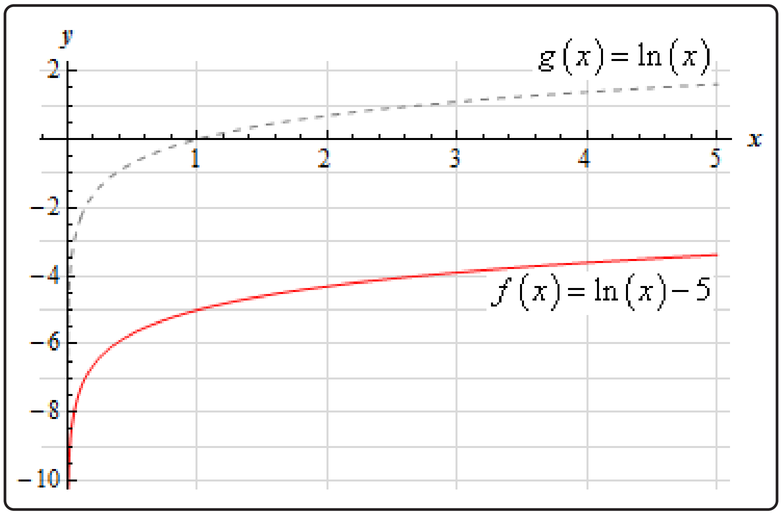
Recall the basic Algebraic transformations. If we know the graph of $g(x)$ then the graph of $g(x) + c$ is simply the graph of $g(x)$ shifted down by c units if $c < 0$ or shifted up by c units if $c > 0$.

So, in our case if $g(x) = \ln(x)$ we can see that,

$$f(x) = \ln(x) - 5 = g(x) - 5$$

and so the graph we're being asked to sketch is the graph of the natural logarithm function shifted down by 5 units.

Here is the graph of $f(x) = \ln(x) - 5$ and note that to help see the transformation we have also sketched in the graph of $g(x) = \ln(x)$.



5. Without using a graphing calculator sketch the graph of $h(x) = \cos\left(x + \frac{\pi}{2}\right)$.

Hint

Recall that the graph of $g(x + c)$ is simply the graph of $g(x)$ shifted right by c units if $c < 0$ or shifted left by c units if $c > 0$.

Solution

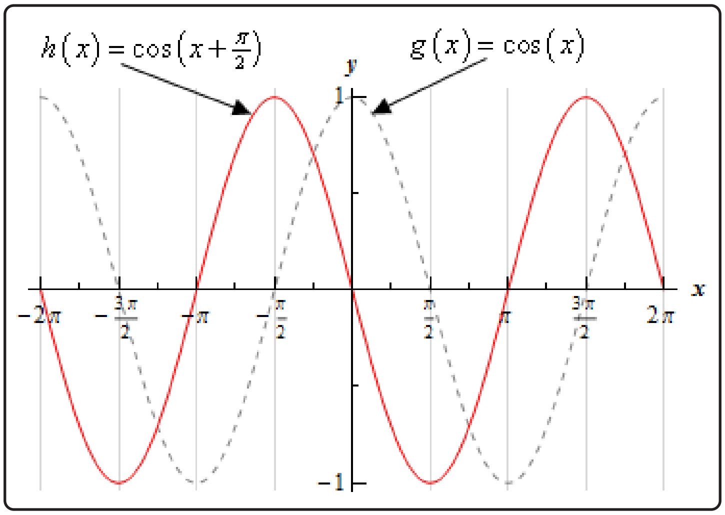
Recall the basic Algebraic transformations. If we know the graph of $g(x)$ then the graph of $g(x + c)$ is simply the graph of $g(x)$ shifted right by c units if $c < 0$ or shifted left by c units if $c > 0$.

So, in our case if $g(x) = \cos(x)$ we can see that,

$$h(x) = \cos\left(x + \frac{\pi}{2}\right) = g\left(x + \frac{\pi}{2}\right)$$

and so the graph we're being asked to sketch is the graph of the cosine function shifted left by $\frac{\pi}{2}$ units.

Here is the graph of $h(x) = \cos\left(x + \frac{\pi}{2}\right)$ and note that to help see the transformation we have also sketched in the graph of $g(x) = \cos(x)$.



6. Without using a graphing calculator sketch the graph of $h(x) = (x-3)^2 + 4$.

Hint

The Algebraic transformations that we used to help us graph the first few graphs in this section can be used together to shift the graph of a function both up/down and right/left at the same time.

Solution

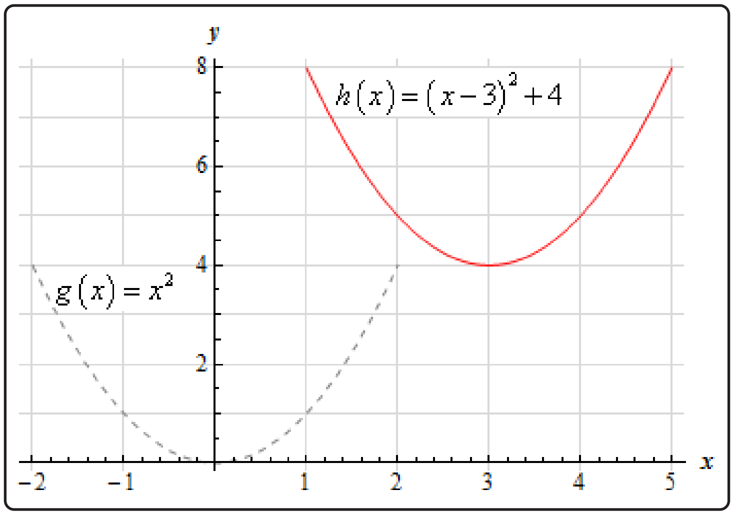
The Algebraic transformations we were using in the first few problems of this section can be combined to shift a graph up/down and right/left at the same time. If we know the graph of $g(x)$ then the graph of $g(x+c)+k$ is simply the graph of $g(x)$ shifted right by c units if $c < 0$ or shifted left by c units if $c > 0$ and shifted up by k units if $k > 0$ or shifted down by k units if $k < 0$.

So, in our case if $g(x) = x^2$ we can see that,

$$h(x) = (x-3)^2 + 4 = g(x-3) + 4$$

and so the graph we're being asked to sketch is the graph of $g(x) = x^2$ shifted right by 3 units and up by 4 units.

Here is the graph of $h(x) = (x-3)^2 + 4$ and note that to help see the transformation we have also sketched in the graph of $g(x) = x^2$.



7. Without using a graphing calculator sketch the graph of $W(x) = \mathbf{e}^{x+2} - 3$.

Hint

The Algebraic transformations that we used to help us graph the first few graphs in this section can be used together to shift the graph of a function both up/down and right/left at the same time.

Solution

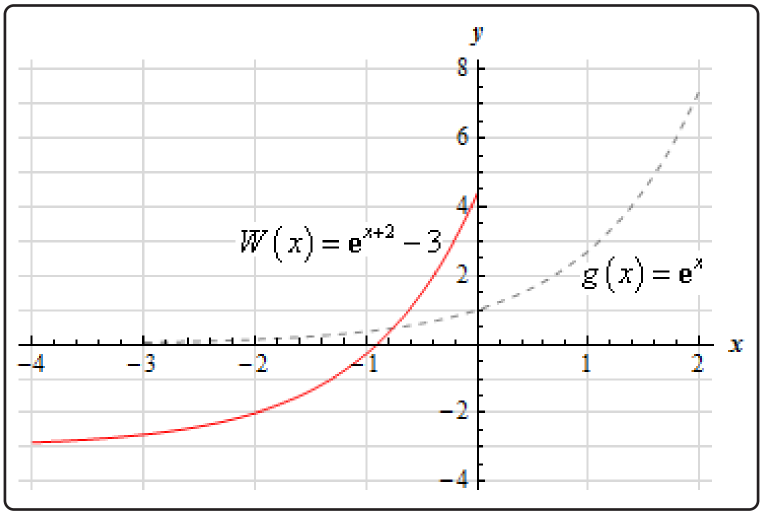
The Algebraic transformations we were using in the first few problems of this section can be combined to shift a graph up/down and right/left at the same time. If we know the graph of $g(x)$ then the graph of $g(x+c)+k$ is simply the graph of $g(x)$ shifted right by c units if $c<0$ or shifted left by c units if $c>0$ and shifted up by k units if $k>0$ or shifted down by k units if $k<0$.

So, in our case if $g(x) = \mathbf{e}^{x}$ we can see that,

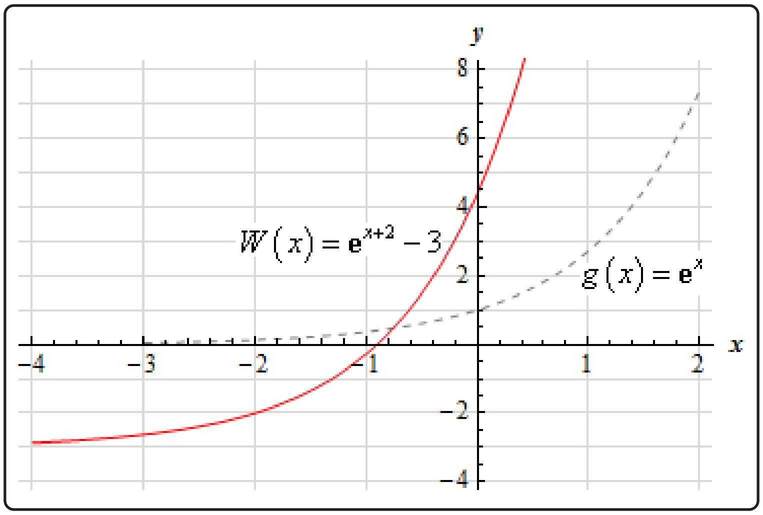
$$W(x) = \mathbf{e}^{x+2} - 3 = g(x+2) - 3$$

and so the graph we're being asked to sketch is the graph of $g(x) = \mathbf{e}^{x}$ shifted left by 2 units and down by 3 units.

Here is the graph of $W(x) = \mathbf{e}^{x+2} - 3$ and note that to help see the transformation we have also sketched in the graph of $g(x) = \mathbf{e}^{x}$.



In this case the resulting sketch of $W(x)$ that we get by shifting the graph of $g(x)$ is not really the best, as it pretty much cuts off at $x = 0$ so in this case we should probably extend the graph of $W(x)$ a little. Here is a better sketch of the graph.



8. Without using a graphing calculator sketch the graph of $f(y) = (y-1)^2 + 2$.

Hint

The Algebraic transformations can also be used to help us sketch graphs of functions in the form $x = f(y)$, but we do need to remember that we're now working with functions in which the variables have been interchanged.

Solution

Even though our function is in the form $x = f(y)$ we can still use the Algebraic transformations to help us sketch this graph. We do need to be careful however and remember that we're working with interchanged variables and so the transformations will also switch.

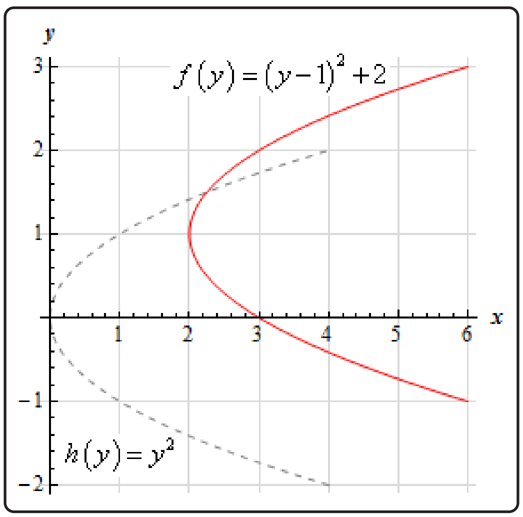
In this case if we know the graph of $h(y)$ then the graph of $h(y+c)+k$ is simply the graph of $h(x)$ shifted up by c units if $c<0$ or shifted down by c units if $c>0$ and shifted right by k units if $k>0$ or shifted left by k units if $k<0$.

So, in our case if $h(y) = y^2$ we can see that,

$$f(y) = (y-1)^2 + 2 = h(y-1) + 2$$

and so the graph we're being asked to sketch is the graph of $h(y) = y^2$ shifted up by 1 units and right by 2 units.

Here is the graph of $f(y) = (y-1)^2 + 2$ and note that to help see the transformation we have also sketched in the graph of $h(y) = y^2$.



9. Without using a graphing calculator sketch the graph of $R(x) = -\sqrt{x}$.

Hint

Recall that the graph of $-f(x)$ is the graph of $f(x)$ reflected about the x-axis.

Solution

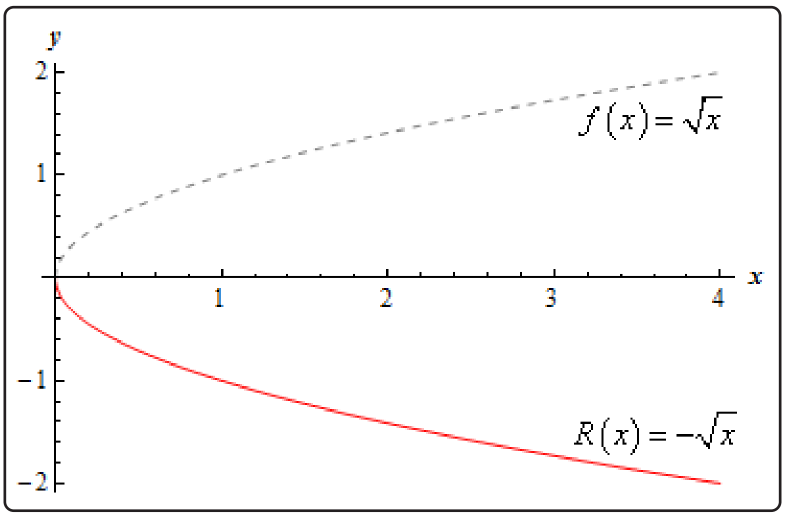
Recall the basic Algebraic transformations. If we know the graph of $f(x)$ then the graph of $-f(x)$ is simply the graph of $f(x)$ reflected about the x-axis.

So, in our case if $f(x) = \sqrt{x}$ we can see that,

$$R(x) = -\sqrt{x} = -f(x)$$

and so the graph we're being asked to sketch is the graph of the square root function reflected about the x-axis.

Here is the graph of $R(x) = -\sqrt{x}$ (the solid curve) and note that to help see the transformation we have also sketched in the graph of $f(x) = \sqrt{x}$ (the dashed curve).



10. Without using a graphing calculator sketch the graph of $g(x) = \sqrt{-x}$.

Hint

Recall that the graph of $f(-x)$ is the graph of $f(x)$ reflected about the y-axis.

Solution

First, do not get excited about the minus sign under the root. We all know that we won't get real numbers if we take the square root of a negative number, but that minus sign doesn't necessarily mean that we'll be taking the square root of negative numbers. If we plug in positive value of x then clearly we will be taking the square root of negative numbers, but if we plug in negative values of x we will now be taking the square root of positive numbers and so there really is nothing wrong with the function as written. We'll just be using a different set of x's than what we may be used to working with when dealing with square roots.

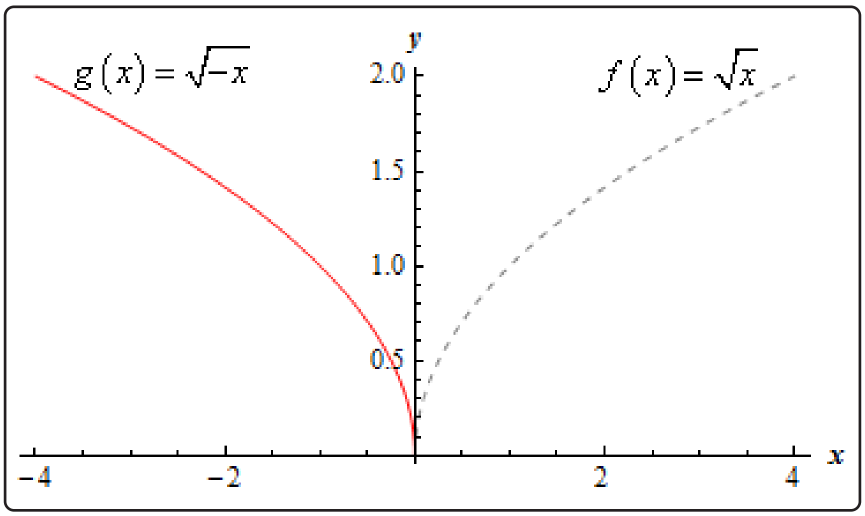
Now, recall the basic Algebraic transformations. If we know the graph of $f(x)$ then the graph of $f(-x)$ is simply the graph of $f(x)$ reflected about the y-axis.

So, in our case if $f(x) = \sqrt{x}$ we can see that,

$$g(x) = \sqrt{-x} = f(-x)$$

and so the graph we're being asked to sketch is the graph of the square root function reflected about the y-axis.

Here is the graph of $g(x) = \sqrt{-x}$ and note that to help see the transformation we have also sketched in the graph of $f(x) = \sqrt{x}$.



11. Without using a graphing calculator sketch the graph of $h(x) = 2x^2 - 3x + 4$.

Hint

Recall that the graph of $f(x) = ax^2 + bx + c$ is the graph of a parabola with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ that opens upwards if $a > 0$ and downwards if $a < 0$ and y-intercept at $(0, c)$.

Solution

We know that the graph of $f(x) = ax^2 + bx + c$ will be a parabola that opens upwards if

$a > 0$ and opens downwards if $a < 0$. We also know that its vertex is at,

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

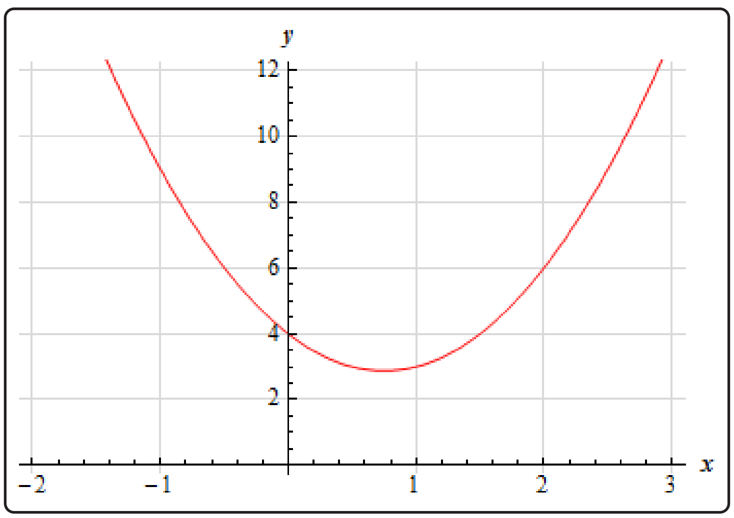
The y-intercept of the parabola is the point $(0, f(0)) = (0, c)$ and the x-intercepts (if any) are found by solving $f(x) = 0$. So, or our case we know we have a parabola that opens upwards and that its vertex is at,

$$\left(-\frac{-3}{2(2)}, f\left(-\frac{-3}{2(2)}\right)\right) = \left(\frac{3}{4}, f\left(\frac{3}{4}\right)\right) = \left(\frac{3}{4}, \frac{23}{8}\right) = (0.75,\ 2.875)$$

We can also see that the y-intercept is $(0, 4)$. Because the vertex is above the x-axis and the parabola opens upwards we can see that there will be no x-intercepts.

It is usually best to have at least one point on either side of the vertex and we know that parabolas are symmetric about the vertical line running through the vertex. Therefore, because we know that the y-intercept is 0.75 units to the left of the vertex that we must also have a point that is 0.75 to the right of the vertex with the same y-value and this point is : $(1.5, 4)$.

Here is a sketch of this parabola.



12. Without using a graphing calculator sketch the graph of $f(y) = -4y^2 + 8y + 3$.

Hint

Recall that the graph of $f(y) = ay^2 + by + c$ is the graph of a parabola with vertex $\left(f\left(-\frac{b}{2a}\right), -\frac{b}{2a}\right)$ that opens towards the right if $a > 0$ and towards the left if $a < 0$ and x-intercept at $(c, 0)$.

Solution

We know that the graph of $f(y) = ay^2 + by + c$ will be a parabola that opens towards the right if $a > 0$ and opens towards the left if $a < 0$. We also know that its vertex is at,

$$\left(f\left(-\frac{b}{2a}\right), -\frac{b}{2a}\right)$$

The x-intercept of the parabola is the point $(f(0), 0) = (c, 0)$ and the y-intercepts (if any) are found by solving $f(y) = 0$

So, or our case we know we have a parabola that opens towards the left and that its vertex is at,

$$\left(f\left(-\frac{8}{2(-4)}\right), -\frac{8}{2(-4)}\right) = (f(1), 1) = (7, 1)$$

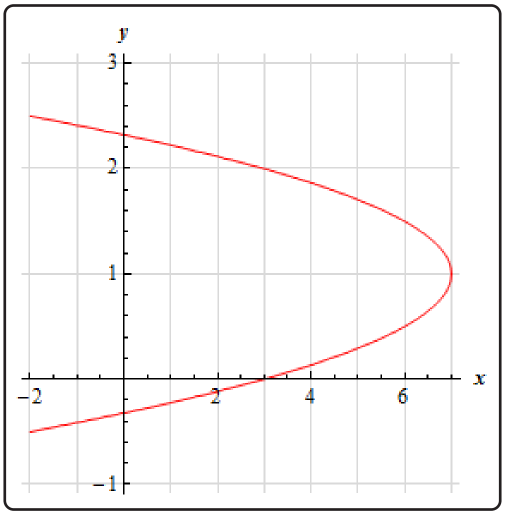
We can also see that the x-intercept is $(3, 0)$.

To find the y-intercepts all we need to do is solve : $-4y^2 + 8y + 3 = 0$.

$$y = \frac{-8 \pm \sqrt{8^2 - 4(-4)(3)}}{2(-4)} = \frac{-8 \pm \sqrt{112}}{-8} = \frac{-8 \pm 4\sqrt{7}}{-8}$$
$$= \frac{2 \pm \sqrt{7}}{2} = -0.3229, \quad 2.3229$$

So, the two y-intercepts are : $(0, -0.3229)$ and $(0, 2.3229)$.

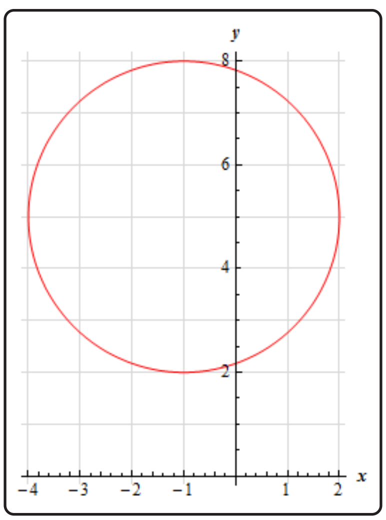
Here is a sketch of this parabola.



13. Without using a graphing calculator sketch the graph of $(x+1)^2 + (y-5)^2 = 9$.

Solution

This is just a circle in standard form and so we can see that it has a center of $(-1, 5)$ and a radius of 3. Here is a quick sketch of the circle.



14. Without using a graphing calculator sketch the graph of $x^2 - 4x + y^2 - 6y - 87 = 0$.

Hint

Complete the square a couple of times to put this into standard from. This will allow you to identify the type of graph this will be.

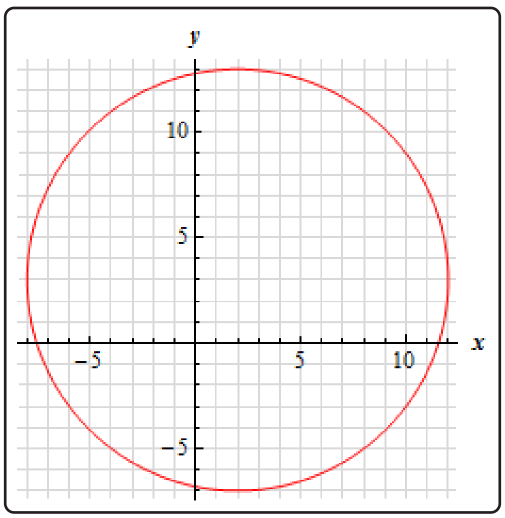
Solution

The first thing that we should do is complete the square on the x's and the y's to see what we've got here. This could be a circle, ellipse, or hyperbola and completing the square a couple of times will put it into standard form and we'll be able to identify the graph at that point.

Here is the completing the square work.

$$\begin{aligned} x^2 - 4x + (4-4) + y^2 - 6y + (9-9) - 87 &= 0 \\ (x-2)^2 + (y-3)^2 - 100 &= 0 \\ (x-2)^2 + (y-3)^2 &= 100 \end{aligned}$$

So, we've got a circle with center $(2, 3)$ and radius 10. Here is a sketch of the circle.



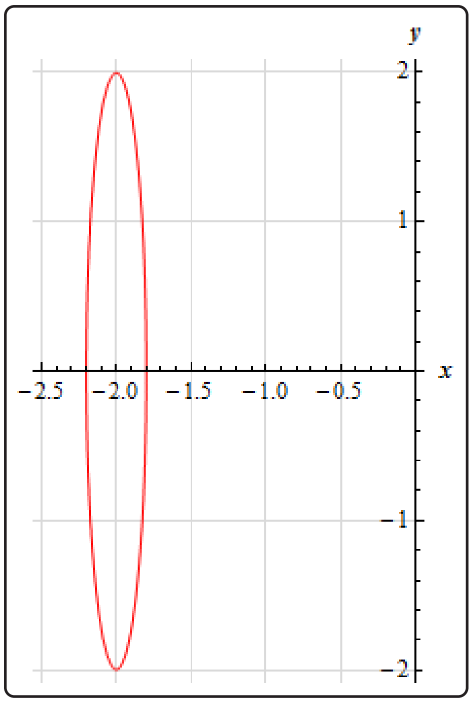
15. Without using a graphing calculator sketch the graph of $25(x+2)^2 + \frac{y^2}{4} = 1$.

Solution

This is just an ellipse that is almost in standard form. With a little rewrite we can put it into standard form as follows,

$$\frac{(x+2)^2}{{}^1/_{25}} + \frac{y^2}{4} = 1$$

We can now see that the ellipse has a center of $(-2, 0)$ while the left/right most points will be $\frac{1}{5} = 0.2$ units away from the center and the top/bottom most points will be 2 units away from the center. Here is a quick sketch of the ellipse.

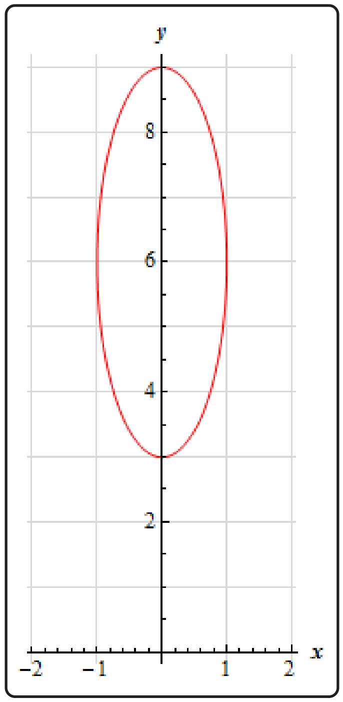


16. Without using a graphing calculator sketch the graph of $x^2 + \frac{(y-6)^2}{9} = 1$.

Solution

This is just an ellipse that is in standard form (if it helps rewrite the first term as $\frac{x^2}{1}$) and so we can see that it has a center of $(0, 6)$ while the left/right most points will be 1 unit away from the center and the top/bottom most points will be 3 units away from the center.

Here is a quick sketch of the ellipse.

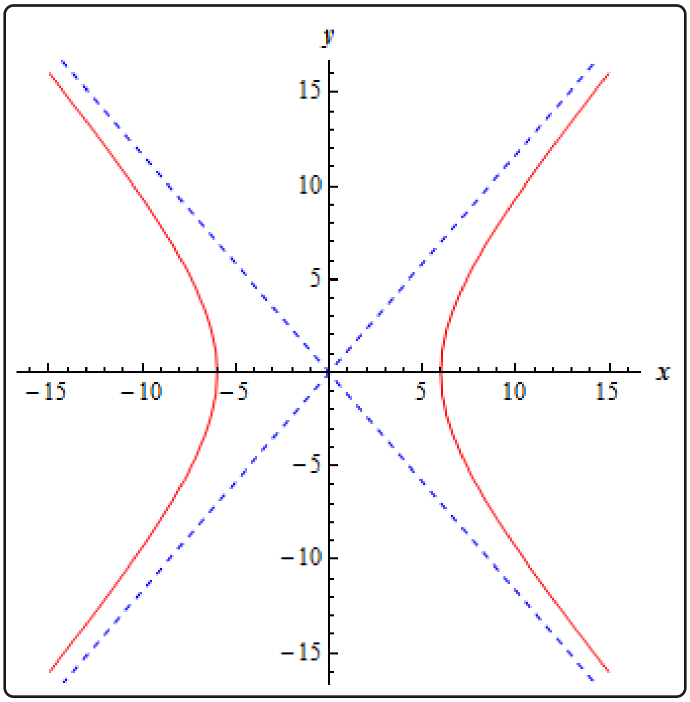


17. Without using a graphing calculator sketch the graph of $\dfrac{x^2}{36} - \dfrac{y^2}{49} = 1$.

Solution

This is a hyperbola in standard form with the minus sign in front of the y term and so will open right and left. The center of the hyperbola is at $(0,0)$, the two vertices are at $(-6,0)$ and $(6,0)$, and the slope of the two asymptotes are $\pm\frac{7}{6}$.

Here is a quick sketch of the hyperbola.

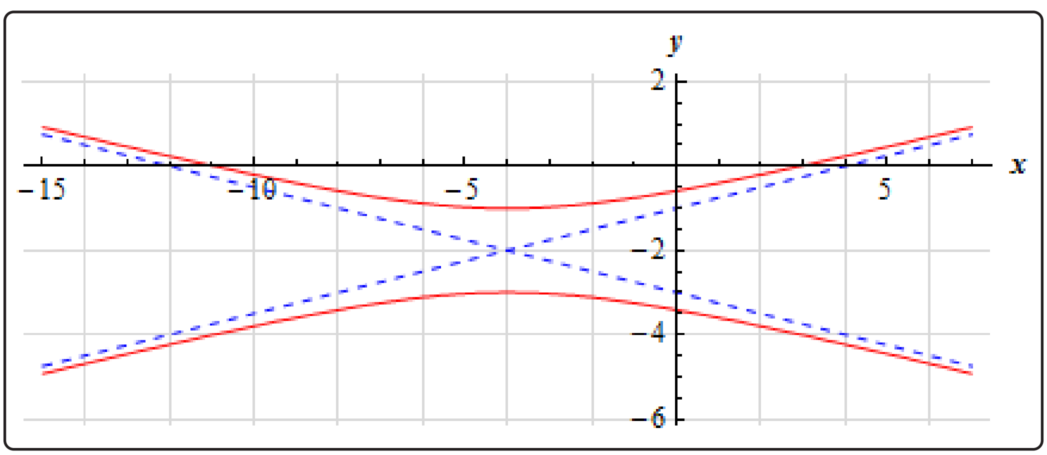


18. Without using a graphing calculator sketch the graph of $(y+2)^2 - \frac{(x+4)^2}{16} = 1$.

Solution

This is a hyperbola in standard form with the minus sign in front of the x term and so will open up and down. The center of the hyperbola is at $(-4, -2)$, the two vertices are at $(-4, -1)$ and $(-4, -3)$, and the slope of the two asymptotes are $\pm\frac{1}{4}$.

Here is a quick sketch of the hyperbola.



2 Limits

The topic that we will be examining in this chapter is that of Limits. This is the first of three major topics that we will be covering in this course. While we will be spending the least amount of time on limits in comparison to the other two topics limits are very important in the study of Calculus. We will be seeing limits in a variety of places once we move out of this chapter. In particular we will see that limits are part of the formal definition of the other two major topics.

In this chapter we will discuss just what a limit tells us about a function as well as how they can be used to get the rate of change of a function as well as the slope of the line tangent to the graph of a function (although we'll be seeing other, easier, ways of doing these later). We will investigate limit properties as well as how a variety of techniques to employ when attempting to compute a limit. We will also look at limits whose "value" is infinity and how to compute limits at infinity.

In addition, we'll introduce the concept of continuity and how continuity is used in the Intermediate Value Theorem. The Intermediate Value Theorem is an important idea that has a variety of "real world" applications including showing that a function has a root (*i.e.* is equal to zero) in some interval.

Finally, we'll close out the chapter with the formal/precise definition of the Limit, sometimes called the delta-epsilon definition.

The following sections are the practice problems, with solutions, for this material.

2.1 Tangent Lines and Rates of Change

1. For the function $f(x) = 3(x+2)^2$ and the point P given by $x = -3$ answer each of the following questions.

   (a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points P and Q.

(i) -3.5	**(ii)** -3.1	**(iii)** -3.01	**(iv)** -3.001	**(v)** -3.0001
(vi) -2.5	**(vii)** -2.9	**(viii)** -2.99	**(ix)** -2.999	**(x)** -2.9999

   (b) Use the information from (a) to estimate the slope of the tangent line to $f(x)$ at $x = -3$ and write down the equation of the tangent line.

Solutions

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points P and Q.

(i) -3.5	**(ii)** -3.1	**(iii)** -3.01	**(iv)** -3.001	**(v)** -3.0001
(vi) -2.5	**(vii)** -2.9	**(viii)** -2.99	**(ix)** -2.999	**(x)** -2.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$m_{PQ} = \frac{f(x) - f(-3)}{x - (-3)} = \frac{3(x+2)^2 - 3}{x+3}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x. All of the values in the table below are accurate to 8 decimal places, but in this case the values terminated prior to 8 decimal places and so the "trailing" zeros are not shown.

x	m_{PQ}	x	m_{PQ}
-3.5	-7.5	-2.5	-4.5
-3.1	-6.3	-2.9	-5.7
-3.01	-6.03	-2.99	-5.97
-3.001	-6.003	-2.999	-5.997
-3.0001	-6.0003	-2.9999	-5.9997

(b) Use the information from (a) to estimate the slope of the tangent line to $f(x)$ at $x = -3$ and write down the equation of the tangent line.

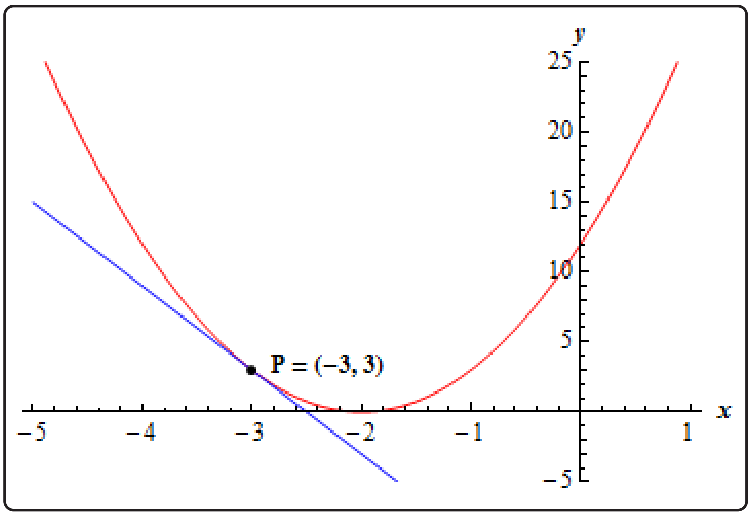
Solution

From the table of values above we can see that the slope of the secant lines appears to be moving towards a value of -6 from both sides of $x = -3$ and so we can estimate that the slope of the tangent line is : $\boxed{m = -6}$.

The equation of the tangent line is then,

$$y = f(-3) + m(x - (-3)) = 3 - 6(x + 3) \qquad \Rightarrow \qquad \boxed{y = -6x - 15}$$

Here is a graph of the function and the tangent line.



2. For the function $g(x) = \sqrt{4x + 8}$ and the point P given by $x = 2$ answer each of the following questions.

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points P and Q.

(i) 2.5	**(ii)** 2.1	**(iii)** 2.01	**(iv)** 2.001	**(v)** 2.0001
(vi) 1.5	**(vii)** 1.9	**(viii)** 1.99	**(ix)** 1.999	**(x)** 1.9999

(b) Use the information from (a) to estimate the slope of the tangent line to $g(x)$ at $x = 2$ and write down the equation of the tangent line.

Solutions

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points P and Q.

(i) 2.5	**(ii)** 2.1	**(iii)** 2.01	**(iv)** 2.001	**(v)** 2.0001
(vi) 1.5	**(vii)** 1.9	**(viii)** 1.99	**(ix)** 1.999	**(x)** 1.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$m_{PQ} = \frac{g(x) - g(2)}{x - 2} = \frac{\sqrt{4x+8} - 4}{x - 2}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x. All of the values in the table below are accurate to 8 decimal places.

x	m_{PQ}	x	m_{PQ}
2.5	0.48528137	1.5	0.51668523
2.1	0.49691346	1.9	0.50316468
2.01	0.49968789	1.99	0.50031289
2.001	0.49996875	1.999	0.50003125
2.0001	0.49999688	1.9999	0.50000313

(b) Use the information from (a) to estimate the slope of the tangent line to $g(x)$ at $x = 2$ and write down the equation of the tangent line.

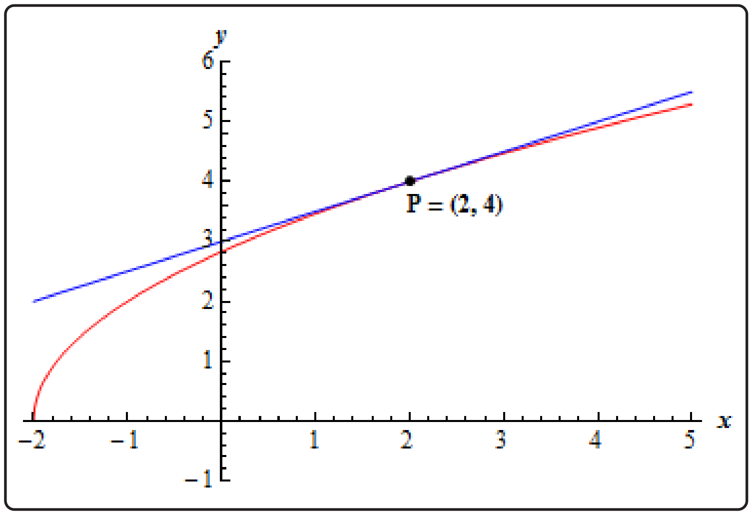
Solution

From the table of values above we can see that the slope of the secant lines appears to be moving towards a value of 0.5 from both sides of $x = 2$ and so we can estimate that the slope of the tangent line is : $\boxed{m = 0.5 = \frac{1}{2}}$.

The equation of the tangent line is then,

$$y = g(2) + m(x-2) = 4 + \frac{1}{2}(x-2) \qquad \Rightarrow \qquad \boxed{y = \frac{1}{2}x + 3}$$

Here is a graph of the function and the tangent line.



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3. For the function $W(x) = \ln(1 + x^4)$ and the point P given by $x = 1$ answer each of the following questions.

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points P and Q.

(i) 1.5	**(ii)** 1.1	**(iii)** 1.01	**(iv)** 1.001	**(v)** 1.0001
(vi) 0.5	**(vii)** 0.9	**(viii)** 0.99	**(ix)** 0.999	**(x)** 0.9999

(b) Use the information from (a) to estimate the slope of the tangent line to $W(x)$ at $x = 1$ and write down the equation of the tangent line.

Solutions

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points P and Q.

(i) 1.5	**(ii)** 1.1	**(iii)** 1.01	**(iv)** 1.001	**(v)** 1.0001
(vi) 0.5	**(vii)** 0.9	**(viii)** 0.99	**(ix)** 0.999	**(x)** 0.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$m_{PQ} = \frac{W(x) - W(1)}{x - 1} = \frac{\ln(1 + x^4) - \ln(2)}{x - 1}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x. All of the values in the table below are accurate to 8 decimal places.

x	m_{PQ}	x	m_{PQ}
1.5	2.21795015	0.5	1.26504512
1.1	2.08679449	0.9	1.88681740
1.01	2.00986668	0.99	1.98986668
1.001	2.00099867	0.999	1.99899867
1.0001	2.00009999	0.9999	1.99989999

(b) Use the information from (a) to estimate the slope of the tangent line to $W(x)$ at $x = 1$ and write down the equation of the tangent line.

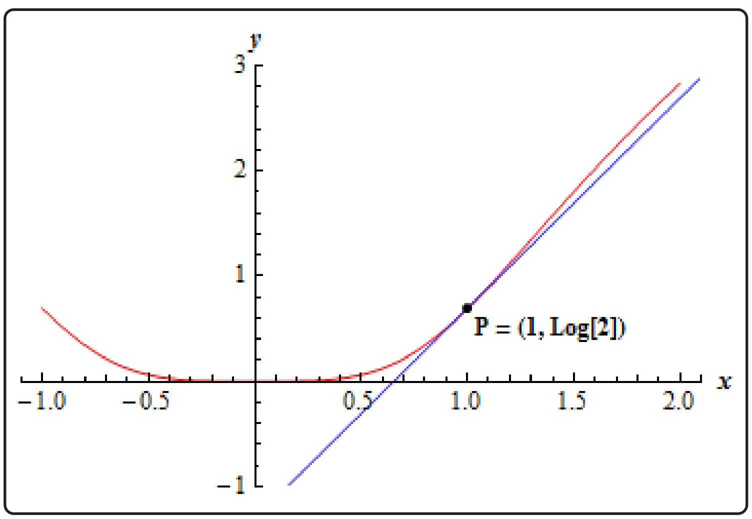
Solution

From the table of values above we can see that the slope of the secant lines appears to be moving towards a value of 2 from both sides of $x = 1$ and so we can estimate that the slope of the tangent line is : $\boxed{m = 2}$.

The equation of the tangent line is then,

$$y = W(1) + m(x-1) = \boxed{\ln(2) + 2(x-1)}$$

Here is a graph of the function and the tangent line.



4. The volume of air in a balloon is given by $V(t) = \dfrac{6}{4t+1}$ answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the volume of air in the balloon between $t = 0.25$ and the following values of t.

(i) 1 **(ii)** 0.5 **(iii)** 0.251 **(iv)** 0.2501 **(v)** 0.25001

(vi) 0 **(vii)** 0.1 **(viii)** 0.249 **(ix)** 0.2499 **(x)** 0.24999

(b) Use the information from (a) to estimate the instantaneous rate of change of the volume of air in the balloon at $t = 0.25$.

Solutions

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the volume of air in the balloon between $t = 0.25$ and the following values of t.

(i) 1 **(ii)** 0.5 **(iii)** 0.251 **(iv)** 0.2501 **(v)** 0.25001

(vi) 0 **(vii)** 0.1 **(viii)** 0.249 **(ix)** 0.2499 **(x)** 0.24999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$A.R.C. = \frac{V(t) - V(0.25)}{t - 0.25} = \frac{\frac{6}{4t+1} - 3}{t - 0.25}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x. All of the values in the table below are accurate to 8 decimal places. In several of the initial values in the table the values terminated and so the "trailing" zeroes were not shown.

x	$A.R.C.$	x	$A.R.C.$
1	-2.4	0	-12
0.5	-4	0.1	-8.57142857
0.251	-5.98802395	0.249	-6.01202405
0.2501	-5.99880024	0.2499	-6.00120024
0.25001	-5.99988000	0.24999	-6.00012000

(b) Use the information from (a) to estimate the instantaneous rate of change of the volume of air in the balloon at $t = 0.25$.

Solution

From the table of values above we can see that the average rate of change of the volume of air is moving towards a value of -6 from both sides of $t = 0.25$ and so we can estimate that the instantaneous rate of change of the volume of air in the balloon is -6.

---

5. The population (in hundreds) of fish in a pond is given by $P(t) = 2t + \sin(2t - 10)$ answer each of the following questions.

   (a) Compute (accurate to at least 8 decimal places) the average rate of change of the population of fish between $t = 5$ and the following values of t. Make sure your calculator is set to radians for the computations.

(i) 5.5	**(ii)** 5.1	**(iii)** 5.01	**(iv)** 5.001	**(v)** 5.0001
(vi) 4.5	**(vii)** 4.9	**(viii)** 4.99	**(ix)** 4.999	**(x)** 4.9999

   (b) Use the information from (a) to estimate the instantaneous rate of change of the population of the fish at $t = 5$.

Solutions

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the population of fish between $t = 5$ and the following values of t. Make sure your calculator is set to radians for the computations.

(i) 5.5	**(ii)** 5.1	**(iii)** 5.01	**(iv)** 5.001	**(v)** 5.0001
(vi) 4.5	**(vii)** 4.9	**(viii)** 4.99	**(ix)** 4.999	**(x)** 4.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$A.R.C. = \frac{P(t) - P(5)}{t - 5} = \frac{2t + \sin(2t - 10) - 10}{t - 5}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x. All of the values in the table below are accurate to 8 decimal places.

x	$A.R.C.$	x	$A.R.C.$
5.5	3.68294197	4.5	3.68294197
5.1	3.98669331	4.9	3.98669331
5.01	3.99986667	4.99	3.99986667
5.001	3.99999867	4.999	3.99999867
5.0001	3.99999999	4.9999	3.99999999

(b) Use the information from (a) to estimate the instantaneous rate of change of the population of the fish at $t = 5$.

Solution

From the table of values above we can see that the average rate of change of the population of fish is moving towards a value of 4 from both sides of $t = 5$ and so we can estimate that the instantaneous rate of change of the population of the fish is 400 (remember the population is in hundreds).

6. The position of an object is given by $s(t) = \cos^2\left(\dfrac{3t-6}{2}\right)$ answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 2$ and the following values of t. Make sure your calculator is set to radians for the computations.

(i) 2.5 **(ii)** 2.1 **(iii)** 2.01 **(iv)** 2.001 **(v)** 2.0001

(vi) 1.5 **(vii)** 1.9 **(viii)** 1.99 **(ix)** 1.999 **(x)** 1.9999

(b) Use the information from (a) to estimate the instantaneous velocity of the object at $t = 2$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

Solutions

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 2$ and the following values of t. Make sure your calculator is set to radians for the computations.

(i) 2.5 **(ii)** 2.1 **(iii)** 2.01 **(iv)** 2.001 **(v)** 2.0001

(vi) 1.5 **(vii)** 1.9 **(viii)** 1.99 **(ix)** 1.999 **(x)** 1.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$A.V. = \frac{s(t) - s(2)}{t-2} = \frac{\cos^2\left(\frac{3t-6}{2}\right) - 1}{t-2}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x. All of the values in the table below are accurate to 8 decimal places.

x	$A.V.$	x	$A.V.$
2.5	−0.92926280	1.5	0.92926280
2.1	−0.22331755	1.9	0.22331755
2.01	−0.02249831	1.99	0.02249831
2.001	−0.00225000	1.999	0.00225000
2.0001	−0.00022500	1.9999	0.00022500

(b) Use the information from (a) to estimate the instantaneous velocity of the object at $t = 2$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

Solution

From the table of values above we can see that the average velocity of the object is moving towards a value of 0 from both sides of $t = 2$ and so we can estimate that the instantaneous velocity is 0 and so the object will not be moving at $t = 2$.

7. The position of an object is given by $s(t) = (8-t)(t+6)^{\frac{3}{2}}$. Note that a negative position here simply means that the position is to the left of the "zero position" and is perfectly acceptable. Answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 10$ and the following values of t.

(i) 10.5	**(ii)** 10.1	**(iii)** 10.01	**(iv)** 10.001	**(v)** 10.0001
(vi) 9.5	**(vii)** 9.9	**(viii)** 9.99	**(ix)** 9.999	**(x)** 9.9999

(b) Use the information from (a) to estimate the instantaneous velocity of the object at $t = 10$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

Solutions

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 10$ and the following values of t.

(i) 10.5	**(ii)** 10.1	**(iii)** 10.01	**(iv)** 10.001	**(v)** 10.0001
(vi) 9.5	**(vii)** 9.9	**(viii)** 9.99	**(ix)** 9.999	**(x)** 9.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$A.V. = \frac{s(t) - s(10)}{t-10} = \frac{(8-t)(t+6)^{\frac{3}{2}} + 128}{t-10}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x. All of the values in the table below are accurate to 8 decimal places.

x	$A.V.$	x	$A.V.$
10.5	−79.11658419	9.5	−72.92931693
10.1	−76.61966704	9.9	−75.38216890
10.01	−76.06188418	9.99	−75.93813418
10.001	−76.00618759	9.999	−75.99381259
10.0001	−76.00061875	9.9999	−75.99938125

(b) Use the information from (a) to estimate the instantaneous velocity of the object at $t = 10$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

Solution

From the table of values above we can see that the average velocity of the object is moving towards a value of -76 from both sides of $t = 10$ and so we can estimate that the instantaneous velocity is -76 and so the object will be moving to the left at $t = 10$.

2.2 The Limit

1. For the function $f(x) = \dfrac{8-x^3}{x^2-4}$ answer each of the following questions.

(a) Evaluate the function at the following values of x compute (accurate to at least 8 decimal places).

(i) 2.5 **(ii)** 2.1 **(iii)** 2.01 **(iv)** 2.001 **(v)** 2.0001

(vi) 1.5 **(vii)** 1.9 **(viii)** 1.99 **(ix)** 1.999 **(x)** 1.9999

(b) Use the information from **(a)** to estimate the value of $\lim\limits_{x\to 2}\dfrac{8-x^3}{x^2-4}$.

Solutions

(a) Evaluate the function at the following values of x compute (accurate to at least 8 decimal places).

(b) **(i)** 2.5 **(ii)** 2.1 **(iii)** 2.01 **(iv)** 2.001 **(v)** 2.0001

(vi) 1.5 **(vii)** 1.9 **(viii)** 1.99 **(ix)** 1.999 **(x)** 1.9999

Solution

Here is a table of values of the function at the given points accurate to 8 decimal places.

x	$f(x)$	x	$f(x)$
2.5	−3.38888889	1.5	−2.64285714
2.1	−3.07560976	1.9	−2.92564103
2.01	−3.00750623	1.99	−2.99250627
2.001	−3.00075006	1.999	−2.99925006
2.0001	−3.00007500	1.9999	−2.99992500

(c) Use the information from **(a)** to estimate the value of $\lim\limits_{x\to 2}\dfrac{8-x^3}{x^2-4}$.

Solution

From the table of values above it looks like we can estimate that,

$$\lim_{x\to 2}\frac{8-x^3}{x^2-4} = -3$$

2. For the function $R(t) = \dfrac{2 - \sqrt{t^2 + 3}}{t + 1}$ answer each of the following questions.

(a) Evaluate the function at the following values of t compute (accurate to at least 8 decimal places).

(i) -0.5	**(ii)** -0.9	**(iii)** -0.99	**(iv)** -0.999	**(v)** -0.9999
(vi) -1.5	**(vii)** -1.1	**(viii)** -1.01	**(ix)** -1.001	**(x)** -1.0001

(b) Use the information from **(a)** to estimate the value of $\displaystyle\lim_{t \to -1} \frac{2 - \sqrt{t^2 + 3}}{t + 1}$.

Solutions

(a) Evaluate the function at the following values of t compute (accurate to at least 8 decimal places).

(i) -0.5	**(ii)** -0.9	**(iii)** -0.99	**(iv)** -0.999	**(v)** -0.9999
(vi) -1.5	**(vii)** -1.1	**(viii)** -1.01	**(ix)** -1.001	**(x)** -1.0001

Solution

Here is a table of values of the function at the given points accurate to 8 decimal places.

t	$R(t)$	t	$R(t)$
-0.5	0.39444872	-1.5	0.58257569
-0.9	0.48077870	-1.1	0.51828453
-0.99	0.49812031	-1.01	0.50187032
-0.999	0.49981245	-1.001	0.50018745
-0.9999	0.49998125	-1.0001	0.50001875

(b) Use the information from **(a)** to estimate the value of $\displaystyle\lim_{t \to -1} \frac{2 - \sqrt{t^2 + 3}}{t + 1}$.

Solution

From the table of values above it looks like we can estimate that,

$$\lim_{t \to -1} \frac{2 - \sqrt{t^2 + 3}}{t + 1} = \frac{1}{2}$$

3. For the function $g(\theta) = \dfrac{\sin(7\theta)}{\theta}$ answer each of the following questions.

(a) Evaluate the function at the following values of θ compute (accurate to at least 8 decimal places). Make sure your calculator is set to radians for the computations.

(i) 0.5	**(ii)** 0.1	**(iii)** 0.01	**(iv)** 0.001	**(v)** 0.0001
(vi) -0.5	**(vii)** -0.1	**(viii)** -0.01	**(ix)** -0.001	**(x)** -0.0001

(b) Use the information from **(a)** to estimate the value of $\displaystyle\lim_{\theta \to 0} \frac{\sin(7\theta)}{\theta}$.

Solutions

(a) Evaluate the function at the following values of θ compute (accurate to at least 8 decimal places). Make sure your calculator is set to radians for the computations.

(i) 0.5	**(ii)** 0.1	**(iii)** 0.01	**(iv)** 0.001	**(v)** 0.0001
(vi) -0.5	**(vii)** -0.1	**(viii)** -0.01	**(ix)** -0.001	**(x)** -0.0001

Solution

Here is a table of values of the function at the given points accurate to 8 decimal places.

θ	$g(\theta)$	θ	$g(\theta)$
0.5	-0.70156646	-0.5	-0.70156646
0.1	6.44217687	-0.1	6.44217687
0.01	6.99428473	-0.01	6.99428473
0.001	6.99994283	-0.001	6.99994283
0.0001	6.99999943	-0.0001	6.99999943

(b) Use the information from **(a)** to estimate the value of $\displaystyle\lim_{\theta \to 0} \frac{\sin(7\theta)}{\theta}$.

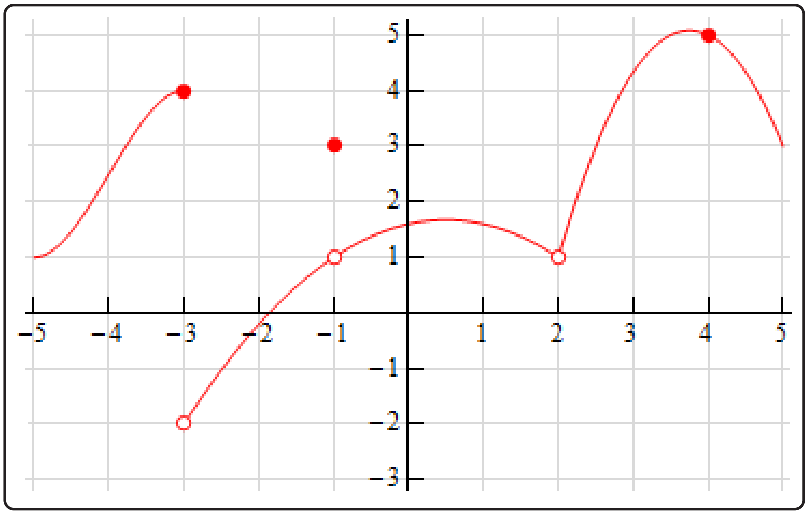
Solution

From the table of values above it looks like we can estimate that,

$$\lim_{\theta \to 0} \frac{\sin(7\theta)}{\theta} = 7$$

4. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim\limits_{x\to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -3$ **(b)** $a = -1$ **(c)** $a = 2$ **(d)** $a = 4$



Solutions

(a) $a = -3$

Solution

From the graph we can see that,

$$\boxed{f(-3) = 4}$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = -3$ from both sides the graph is approaching different values (4 from the left and -2 from the right). Because of this we get,

$$\boxed{\lim_{x\to -3} f(x) \quad \text{does not exist}}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(b) $a = -1$

Solution

From the graph we can see that,

$$\boxed{f(-1) = 3}$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -1$ from both sides the graph is approaching the same value, 1, and so we get,

$$\boxed{\lim_{x \to -1} f(x) = 1}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(c) $a = 2$

Solution

Because there is no closed dot for $x = 2$ we can see that,

$$\boxed{f(2) \text{ does not exist}}$$

We can also see that as we approach $x = 2$ from both sides the graph is approaching the same value, 1, and so we get,

$$\boxed{\lim_{x \to 2} f(x) = 1}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Therefore, even though the function doesn't exist at this point the limit can still have a value.

(d) $a = 4$

Solution

From the graph we can see that,

$$\boxed{f(4) = 5}$$

because the closed dot is at the value of $y = 5$.

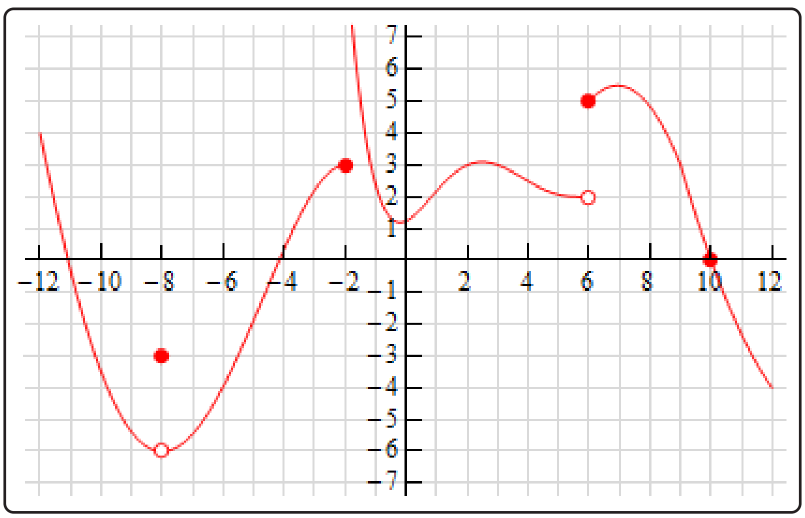
We can also see that as we approach $x = 4$ from both sides the graph is approaching the same value, 5, and so we get,

$$\boxed{\lim_{x \to 4} f(x) = 5}$$

---

5. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim\limits_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -8$ **(b)** $a = -2$ **(c)** $a = 6$ **(d)** $a = 10$



Solutions

(a) $a = -8$

Solution

From the graph we can see that,

$$\boxed{f(-8) = -3}$$

because the closed dot is at the value of $y = -3$.

We can also see that as we approach $x = -8$ from both sides the graph is approaching the same value, -6, and so we get,

$$\boxed{\lim_{x \to -8} f(x) = -6}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(b) $a = -2$

Solution

From the graph we can see that,

$$\boxed{f(-2) = 3}$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -2$ from both sides the graph is approaching different values (3 from the left and doesn't approach any value from the right). Because of this we get,

$$\boxed{\lim_{x \to -2} f(x) \quad \text{does not exist}}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(c) $a = 6$

Solution

From the graph we can see that,

$$\boxed{f(6) = 5}$$

because the closed dot is at the value of $y = 5$.

We can also see that as we approach $x = 6$ from both sides the graph is approaching different values (2 from the left and 5 from the right). Because of this we get,

$$\boxed{\lim_{x \to 6} f(x) \quad \text{does not exist}}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(d) $a = 10$

Solution

From the graph we can see that,

$$\boxed{f(10) = 0}$$

because the closed dot is at the value of $y = 0$.

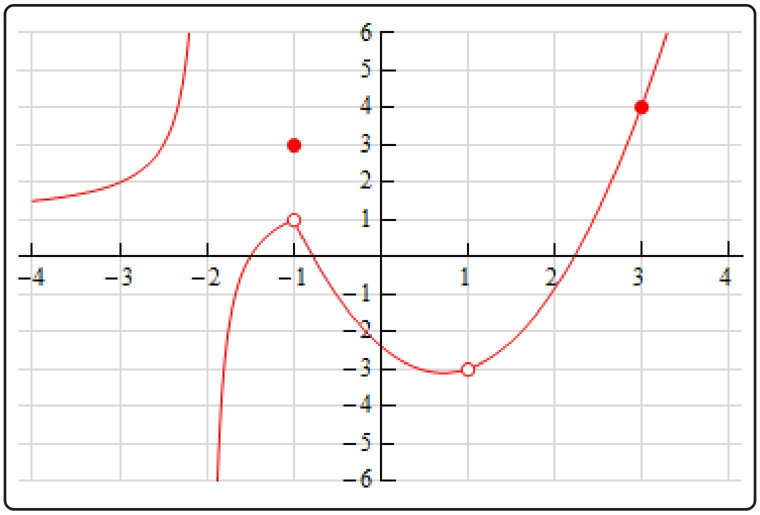
We can also see that as we approach $x = 10$ from both sides the graph is approaching the same value, 0, and so we get,

$$\boxed{\lim_{x \to 10} f(x) = 0}$$

---

6. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim\limits_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -2$ **(b)** $a = -1$ **(c)** $a = 1$ **(d)** $a = 3$



Solutions

(a) $a = -2$

Solution

Because there is no closed dot for $x = -2$ we can see that,

$$\boxed{f(-2) \text{ does not exist}}$$

We can also see that as we approach $x = -2$ from both sides the graph is not approaching a value from either side and so we get,

$$\boxed{\lim_{x \to -2} f(x) \text{ does not exist}}$$

(b) $a = -1$

Solution

From the graph we can see that,

$$\boxed{f(-1) = 3}$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -1$ from both sides the graph is approaching the same value, 1, and so we get,

$$\boxed{\lim_{x \to -1} f(x) = 1}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(c) $a = 1$

Solution

Because there is no closed dot for $x = 1$ we can see that,

$$\boxed{f(1) \text{ does not exist}}$$

We can also see that as we approach $x = 1$ from both sides the graph is approaching the same value, -3, and so we get,

$$\boxed{\lim_{x \to 1} f(x) = -3}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Therefore, even though the function doesn't exist at this point the limit can still have a value.

(d) $a = 3$

Solution

From the graph we can see that,

$$\boxed{f(3) = 4}$$

because the closed dot is at the value of $y = 4$.

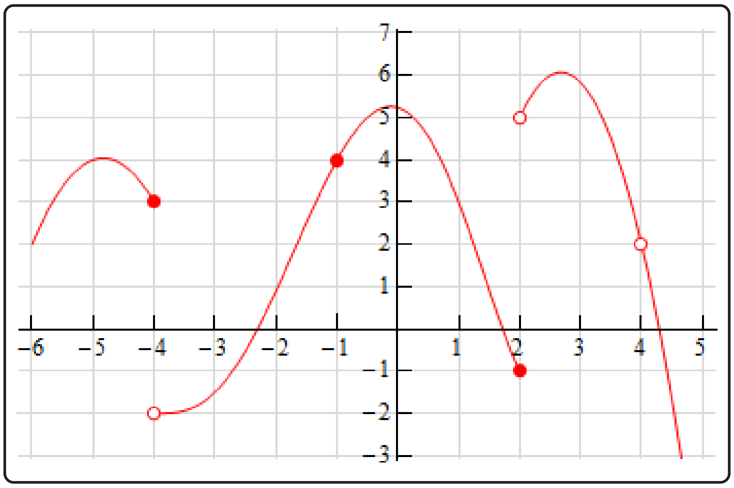
We can also see that as we approach $x = 3$ from both sides the graph is approaching the same value, 4, and so we get,

$$\boxed{\lim_{x \to 3} f(x) = 4}$$

2.3 One-Sided Limits

1. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$, $\lim\limits_{x \to a^-} f(x)$, $\lim\limits_{x \to a^+} f(x)$, and $\lim\limits_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -4$ **(b)** $a = -1$ **(c)** $a = 2$ **(d)** $a = 4$



Solutions

(a) $a = -4$

Solution

From the graph we can see that,

$$\boxed{f(-4) = 3}$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -4$ from the left the graph is approaching a value of 3 and as we approach from the right the graph is approaching a value of -2. Therefore, we get,

$$\boxed{\lim_{x \to -4^-} f(x) = 3 \qquad \& \qquad \lim_{x \to -4^+} f(x) = -2}$$

Now, because the two one-sided limits are different we know that,

$$\boxed{\lim_{x \to -4} f(x) \text{ does not exist}}$$

(b) $a = -1$

Solution

From the graph we can see that,

$$\boxed{f(-1) = 4}$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = -1$ from both sides the graph is approaching the same value, 4, and so we get,

$$\boxed{\lim_{x \to -1^-} f(x) = 4 \qquad \& \qquad \lim_{x \to -1^+} f(x) = 4}$$

The two one-sided limits are the same so we know,

$$\boxed{\lim_{x \to -1} f(x) = 4}$$

(c) $a = 2$

Solution

From the graph we can see that,

$$\boxed{f(2) = -1}$$

because the closed dot is at the value of $y = -1$.

We can also see that as we approach $x = 2$ from the left the graph is approaching a value of -1 and as we approach from the right the graph is approaching a value of 5. Therefore, we get,

$$\boxed{\lim_{x \to 2^-} f(x) = -1 \qquad \& \qquad \lim_{x \to 2^+} f(x) = 5}$$

Now, because the two one-sided limits are different we know that,

$$\boxed{\lim_{x\to 2} f(x) \text{ does not exist}}$$

(d) $a = 4$

Solution

Because there is no closed dot for $x = 4$ we can see that,

$$\boxed{f(4) \text{ does not exist}}$$

We can also see that as we approach $x = 4$ from both sides the graph is approaching the same value, 2, and so we get,

$$\boxed{\lim_{x\to 4^-} f(x) = 2 \qquad \& \qquad \lim_{x\to 4^+} f(x) = 2}$$

The two one-sided limits are the same so we know,

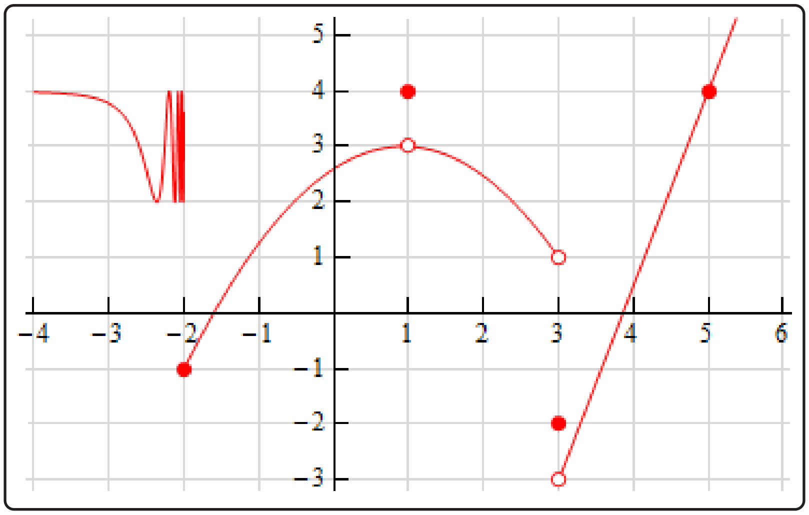
$$\boxed{\lim_{x\to 4} f(x) = 2}$$

Always recall that the value of a limit (including one-sided limits) does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Therefore, even though the function doesn't exist at this point the limit and one-sided limits can still have a value.

---

2. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$, $\lim\limits_{x \to a^-} f(x)$, $\lim\limits_{x \to a^+} f(x)$, and $\lim\limits_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -2$ **(b)** $a = 1$ **(c)** $a = 3$ **(d)** $a = 5$



Solutions

(a) $a = -2$

Solution

From the graph we can see that,

$$\boxed{f(-2) = -1}$$

because the closed dot is at the value of $y = -1$.

We can also see that as we approach $x = -2$ from the left the graph is not approaching a single value, but instead oscillating wildly, and as we approach from the right the graph is approaching a value of -1. Therefore, we get,

$$\boxed{\lim_{x \to -2^-} f(x) \text{ does not exist} \qquad \& \qquad \lim_{x \to -2^+} f(x) = -1}$$

Recall that in order for limit to exist the function must be approaching a single value and so, in this case, because the graph to the left of $x = -2$ is not approaching a single value the left-hand limit will not exist. This does not mean that the right-hand limit will not exist. In this case the graph to the right of $x = -2$ is approaching a single value the right-hand limit will exist.

Now, because the two one-sided limits are different we know that,

$$\boxed{\lim_{x \to -2} f(x) \text{ does not exist}}$$

(b) $a = 1$

Solution

From the graph we can see that,

$$\boxed{f(1) = 4}$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = 1$ from both sides the graph is approaching the same value, 3, and so we get,

$$\boxed{\lim_{x \to 1^-} f(x) = 3 \qquad \& \qquad \lim_{x \to 1^+} f(x) = 3}$$

The two one-sided limits are the same so we know,

$$\boxed{\lim_{x \to 1} f(x) = 3}$$

(c) $a = 3$

Solution

From the graph we can see that,

$$\boxed{f(3) = -2}$$

because the closed dot is at the value of $y = -2$.

We can also see that as we approach $x = 2$ from the left the graph is approaching a value of 1 and as we approach from the right the graph is approaching a value of

-3. Therefore, we get,

$$\boxed{\lim_{x \to 3^-} f(x) = 1 \quad \& \quad \lim_{x \to 3^+} f(x) = -3}$$

Now, because the two one-sided limits are different we know that,

$$\boxed{\lim_{x \to 3} f(x) \text{ does not exist}}$$

(d) $a = 5$

Solution

From the graph we can see that,

$$\boxed{f(5) = 4}$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = 5$ from both sides the graph is approaching the same value, 4, and so we get,

$$\boxed{\lim_{x \to 5^-} f(x) = 4 \quad \& \quad \lim_{x \to 5^+} f(x) = 4}$$

The two one-sided limits are the same so we know,

$$\boxed{\lim_{x \to 5} f(x) = 4}$$

---

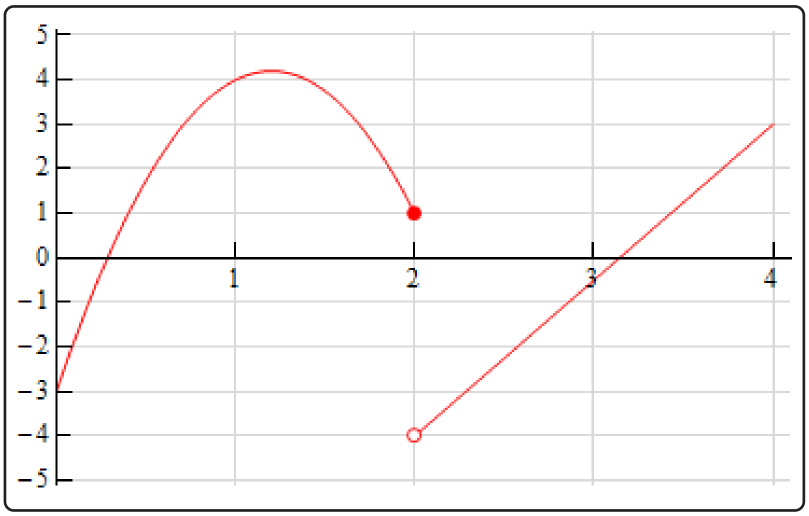
3. Sketch a graph of a function that satisfies each of the following conditions.

$$\lim_{x \to 2^-} f(x) = 1 \qquad \lim_{x \to 2^+} f(x) = -4 \qquad f(2) = 1$$

Solution

There are literally an infinite number of possible graphs that we could give here for an answer. However, all of them must have a closed dot on the graph at the point $(2, 1)$, the graph must be approaching a value of 1 as it approaches $x = 2$ from the left (as indicated by the left-hand limit) and it must be approaching a value of -4 as it approaches $x = 2$ from the right (as indicated by the right-hand limit).

Here is a sketch of one possible graph that meets these conditions.



4. Sketch a graph of a function that satisfies each of the following conditions.

$$\lim_{x \to 3^-} f(x) = 0 \qquad \lim_{x \to 3^+} f(x) = 4 \qquad f(3) \text{ does not exist}$$
$$\lim_{x \to -1} f(x) = -3 \qquad f(-1) = 2$$

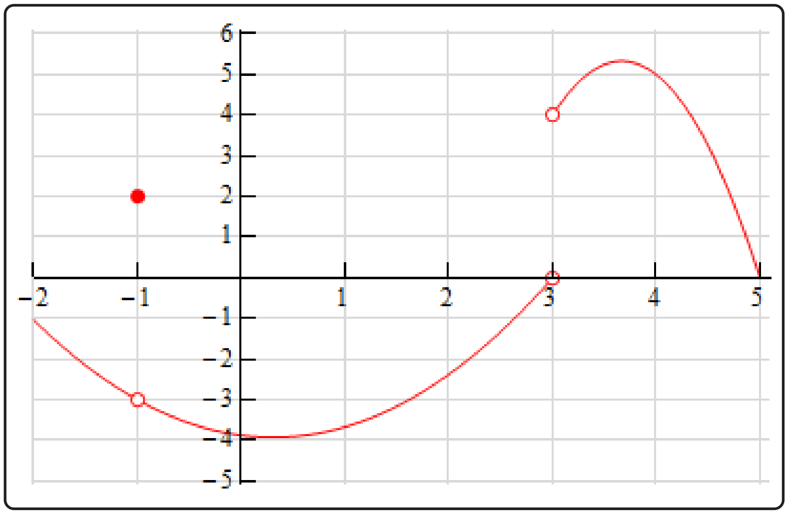
Solution

There are literally an infinite number of possible graphs that we could give here for an answer. However, all of them must the following two sets of criteria.

First, at $x = 3$ there cannot be a closed dot on the graph (as indicated by the fact that the function does not exist here), the graph must be approaching a value of 0 as it approaches $x = 3$ from the left (as indicated by the left-hand limit) and it must be approaching a value of 4 as it approaches $x = 3$ from the right (as indicated by the right-hand limit).

Next, the graph must have a closed dot at the point $(-1, 2)$ and the graph must be approaching a value of -3 as it approaches $x = -1$ from both sides (as indicated by the fact that value of the overall limit is -3 at this point).

Here is a sketch of one possible graph that meets these conditions.



2.4 Limits Properties

1. Given $\lim\limits_{x \to 8} f(x) = -9$, $\lim\limits_{x \to 8} g(x) = 2$ and $\lim\limits_{x \to 8} h(x) = 4$ use the limit properties given in this section to compute each of the following limits. If it is not possible to compute any of the limits clearly explain why not.

(a) $\lim\limits_{x \to 8} [2f(x) - 12h(x)]$

(b) $\lim\limits_{x \to 8} [3h(x) - 6]$

(c) $\lim\limits_{x \to 8} [g(x)h(x) - f(x)]$

(d) $\lim\limits_{x \to 8} [f(x) - g(x) + h(x)]$

Hint

For each of these all we need to do is use the limit properties on the limit until the given limits appear and we can then compute the value.

Solutions

(a) $\lim\limits_{x \to 8} [2f(x) - 12h(x)]$

Solution

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned} \lim_{x \to 8} [2f(x) - 12h(x)] &= \lim_{x \to 8} [2f(x)] - \lim_{x \to 8} [12h(x)] && \text{Property 2} \\ &= 2\lim_{x \to 8} f(x) - 12\lim_{x \to 8} h(x) && \text{Property 1} \\ &= 2(-9) - 12(4) && \text{Plug in limits} \\ &= \boxed{-66} \end{aligned}$$

(b) $\displaystyle\lim_{x\to 8}\left[3h\left(x\right)-6\right]$

Solution

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}\lim_{x\to 8}\left[3h\left(x\right)-6\right] &= \lim_{x\to 8}\left[3h\left(x\right)\right]-\lim_{x\to 8}6 && \text{Property 2}\\ &= 3\lim_{x\to 8}h\left(x\right)-\lim_{x\to 8}6 && \text{Property 1}\\ &= 3\left(4\right)-6 && \text{Plug in limits \& Property 7}\\ &= \boxed{6}\end{aligned}$$

(c) $\displaystyle\lim_{x\to 8}\left[g\left(x\right)h\left(x\right)-f\left(x\right)\right]$

Solution

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}\lim_{x\to 8}\left[g\left(x\right)h\left(x\right)-f\left(x\right)\right] &= \lim_{x\to 8}\left[g\left(x\right)h\left(x\right)\right]-\lim_{x\to 8}f\left(x\right) && \text{Property 2}\\ &= \left[\lim_{x\to 8}g\left(x\right)\right]\left[\lim_{x\to 8}h\left(x\right)\right]-\lim_{x\to 8}f\left(x\right) && \text{Property 3}\\ &= \left(2\right)\left(4\right)-\left(-9\right) && \text{Plug in limits}\\ &= \boxed{17}\end{aligned}$$

(d) $\lim\limits_{x \to 8} \left[f(x) - g(x) + h(x)\right]$

Solution

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned} \lim_{x \to 8} \left[f(x) - g(x) + h(x)\right] &= \lim_{x \to 8} f(x) - \lim_{x \to 8} g(x) + \lim_{x \to 8} h(x) && \text{Property 2} \\ &= -9 - 2 + 4 && \text{Plug in limits} \\ &= \boxed{-7} \end{aligned}$$

---

2. Given $\lim\limits_{x \to -4} f(x) = 1$, $\lim\limits_{x \to -4} g(x) = 10$ and $\lim\limits_{x \to -4} h(x) = -7$ use the limit properties given in this section to compute each of the following limits. If it is not possible to compute any of the limits clearly explain why not.

(a) $\lim\limits_{x \to -4} \left[\dfrac{f(x)}{g(x)} - \dfrac{h(x)}{f(x)}\right]$

(b) $\lim\limits_{x \to -4} \left[f(x)\, g(x)\, h(x)\right]$

(c) $\lim\limits_{x \to -4} \left[\dfrac{1}{h(x)} + \dfrac{3 - f(x)}{g(x) + h(x)}\right]$

(d) $\lim\limits_{x \to -4} \left[2h(x) - \dfrac{1}{h(x) + 7f(x)}\right]$

Hint

For each of these all we need to do is use the limit properties on the limit until the given limits appear and we can then compute the value.

Solutions

(a) $\displaystyle\lim_{x\to-4}\left[\frac{f(x)}{g(x)}-\frac{h(x)}{f(x)}\right]$

Solution

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
\lim_{x\to-4}\left[\frac{f(x)}{g(x)}-\frac{h(x)}{f(x)}\right] &= \lim_{x\to-4}\frac{f(x)}{g(x)}-\lim_{x\to-4}\frac{h(x)}{f(x)} && \text{Property 2}\\
&= \frac{\lim\limits_{x\to-4}f(x)}{\lim\limits_{x\to-4}g(x)}-\frac{\lim\limits_{x\to-4}h(x)}{\lim\limits_{x\to-4}f(x)} && \text{Property 4}\\
&= \frac{1}{10}-\frac{-7}{1} && \text{Plug in limits}\\
&= \boxed{\frac{71}{10}}
\end{aligned}$$

Note that were able to use Property 4 in the second step only because after we evaluated the limit of the denominators (both of them) we found that the limits of the denominators were not zero.

(b) $\displaystyle\lim_{x\to-4}\left[f(x)\,g(x)\,h(x)\right]$

Solution

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
\lim_{x\to-4}\left[f(x)\,g(x)\,h(x)\right] &= \left[\lim_{x\to-4}f(x)\right]\left[\lim_{x\to-4}g(x)\right]\left[\lim_{x\to-4}h(x)\right] && \text{Property 3}\\
&= (1)(10)(-7) && \text{Plug in limits}\\
&= \boxed{-70}
\end{aligned}$$

Note that the properties 2 & 3 in this section were only given with two functions but they can easily be extended out to more than two functions as we did here for property 3.

(c) $\displaystyle\lim_{x\to -4}\left[\frac{1}{h(x)}+\frac{3-f(x)}{g(x)+h(x)}\right]$

Solution

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
\lim_{x\to -4}\left[\frac{1}{h(x)}+\frac{3-f(x)}{g(x)+h(x)}\right] &= \lim_{x\to -4}\frac{1}{h(x)}+\lim_{x\to -4}\frac{3-f(x)}{g(x)+h(x)} && \text{Property 2}\\
&= \frac{\lim\limits_{x\to -4} 1}{\lim\limits_{x\to -4} h(x)}+\frac{\lim\limits_{x\to -4}\left[3-f(x)\right]}{\lim\limits_{x\to -4}\left[g(x)+h(x)\right]} && \text{Property 4}\\
&= \frac{\lim\limits_{x\to -4} 1}{\lim\limits_{x\to -4} h(x)}+\frac{\lim\limits_{x\to -4} 3-\lim\limits_{x\to -4} f(x)}{\lim\limits_{x\to -4} g(x)+\lim\limits_{x\to -4} h(x)} && \text{Property 2}\\
&= \frac{1}{-7}+\frac{3-1}{10-7} && \text{Plug in limits \& Property 1}\\
&= \boxed{\frac{11}{21}}
\end{aligned}$$

Note that were able to use Property 4 in the second step only because after we evaluated the limit of the denominators (both of them) we found that the limits of the denominators were not zero.

(d) $\displaystyle\lim_{x\to -4}\left[2h(x)-\frac{1}{h(x)+7f(x)}\right]$

Solution

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
\lim_{x\to -4}\left[2h(x)-\frac{1}{h(x)+7f(x)}\right] &= \lim_{x\to -4} 2h(x) - \lim_{x\to -4}\frac{1}{h(x)+7f(x)} && \text{Property 2}\\
&= \lim_{x\to -4} 2h(x) - \frac{\lim\limits_{x\to -4} 1}{\lim\limits_{x\to -4}\left[h(x)+7f(x)\right]} && \text{Property 4}
\end{aligned}$$

At this point let's step back a minute. In the previous parts we didn't worry about using property 4 on a rational expression. However, in this case let's be a little more careful. We can only use property 4 if the limit of the denominator is not zero. Let's check that limit and see what we get.

$$\begin{aligned}
\lim_{x\to -4}\left[h(x)+7f(x)\right] &= \lim_{x\to -4} h(x) + \lim_{x\to -4}\left[7f(x)\right] && \text{Property 2}\\
&= \lim_{x\to -4} h(x) + 7\lim_{x\to -4} f(x) && \text{Property 1}\\
&= -7+7(1) \quad \text{Plug in limits \&} && \text{Property 1}\\
&= 0
\end{aligned}$$

Okay, we can see that the limit of the denominator in the second term will be zero so we cannot actually use property 4 on that term. This means that this limit cannot be done and note that the fact that we could determine a value for the limit of the first term will not change this fact. This limit cannot be done.

---

3. Given $\lim\limits_{x\to 0} f(x) = 6$, $\lim\limits_{x\to 0} g(x) = -4$ and $\lim\limits_{x\to 0} h(x) = -1$ use the limit properties given in this section to compute each of the following limits. If it is not possible to compute any of the limits clearly explain why not.

   (a) $\lim\limits_{x\to 0}\left[f(x)+h(x)\right]^3$

   (b) $\lim\limits_{x\to 0}\sqrt{g(x)\,h(x)}$

   (c) $\lim\limits_{x\to 0}\sqrt[3]{11+\left[g(x)\right]^2}$

   (d) $\lim\limits_{x\to 0}\sqrt{\dfrac{f(x)}{h(x)-g(x)}}$

Hint

For each of these all we need to do is use the limit properties on the limit until the given limits appear and we can then compute the value.

Solutions

(a) $\lim\limits_{x\to 0} \left[f(x) + h(x)\right]^3$

Solution

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$
\begin{aligned}
\lim_{x\to 0} \left[f(x) + h(x)\right]^3 &= \left[\lim_{x\to 0} \left(f(x) + h(x)\right)\right]^3 && \text{Property 5} \\
&= \left[\lim_{x\to 0} f(x) + \lim_{x\to 0} h(x)\right]^3 && \text{Property 2} \\
&= \left[6 - 1\right]^3 && \text{Plug in limits} \\
&= \boxed{125}
\end{aligned}
$$

(b) $\lim\limits_{x\to 0} \sqrt{g(x)\, h(x)}$

Solution

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$
\begin{aligned}
\lim_{x\to 0} \sqrt{g(x)\, h(x)} &= \sqrt{\lim_{x\to 0} g(x)\, h(x)} && \text{Property 6} \\
&= \sqrt{\left[\lim_{x\to 0} g(x)\right]\left[\lim_{x\to 0} h(x)\right]} && \text{Property 3} \\
&= \sqrt{(-4)(-1)} && \text{Plug in limits} \\
&= \boxed{2}
\end{aligned}
$$

(c) $\lim\limits_{x \to 0} \sqrt[3]{11 + [g(x)]^2}$

Solution

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
\lim_{x \to 0} \sqrt[3]{11 + [g(x)]^2} &= \sqrt[3]{\lim_{x \to 0}\left(11 + [g(x)]^2\right)} && \text{Property 6}\\
&= \sqrt[3]{\lim_{x \to 0} 11 + \lim_{x \to 0} [g(x)]^2} && \text{Property 2}\\
&= \sqrt[3]{\lim_{x \to 0} 11 + \left[\lim_{x \to 0} g(x)\right]^2} && \text{Property 5}\\
&= \sqrt[3]{11 + (-4)^2} && \text{Plug in limits \& Property 7}\\
&= \boxed{3}
\end{aligned}$$

(d) $\lim\limits_{x \to 0} \sqrt{\dfrac{f(x)}{h(x) - g(x)}}$

Solution

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
\lim_{x \to 0} \sqrt{\frac{f(x)}{h(x) - g(x)}} &= \sqrt{\lim_{x \to 0} \frac{f(x)}{h(x) - g(x)}} && \text{Property 6}\\
&= \sqrt{\frac{\lim\limits_{x \to 0} f(x)}{\lim\limits_{x \to 0} (h(x) - g(x))}} && \text{Property 4}\\
&= \sqrt{\frac{\lim\limits_{x \to 0} f(x)}{\lim\limits_{x \to 0} h(x) - \lim\limits_{x \to 0} g(x)}} && \text{Property 2}\\
&= \sqrt{\frac{6}{-1 - (-4)}} && \text{Plug in limits}\\
&= \sqrt{2}
\end{aligned}$$

Note that were able to use Property 4 in the second step only because after we evaluated the limit of the denominators (both of them) we found that the limits of the denominators were not zero.

---

4. Use the limit properties given in this section to compute the following limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

$$\lim_{t \to -2} \left(14 - 6t + t^3\right)$$

Hint

All we need to do is use the limit properties on the limit until we can use Properties 7, 8 and/or 9 from this section to compute the limit.'

Solution

$$\begin{aligned} \lim_{t \to -2} \left(14 - 6t + t^3\right) &= \lim_{t \to -2} 14 - \lim_{t \to -2} 6t + \lim_{t \to -2} t^3 && \text{Property 2} \\ &= \lim_{t \to -2} 14 - 6 \lim_{t \to -2} t + \lim_{t \to -2} t^3 && \text{Property 1} \\ &= 14 - 6\left(-2\right) + \left(-2\right)^3 && \text{Properties 7, 8, \& 9} \\ &= \boxed{18} \end{aligned}$$

---

5. Use the limit properties given in this section to compute the following limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

$$\lim_{x \to 6} \left(3x^2 + 7x - 16\right)$$

Hint

All we need to do is use the limit properties on the limit until we can use Properties 7, 8 and/or 9 from this section to compute the limit.

Solution

$$\begin{aligned}\lim_{x\to 6}\left(3x^2+7x-16\right) &= \lim_{x\to 6} 3x^2 + \lim_{x\to 6} 7x - \lim_{x\to 6} 16 && \text{Property 2}\\ &= 3\lim_{x\to 6} x^2 + 7\lim_{x\to 6} x - \lim_{x\to 6} 16 && \text{Property 1}\\ &= 3\left(6^2\right) + 7\left(6\right) - 16 && \text{Properties 7, 8, \& 9}\\ &= \boxed{134}\end{aligned}$$

6. Use the limit properties given in this section to compute the following limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

$$\lim_{w\to 3}\frac{w^2-8w}{4-7w}$$

Hint

All we need to do is use the limit properties on the limit until we can use Properties 7, 8 and/or 9 from this section to compute the limit.

Solution

$$\begin{aligned}\lim_{w\to 3}\frac{w^2-8w}{4-7w} &= \frac{\lim\limits_{w\to 3}\left(w^2-8w\right)}{\lim\limits_{w\to 3}\left(4-7w\right)} && \text{Property 4}\\ &= \frac{\lim\limits_{w\to 3} w^2 - \lim\limits_{w\to 3} 8w}{\lim\limits_{w\to 3} 4 - \lim\limits_{w\to 3} 7w} && \text{Property 2}\\ &= \frac{\lim\limits_{w\to 3} w^2 - 8\lim\limits_{w\to 3} w}{\lim\limits_{w\to 3} 4 - 7\lim\limits_{w\to 3} w} && \text{Property 1}\\ &= \frac{3^2-8\left(3\right)}{4-7\left(3\right)} && \text{Properties 7, 8, \& 9}\\ &= \boxed{\frac{15}{17}}\end{aligned}$$

Note that we were able to use property 4 in the first step because after evaluating the limit in the denominator we found that it wasn't zero.

7. Use the limit properties given in this section to compute the following limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

$$\lim_{x \to -5} \frac{x+7}{x^2+3x-10}$$

Hint

All we need to do is use the limit properties on the limit until we can use Properties 7, 8 and/or 9 from this section to compute the limit.

Solution

$$\lim_{x \to -5} \frac{x+7}{x^2+3x-10} = \frac{\lim\limits_{x \to -5}(x+7)}{\lim\limits_{x \to -5}(x^2+3x-10)} \qquad \text{Property 4}$$

Okay, at this point let's step back a minute. We used property 4 here and we know that we can only do that if the limit of the denominator is not zero. So, let's check that out and see what we get.

$$\begin{aligned}
\lim_{x \to -5}\left(x^2+3x-10\right) &= \lim_{x \to -5} x^2 + \lim_{x \to -5} 3x - \lim_{x \to -5} 10 && \text{Property 2} \\
&= \lim_{x \to -5} x^2 + 3\lim_{x \to -5} x - \lim_{x \to -5} 10 && \text{Property 1} \\
&= (-5)^2 + 3(-5) - 10 && \text{Properties 7, 8, \& 9} \\
&= 0
\end{aligned}$$

So, the limit of the denominator is zero so we couldn't use property 4 in this case. Therefore, we cannot do this limit at this point (note that it will be possible to do this limit after the next section).

8. Use the limit properties given in this section to compute the following limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

$$\lim_{z \to 0} \sqrt{z^2+6}$$

Hint

All we need to do is use the limit properties on the limit until we can use Properties 7, 8 and/or 9 from this section to compute the limit.

Solution

$$\begin{aligned}\lim_{z\to 0}\sqrt{z^2+6} &= \sqrt{\lim_{z\to 0}(z^2+6)} && \text{Property 6}\\ &= \sqrt{\lim_{z\to 0} z^2 + \lim_{z\to 0} 6} && \text{Property 2}\\ &= \sqrt{0^2+6} && \text{Properties 7 \& 9}\\ &= \sqrt{6}\end{aligned}$$

---

9. Use the limit properties given in this section to compute the following limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

$$\lim_{x\to 10}\left(4x+\sqrt[3]{x}-2\right)$$

Hint

All we need to do is use the limit properties on the limit until we can use Properties 7, 8 and/or 9 from this section to compute the limit.

Solution

$$\begin{aligned}\lim_{x\to 10}\left(4x+\sqrt[3]{x-2}\right) &= \lim_{x\to 10} 4x + \lim_{x\to 10}\sqrt[3]{x-2} && \text{Property 2}\\ &= \lim_{x\to 10} 4x + \sqrt[3]{\lim_{x\to 10}(x-2)} && \text{Property 6}\\ &= \lim_{x\to 10} 4x + \sqrt[3]{\lim_{x\to 10} x - \lim_{x\to 10} 2} && \text{Property 2}\\ &= 4\lim_{x\to 10} x + \sqrt[3]{\lim_{x\to 10} x - \lim_{x\to 10} 2} && \text{Property 1}\\ &= 4(10) + \sqrt[3]{10-2} && \text{Properties 7 \& 8}\\ &= 42\end{aligned}$$

2.5 Computing Limits

1. Evaluate $\lim\limits_{x \to 2} \left(8 - 3x + 12x^2\right)$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. We know that the first thing that we should try to do is simply plug in the value and see if we can compute the limit.

$$\lim_{x \to 2} \left(8 - 3x + 12x^2\right) = 8 - 3(2) + 12(4) = \boxed{50}$$

2. Evaluate $\lim\limits_{t \to -3} \dfrac{6 + 4t}{t^2 + 1}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. We know that the first thing that we should try to do is simply plug in the value and see if we can compute the limit.

$$\lim_{t \to -3} \frac{6 + 4t}{t^2 + 1} = \frac{-6}{10} = \boxed{-\frac{3}{5}}$$

3. Evaluate $\lim\limits_{x \to -5} \dfrac{x^2 - 25}{x^2 + 2x - 15}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{x \to -5} \frac{x^2 - 25}{x^2 + 2x - 15} = \lim_{x \to -5} \frac{(x-5)(x+5)}{(x-3)(x+5)} = \lim_{x \to -5} \frac{x - 5}{x - 3} = \boxed{\frac{5}{4}}$$

4. Evaluate $\lim\limits_{z \to 8} \dfrac{2z^2 - 17z + 8}{8 - z}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{z \to 8} \frac{2z^2 - 17z + 8}{8 - z} = \lim_{z \to 8} \frac{(2z - 1)(z - 8)}{-(z - 8)} = \lim_{z \to 8} \frac{2z - 1}{-1} = \boxed{-15}$$

5. Evaluate $\lim\limits_{y \to 7} \dfrac{y^2 - 4y - 21}{3y^2 - 17y - 28}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{y \to 7} \frac{y^2 - 4y - 21}{3y^2 - 17y - 28} = \lim_{y \to 7} \frac{(y - 7)(y + 3)}{(3y + 4)(y - 7)} = \lim_{y \to 7} \frac{y + 3}{3y + 4} = \frac{10}{25} = \boxed{\frac{2}{5}}$$

6. Evaluate $\lim\limits_{h \to 0} \dfrac{(6 + h)^2 - 36}{h}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{h \to 0} \frac{(6 + h)^2 - 36}{h} = \lim_{h \to 0} \frac{36 + 12h + h^2 - 36}{h} = \lim_{h \to 0} \frac{h(12 + h)}{h}$$

$$= \lim_{h \to 0} (12 + h) = \boxed{12}$$

7. Evaluate $\lim\limits_{z \to 4} \dfrac{\sqrt{z}-2}{z-4}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. If you're really good at factoring you can factor this and simplify. Another method that can be used however is to rationalize the numerator, so let's do that for this problem.

$$\begin{aligned}\lim_{z \to 4} \frac{\sqrt{z}-2}{z-4} = \lim_{z \to 4} \frac{\left(\sqrt{z}-2\right)}{(z-4)} \frac{\left(\sqrt{z}+2\right)}{\left(\sqrt{z}+2\right)} &= \lim_{z \to 4} \frac{z-4}{(z-4)\left(\sqrt{z}+2\right)} \\ &= \lim_{z \to 4} \frac{1}{\sqrt{z}+2} = \boxed{\frac{1}{4}}\end{aligned}$$

8. Evaluate $\lim\limits_{x \to -3} \dfrac{\sqrt{2x+22}-4}{x+3}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. Simply factoring will not do us much good here so in this case it looks like we'll need to rationalize the numerator.

$$\begin{aligned}\lim_{x \to -3} \frac{\sqrt{2x+22}-4}{x+3} &= \lim_{x \to -3} \frac{\left(\sqrt{2x+22}-4\right)}{(x+3)} \frac{\left(\sqrt{2x+22}+4\right)}{\left(\sqrt{2x+22}+4\right)} \\ &= \lim_{x \to -3} \frac{2x+22-16}{(x+3)\left(\sqrt{2x+22}+4\right)} \\ &= \lim_{x \to -3} \frac{2(x+3)}{(x+3)\left(\sqrt{2x+22}+4\right)} \\ &= \lim_{x \to -3} \frac{2}{\sqrt{2x+22}+4} = \frac{2}{8} = \boxed{\frac{1}{4}}\end{aligned}$$

9. Evaluate $\lim\limits_{x \to 0} \dfrac{x}{3 - \sqrt{x+9}}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. Simply factoring will not do us much good here so in this case it looks like we'll need to rationalize the denominator.

$$\begin{aligned}\lim_{x \to 0} \frac{x}{3 - \sqrt{x+9}} &= \lim_{x \to 0} \frac{x}{\left(3 - \sqrt{x+9}\right)} \frac{\left(3 + \sqrt{x+9}\right)}{\left(3 + \sqrt{x+9}\right)} = \lim_{x \to 0} \frac{x\left(3 + \sqrt{x+9}\right)}{9 - (x+9)} \\ &= \lim_{x \to 0} \frac{x\left(3 + \sqrt{x+9}\right)}{-x} = \lim_{x \to 0} \frac{3 + \sqrt{x+9}}{-1} = \boxed{-6}\end{aligned}$$

10. Given the function

$$f(x) = \begin{cases} 7 - 4x & x < 1 \\ x^2 + 2 & x \ge 1 \end{cases}$$

Evaluate the following limits, if they exist.

(a) $\lim\limits_{x \to -6} f(x)$ **(b)** $\lim\limits_{x \to 1} f(x)$

Hint

Recall that when looking at overall limits (as opposed to one-sided limits) we need to make sure that the value of the function must be approaching the same value from both sides. In other words, the two one sided limits must both exist and be equal.

Solutions

(a) $\lim\limits_{x \to -6} f(x)$

Solution

For this part we know that $-6 < 1$ and so there will be values of x on both sides of -6 in the range $x < 1$ and so we can assume that, in the limit, we will have $x < 1$. This will allow us to use the piece of the function in that range and then just use standard limit techniques to compute the limit.

$$\lim_{x \to -6} f(x) = \lim_{x \to -6} (7 - 4x) = \boxed{31}$$

(b) $\displaystyle\lim_{x \to 1} f(x)$

Solution

This part is going to be different from the previous part. We are looking at the limit at $x = 1$ and that is the "cut-off" point in the piecewise functions. Recall from the discussion in the section, that this means that we are going to have to look at the two one sided limits.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (7 - 4x) = \underline{3} \qquad \text{because } x \to 1^- \text{ implies that } x < 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(x^2 + 2\right) = \underline{3} \qquad \text{because } x \to 1^+ \text{ implies that } x > 1$$

So, in this case, we can see that,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 3$$

and so we know that the overall limit must exist and,

$$\lim_{x \to 1} f(x) = \boxed{3}$$

11. Given

$$h(z) = \begin{cases} 6z & z \le -4 \\ 1 - 9z & z > -4 \end{cases}$$

Evaluate the following limits, if they exist.

(a) $\displaystyle\lim_{z \to 7} h(z)$

(b) $\displaystyle\lim_{z \to -4} h(z)$

Hint

Recall that when looking at overall limits (as opposed to one-sided limits) we need to make sure that the value of the function must be approaching the same value from both sides. In other words, the two one sided limits must both exist and be equal.

Solutions

(a) $\lim\limits_{z \to 7} h(z)$

Solution

For this part we know that $7 > -4$ and so there will be values of z on both sides of 7 in the range $z > -4$ and so we can assume that, in the limit, we will have $z > -4$. This will allow us to use the piece of the function in that range and then just use standard limit techniques to compute the limit.

$$\lim_{z \to 7} h(z) = \lim_{z \to 7} (1 - 9z) = \boxed{-62}$$

(b) $\lim\limits_{z \to -4} h(z)$

Solution

This part is going to be different from the previous part. We are looking at the limit at $z = -4$ and that is the "cut-off" point in the piecewise functions. Recall from the discussion in the section, that this means that we are going to have to look at the two one sided limits.

$$\lim_{z \to -4^-} h(z) = \lim_{z \to -4^-} 6z = \underline{-24} \qquad \text{because } z \to -4^- \text{ implies that } z < -4$$

$$\lim_{z \to -4^+} h(z) = \lim_{z \to -4^+} (1 - 9z) = \underline{37} \qquad \text{because } z \to -4^+ \text{ implies that } z > -4$$

So, in this case, we can see that,

$$\lim_{z \to -4^-} h(z) = -24 \neq 37 = \lim_{z \to -4^+} h(z)$$

and so we know that the overall limit **does not exist**.

---

12. Evaluate $\lim\limits_{x \to 5} (10 + |x - 5|)$, if it exists.

Hint

Recall the mathematical definition of the absolute value function and that it is in fact a piecewise function.

Solution

Recall the definition of the absolute value function.

$$|p| = \begin{cases} p & p \geq 0 \\ -p & p < 0 \end{cases}$$

So, because the function inside the absolute value is zero at $x = 5$ we can see that,

$$|x-5| = \begin{cases} x-5 & x \geq 5 \\ -(x-5) & x < 5 \end{cases}$$

This means that we are being asked to compute the limit at the "cut-off" point in a piecewise function and so, as we saw in this section, we'll need to look at two one-sided limits in order to determine if this limit exists (and its value if it does exist).

$$\begin{aligned} \lim_{x \to 5^-} (10 + |x-5|) &= \lim_{x \to 5^-} (10 - (x-5)) \\ &= \lim_{x \to 5^-} (15 - x) = 10 \qquad \text{recall } x \to 5^- \text{ implies } x < 5 \\ \lim_{x \to 5^+} (10 + |x-5|) &= \lim_{x \to 5^+} (10 + (x-5)) \\ &= \lim_{x \to 5^+} (5 + x) = 10 \qquad \text{recall } x \to 5^+ \text{ implies } x > 5 \end{aligned}$$

So, for this problem, we can see that,

$$\lim_{x \to 5^-} (10 + |x-5|) = \lim_{x \to 5+} (10 + |x-5|) = 10$$

and so the overall limit must exist and,

$$\lim_{x \to 5} (10 + |x-5|) = \boxed{10}$$

13. Evaluate $\displaystyle\lim_{t \to -1} \frac{t+1}{|t+1|}$, if it exists.

Hint

Recall the mathematical definition of the absolute value function and that it is in fact a piecewise function.

Solution

Recall the definition of the absolute value function.

$$|p| = \begin{cases} p & p \geq 0 \\ -p & p < 0 \end{cases}$$

So, because the function inside the absolute value is zero at $t = -1$ we can see that,

$$|t+1| = \begin{cases} t+1 & t \geq -1 \\ -(t+1) & t < -1 \end{cases}$$

This means that we are being asked to compute the limit at the "cut-off" point in a piecewise function and so, as we saw in this section, we'll need to look at two one-sided limits in order to determine if this limit exists (and its value if it does exist).

$$\lim_{t \to -1^-} \frac{t+1}{|t+1|} = \lim_{t \to -1^-} \frac{t+1}{-(t+1)} = \lim_{t \to -1^-} -1 = -1 \qquad \text{recall } t \to -1^- \text{ implies } t < -1$$

$$\lim_{t \to -1^+} \frac{t+1}{|t+1|} = \lim_{t \to -1^+} \frac{t+1}{t+1} = \lim_{t \to -1^+} 1 = 1 \qquad \text{recall } t \to -1^+ \text{ implies } t > -1$$

So, for this problem, we can see that,

$$\lim_{t \to -1^-} \frac{t+1}{|t+1|} = -1 \neq 1 = \lim_{t \to -1^+} \frac{t+1}{|t+1|}$$

and so the overall limit **does not exist**.

---

14. Given that $7x \leq f(x) \leq 3x^2 + 2$ for all x determine the value of $\lim\limits_{x \to 2} f(x)$.

Hint

Recall the Squeeze Theorem.

Solution

This problem is set up to use the Squeeze Theorem. First, we already know that $f(x)$ is always between two other functions. Now all that we need to do is verify that the two "outer" functions have the same limit at $x = 2$ and if they do we can use the Squeeze Theorem to get the answer.

$$\lim_{x \to 2} 7x = 14 \qquad\qquad \lim_{x \to 2} \left(3x^2 + 2\right) = 14$$

So, we have,

$$\lim_{x \to 2} 7x = \lim_{x \to 2} \left(3x^2 + 2\right) = 14$$

and so by the Squeeze Theorem we must also have,

$$\lim_{x \to 2} f(x) = 14$$

15. Use the Squeeze Theorem to determine the value of $\lim_{x \to 0} x^4 \sin\left(\frac{\pi}{x}\right)$.

Hint

Recall how we worked the Squeeze Theorem problem in this section to find the lower and upper functions we need in order to use the Squeeze Theorem.

Solution

We first need to determine lower/upper functions. We'll start off by acknowledging that provided $x \neq 0$ (which we know it won't be because we are looking at the limit as $x \to 0$) we will have,

$$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$$

Now, simply multiply through this by x^4 to get,

$$-x^4 \le x^4 \sin\left(\frac{\pi}{x}\right) \le x^4$$

Before proceeding note that we can only do this because we know that $x^4 > 0$ for $x \neq 0$. Recall that if we multiply through an inequality by a negative number we would have had to switch the signs. So, for instance, had we multiplied through by x^3 we would have had issues because this is positive if $x > 0$ and negative if $x < 0$.

Now, let's get back to the problem. We have a set of lower/upper functions and clearly,

$$\lim_{x \to 0} x^4 = \lim_{x \to 0} \left(-x^4\right) = 0$$

Therefore, by the Squeeze Theorem we must have,

$$\lim_{x \to 0} x^4 \sin\left(\frac{\pi}{x}\right) = 0$$

2.6 Infinite Limits

1. For $f(x) = \dfrac{9}{(x-3)^5}$ evaluate,

(a) $\lim\limits_{x \to 3^-} f(x)$ **(b)** $\lim\limits_{x \to 3^+} f(x)$ **(c)** $\lim\limits_{x \to 3} f(x)$

Solutions

(a) $\lim\limits_{x \to 3^-} f(x)$

Solution

Let's start off by acknowledging that for $x \to 3^-$ we know $x < 3$.

For the numerator we can see that, in the limit, it will just be 9.

The denominator takes a little more work. Clearly, in the limit, we have,

$$x - 3 \to 0$$

but we can actually go a little farther. Because we know that $x < 3$ we also know that,

$$x - 3 < 0$$

More compactly, we can say that in the limit we will have,

$$x - 3 \to 0^-$$

Raising this to the fifth power will not change this behavior and so, in the limit, the denominator will be,

$$(x-3)^5 \to 0^-$$

We can now do the limit of the function. In the limit, the numerator is a fixed positive constant and the denominator is an increasingly small negative number. In the limit, the quotient must then be an increasing large negative number or,

$$\boxed{\lim_{x \to 3^-} \frac{9}{(x-3)^5} = -\infty}$$

Note that this also means that there is a vertical asymptote at $x = 3$.

(b) $\lim\limits_{x\to 3^+} f(x)$

Solution

Let's start off by acknowledging that for $x \to 3^+$ we know $x > 3$.

As in the first part the numerator, in the limit, it will just be 9.

The denominator will also work similarly to the first part. In the limit, we have,

$$x - 3 \to 0$$

and because we know that $x > 3$ we also know that,

$$x - 3 > 0$$

More compactly, we can say that in the limit we will have,

$$x - 3 \to 0^+$$

Raising this to the fifth power will not change this behavior and so, in the limit, the denominator will be,

$$(x-3)^5 \to 0^+$$

We can now do the limit of the function. In the limit, the numerator is a fixed positive constant and the denominator is an increasingly small positive number. In the limit, the quotient must then be an increasing large positive number or,

$$\boxed{\lim_{x\to 3^+} \frac{9}{(x-3)^5} = \infty}$$

Note that this also means that there is a vertical asymptote at $x = 3$, which we already knew from the first part.

(c) $\lim\limits_{x \to 3} f(x)$

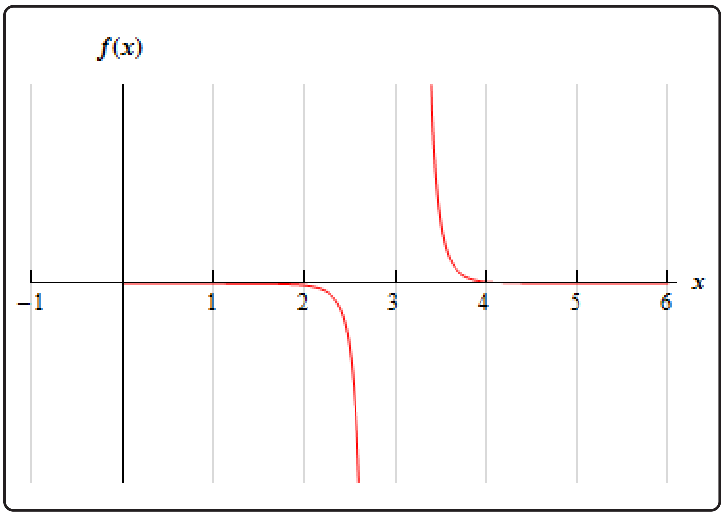
Solution

In this case we can see from the first two parts that,

$$\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$$

and so, from our basic limit properties we can see that $\lim\limits_{x \to 3} f(x)$ **does not exist**.

For the sake of completeness and to verify the answers for this problem here is a quick sketch of the function.



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2. For $h(t) = \dfrac{2t}{6+t}$ evaluate,

(a) $\lim\limits_{t \to -6^-} h(t)$ **(b)** $\lim\limits_{t \to -6^+} h(t)$ **(c)** $\lim\limits_{t \to -6} h(t)$

Solutions

(a) $\lim\limits_{t \to -6^-} h(t)$

Solution

Let's start off by acknowledging that for $t \to -6^-$ we know $t < -6$.

For the numerator we can see that, in the limit, we will get -12.

The denominator takes a little more work. Clearly, in the limit, we have,

$$6 + t \to 0$$

but we can actually go a little farther. Because we know that $t < -6$ we also know that,

$$6 + t < 0$$

More compactly, we can say that in the limit we will have,

$$6 + t \to 0^-$$

So, in the limit, the numerator is approaching a negative number and the denominator is an increasingly small negative number. The quotient must then be an increasing large positive number or,

$$\boxed{\lim_{t \to -6^-} \frac{2t}{6+t} = \infty}$$

Note that this also means that there is a vertical asymptote at $t = -6$.

(b) $\lim\limits_{t \to -6^+} h(t)$

Solution

Let's start off by acknowledging that for $t \to -6^+$ we know $t > -6$.

For the numerator we can see that, in the limit, we will get -12.

The denominator will also work similarly to the first part. In the limit, we have,

$$6 + t \to 0$$

but we can actually go a little farther. Because we know that $t > -6$ we also know that,

$$6 + t > 0$$

More compactly, we can say that in the limit we will have,

$$6 + t \to 0^+$$

So, in the limit, the numerator is approaching a negative number and the denominator is an increasingly small positive number. The quotient must then be an increasing large negative number or,

$$\boxed{\lim_{t \to -6^+} \frac{2t}{6+t} = -\infty}$$

Note that this also means that there is a vertical asymptote at $t = -6$, which we already knew from the first part.

(c) $\lim\limits_{t \to -6} h(t)$

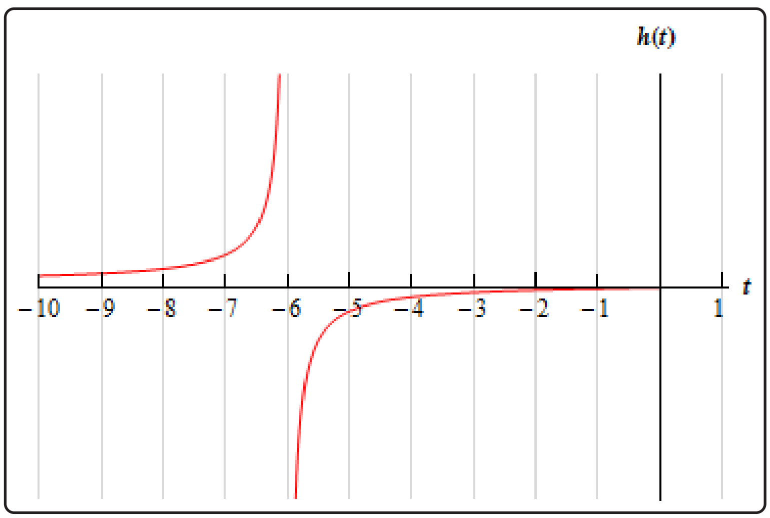
Solution

In this case we can see from the first two parts that,

$$\lim_{t \to -6^-} h(t) \neq \lim_{t \to -6^+} h(t)$$

and so, from our basic limit properties we can see that $\lim\limits_{t \to -6} h(t)$ **does not exist**.

For the sake of completeness and to verify the answers for this problem here is a quick sketch of the function.



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3. For $g(z) = \dfrac{z+3}{(z+1)^2}$ evaluate,

(a) $\lim\limits_{z \to -1^-} g(z)$ **(b)** $\lim\limits_{z \to -1^+} g(z)$ **(c)** $\lim\limits_{z \to -1} g(z)$

Solutions

(a) $\lim\limits_{z \to -1^-} g(z)$

Solution

Let's start off by acknowledging that for $z \to -1^-$ we know $z < -1$.

For the numerator we can see that, in the limit, we will get 2.

Now let's take care of the denominator. In the limit, we will have,

$$z + 1 \to 0^-$$

and upon squaring the $z+1$ we see that, in the limit, we will have,

$$(z+1)^2 \to 0^+$$

So, in the limit, the numerator is approaching a positive number and the denominator is an increasingly small positive number. The quotient must then be an increasing large positive number or,

$$\boxed{\lim_{z \to -1^-} \frac{z+3}{(z+1)^2} = \infty}$$

Note that this also means that there is a vertical asymptote at $z = -1$.

(b) $\lim\limits_{z \to -1^+} g(z)$

Solution

Let's start off by acknowledging that for $z \to -1^+$ we know $z > 1$.

For the numerator we can see that, in the limit, we will get 2.

Now let's take care of the denominator. In the limit, we will have,

$$z + 1 \to 0^+$$

and upon squaring the $z+1$ we see that, in the limit, we will have,

$$(z+1)^2 \to 0^+$$

So, in the limit, the numerator is approaching a positive number and the denominator is an increasingly small positive number. The quotient must then be an increasing large positive number or,

$$\boxed{\lim_{z \to -1^+} \frac{z+3}{(z+1)^2} = \infty}$$

Note that this also means that there is a vertical asymptote at $z = -1$, which we already knew from the first part.

(c) $\lim\limits_{z \to -1} g(z)$

Solution

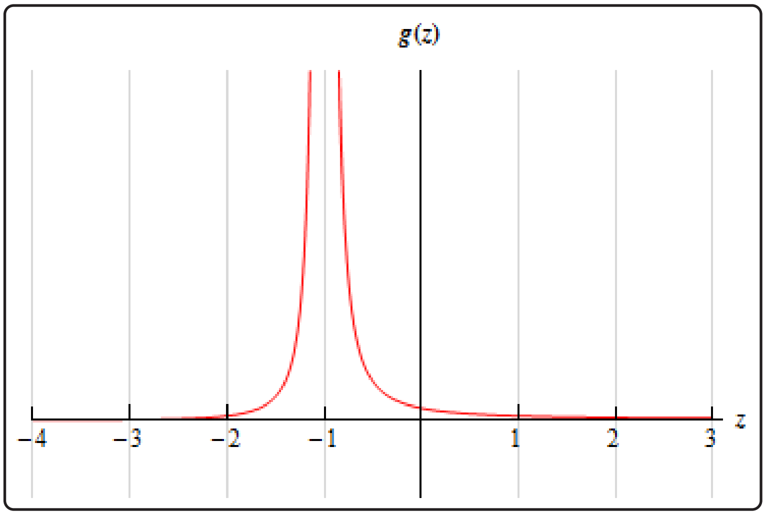
In this case we can see from the first two parts that,

$$\lim_{z \to -1^-} g(z) = \lim_{z \to -1^+} g(z) = \infty$$

and so, from our basic limit properties we can see that,

$$\boxed{\lim_{z \to -1} g(z) = \infty}$$

For the sake of completeness and to verify the answers for this problem here is a quick sketch of the function.



4. For $g(x) = \dfrac{x+7}{x^2-4}$ evaluate,

(a) $\lim\limits_{x \to 2^-} g(x)$ **(b)** $\lim\limits_{x \to 2^+} g(x)$ **(c)** $\lim\limits_{x \to 2} g(x)$

Solutions

(a) $\lim\limits_{x \to 2^-} g(x)$

Solution

Let's start off by acknowledging that for $x \to 2^-$ we know $x < 2$.

For the numerator we can see that, in the limit, we will get 9.

Now let's take care of the denominator. First, we know that if we square a number less than 2 (and greater than -2, which it is safe to assume we have here because we're doing the limit) we will get a number that is less than 4 and so, in the limit, we will have,

$$x^2 - 4 \to 0^-$$

So, in the limit, the numerator is approaching a positive number and the denominator is an increasingly small negative number. The quotient must then be an increasing large negative number or,

$$\boxed{\lim_{x \to 2^-} \frac{x+7}{x^2-4} = -\infty}$$

Note that this also means that there is a vertical asymptote at $x = 2$.

(b) $\lim\limits_{x \to 2^+} g(x)$

Solution

Let's start off by acknowledging that for $x \to 2^+$ we know $x > 2$.

For the numerator we can see that, in the limit, we will get 9.

Now let's take care of the denominator. First, we know that if we square a number greater than 2 we will get a number that is greater than 4 and so, in the limit, we will have,

$$x^2 - 4 \to 0^+$$

So, in the limit, the numerator is approaching a positive number and the denominator is an increasingly small positive number. The quotient must then be an increasing large positive number or,

$$\boxed{\lim_{x \to 2^+} \frac{x+7}{x^2-4} = \infty}$$

Note that this also means that there is a vertical asymptote at $x = 2$, which we already knew from the first part.

(c) $\lim\limits_{x \to 2} g(x)$

Solution

In this case we can see from the first two parts that,

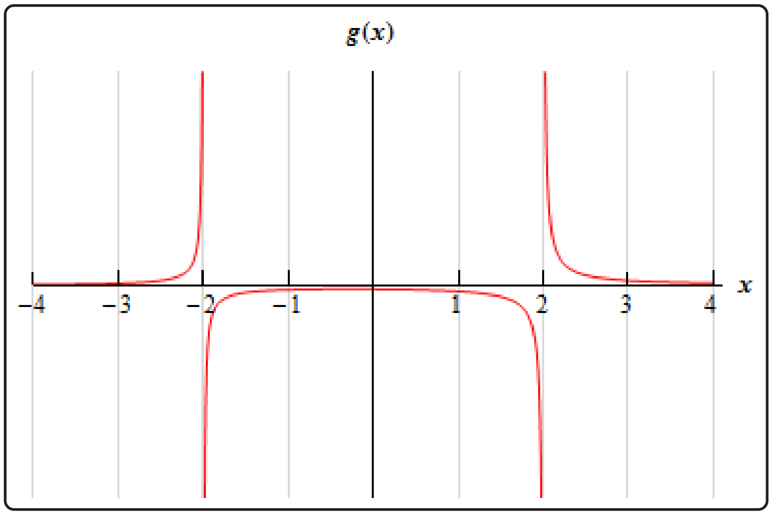
$$\lim_{x \to 2^-} g(x) \neq \lim_{x \to 2^+} g(x)$$

and so, from our basic limit properties we can see that

$$\lim_{x \to 2} g(x)$$

does not exist.

For the sake of completeness and to verify the answers for this problem here is a quick sketch of the function.



As we're sure that you had already noticed there would be another vertical asymptote at $x = -2$ for this function. For the practice you might want to make sure that you can also do the limits for that point.

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5. For $h(x) = \ln(-x)$ evaluate,

(a) $\displaystyle\lim_{x\to 0^-} h(x)$ **(b)** $\displaystyle\lim_{x\to 0^+} h(x)$ **(c)** $\displaystyle\lim_{x\to 0} h(x)$

Hint

Do not get excited about the $-x$ inside the logarithm. Just recall what you know about natural logarithms, where they exist and don't exist and the limits of the natural logarithm at $x = 0$.

Solutions

(a) $\displaystyle\lim_{x\to 0^-} h(x)$

Solution

Okay, let's start off by acknowledging that for $x \to 0^-$ we know $x < 0$ and so $-x > 0$ or,

$$-x \to 0^+$$

What this means for us is that this limit *does* make sense! We know that we can't have negative arguments in a logarithm, but because of the negative sign in this particular logarithm that means that we *can* use negative x's in this function (positive

x's on the other hand will now cause problems of course...).

By Example 6 in the notes for this section we know that as the argument of a logarithm approaches zero from the right (as ours does in this limit) then the logarithm will approach $-\infty$.

Therefore, the answer for this part is,

$$\boxed{\lim_{x \to 0^-} \ln(-x) = -\infty}$$

(b) $\lim\limits_{x \to 0^+} h(x)$

Solution

In this part we know that for $x \to 0^+$ we have $x > 0$ and so $-x < 0$. At this point we can stop. We know that we can't have negative arguments in a logarithm and for this limit that is exactly what we'll get and so $\lim\limits_{x \to 0^+} h(x)$ **does not exist**.

(c) $\lim\limits_{x \to 0} h(x)$

Solution

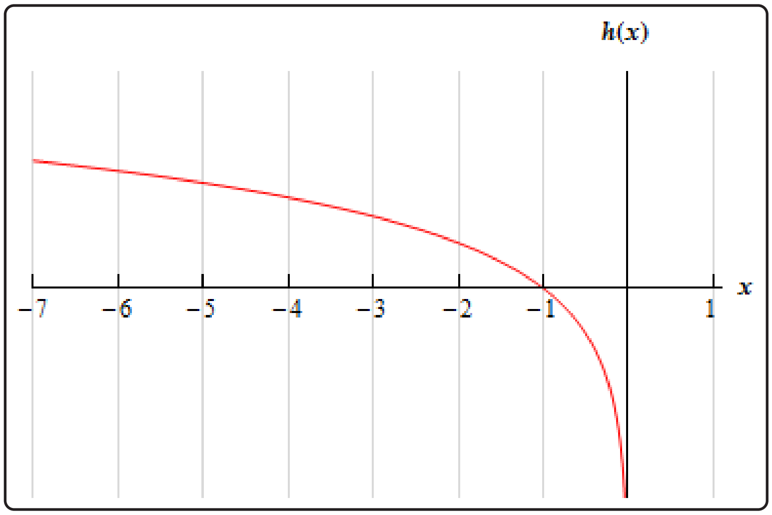
The answer for this part is $\lim\limits_{x \to 0} h(x)$ **does not exist**. We can use two lines of reasoning to justify this.

First, we are unable to look at both sides of the point in question and so there is no possible way for the limit to exist. The second line of reasoning is really the same as the first but put in different terms. From the first two parts that,

$$\lim_{x \to 0^-} h(x) \neq \lim_{x \to 0^+} h(x)$$

and so, from our basic limit properties we can see that $\lim\limits_{x \to 0} h(x)$ **does not exist**.

For the sake of completeness and to verify the answers for this problem here is a quick sketch of the function.



6. For $R(y) = \tan(y)$ evaluate,

(a) $\displaystyle\lim_{y \to \frac{3\pi}{2}^-} R(y)$ **(b)** $\displaystyle\lim_{y \to \frac{3\pi}{2}^+} R(y)$ **(c)** $\displaystyle\lim_{y \to \frac{3\pi}{2}} R(y)$

Hint

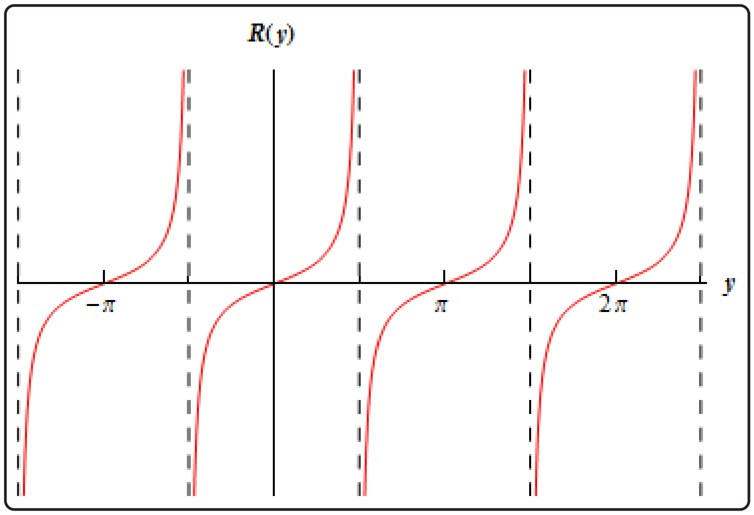
Don't forget the graph of the tangent function.

Solutions

(a) $\displaystyle\lim_{y \to \frac{3\pi}{2}^-} R(y)$

Solution

The easiest way to do this problem is from the graph of the tangent function so here is a quick sketch of the tangent function over several periods.



From the sketch we can see that,

$$\boxed{\lim_{y\to\frac{3\pi}{2}^-} \tan\left(y\right) = \infty}$$

(b) $\displaystyle\lim_{y\to\frac{3\pi}{2}^+} R\left(y\right)$

Solution

From the graph in the first part we can see that,

$$\boxed{\lim_{y\to\frac{3\pi}{2}^+} \tan\left(y\right) = -\infty}$$

(c) $\displaystyle\lim_{y\to\frac{3\pi}{2}} R\left(y\right)$

Solution

From the first two parts that,

$$\lim_{y\to\frac{3\pi}{2}^-} R\left(y\right) \neq \lim_{y\to\frac{3\pi}{2}^+} R\left(y\right)$$

and so, from our basic limit properties we can see that $\lim\limits_{y \to \frac{3\pi}{2}} R(y)$ **does not exist**.

7. Find all the vertical asymptotes of $f(x) = \dfrac{7x}{(10-3x)^4}$.

Hint

Remember how vertical asymptotes are defined and use the examples above to help determine where they are liable to be for the given function. Once you have the locations for the possible vertical asymptotes verify that they are in fact vertical asymptotes.

Solution

Recall that vertical asymptotes will occur at $x = a$ if any of the limits (one-sided or overall limit) at $x = a$ are plus or minus infinity.

From previous examples we can see that for rational expressions vertical asymptotes will occur where there is division by zero. Therefore, it looks like the only possible vertical asymptote will be at $x = \frac{10}{3}$.

Now let's verify that this is in fact a vertical asymptote by evaluating the two one-sided limits,

$$\lim_{x \to \frac{10}{3}^-} \frac{7x}{(10-3x)^4} \qquad \text{and} \qquad \lim_{x \to \frac{10}{3}^+} \frac{7x}{(10-3x)^4}$$

In either case as $x \to \frac{10}{3}$ (from both left and right) the numerator goes to $\frac{70}{3}$.

For the one-sided limits we have the following information,

$$\begin{aligned} x &\to \frac{10}{3}^- & \Rightarrow \quad x &< \frac{10}{3} & \Rightarrow \quad \frac{10}{3} - x &> 0 & \Rightarrow \quad 10 - 3x &> 0 \\ x &\to \frac{10}{3}^+ & \Rightarrow \quad x &> \frac{10}{3} & \Rightarrow \quad \frac{10}{3} - x &< 0 & \Rightarrow \quad 10 - 3x &< 0 \end{aligned}$$

Now, because of the exponent on the denominator is even we can see that for either of the one-sided limits we will have,

$$(10-3x)^4 \to 0^+$$

So, in either case, in the limit, the numerator approaches a fixed positive number and the denominator is positive and increasingly small. Therefore, we will have,

$$\lim_{x \to \frac{10}{3}^-} \frac{7x}{(10-3x)^4} = \infty \qquad \lim_{x \to \frac{10}{3}^+} \frac{7x}{(10-3x)^4} = \infty \qquad \lim_{x \to \frac{10}{3}} \frac{7x}{(10-3x)^4} = \infty$$

Any of these limits indicate that there is in fact a vertical asymptote at $x = \frac{10}{3}$.

8. Find all the vertical asymptotes of $g\left(x\right)=\dfrac{-8}{\left(x+5\right)\left(x-9\right)}$.

Hint

Remember how vertical asymptotes are defined and use the examples above to help determine where they are liable to be for the given function. Once you have the locations for the possible vertical asymptotes verify that they are in fact vertical asymptotes.

Solution

Recall that vertical asymptotes will occur at $x=a$ if any of the limits (one-sided or overall limit) at $x=a$ are plus or minus infinity.

From previous examples we can see that for rational expressions vertical asymptotes will occur where there is division by zero. Therefore, it looks like we will have possible vertical asymptote at $x=-5$ and $x=9$.

Now let's verify that these are in fact vertical asymptotes by evaluating the two one-sided limits for each of them.

Let's start with $x=-5$. We'll need to evaluate,

$$\lim_{x\to-5^-}\frac{-8}{\left(x+5\right)\left(x-9\right)}\qquad\text{and}\qquad\lim_{x\to-5^+}\frac{-8}{\left(x+5\right)\left(x-9\right)}$$

In either case as $x\to-5$ (from both left and right) the numerator is a constant -8.

For the one-sided limits we have the following information,

$$\begin{array}{lllll} x\to-5^- & \Rightarrow & x<-5 & \Rightarrow & x+5<0 \\ x\to-5^+ & \Rightarrow & x>-5 & \Rightarrow & x+5>0 \end{array}$$

Also, note that for x's close enough to -5 (which because we're looking at $x\to-5$ is safe enough to assume), we will have $x-9<0$.

So, in the left-hand limit, the numerator is a fixed negative number and the denominator is positive (a product of two negative numbers) and increasingly small. Likewise, for the right-hand limit, the denominator is negative (a product of a positive and negative number) and increasingly small. Therefore, we will have,

$$\lim_{x\to-5^-}\frac{-8}{\left(x+5\right)\left(x-9\right)}=-\infty\qquad\text{and}\qquad\lim_{x\to-5^+}\frac{-8}{\left(x+5\right)\left(x-9\right)}=\infty$$

Now for $x=9$. Again, the numerator is a constant -8. We also have,

$$\begin{array}{lllll} x\to9^- & \Rightarrow & x<9 & \Rightarrow & x-9<0 \\ x\to9^+ & \Rightarrow & x>9 & \Rightarrow & x-9>0 \end{array}$$

Finally, for x's close enough to 9 (which because we're looking at $x \to 9$ is safe enough to assume), we will have $x + 5 > 0$.

So, in the left-hand limit, the numerator is a fixed negative number and the denominator is negative (a product of a positive and negative number) and increasingly small. Likewise, for the right-hand limit, the denominator is positive (a product of two positive numbers) and increasingly small. Therefore, we will have,

$$\lim_{x \to 9^-} \frac{-8}{(x+5)(x-9)} = \infty \qquad \text{and} \qquad \lim_{x \to 9^+} \frac{-8}{(x+5)(x-9)} = -\infty$$

So, as all of these limits show we do in fact have vertical asymptotes at $x = -5$ and $x = 9$.

2.7 Limits At Infinity, Part I

1. For $f(x) = 4x^7 - 18x^3 + 9$ evaluate each of the following limits.

(a) $\lim\limits_{x \to -\infty} f(x)$ **(b)** $\lim\limits_{x \to \infty} f(x)$

Solutions

(a) $\lim\limits_{x \to -\infty} f(x)$

Solution

To do this all we need to do is factor out the largest power of x from the whole polynomial and then use basic limit properties along with Fact 1 from this section to evaluate the limit.

$$\begin{aligned}\lim_{x \to -\infty}\left(4x^7 - 18x^3 + 9\right) &= \lim_{x \to -\infty}\left[x^7\left(4 - \frac{18}{x^4} + \frac{9}{x^7}\right)\right] \\ &= \left(\lim_{x \to -\infty} x^7\right)\left[\lim_{x \to -\infty}\left(4 - \frac{18}{x^4} + \frac{9}{x^7}\right)\right] \\ &= (-\infty)(4) = \boxed{-\infty}\end{aligned}$$

(b) $\lim\limits_{x \to \infty} f(x)$

Solution

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't need to be redone here. We can pick up the problem right before we actually took the limits and then proceed.

$$\lim_{x \to \infty}\left(4x^7 - 18x^3 + 9\right) = \left(\lim_{x \to \infty} x^7\right)\left[\lim_{x \to \infty}\left(4 - \frac{18}{x^4} + \frac{9}{x^7}\right)\right] = (\infty)(4) = \boxed{\infty}$$

---

2. For $h(t) = \sqrt[3]{t} + 12t - 2t^2$ evaluate each of the following limits.

(a) $\lim\limits_{t \to -\infty} h(t)$ **(b)** $\lim\limits_{t \to \infty} h(t)$

Solutions

(a) $\lim\limits_{t \to -\infty} h(t)$

Solution

To do this all we need to do is factor out the largest power of x from the whole polynomial and then use basic limit properties along with Fact 1 from this section to evaluate the limit.

Note as well that we'll convert the root over to a fractional exponent in order to allow it to be easier to deal with. Also note that this limit is a perfectly acceptable limit because the root is a cube root and we *can* take cube roots of negative numbers! We would only have run into problems had the index on the root been an even number.

$$\begin{aligned}\lim_{t \to -\infty} \left(t^{\frac{1}{3}} + 12t - 2t^2\right) &= \lim_{t \to -\infty} \left[t^2 \left(\frac{1}{t^{\frac{5}{3}}} + \frac{12}{t} - 2\right)\right] \\ &= \left(\lim_{t \to -\infty} t^2\right) \left[\lim_{t \to -\infty} \left(\frac{1}{t^{\frac{5}{3}}} + \frac{12}{t} - 2\right)\right] \\ &= (\infty)(-2) = \boxed{-\infty}\end{aligned}$$

(b) $\lim\limits_{tS \to \infty} h(t)$

Solution

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't need to be redone here. We can pick up the problem right before we actually took the limits and then proceed.

$$\begin{aligned}\lim_{t \to \infty} \left(t^{\frac{1}{3}} + 12t - 2t^2\right) &= \left(\lim_{t \to \infty} t^2\right) \left[\lim_{t \to \infty} \left(\frac{1}{t^{\frac{5}{3}}} + \frac{12}{t} - 2\right)\right] \\ &= (\infty)(-2) = \boxed{-\infty}\end{aligned}$$

3. For $f(x) = \dfrac{8-4x^2}{9x^2+5x}$ answer each of the following questions.

(a) Evaluate $\displaystyle\lim_{x\to-\infty} f(x)$.

(b) Evaluate $\displaystyle\lim_{x\to\infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solutions

(a) Evaluate $\displaystyle\lim_{x\to-\infty} f(x)$.

Solution

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with Fact 1 from this section to evaluate the limit.

$$\lim_{x\to-\infty}\frac{8-4x^2}{9x^2+5x} = \lim_{x\to-\infty}\frac{x^2\left(\frac{8}{x^2}-4\right)}{x^2\left(9+\frac{5}{x}\right)} = \lim_{x\to-\infty}\frac{\frac{8}{x^2}-4}{9+\frac{5}{x}} = \boxed{\frac{-4}{9}}$$

(b) Evaluate $\displaystyle\lim_{x\to\infty} f(x)$.

Solution

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't really need to be redone here. However, it is easy enough to add them in so we'll go ahead and include them.

$$\lim_{x\to\infty}\frac{8-4x^2}{9x^2+5x} = \lim_{x\to\infty}\frac{x^2\left(\frac{8}{x^2}-4\right)}{x^2\left(9+\frac{5}{x}\right)} = \lim_{x\to\infty}\frac{\frac{8}{x^2}-4}{9+\frac{5}{x}} = \boxed{\frac{-4}{9}}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solution

We know that there will be a horizontal asymptote for $x\to-\infty$ if $\displaystyle\lim_{x\to-\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x\to\infty$ if

$\lim\limits_{x \to \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote.

$$y = -\frac{4}{9}$$

for both $x \to -\infty$ and $x \to \infty$.

---

4. For $f(x) = \dfrac{3x^7 - 4x^2 + 1}{5 - 10x^2}$ answer each of the following questions.

(a) Evaluate $\lim\limits_{x \to -\infty} f(x)$.

(b) Evaluate $\lim\limits_{x \to \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solutions

(a) Evaluate $\lim\limits_{x \to -\infty} f(x)$.

Solution

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with Fact 1 from this section to evaluate the limit.

$$\lim_{x \to -\infty} \frac{3x^7 - 4x^2 + 1}{5 - 10x^2} = \lim_{x \to -\infty} \frac{x^2\left(3x^5 - 4 + \frac{1}{x^2}\right)}{x^2\left(\frac{5}{x^2} - 10\right)} = \lim_{x \to -\infty} \frac{3x^5 - 4 + \frac{1}{x^2}}{\frac{5}{x^2} - 10}$$

$$= \frac{-\infty}{-10} = \boxed{\infty}$$

(b) Evaluate $\lim\limits_{x \to \infty} f(x)$.

Solution

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't really need to be redone here. However, it is easy enough to add them in so we'll go ahead and include them.

$$\lim_{x\to\infty}\frac{3x^7-4x^2+1}{5-10x^2}=\lim_{x\to\infty}\frac{x^2\left(3x^5-4+\frac{1}{x^2}\right)}{x^2\left(\frac{5}{x^2}-10\right)}=\lim_{x\to\infty}\frac{3x^5-4+\frac{1}{x^2}}{\frac{5}{x^2}-10}$$
$$=\frac{\infty}{-10}=\boxed{-\infty}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solution

We know that there will be a horizontal asymptote for $x\to-\infty$ if $\lim\limits_{x\to-\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x\to\infty$ if $\lim\limits_{x\to\infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that this function will have **no horizontal asymptotes** since neither of the two limits are finite.

---

5. For $f(x)=\dfrac{20x^4-7x^3}{2x+9x^2+5x^4}$ answer each of the following questions.

(a) Evaluate $\lim\limits_{x\to-\infty} f(x)$.

(b) Evaluate $\lim\limits_{x\to\infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solutions

(a) Evaluate $\lim\limits_{x\to-\infty} f(x)$.

Solution

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with Fact 1 from this section to evaluate the limit.

$$\lim_{x\to-\infty}\frac{20x^4-7x^3}{2x+9x^2+5x^4}=\lim_{x\to-\infty}\frac{x^4\left(20-\frac{7}{x}\right)}{x^4\left(\frac{2}{x^3}+\frac{9}{x^2}+5\right)}=\lim_{x\to-\infty}\frac{20-\frac{7}{x}}{\frac{2}{x^3}+\frac{9}{x^2}+5}$$
$$=\frac{20}{5}=\boxed{4}$$

(b) Evaluate $\lim\limits_{x \to \infty} f(x)$.

Solution

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't really need to be redone here. However, it is easy enough to add them in so we'll go ahead and include them.

$$\lim_{x \to \infty} \frac{20x^4 - 7x^3}{2x + 9x^2 + 5x^4} = \lim_{x \to \infty} \frac{x^4\left(20 - \frac{7}{x}\right)}{x^4\left(\frac{2}{x^3} + \frac{9}{x^2} + 5\right)} = \lim_{x \to \infty} \frac{20 - \frac{7}{x}}{\frac{2}{x^3} + \frac{9}{x^2} + 5}$$
$$= \frac{20}{5} = \boxed{4}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solution

We know that there will be a horizontal asymptote for $x \to -\infty$ if $\lim\limits_{x \to -\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x \to \infty$ if $\lim\limits_{x \to \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote.

$$y = 4$$

for both $x \to -\infty$ and $x \to \infty$.

6. For $f(x) = \dfrac{x^3 - 2x + 11}{3 - 6x^5}$ answer each of the following questions.

(a) Evaluate $\lim\limits_{x \to -\infty} f(x)$.

(b) Evaluate $\lim\limits_{x \to \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solutions

(a) Evaluate $\lim\limits_{x \to -\infty} f(x)$.

Solution

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with Fact 1 from this section to evaluate the limit.

$$\lim_{x \to -\infty} \frac{x^3 - 2x + 11}{3 - 6x^5} = \lim_{x \to -\infty} \frac{x^5\left(\frac{1}{x^2} - \frac{2}{x^4} + \frac{11}{x^5}\right)}{x^5\left(\frac{3}{x^5} - 6\right)} = \lim_{x \to -\infty} \frac{\frac{1}{x^2} - \frac{2}{x^4} + \frac{11}{x^5}}{\frac{3}{x^5} - 6}$$
$$= \frac{0}{-6} = \boxed{0}$$

(b) Evaluate $\lim\limits_{x \to \infty} f(x)$.

Solution

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't really need to be redone here. However, it is easy enough to add them in so we'll go ahead and include them.

$$\lim_{x \to \infty} \frac{x^3 - 2x + 11}{3 - 6x^5} = \lim_{x \to \infty} \frac{x^5\left(\frac{1}{x^2} - \frac{2}{x^4} + \frac{11}{x^5}\right)}{x^5\left(\frac{3}{x^5} - 6\right)} = \lim_{x \to \infty} \frac{\frac{1}{x^2} - \frac{2}{x^4} + \frac{11}{x^5}}{\frac{3}{x^5} - 6}$$
$$= \frac{0}{-6} = \boxed{0}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solution

We know that there will be a horizontal asymptote for $x \to -\infty$ if $\lim\limits_{x \to -\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x \to \infty$ if $\lim\limits_{x \to \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote.

$$y = 0$$

for both $x \to -\infty$ and $x \to \infty$.

7. For $f(x) = \dfrac{x^6 - x^4 + x^2 - 1}{7x^6 + 4x^3 + 10}$ answer each of the following questions.

(a) Evaluate $\lim\limits_{x \to -\infty} f(x)$.

(b) Evaluate $\lim\limits_{x \to \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solutions

(a) Evaluate $\lim\limits_{x \to -\infty} f(x)$.

Solution

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with Fact 1 from this section to evaluate the limit.

$$\lim_{x \to -\infty} \frac{x^6 - x^4 + x^2 - 1}{7x^6 + 4x^3 + 10} = \lim_{x \to -\infty} \frac{x^6\left(1 - \frac{1}{x^2} + \frac{1}{x^4} - \frac{1}{x^6}\right)}{x^6\left(7 + \frac{4}{x^3} + \frac{10}{x^6}\right)}$$
$$= \lim_{x \to -\infty} \frac{1 - \frac{1}{x^2} + \frac{1}{x^4} - \frac{1}{x^6}}{7 + \frac{4}{x^3} + \frac{10}{x^6}} = \boxed{\frac{1}{7}}$$

(b) Evaluate $\lim\limits_{x \to \infty} f(x)$.

Solution

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't really need to be redone here. However, it is easy enough to add them in so we'll go ahead and include them.

$$\lim_{x \to \infty} \frac{x^6 - x^4 + x^2 - 1}{7x^6 + 4x^3 + 10} = \lim_{x \to \infty} \frac{x^6\left(1 - \frac{1}{x^2} + \frac{1}{x^4} - \frac{1}{x^6}\right)}{x^6\left(7 + \frac{4}{x^3} + \frac{10}{x^6}\right)}$$
$$= \lim_{x \to \infty} \frac{1 - \frac{1}{x^2} + \frac{1}{x^4} - \frac{1}{x^6}}{7 + \frac{4}{x^3} + \frac{10}{x^6}} = \boxed{\frac{1}{7}}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solution

We know that there will be a horizontal asymptote for $x \to -\infty$ if $\displaystyle\lim_{x \to -\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x \to \infty$ if $\displaystyle\lim_{x \to \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote.

$$y = \frac{1}{7}$$

for both $x \to -\infty$ and $x \to \infty$.

---

8. For $f(x) = \dfrac{\sqrt{7 + 9x^2}}{1 - 2x}$ answer each of the following questions.

(a) Evaluate $\displaystyle\lim_{x \to -\infty} f(x)$.

(b) Evaluate $\displaystyle\lim_{x \to \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solutions

(a) Evaluate $\displaystyle\lim_{x \to -\infty} f(x)$.

Solution

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with Fact 1 from this section to evaluate the limit.

In this case the largest power of x in the denominator is just x and so we will need to factor an x out of both the denominator and the numerator. Recall as well that this means we'll need to factor an x^2 out of the root in the numerator so that we'll have an x in the numerator when we are done.

So, let's do the first couple of steps in this process to get us started.

$$\begin{aligned}
\lim_{x \to -\infty} \frac{\sqrt{7 + 9x^2}}{1 - 2x} = \lim_{x \to -\infty} \frac{\sqrt{x^2\left(\frac{7}{x^2} + 9\right)}}{x\left(\frac{1}{x} - 2\right)} &= \lim_{x \to -\infty} \frac{\sqrt{x^2}\sqrt{\frac{7}{x^2} + 9}}{x\left(\frac{1}{x} - 2\right)} \\
&= \lim_{x \to -\infty} \frac{|x|\sqrt{\frac{7}{x^2} + 9}}{x\left(\frac{1}{x} - 2\right)}
\end{aligned}$$

Recall from the discussion in this section that.

$$\sqrt{x^2} = |x|$$

and we *do* need to be careful with that.

Now, because we are looking at the limit $x \to -\infty$ it is safe to assume that $x < 0$. Therefore, from the definition of the absolute value we get.

$$|x| = -x$$

and the limit is then.

$$\lim_{x \to -\infty} \frac{\sqrt{7+9x^2}}{1-2x} = \lim_{x \to -\infty} \frac{-x\sqrt{\frac{7}{x^2}+9}}{x\left(\frac{1}{x}-2\right)} = \lim_{x \to -\infty} \frac{-\sqrt{\frac{7}{x^2}+9}}{\frac{1}{x}-2} = \frac{-\sqrt{9}}{-2} = \boxed{\frac{3}{2}}$$

(b) Evaluate $\lim\limits_{x \to \infty} f(x)$.

Solution

For this part all of the mathematical manipulations we did in the first part up to dealing with the absolute value did not depend upon the limit itself and so don't really need to be redone here. So, up to that part we have.

$$\lim_{x \to \infty} \frac{\sqrt{7+9x^2}}{1-2x} = \lim_{x \to \infty} \frac{|x|\sqrt{\frac{7}{x^2}+9}}{x\left(\frac{1}{x}-2\right)}$$

In this part we are looking at the limit $x \to \infty$ and so it will be safe to assume in this part that $x > 0$. Therefore, from the definition of the absolute value we get.

$$|x| = x$$

and the limit is then.

$$\lim_{x \to \infty} \frac{\sqrt{7+9x^2}}{1-2x} = \lim_{x \to \infty} \frac{x\sqrt{\frac{7}{x^2}+9}}{x\left(\frac{1}{x}-2\right)} = \lim_{x \to \infty} \frac{\sqrt{\frac{7}{x^2}+9}}{\frac{1}{x}-2} = \frac{\sqrt{9}}{-2} = \boxed{-\frac{3}{2}}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solution

We know that there will be a horizontal asymptote for $x \to -\infty$ if $\lim\limits_{x \to -\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x \to \infty$ if $\lim\limits_{x \to \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote.

$$y = \frac{3}{2}$$

for $x \to -\infty$ and we have the horizontal asymptote.

$$y = -\frac{3}{2}$$

for $x \to \infty$.

---

9. For $f(x) = \dfrac{x+8}{\sqrt{2x^2+3}}$ answer each of the following questions.

(a) Evaluate $\lim\limits_{x \to -\infty} f(x)$.

(b) Evaluate $\lim\limits_{x \to \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solutions

(a) Evaluate $\lim\limits_{x \to -\infty} f(x)$.

Solution

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with Fact 1 from this section to evaluate the limit.

For the denominator we need to be a little careful. The power of x in the denominator needs to be outside of the root so it can cancel against the x's in the numerator. The largest power of x outside of the root that we can get (and leave something we can deal with in the root) will be just x. We get this by factoring an x^2 out of the root.

So, let's do the first couple of steps in this process to get us started.

$$\lim_{x \to -\infty} \frac{x+8}{\sqrt{2x^2+3}} = \lim_{x \to -\infty} \frac{x\left(1+\frac{8}{x}\right)}{\sqrt{x^2\left(2+\frac{3}{x^2}\right)}} = \lim_{x \to -\infty} \frac{x\left(1+\frac{8}{x}\right)}{\sqrt{x^2}\sqrt{2+\frac{3}{x^2}}}$$
$$= \lim_{x \to -\infty} \frac{x\left(1+\frac{8}{x}\right)}{|x|\sqrt{2+\frac{3}{x^2}}}$$

Recall from the discussion in this section that.

$$\sqrt{x^2} = |x|$$

and we *do* need to be careful with that.

Now, because we are looking at the limit $x \to -\infty$ it is safe to assume that $x < 0$. Therefore, from the definition of the absolute value we get.

$$|x| = -x$$

and the limit is then.

$$\lim_{x \to -\infty} \frac{x+8}{\sqrt{2x^2+3}} = \lim_{x \to -\infty} \frac{x\left(1+\frac{8}{x}\right)}{-x\sqrt{2+\frac{3}{x^2}}} = \lim_{x \to -\infty} \frac{1+\frac{8}{x}}{-\sqrt{2+\frac{3}{x^2}}} = \boxed{\frac{1}{-\sqrt{2}}}$$

(b) Evaluate $\lim\limits_{x \to \infty} f(x)$.

Solution

For this part all of the mathematical manipulations we did in the first part up to dealing with the absolute value did not depend upon the limit itself and so don't really need to be redone here. So, up to that part we have.

$$\lim_{x \to \infty} \frac{x+8}{\sqrt{2x^2+3}} = \lim_{x \to \infty} \frac{x\left(1+\frac{8}{x}\right)}{|x|\sqrt{2+\frac{3}{x^2}}}$$

In this part we are looking at the limit $x \to \infty$ and so it will be safe to assume in this part that $x > 0$. Therefore, from the definition of the absolute value we get.

$$|x| = x$$

and the limit is then.

$$\lim_{x\to\infty}\frac{x+8}{\sqrt{2x^2+3}}=\lim_{x\to\infty}\frac{x\left(1+\frac{8}{x}\right)}{x\sqrt{2+\frac{3}{x^2}}}=\lim_{x\to\infty}\frac{1+\frac{8}{x}}{\sqrt{2+\frac{3}{x^2}}}=\boxed{\frac{1}{\sqrt{2}}}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solution

We know that there will be a horizontal asymptote for $x\to-\infty$ if $\lim\limits_{x\to-\infty}f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x\to\infty$ if $\lim\limits_{x\to\infty}f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote.

$$y=-\frac{1}{\sqrt{2}}$$

for $x\to-\infty$ and we have the horizontal asymptote.

$$y=\frac{1}{\sqrt{2}}$$

for $x\to\infty$.

---

10. For $f(x)=\dfrac{8+x-4x^2}{\sqrt{6+x^2+7x^4}}$ answer each of the following questions.

(a) Evaluate $\lim\limits_{x\to-\infty}f(x)$.

(b) Evaluate $\lim\limits_{x\to\infty}f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solutions

(a) Evaluate $\lim\limits_{x\to-\infty}f(x)$.

Solution

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with Fact 1 from this section to evaluate the limit.

For the denominator we need to be a little careful. The power of x in the denominator needs to be outside of the root so it can cancel against the x's in the numerator. The largest power of x outside of the root that we can get (and leave something we can deal with in the root) will be just x^2. We get this by factoring an x^4 out of the root.

So, let's do the first couple of steps in this process to get us started.

$$\lim_{x \to -\infty} \frac{8 + x - 4x^2}{\sqrt{6 + x^2 + 7x^4}} = \lim_{x \to -\infty} \frac{x^2\left(\frac{8}{x^2} + \frac{1}{x} - 4\right)}{\sqrt{x^4\left(\frac{6}{x^4} + \frac{1}{x^2} + 7\right)}}$$
$$= \lim_{x \to -\infty} \frac{x^2\left(\frac{8}{x^2} + \frac{1}{x} - 4\right)}{\sqrt{x^4}\sqrt{\frac{6}{x^4} + \frac{1}{x^2} + 7}} = \lim_{x \to -\infty} \frac{x^2\left(\frac{8}{x^2} + \frac{1}{x} - 4\right)}{|x^2|\sqrt{\frac{6}{x^4} + \frac{1}{x^2} + 7}}$$

Recall from the discussion in this section that.

$$\sqrt{x^2} = |x|$$

So, in this case we'll have

$$\sqrt{x^4} = \left|x^2\right| = x^2$$

and note that we can get rid of the absolute value bars because we know that $x^2 \geq 0$. So, let's finish the limit up.

$$\lim_{x \to -\infty} \frac{8 + x - 4x^2}{\sqrt{6 + x^2 + 7x^4}} = \lim_{x \to -\infty} \frac{x^2\left(\frac{8}{x^2} + \frac{1}{x} - 4\right)}{x^2\sqrt{\frac{6}{x^4} + \frac{1}{x^2} + 7}} = \lim_{x \to -\infty} \frac{\frac{8}{x^2} + \frac{1}{x} - 4}{\sqrt{\frac{6}{x^4} + \frac{1}{x^2} + 7}}$$
$$= \boxed{\frac{-4}{\sqrt{7}}}$$

(b) Evaluate $\lim\limits_{x \to \infty} f(x)$.

Solution

Unlike the previous two problems with roots in them all of the mathematical manipulations in this case did not depend upon the actual limit because we were factoring an x^2 out which will always be positive and so there will be no reason to redo all of that work.

Here is this limit (with most of the work excluded).

For this part all of the mathematical manipulations we did in the first part up to dealing with the absolute value did not depend upon the limit itself and so don't really need to be redone here. So, up to that part we have.

$$\lim_{x\to\infty}\frac{8+x-4x^2}{\sqrt{6+x^2+7x^4}}=\lim_{x\to\infty}\frac{x^2\left(\frac{8}{x^2}+\frac{1}{x}-4\right)}{x^2\sqrt{\frac{6}{x^4}+\frac{1}{x^2}+7}}=\lim_{x\to\infty}\frac{\frac{8}{x^2}+\frac{1}{x}-4}{\sqrt{\frac{6}{x^4}+\frac{1}{x^2}+7}}=\boxed{\frac{-4}{\sqrt{7}}}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

Solution

We know that there will be a horizontal asymptote for $x \to -\infty$ if $\lim\limits_{x\to-\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x \to \infty$ if $\lim\limits_{x\to\infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote.

$$y=-\frac{4}{\sqrt{7}}$$

For both $x \to -\infty$ and $x \to \infty$.

2.8 Limits At Infinity, Part II

1. For $f\left(x\right) = \mathbf{e}^{8+2x-x^3}$ evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to -\infty} f\left(x\right)$ **(b)** $\displaystyle\lim_{x\to \infty} f\left(x\right)$

Solutions

(a) $\displaystyle\lim_{x\to -\infty} f\left(x\right)$

Solution

First notice that,

$$\lim_{x\to -\infty}\left(8+2x-x^3\right) = \infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling Example 1 from this section, we know that because the exponent goes to infinity in the limit the answer is,

$$\lim_{x\to -\infty}\mathbf{e}^{8+2x-x^3} = \boxed{\infty}$$

(b) $\displaystyle\lim_{x\to \infty} f\left(x\right)$

Solution

First notice that,

$$\lim_{x\to \infty}\left(8+2x-x^3\right) = -\infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling Example 1 from this section, we know that because the exponent goes to negative infinity in the limit the answer is,

$$\lim_{x\to \infty}\mathbf{e}^{8+2x-x^3} = \boxed{0}$$

---

2. For $f(x) = \mathbf{e}^{\frac{6x^2+x}{5+3x}}$ evaluate each of the following limits.

(a) $\lim\limits_{x \to -\infty} f(x)$ **(b)** $\lim\limits_{x \to \infty} f(x)$

Solutions

(a) $\lim\limits_{x \to -\infty} f(x)$

Solution

First notice that,

$$\lim_{x \to -\infty} \frac{6x^2 + x}{5 + 3x} = -\infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling Example 1 from this section, we know that because the exponent goes to negative infinity in the limit the answer is,

$$\lim_{x \to -\infty} \mathbf{e}^{\frac{6x^2+x}{5+3x}} = \boxed{0}$$

(b) $\lim\limits_{x \to \infty} f(x)$

Solution

First notice that,

$$\lim_{x \to \infty} \frac{6x^2 + x}{5 + 3x} = \infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling Example 1 from this section, we know that because the exponent goes to infinity in the limit the answer is,

$$\lim_{x \to \infty} \mathbf{e}^{\frac{6x^2+x}{5+3x}} = \boxed{\infty}$$

3. For $f(x) = 2\mathbf{e}^{6x} - \mathbf{e}^{-7x} - 10\mathbf{e}^{4x}$ evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to -\infty} f(x)$ **(b)** $\displaystyle\lim_{x\to \infty} f(x)$

Hint

Remember that if there are two terms that seem to be suggesting that the function should be going in opposite directions that you'll need to factor out of the function that term that is going to infinity faster to "prove" the limit.

Solutions

(a) $\displaystyle\lim_{x\to -\infty} f(x)$

Solution

For this limit the exponentials with positive exponents will simply go to zero and there is only one exponential with a negative exponent (which will go to infinity) and so there isn't much to do with this limit.

$$\lim_{x\to -\infty}\left(2\mathbf{e}^{6x} - \mathbf{e}^{-7x} - 10\mathbf{e}^{4x}\right) = 0 - \infty - 0 = \boxed{-\infty}$$

(b) $\displaystyle\lim_{x\to \infty} f(x)$

Solution

Here we have two exponents with positive exponents and so both will go to infinity in the limit. However, each term has opposite signs and so each term seems to be suggesting different answers for the limit.

In order to determine which "wins out" so to speak all we need to do is factor out the term with the largest exponent and then use basic limit properties.

$$\lim_{x\to \infty}\left(2\mathbf{e}^{6x} - \mathbf{e}^{-7x} - 10\mathbf{e}^{4x}\right) = \lim_{x\to \infty}\left[\mathbf{e}^{6x}\left(2 - \mathbf{e}^{-13x} - 10\mathbf{e}^{-2x}\right)\right] = (\infty)(2)$$
$$= \boxed{\infty}$$

---

4. For $f(x) = 3\mathbf{e}^{-x} - 8\mathbf{e}^{-5x} - \mathbf{e}^{10x}$ evaluate each of the following limits.

(a) $\lim\limits_{x \to -\infty} f(x)$ **(b)** $\lim\limits_{x \to \infty} f(x)$

Hint

Remember that if there are two terms that seem to be suggesting that the function should be going in opposite directions that you'll need to factor out of the function that term that is going to infinity faster to "prove" the limit.

Solutions

(a) $\lim\limits_{x \to -\infty} f(x)$

Solution

Here we have two exponents with negative exponents and so both will go to infinity in the limit. However, each term has opposite signs and so each term seems to be suggesting different answers for the limit.

In order to determine which "wins out" so to speak all we need to do is factor out the term with the most negative exponent and then use basic limit properties.

$$\lim_{x \to -\infty} \left(3\mathbf{e}^{-x} - 8\mathbf{e}^{-5x} - \mathbf{e}^{10x}\right) = \lim_{x \to -\infty} \left[\mathbf{e}^{-5x}\left(3\mathbf{e}^{4x} - 8 - \mathbf{e}^{15x}\right)\right]$$
$$= (\infty)(-8) = \boxed{-\infty}$$

(b) $\lim\limits_{x \to \infty} f(x)$

Solution

For this limit the exponentials with negative exponents will simply go to zero and there is only one exponential with a positive exponent (which will go to infinity) and so there isn't much to do with this limit.

$$\lim_{x \to \infty} \left(3\mathbf{e}^{-x} - 8\mathbf{e}^{-5x} - \mathbf{e}^{10x}\right) = 0 - 0 - \infty = \boxed{-\infty}$$

5. For $f(x) = \dfrac{\mathbf{e}^{-3x} - 2\mathbf{e}^{8x}}{9\mathbf{e}^{8x} - 7\mathbf{e}^{-3x}}$ evaluate each of the following limits.

(a) $\displaystyle\lim_{x \to -\infty} f(x)$ **(b)** $\displaystyle\lim_{x \to \infty} f(x)$

Hint

Remember that you'll need to factor the term in the denominator that is causing the denominator to go to infinity from both the numerator and denominator in order to evaluate this limit.

Solutions

(a) $\displaystyle\lim_{x \to -\infty} f(x)$

Solution

The exponential with the negative exponent is the only term in the denominator going to infinity for this limit and so we'll need to factor the exponential with the negative exponent in the denominator from both the numerator and denominator to evaluate this limit.

$$\begin{aligned}\lim_{x \to -\infty} \frac{\mathbf{e}^{-3x} - 2\mathbf{e}^{8x}}{9\mathbf{e}^{8x} - 7\mathbf{e}^{-3x}} &= \lim_{x \to -\infty} \frac{\mathbf{e}^{-3x}\left(1 - 2\mathbf{e}^{11x}\right)}{\mathbf{e}^{-3x}\left(9\mathbf{e}^{11x} - 7\right)} = \lim_{x \to -\infty} \frac{1 - 2\mathbf{e}^{11x}}{9\mathbf{e}^{11x} - 7} \\ &= \frac{1-0}{0-7} = \boxed{-\frac{1}{7}}\end{aligned}$$

(b) $\displaystyle\lim_{x \to \infty} f(x)$

Solution

The exponential with the positive exponent is the only term in the denominator going to infinity for this limit and so we'll need to factor the exponential with the positive exponent in the denominator from both the numerator and denominator to evaluate this limit.

$$\begin{aligned}\lim_{x \to \infty} \frac{\mathbf{e}^{-3x} - 2\mathbf{e}^{8x}}{9\mathbf{e}^{8x} - 7\mathbf{e}^{-3x}} &= \lim_{x \to \infty} \frac{\mathbf{e}^{8x}\left(\mathbf{e}^{-11x} - 2\right)}{\mathbf{e}^{8x}\left(9 - 7\mathbf{e}^{-11x}\right)} = \lim_{x \to \infty} \frac{\mathbf{e}^{-11x} - 2}{9 - 7\mathbf{e}^{-11x}} \\ &= \frac{0-2}{9-0} = \boxed{-\frac{2}{9}}\end{aligned}$$

6. For $f\left(x\right)=\dfrac{\mathbf{e}^{-7x}-2\mathbf{e}^{3x}-\mathbf{e}^{x}}{\mathbf{e}^{-x}+16\mathbf{e}^{10x}+2\mathbf{e}^{-4x}}$ evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to-\infty} f\left(x\right)$ **(b)** $\displaystyle\lim_{x\to\infty} f\left(x\right)$

Hint

Remember that you'll need to factor the term in the denominator that is causing the denominator to go to infinity fastest from both the numerator and denominator in order to evaluate this limit.

Solutions

(a) $\displaystyle\lim_{x\to-\infty} f\left(x\right)$

Solution

The exponentials with the negative exponents are the only terms in the denominator going to infinity for this limit and so we'll need to factor the exponential with the most negative exponent in the denominator (because it will be going to infinity fastest) from both the numerator and denominator to evaluate this limit.

$$\begin{aligned}\lim_{x\to-\infty}\frac{\mathbf{e}^{-7x}-2\mathbf{e}^{3x}-\mathbf{e}^{x}}{\mathbf{e}^{-x}+16\mathbf{e}^{10x}+2\mathbf{e}^{-4x}}&=\lim_{x\to-\infty}\frac{\mathbf{e}^{-4x}\left(\mathbf{e}^{-3x}-2\mathbf{e}^{7x}-\mathbf{e}^{5x}\right)}{\mathbf{e}^{-4x}\left(\mathbf{e}^{3x}+16\mathbf{e}^{14x}+2\right)}\\&=\lim_{x\to-\infty}\frac{\mathbf{e}^{-3x}-2\mathbf{e}^{7x}-\mathbf{e}^{5x}}{\mathbf{e}^{3x}+16\mathbf{e}^{14x}+2}=\frac{\infty-0-0}{0+0+2}=\boxed{\infty}\end{aligned}$$

(b) $\displaystyle\lim_{x\to\infty} f\left(x\right)$

Solution

The exponentials with the positive exponents are the only terms in the denominator going to infinity for this limit and so we'll need to factor the exponential with the most positive exponent in the denominator (because it will be going to infinity fastest) from both the numerator and denominator to evaluate this limit.

$$\begin{aligned}\lim_{x\to\infty}\frac{\mathbf{e}^{-7x}-2\mathbf{e}^{3x}-\mathbf{e}^{x}}{\mathbf{e}^{-x}+16\mathbf{e}^{10x}+2\mathbf{e}^{-4x}}&=\lim_{x\to\infty}\frac{\mathbf{e}^{10x}\left(\mathbf{e}^{-17x}-2\mathbf{e}^{-7x}-\mathbf{e}^{-9x}\right)}{\mathbf{e}^{10x}\left(\mathbf{e}^{-11x}+16+2\mathbf{e}^{-14x}\right)}\\&=\lim_{x\to\infty}\frac{\mathbf{e}^{-17x}-2\mathbf{e}^{-7x}-\mathbf{e}^{-9x}}{\mathbf{e}^{-11x}+16+2\mathbf{e}^{-14x}}=\frac{0-0-0}{0+16+0}=\boxed{0}\end{aligned}$$

7. Evaluate $\lim\limits_{t \to -\infty} \ln\left(4 - 9t - t^3\right)$.

Solution

First notice that,

$$\lim_{t \to -\infty} \left(4 - 9t - t^3\right) = \infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling Example 5 from this section, we know that because the argument goes to infinity in the limit the answer is,

$$\lim_{t \to -\infty} \ln\left(4 - 9t - t^3\right) = \boxed{\infty}$$

8. Evaluate $\lim\limits_{z \to -\infty} \ln\left(\dfrac{3z^4 - 8}{2 + z^2}\right)$.

Solution

First notice that,

$$\lim_{z \to -\infty} \frac{3z^4 - 8}{2 + z^2} = \infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling Example 5 from this section, we know that because the argument goes to infinity in the limit the answer is,

$$\lim_{z \to -\infty} \ln\left(\frac{3z^4 - 8}{2 + z^2}\right) = \boxed{\infty}$$

9. Evaluate $\lim\limits_{x \to \infty} \ln\left(\dfrac{11 + 8x}{x^3 + 7x}\right)$.

Solution

First notice that,

$$\lim_{x \to \infty} \frac{11 + 8x}{x^3 + 7x} = 0$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Also, note that because we are evaluating the limit $x \to \infty$ it is safe to assume that $x > 0$ and so we can further say that,

$$\frac{11+8x}{x^3+7x} \to 0^+$$

Now, recalling Example 5 from this section, we know that because the argument goes to zero from the right in the limit the answer is,

$$\lim_{x\to\infty} \ln\left(\frac{11+8x}{x^3+7x}\right) = \boxed{-\infty}$$

---

10. Evaluate $\lim\limits_{x\to-\infty} \tan^{-1}\left(7-x+3x^5\right)$.

Solution

First notice that,

$$\lim_{x\to-\infty} \left(7-x+3x^5\right) = -\infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling Example 7 from this section, we know that because the argument goes to negative infinity in the limit the answer is,

$$\lim_{x\to-\infty} \tan^{-1}\left(7-x+3x^5\right) = \boxed{-\frac{\pi}{2}}$$

---

11. Evaluate $\lim\limits_{t\to\infty} \tan^{-1}\left(\dfrac{4+7t}{2-t}\right)$.

Solution

First notice that,

$$\lim_{t\to\infty} \frac{4+7t}{2-t} = -7$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Then answer is then,

$$\lim_{t\to\infty} \tan^{-1}\left(\frac{4+7t}{2-t}\right) = \boxed{\tan^{-1}(-7)}$$

Do not get so used the “special case” limits that we tend to usually do in the problems at the end of a section that you decide that you must have done something wrong when you run across a problem that doesn’t fall in the “special case” category.

---

12. Evaluate $\lim\limits_{w\to\infty} \tan^{-1}\left(\dfrac{3w^2-9w^4}{4w-w^3}\right)$.

Solution

First notice that,

$$\lim_{w\to\infty} \frac{3w^2-9w^4}{4w-w^3} = \infty$$

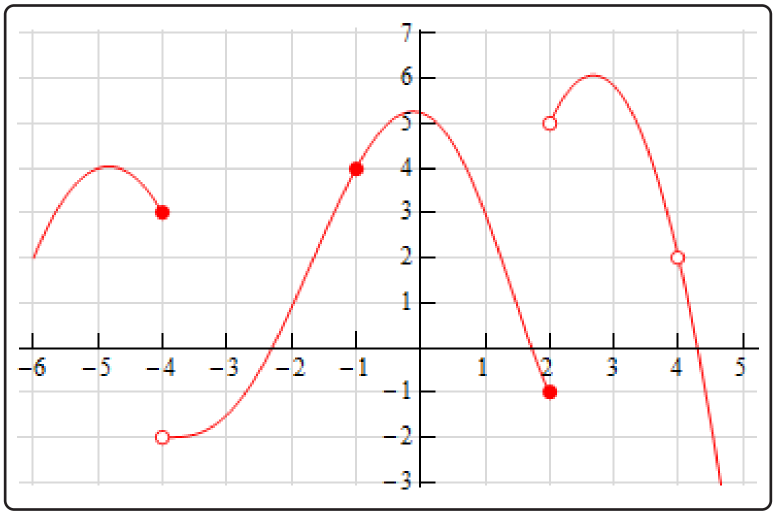
If you aren’t sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling Example 7 from this section, we know that because the argument goes to infinity in the limit the answer is,

$$\lim_{w\to\infty} \tan^{-1}\left(\frac{3w^2-9w^4}{4w-w^3}\right) = \boxed{\frac{\pi}{2}}$$

2.9 Continuity

1. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



Solution

Before starting the solution recall that in order for a function to be continuous at $x = a$ both $f(a)$ and $\lim\limits_{x \to a} f(x)$ must exist and we must have,

$$\lim_{x \to a} f(x) = f(a)$$

Using this idea it should be fairly clear where the function is not continuous.

First notice that at $x = -4$ we have,

$$\lim_{x \to -4^-} f(x) = 3 \neq -2 = \lim_{x \to -4^+} f(x)$$

and therefore, we also know that $\lim\limits_{x \to -4} f(x)$ doesn't exist. We can therefore conclude that $f(x)$ is **discontinuous** at $x = -4$ because the limit does not exist.

Likewise, at $x = 2$ we have,

$$\lim_{x \to 2^-} f(x) = -1 \neq 5 = \lim_{x \to 2^+} f(x)$$

and therefore, we also know that

$$\lim_{x \to 2} f(x)$$

doesn't exist. So again, because the limit does not exist, we can see that $f(x)$ is **discontinuous** at $x = 2$.

Finally let's take a look at $x = 4$. Here we can see that,

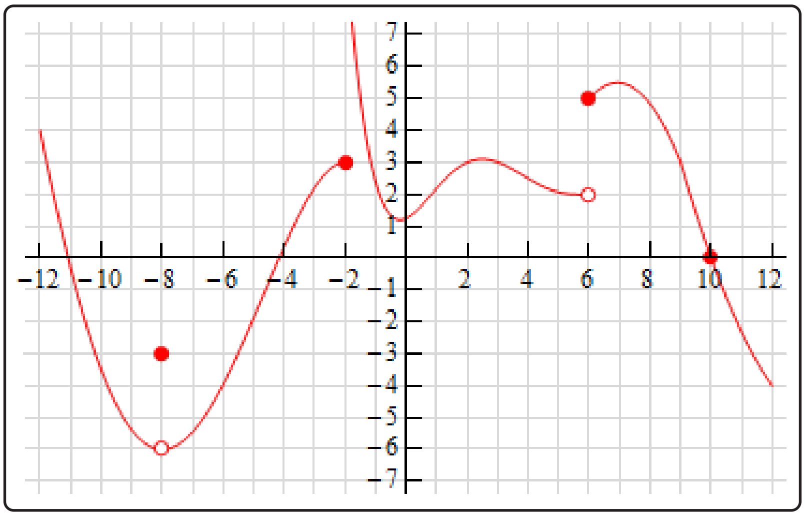
$$\lim_{x \to 4^-} f(x) = 2 = \lim_{x \to 4^+} f(x)$$

and therefore, we also know that $\lim\limits_{x \to 4} f(x) = 2$. However, we can also see that $f(4)$ doesn't exist and so once again $f(x)$ is **discontinuous** at $x = 4$ because this time the function does not exist at $x = 4$.

All other points on this graph will have both the function and limit exist and we'll have $\lim\limits_{x \to a} f(x) = f(a)$ and so will be continuous.

In summary then the points of discontinuity for this graph are : $x = -4$, $x = 2$ and $x = 4$.

2. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



Solution

Before starting the solution recall that in order for a function to be continuous at $x = a$ both $f(a)$ and $\lim\limits_{x \to a} f(x)$ must exist and we must have,

$$\lim_{x \to a} f(x) = f(a)$$

Using this idea it should be fairly clear where the function is not continuous.

First notice that at $x = -8$ we have,

$$\lim_{x \to -8^-} f(x) = -6 = \lim_{x \to -8^+} f(x)$$

and therefore, we also know that $\lim\limits_{x \to -8} f(x) = -6$. We can also see that $f(-8) = -3$ and so we have,

$$-6 = \lim_{x \to -8} f(x) \neq f(-8) = -3$$

Because the function and limit have different values we can conclude that $f(x)$ is **discontinuous** at $x = -8$.

Next let's take a look at $x = -2$ we have,

$$\lim_{x \to -2^-} f(x) = 3 \neq \infty = \lim_{x \to -2^+} f(x)$$

and therefore, we also know that $\lim\limits_{x \to -2} f(x)$ doesn't exist. We can therefore conclude that $f(x)$ is **discontinuous** at $x = -2$ because the limit does not exist.

Finally let's take a look at $x = 6$. Here we can see we have,

$$\lim_{x \to 6^-} f(x) = 2 \neq 5 = \lim_{x \to 6^+} f(x)$$

and therefore, we also know that

$$\lim_{x \to 6} f(x)$$

doesn't exist. So, once again, because the limit does not exist, we can conclude that $f(x)$ is **discontinuous** at $x = 6$.

All other points on this graph will have both the function and limit exist and we'll have $\lim\limits_{x \to a} f(x) = f(a)$ and so will be continuous.

In summary then the points of discontinuity for this graph are : $x = -8$, $x = -2$ and $x = 6$.

---

3. Using only Properties 1 - 9 from the Limit Properties section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a)** $x = -1$, **(b)** $x = 0$, **(c)** $x = 3$

$$f(x) = \frac{4x + 5}{9 - 3x}$$

Solutions

(a) $x = -1$

Solution

Before starting off with the solution to this part notice that we CAN NOT do what we've commonly done to evaluate limits to this point. In other words, we can't just plug in the point to evaluate the limit. Doing this implicitly assumes that the function is continuous at the point and that is what we are being asked to determine here.

Therefore, the only way for us to compute the limit is to go back to the properties from the Limit Properties section and compute the limit as we did back in that section. We won't be putting all the details here so if you need a little refresher on doing this you should go back to the problems from that section and work a few of them.

So, here we go.

$$\lim_{x\to-1} f(x) = \lim_{x\to-1} \frac{4x+5}{9-3x} = \frac{\lim\limits_{x\to-1} (4x+5)}{\lim\limits_{x\to-1} (9-3x)} = \frac{4\lim\limits_{x\to-1} x + \lim\limits_{x\to-1} 5}{\lim\limits_{x\to-1} 9 - 3\lim\limits_{x\to-1} x}$$
$$= \frac{4(-1)+5}{9-3(-1)} = f(-1)$$

So, we can see that $\lim\limits_{x\to-1} f(x) = f(-1)$ and so the function **is continuous** at $x = -1$.

(b) $x = 0$

Solution

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**.

Here is the work for this part.

$$\lim_{x\to0} f(x) = \lim_{x\to0} \frac{4x+5}{9-3x} = \frac{\lim\limits_{x\to0} (4x+5)}{\lim\limits_{x\to0} (9-3x)} = \frac{4\lim\limits_{x\to0} x + \lim\limits_{x\to0} 5}{\lim\limits_{x\to0} 9 - 3\lim\limits_{x\to0} x} = \frac{4(0)+5}{9-3(0)} = f(0)$$

So, we can see that $\lim\limits_{x\to0} f(x) = f(0)$ and so the function **is continuous** at $x = 0$.

(c) $x = 3$

Solution

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**. Although there is also of course the problem here that $f(3)$ doesn't exist and so we couldn't plug in the value even if we wanted to.

This also tells us what we need to know however. As noted in the notes for this section if either the function or the limit do not exist then the function is not continuous at the point. Therefore, we can see that the function **is not continuous** at

$x = 3$.

For practice you might want to verify that,

$$\lim_{x \to 3^-} f(x) = \infty \qquad \lim_{x \to 3^+} f(x) = -\infty$$

and so $\lim\limits_{x \to 3} f(x)$ also doesn't exist.

---

4. Using only Properties 1 - 9 from the Limit Properties section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a)** $z = -2$, **(b)** $z = 0$, **(c)** $z = 5$

$$g(z) = \frac{6}{z^2 - 3z - 10}$$

Solutions

(a) $z = -2$

Solution

Before starting off with the solution to this part notice that we CAN NOT do what we've commonly done to evaluate limits to this point. In other words, we can't just plug in the point to evaluate the limit. Doing this implicitly assumes that the function is continuous at the point and that is what we are being asked to determine here.

Of course, even if we had tried to plug in the point we would have run into problems as $g(-2)$ doesn't exist and this tell us all we need to know. As noted in the notes for this section if either the function or the limit do not exist then the function is not continuous at the point. Therefore, we can see that the function **is not continuous** at $z = -2$.

For practice you might want to verify that,

$$\lim_{z \to -2^-} g(z) = \infty \qquad \lim_{z \to -2^+} g(z) = -\infty$$

and so $\lim\limits_{z \to -2} g(z)$ also doesn't exist.

(b) $z = 0$

Solution

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**.

Therefore, because we can't just plug the point into the function, the only way for us to compute the limit is to go back to the properties from the Limit Properties section and compute the limit as we did back in that section. We won't be putting all the details here so if you need a little refresher on doing this you should go back to the problems from that section and work a few of them.

Here is the work for this part.

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{6}{z^2 - 3z - 10} = \frac{\lim\limits_{z\to 0} 6}{\lim\limits_{z\to 0}\left(z^2 - 3z - 10\right)} = \frac{\lim\limits_{z\to 0} 6}{\lim\limits_{z\to 0} z^2 - 3\lim\limits_{z\to 0} z - \lim\limits_{z\to 0} 10}$$
$$= \frac{6}{0^2 - 3(0) - 10} = g(0)$$

So, we can see that $\lim\limits_{z\to 0} g(z) = g(0)$ and so the function **is continuous** at $z = 0$.

(c) $z = 5$

Solution

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**. Although there is also of course the problem here that $g(5)$ doesn't exist and so we couldn't plug in the value even if we wanted to.

This also tells us what we need to know however. As noted in the notes for this section if either the function of the limit do not exist then the function is not continuous at the point. Therefore, we can see that the function **is not continuous** at $z = 5$.

For practice you might want to verify that,

$$\lim_{z\to 5^-} g(z) = -\infty \qquad \lim_{z\to 5^+} g(z) = \infty$$

and so $\lim\limits_{z\to 5} g(z)$ also doesn't exist.

5. Using only Properties 1 - 9 from the Limit Properties section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a)** $x = 4$, **(b)** $x = 6$

$$g(x) = \begin{cases} 2x & x < 6 \\ x-1 & x \geq 6 \end{cases}$$

Solutions

(a) $x = 4$

Solution

Before starting off with the solution to this part notice that we CAN NOT do what we've commonly done to evaluate limits to this point. In other words, we can't just plug in the point to evaluate the limit. Doing this implicitly assumes that the function is continuous at the point and that is what we are being asked to determine here.

Therefore, the only way for us to compute the limit is to go back to the properties from the Limit Properties section and compute the limit as we did back in that section. We won't be putting all the details here so if you need a little refresher on doing this you should go back to the problems from that section and work a few of them.

For this part we can notice that because there are values of x on both sides of $x = 4$ in the range $x < 6$ we won't need to worry about one-sided limits here. Here is the work for this part.

$$\lim_{x \to 4} g(x) = \lim_{x \to 4} (2x) = 2 \lim_{x \to 4} x = 2(4) = g(4)$$

So, we can see that

$$\lim_{x \to 4} g(x) = g(4)$$

and so the function **is continuous** at $x = 4$.

(b) $x = 6$

Solution

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**.

For this part we have the added complication that the point we're interested in is also the "cut-off" point of the piecewise function and so we'll need to take a look at the two one sided limits to compute the overall limit and again because we are being asked to determine if the function is continuous at this point we'll need to resort to basic limit properties to compute the one-sided limits and not just plug in the point (which assumes continuity again...).

Here is the work for this part.

$$\lim_{x\to 6^-} g(x) = \lim_{x\to 6^-} (2x) = 2 \lim_{x\to 6^-} x = 2(6) = 12$$
$$\lim_{x\to 6^+} g(x) = \lim_{x\to 6^+} (x-1) = \lim_{x\to 6^+} x - \lim_{x\to 6^+} 1 = 6 - 1 = 5$$

So we can see that, $\lim\limits_{x\to 6^-} g(x) \neq \lim\limits_{x\to 6^+} g(x)$ and so $\lim\limits_{x\to 6} g(x)$ does not exist.

Now, as discussed in the notes for this section, in order for a function to be continuous at a point both the function and the limit must exist. Therefore, this function **is not continuous** at $x = 6$ because

$$\lim_{x\to 6} g(x)$$

does not exist.

---

6. Using only Properties 1 - 9 from the Limit Properties section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a)** $t = -2$, **(b)** $t = 10$

$$h(t) = \begin{cases} t^2 & t < -2 \\ t+6 & t \ge -2 \end{cases}$$

Solutions

(a) $t = -2$

Solution

Before starting off with the solution to this part notice that we CAN NOT do what we've commonly done to evaluate limits to this point. In other words, we can't just plug in the point to evaluate the limit. Doing this implicitly assumes that the function is continuous at the point and that is what we are being asked to determine here.

Therefore, the only way for us to compute the limit is to go back to the properties from the Limit Properties section and compute the limit as we did back in that section. We won't be putting all the details here so if you need a little refresher on doing this you should go back to the problems from that section and work a few of them.

Also notice that for this part we have the added complication that the point we're interested in is also the "cut-off" point of the piecewise function and so we'll need to take a look at the two one sided limits to compute the overall limit and again because we are being asked to determine if the function is continuous at this point we'll need to resort to basic limit properties to compute the one-sided limits and not just plug in the point (which assumes continuity again...).

Here is the work for this part.

$$\lim_{t \to -2^-} h(t) = \lim_{t \to -2^-} t^2 = (-2)^2 = 4$$
$$\lim_{t \to -2^+} g(t) = \lim_{t \to -2^+} (t+6) = \lim_{t \to -2^+} t + \lim_{t \to -2^+} 6 = -2 + 6 = 4$$

So we can see that $\lim\limits_{t \to -2^-} h(t) = \lim\limits_{t \to -2^+} h(t) = 4$ and so $\lim\limits_{t \to -2} h(t) = 4$.

Next, a quick computation shows us that $h(-2) = -2 + 6 = 4$ and so we can see that

$$\lim_{t \to -2} h(t) = h(-2)$$

and so the function **is continuous** at $t = -2$.

(b) $t = 10$

Solution

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**.

For this part we can notice that because there are values of t on both sides of $t = 10$ in the range $t \geq -2$ we won't need to worry about one-sided limits here. Here is the work for this part.

Here is the work for this part.

$$\lim_{t \to 10} h(t) = \lim_{t \to 10} (t+6) = \lim_{t \to 10} t + \lim_{t \to 10} 6 = 10 + 6 = h(10)$$

So, we can see that $\lim\limits_{t \to 10} h(t) = h(10)$ and so the function **is continuous** at $t = 10$.

---

7. Using only Properties 1 - 9 from the Limit Properties section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a)** $x = -6$, **(b)** $x = 1$

$$g\left(x\right) = \begin{cases} 1 - 3x & x < -6 \\ 7 & x = -6 \\ x^3 & -6 < x < 1 \\ 1 & x = 1 \\ 2 - x & x > 1 \end{cases}$$

Solutions

(a) $x = -6$

Solution

Before starting off with the solution to this part notice that we CAN NOT do what we've commonly done to evaluate limits to this point. In other words, we can't just plug in the point to evaluate the limit. Doing this implicitly assumes that the function is continuous at the point and that is what we are being asked to determine here.

Therefore, the only way for us to compute the limit is to go back to the properties from the Limit Properties section and compute the limit as we did back in that section. We won't be putting all the details here so if you need a little refresher on doing this you should go back to the problems from that section and work a few of them.

Also notice that for this part we have the added complication that the point we're interested in is also the "cut-off" point of the piecewise function and so we'll need to take a look at the two one sided limits to compute the overall limit and again because we are being asked to determine if the function is continuous at this point we'll need to resort to basic limit properties to compute the one-sided limits and not just plug in the point (which assumes continuity again...).

Here is the work for this part.

$$\lim_{x \to -6^-} g\left(x\right) = \lim_{x \to -6^-} \left(1 - 3x\right) = \lim_{x \to -6^-} 1 - 3 \lim_{x \to -6^-} x = 1 - 3\left(-6\right) = 19$$
$$\lim_{x \to -6^+} g\left(x\right) = \lim_{x \to -6^+} x^3 = \left(-6\right)^3 = -216$$

So, we can see that, $\lim\limits_{x \to -6^-} g\left(x\right) \neq \lim\limits_{x \to -6^+} g\left(x\right)$ and so $\lim\limits_{x \to -6} g\left(x\right)$ does not exist.

Now, as discussed in the notes for this section, in order for a function to be continuous at a point both the function and the limit must exist. Therefore, this function **is not continuous** at $x = -6$ because

$$\lim_{x \to -6} g\left(x\right)$$

does not exist.

(b) $x = 1$

Solution

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**.

Again, note that we are dealing with another "cut-off" point here so we'll need to use one-sided limits again as we did in the previous part.

Here is the work for this part.

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x^3 = 1^3 = 1$$
$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (2 - x) = \lim_{x \to 1^+} 2 - \lim_{x \to 1^+} x = 2 - 1 = 1$$

So, we can see that, $\lim\limits_{x \to 1^-} g(x) = \lim\limits_{x \to 1^+} g(x) = 1$ and so $\lim\limits_{x \to 1} g(x) = 1$.

Next, a quick computation shows us that $g(1) = 1$ and so we can see that $\lim\limits_{x \to 1} g(x) = g(1)$ and so the function **is continuous** at $x = 1$.

---

8. Determine where the following function is discontinuous.

$$f(x) = \frac{x^2 - 9}{3x^2 + 2x - 8}$$

Hint

If we have two continuous functions and form a rational expression out of them recall where the rational expression will be discontinuous. We discussed this in the Limit Properties section, although we were using the phrase "nice enough" there instead of the word "continuity".

Solution

As noted in the hint for this problem when dealing with a rational expression in which both the numerator and denominator are continuous (as we have here since both are polynomials) the only points in which the rational expression will be discontinuous will be where we have division by zero.

Therefore, all we need to do is determine where the denominator is zero and that is fairly easy for this problem.

$$3x^2 + 2x - 8 = (3x - 4)(x + 2) = 0 \qquad \Rightarrow \qquad x = \frac{4}{3}, \quad x = -2$$

The function will therefore be discontinuous at the points : $x = \frac{4}{3}$ and $x = -2$.

---

9. Determine where the following function is discontinuous.

$$R(t) = \frac{8t}{t^2 - 9t - 1}$$

Hint

If we have two continuous functions and form a rational expression out of them recall where the rational expression will be discontinuous. We discussed this in the Limit Properties section, although we were using the phrase "nice enough" there instead of the word "continuity".

Solution

As noted in the hint for this problem when dealing with a rational expression in which both the numerator and denominator are continuous (as we have here since both are polynomials) the only points in which the rational expression will be discontinuous will be where we have division by zero.

Therefore, all we need to do is determine where the denominator is zero and that is fairly easy for this problem.

$$t^2 - 9t - 1 = 0 \qquad \Rightarrow \qquad t = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(-1)}}{2(1)} = \frac{9 \pm \sqrt{85}}{2} = -0.10977,\ 9.10977$$

The function will therefore be discontinuous at the points : $t = \frac{9 \pm \sqrt{85}}{2}$.

---

10. Determine where the following function is discontinuous.

$$h(z) = \frac{1}{2 - 4\cos(3z)}$$

Hint

If we have two continuous functions and form a rational expression out of them recall where the rational expression will be discontinuous. We discussed this in the Limit Properties section, although we were using the phrase "nice enough" there instead of the word "continuity".

Solution

As noted in the hint for this problem when dealing with a rational expression in which both the numerator and denominator are continuous (as we have here since the numerator is just a constant and the denominator is a sum of continuous functions) the only points in which the rational expression will be discontinuous will be where we have division by zero.

Therefore, all we need to do is determine where the denominator is zero. If you don't recall how to solve equations involving trig functions you should go back to the Review chapter and take a look at the Solving Trig Equations sections there.

Here is the solution work for determining where the denominator is zero. Using our calculator we get,

$$2 - 4\cos(3z) = 0 \qquad \rightarrow \qquad 3x = \cos^{-1}\left(\frac{1}{2}\right) = 1.0472$$

The second angle will be in the fourth quadrant and is $2\pi - 1.0472 = 5.2360$.

The denominator will therefore be zero at,

$$\begin{array}{rcl} 3x = 1.0472 + 2\pi n & \text{OR} & 3x = 5.2360 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \\ x = 0.3491 + \frac{2\pi n}{3} & \text{OR} & x = 1.7453 + \frac{2\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots \end{array}$$

The function will therefore be discontinuous at the points,

$$x = 0.3491 + \frac{2\pi n}{3} \quad \text{OR} \quad x = 1.7453 + \frac{2\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots$$

Note as well that this was one of the few trig equations that could be solved exactly if you know your basic unit circle values. Here is the exact solution for the points of discontinuity.

$$x = \frac{\pi}{9} + \frac{2\pi n}{3} \quad \text{OR} \quad x = \frac{5\pi}{9} + \frac{2\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots$$

---

11. Determine where the following function is discontinuous.

$$y(x) = \frac{x}{7 - \mathbf{e}^{2x+3}}$$

Hint

If we have two continuous functions and form a rational expression out of them recall where the rational expression will be discontinuous. We discussed this in the Limit Properties section, although we were using the phrase "nice enough" there instead of the word "continuity".

Solution

As noted in the hint for this problem when dealing with a rational expression in which both the numerator and denominator are continuous (as we have here since the numerator is a polynomial and the denominator is a sum of two continuous functions) the only points in which the rational expression will be discontinuous will be where we have division by zero.

Therefore, all we need to do is determine where the denominator is zero and that is fairly easy for this problem.

$$7 - \mathbf{e}^{2x+3} = 0 \quad \rightarrow \quad \mathbf{e}^{2x+3} = 7 \quad \rightarrow \quad 2x + 3 = \ln(7) \quad \Rightarrow \quad x = \frac{1}{2}(\ln(7) - 3) = -0.5270$$

The function will therefore be discontinuous at : $x = \frac{1}{2}(\ln(7) - 3) = -0.5270$.

---

12. Determine where the following function is discontinuous.

$$g(x) = \tan(2x)$$

Hint

If we have two continuous functions and form a rational expression out of them recall where the rational expression will be discontinuous. We discussed this in the Limit Properties section, although we were using the phrase “nice enough” there instead of the word “continuity”. And, yes we really do have a rational expression here.

Solution

As noted in the hint for this problem when dealing with a rational expression in which both the numerator and denominator are continuous the only points in which the rational expression will not be continuous will be where we have division by zero.

Also, writing the function as,

$$g(x) = \frac{\sin(2x)}{\cos(2x)}$$

we can see that we really do have a rational expression here. Therefore, all we need to do is determine where the denominator (*i.e.* cosine) is zero. If you don’t recall how to solve equations involving trig functions you should go back to the Review chapter and take a look at the Solving Trig Equations sections there.

Here is the solution work for determining where the denominator is zero. Using our basic unit circle knowledge we know where cosine will be zero so we have,

$$2x = \tfrac{\pi}{2} + 2\pi n \qquad \text{OR} \qquad 2x = \tfrac{3\pi}{2} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots$$

The denominator will therefore be zero, and the function will be discontinuous, at,

$$x = \frac{\pi}{4} + \pi n \qquad \text{OR} \qquad x = \frac{3\pi}{4} + \pi n \quad n = 0, \pm 1, \pm 2, \ldots$$

---

13. Use the Intermediate Value Theorem to show that $25 - 8x^2 - x^3 = 0$ has at least one root in the interval $[-2, 4]$. Note that you are NOT asked to find the solution only show that at least one must exist in the indicated interval,

Hint

The hardest part of these problems for most students is just getting started. First, you need to determine the value of "M" that you need to use and then actually use the Intermediate Value Theorem. So, go back to the IVT and compare the conclusions of the theorem and it should be pretty obvious what the M should be and then just check that the hypothesis (i.e. the "requirements" of the theorem) are met and you'll pretty much be done.

Solution

Okay, let's start off by defining,

$$f(x) = 25 - 8x^2 - x^3 \qquad \& \qquad M = 0$$

The problem is then asking us to show that there is a c in $[-2, 4]$ so that,

$$f(c) = 0 = M$$

but this is exactly the second conclusion of the Intermediate Value Theorem. So, let's see that the "requirements" of the theorem are met.

First, the function is a polynomial and so is continuous everywhere and in particular is continuous on the interval $[-2, 4]$. Note that this IS a requirement that MUST be met in order to use the IVT and it is the one requirement that is most often overlooked. If we don't have a continuous function the IVT simply can't be used.

Now all that we need to do is verify that M is between the function values as the endpoints of the interval. So,

$$f(-2) = 1 \qquad\qquad f(4) = -167$$

Therefore, we have,

$$f(4) = -167 < 0 < 1 = f(-2)$$

So, by the Intermediate Value Theorem there must be a number c such that,

$$-2 < c < 4 \qquad \& \qquad f(c) = 0$$

and we have shown what we were asked to show.

---

14. Use the Intermediate Value Theorem to show that $w^2 - 4\ln(5w+2) = 0$ has at least one root in the interval $[0,4]$. Note that you are NOT asked to find the solution only show that at least one must exist in the indicated interval,

Hint

The hardest part of these problems for most students is just getting started. First, you need to determine the value of "M" that you need to use and then actually use the Intermediate Value Theorem. So, go back to the IVT and compare the conclusions of the theorem and it should be pretty obvious what the M should be and then just check that the hypothesis (*i.e.* the "requirements" of the theorem) are met and you'll pretty much be done.

Solution

Okay, let's start off by defining,

$$f(w) = w^2 - 4\ln(5w+2) \qquad \& \qquad M = 0$$

The problem is then asking us to show that there is a c in $[0,4]$ so that,

$$f(c) = 0 = M$$

but this is exactly the second conclusion of the Intermediate Value Theorem. So, let's see that the "requirements" of the theorem are met.

First, the function is a sum of a polynomial (which is continuous everywhere) and a natural logarithm (which is continuous on $w > -\frac{2}{5}$ - *i.e.* where the argument is positive) and so is continuous on the interval $[0,4]$. Note that this IS a requirement that MUST be met in order to use the IVT and it is the one requirement that is most often overlooked. If we don't have a continuous function the IVT simply can't be used.

Now all that we need to do is verify that M is between the function values as the endpoints of the interval. So,

$$f(0) = -2.7726 \qquad f(4) = 3.6358$$

Therefore, we have,

$$f(0) = -2.7726 < 0 < 3.6358 = f(4)$$

So, by the Intermediate Value Theorem there must be a number c such that,

$$0 < c < 4 \qquad \& \qquad f(c) = 0$$

and we have shown what we were asked to show.

---

15. Use the Intermediate Value Theorem to show that $4t + 10\mathbf{e}^{t} - \mathbf{e}^{2t} = 0$ has at least one root in the interval $[1, 3]$. Note that you are NOT asked to find the solution only show that at least one must exist in the indicated interval,

Hint

The hardest part of these problems for most students is just getting started. First, you need to determine the value of "M" that you need to use and then actually use the Intermediate Value Theorem. So, go back to the IVT and compare the conclusions of the theorem and it should be pretty obvious what the M should be and then just check that the hypothesis (*i.e.* the "requirements" of the theorem) are met and you'll pretty much be done.

Solution

Okay, let's start off by defining,

$$f(t) = 4t + 10\mathbf{e}^{t} - \mathbf{e}^{2t} \qquad \& \qquad M = 0$$

The problem is then asking us to show that there is a c in $[1, 3]$ so that,

$$f(c) = 0 = M$$

but this is exactly the second conclusion of the Intermediate Value Theorem. So, let's see that the "requirements" of the theorem are met.

First, the function is a sum and difference of a polynomial and two exponentials (all of which are continuous everywhere) and so is continuous on the interval $[1, 3]$. Note that this IS a requirement that MUST be met in order to use the IVT and it is the one requirement that is most often overlooked. If we don't have a continuous function the IVT simply can't be used.

Now all that we need to do is verify that M is between the function values as the end-points of the interval. So,

$$f(1) = 23.7938 \qquad\qquad f(3) = -190.5734$$

Therefore, we have,

$$f(3) = -190.5734 < 0 < 23.7938 = f(1)$$

So, by the Intermediate Value Theorem there must be a number c such that,

$$1 < c < 3 \qquad \& \qquad f(c) = 0$$

and we have shown what we were asked to show.

2.10 The Definition of the Limit

1. Use the definition of the limit to prove the following limit.

$$\lim_{x \to 3} x = 3$$

Step 1

First, let's just write out what we need to show.

Let $\varepsilon > 0$ be any number. We need to find a number $\delta > 0$ so that,

$$|x - 3| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - 3| < \delta$$

This problem can look a little tricky since the two inequalities both involve $|x - 3|$. Just keep in mind that the first one is really $|f(x) - L| < \varepsilon$ where $f(x) = x$ and $L = 3$ and the second is really $0 < |x - a| < \delta$ where $a = 3$.

Step 2

In this case, despite the "trickiness" of the statement we need to prove in Step 1, this is really a very simple problem.

We need to determine a δ that will allow us to prove the statement in Step 1. However, because both inequalities involve exactly the same absolute value statement so all we need to do is choose $\delta = \varepsilon$.

Step 3

So, let's see if this works.

Start off by first assuming that $\varepsilon > 0$ is any number and choose $\delta = \varepsilon$. We can now assume that

$$0 < |x - 3| < \delta = \varepsilon \qquad \Rightarrow \qquad 0 < |x - 3| < \varepsilon$$

However, if we just look at the right portion of the double inequality we see that this assumption tells us that,

$$|x - 3| < \varepsilon$$

which is exactly what we needed to show give our choice of δ.

Therefore, according to the definition of the limit we have just proved that,

$$\lim_{x \to 3} x = 3$$

2. Use the definition of the limit to prove the following limit.

$$\lim_{x \to -1} (x + 7) = 6$$

Step 1

First, let's just write out what we need to show.

Let $\varepsilon > 0$ be any number. We need to find a number $\delta > 0$ so that,

$$|(x + 7) - 6| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - (-1)| < \delta$$

Or, with a little simplification this becomes,

$$|x + 1| < \varepsilon \qquad \text{whenever} \qquad 0 < |x + 1| < \delta$$

Step 2

This problem is very similar to Problem 1 from this point on.

We need to determine a δ that will allow us to prove the statement in Step 1. However, because both inequalities involve exactly the same absolute value statement all we need to do is choose $\delta = \varepsilon$.

Step 3

So, let's see if this works.

Start off by first assuming that $\varepsilon > 0$ is any number and choose $\delta = \varepsilon$. We can now assume that,

$$0 < |x - (-1)| < \delta = \varepsilon \qquad \Rightarrow \qquad 0 < |x + 1| < \varepsilon$$

This gives,

$$
\begin{aligned}
|(x+7)-6| &= |x+1| && \text{simplify things up a little} \\
&< \varepsilon && \text{using the information we got by assuming } \delta = \varepsilon
\end{aligned}
$$

So, we've shown that,

$$|(x+7)-6| < \varepsilon \qquad \text{whenever} \qquad 0 < |x-(-1)| < \varepsilon$$

and so by the definition of the limit we have just proved that,

$$\lim_{x \to -1} (x+7) = 6$$

---

3. Use the definition of the limit to prove the following limit.

$$\lim_{x \to 2} x^2 = 4$$

Step 1

First, let's just write out what we need to show.

Let $\varepsilon > 0$ be any number. We need to find a number $\delta > 0$ so that,

$$\left|x^2 - 4\right| < \varepsilon \qquad \text{whenever} \qquad 0 < |x-2| < \delta$$

Step 2

Let's start with a little simplification of the first inequality.

$$\left|x^2 - 4\right| = |(x+2)(x-2)| = |x+2|\,|x-2| < \varepsilon$$

We have the $|x-2|$ we expect to see but we also have an $|x+2|$ that we'll need to deal with.

Step 3

To deal with the $|x+2|$ let's first assume that

$$|x-2| < 1$$

As we noted in a similar example in the notes for this section this is a legitimate assumption because the limit is $x \to 2$ and so x's will be getting very close to 2. Therefore, provided x is close enough to 2 we will have $|x-2| < 1$.

Starting with this assumption we get that,

$$-1 < x-2 < 1 \qquad \to \qquad 1 < x < 3$$

If we now add 2 to all parts of this inequality we get,

$$3 < x+2 < 5$$

Noticing that $3 > 0$ we can see that we then also know that $x+2 > 0$ and so provided $|x-2| < 1$ we will have $x+2 = |x+2|$.

All this means is that, provided $|x-2| < 1$, we will also have,

$$|x+2| = x+2 < 5 \qquad \to \qquad |x+2| < 5$$

This in turn means that we have,

$$|x+2|\,|x-2| < 5\,|x-2| \qquad \text{because } |x+2| < 5$$

Therefore, if we were to further assume, for some reason, that we wanted $5\,|x-2| < \varepsilon$ this would tell us that,

$$|x-2| < \frac{\varepsilon}{5}$$

Step 4

Okay, even though it doesn't seem like it we actually have enough to make a choice for δ.

Given any number $\varepsilon > 0$ let's chose

$$\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$$

Again, this means that δ will be the smaller of the two values which in turn means that,

$$\delta \le 1 \qquad \text{AND} \qquad \delta \le \frac{\varepsilon}{5}$$

Now assume that $0 < |x-2| < \delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$.

Step 5

So, let's see if this works.

Given the assumption $0 < |x-2| < \delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$ we know two things. First, we know that $|x-2| < \frac{\varepsilon}{5}$. Second, we also know that $|x-2| < 1$ which in turn implies that $|x+2| < 5$ as we saw in Step 3.

Now, let's do the following,

$$\begin{aligned} \left|x^2-4\right| &= |x+2|\,|x-2| && \text{factoring} \\ &< 5\,|x-2| && \text{because we know } |x+2| < 5 \\ &< 5\left(\frac{\varepsilon}{5}\right) && \text{because we know } |x-2| < \frac{\varepsilon}{5} \\ &= \varepsilon \end{aligned}$$

So, we've shown that,

$$\left|x^2-4\right| < \varepsilon \qquad \text{whenever} \qquad 0 < |x-2| < \min\left\{1, \frac{\varepsilon}{5}\right\}$$

and so by the definition of the limit we have just proved that,

$$\lim_{x \to 2} x^2 = 4$$

4. Use the definition of the limit to prove the following limit.

$$\lim_{x \to -3} \left(x^2+4x+1\right) = -2$$

Step 1

First, let's just write out what we need to show.

Let $\varepsilon > 0$ be any number. We need to find a number $\delta > 0$ so that,

$$\left|x^2+4x+1-(-2)\right| < \varepsilon \qquad \text{whenever} \qquad 0 < |x-(-3)| < \delta$$

Simplifying this a little gives,

$$\left|x^2+4x+3\right| < \varepsilon \qquad \text{whenever} \qquad 0 < |x+3| < \delta$$

Step 2

Let's start with a little simplification of the first inequality.

$$\left|x^2+4x+3\right| = \left|(x+1)(x+3)\right| = |x+1|\,|x+3| < \varepsilon$$

We have the $|x+3|$ we expect to see but we also have an $|x+1|$ that we'll need to deal with.

Step 3

To deal with the $|x+1|$ let's first assume that

$$|x+3| < 1$$

As we noted in a similar example in the notes for this section this is a legitimate assumption because the limit is $x \to -3$ and so x's will be getting very close to -3. Therefore, provided x is close enough to -3 we will have $|x+3| < 1$.

Starting with this assumption we get that,

$$-1 < x+3 < 1 \qquad \to \qquad -4 < x < -2$$

If we now add 1 to all parts of this inequality we get,

$$-3 < x+1 < -1$$

Noticing that $-1 < 0$ we can see that we then also know that $x+1 < 0$ and so provided $|x+3| < 1$ we will have $|x+1| = -(x+1)$. Also, from the inequality above we see that,

$$1 < -(x+1) < 3$$

All this means is that, provided $|x+3| < 1$, we will also have,

$$|x+1| = -(x+1) < 3 \qquad \to \qquad |x+1| < 3$$

This in turn means that we have,

$$|x+1|\,|x+3| < 3\,|x+3| \qquad \text{because } |x+1| < 3$$

Therefore, if we were to further assume, for some reason, that we wanted $3\,|x+3| < \varepsilon$ this would tell us that,

$$|x+3| < \frac{\varepsilon}{3}$$

Step 4

Okay, even though it doesn't seem like it we actually have enough to make a choice for δ. Given any number $\varepsilon > 0$ let's chose

$$\delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$$

Again, this means that δ will be the smaller of the two values which in turn means that,

$$\delta \le 1 \qquad \text{AND} \qquad \delta \le \frac{\varepsilon}{3}$$

Now assume that $0 < |x+3| < \delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$.

Step 5

So, let's see if this works.

Given the assumption $0 < |x+3| < \delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$ we know two things. First, we know that $|x+3| < \frac{\varepsilon}{3}$. Second, we also know that $|x+3| < 1$ which in turn implies that $|x+1| < 3$ as we saw in Step 3.

Now, let's do the following,

$$\begin{aligned} \left|x^2+4x+3\right| &= |x+1|\,|x+3| && \text{factoring} \\ &< 3\,|x+3| && \text{because we know } |x+1| < 3 \\ &< 3\left(\frac{\varepsilon}{3}\right) && \text{because we know } |x+3| < \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

So, we've shown that,

$$\left|x^2+4x+3\right| < \varepsilon \qquad \text{whenever} \qquad 0 < |x+3| < \min\left\{1, \frac{\varepsilon}{3}\right\}$$

and so by the definition of the limit we have just proved that,

$$\lim_{x \to -3} \left(x^2+4x+1\right) = -2$$

5. Use the definition of the limit to prove the following limit.

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$$

Step 1

First, let's just write out what we need to show.

Let $M > 0$ be any number. We need to find a number $\delta > 0$ so that,

$$\frac{1}{(x-1)^2} > M \qquad \text{whenever} \qquad 0 < |x-1| < \delta$$

Step 2

Let's do a little rewrite the first inequality above a little bit.

$$\frac{1}{(x-1)^2} > M \qquad \to \qquad (x-1)^2 < \frac{1}{M} \qquad \to \qquad |x-1| < \frac{1}{\sqrt{M}}$$

From this it looks like we can choose $\delta = \frac{1}{\sqrt{M}}$.

Step 3

So, let's see if this works.

We'll start by assuming that $M > 0$ is any number and chose $\delta = \frac{1}{\sqrt{M}}$. We can now assume that,

$$0 < |x-1| < \delta = \frac{1}{\sqrt{M}} \qquad \Rightarrow \qquad 0 < |x-1| < \frac{1}{\sqrt{M}}$$

So, if we start with the second inequality we get,

$$\begin{aligned}
|x-1| &< \frac{1}{\sqrt{M}} \\
|x-1|^2 &< \frac{1}{M} && \text{squaring both sides} \\
(x-1)^2 &< \frac{1}{M} && \text{because } |x-1|^2 = (x-1)^2 \\
\frac{1}{(x-1)^2} &> M && \text{rewriting things a little bit}
\end{aligned}$$

So, we've shown that,

$$\frac{1}{(x-1)^2} > M \qquad \text{whenever} \qquad 0 < |x-1| < \frac{1}{\sqrt{M}}$$

and so by the definition of the limit we have just proved that,

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$$

6. Use the definition of the limit to prove the following limit.

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Step 1

First, let's just write out what we need to show.

Let $N < 0$ be any number. Remember that because our limit is going to negative infinity here we need N to be negative. Now, we need to find a number $\delta > 0$ so that,

$$\frac{1}{x} < N \qquad \text{whenever} \qquad -\delta < x - 0 < 0$$

Step 2

Let's do a little rewrite on the first inequality above to get,

$$\frac{1}{x} < N \qquad \rightarrow \qquad x > \frac{1}{N}$$

Now, keep in mind that N is negative and so $\frac{1}{N}$ is also negative. From this it looks like we can choose $\delta = -\frac{1}{N}$. Again, because N is negative this makes δ positive, which we need!

Step 3

So, let's see if this works.

We'll start by assuming that $N < 0$ is any number and chose $\delta = -\frac{1}{N}$. We can now assume that,

$$-\delta < x - 0 < 0 \qquad \Rightarrow \qquad \frac{1}{N} < x < 0$$

So, if we start with the second inequality we get,

$$\begin{aligned} x &> \frac{1}{N} \\ \frac{1}{x} &< N \qquad \text{rewriting things a little bit} \end{aligned}$$

So, we've shown that,

$$\frac{1}{x} < N \qquad \text{whenever} \qquad \frac{1}{N} < x < 0$$

and so by the definition of the limit we have just proved that,

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

7. Use the definition of the limit to prove the following limit.

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

Step 1

First, let's just write out what we need to show.

Let $\varepsilon > 0$ be any number. We need to find a number $M > 0$ so that,

$$\left| \frac{1}{x^2} - 0 \right| < \varepsilon \qquad \text{whenever} \qquad x > M$$

Or, with a little simplification this becomes,

$$\left| \frac{1}{x^2} \right| < \varepsilon \qquad \text{whenever} \qquad x > M$$

Step 2

Let's start with the inequality on the left and do a little rewriting on it.

$$\left|\frac{1}{x^2}\right| < \varepsilon \qquad \to \qquad \frac{1}{|x|^2} < \varepsilon \qquad \to \qquad |x|^2 > \frac{1}{\varepsilon} \qquad \to \qquad |x| > \frac{1}{\sqrt{\varepsilon}}$$

From this it looks like we can choose $M = \frac{1}{\sqrt{\varepsilon}}$

Step 3

So, let's see if this works.

Start off by first assuming that $\varepsilon > 0$ is any number and choose $M = \frac{1}{\sqrt{\varepsilon}}$. We can now assume that,

$$x > \frac{1}{\sqrt{\varepsilon}}$$

Starting with this inequality we get,

$$\begin{aligned} x &> \frac{1}{\sqrt{\varepsilon}} & \\ \frac{1}{x} &< \sqrt{\varepsilon} & \text{do a little rewrite} \\ \frac{1}{x^2} &< \varepsilon & \text{square both sides} \\ \left|\frac{1}{x^2}\right| &< \varepsilon & \text{because } \frac{1}{x^2} = \left|\frac{1}{x^2}\right| \end{aligned}$$

So, we've shown that,

$$\left|\frac{1}{x^2} - 0\right| < \varepsilon \qquad \text{whenever} \qquad x > M$$

and so by the definition of the limit we have just proved that,

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

3 Derivatives

In this chapter we will start looking at the next major topic in a calculus class, derivatives. This chapter is devoted almost exclusively to finding/computing derivatives. We will, however, take a look at a single application of derivatives in this chapter. We will be leaving most of the applications of derivatives that we will be discussing to the next chapter.

This chapter will start out with defining just what a derivative is as well as look at a couple of the main interpretations. In the process we will start to understand just how interconnected the main topics of first Calculus course are. In particular, we will see that, in theory, we can't do derivatives unless we can also do limits.

However, having said that we'll also see that using limits to computer derivatives can be a fairly long process that is prone to inadvertent errors if we get in a hurry and, in some cases, will be all but impossible to do. Therefore, after discussing the definition of the derivative we'll move off to looking at some formulas for computing derivatives that will allow us to avoid having to use limits to compute derivatives. Note however that won't mean that we can just forget all about using limits to compute derivatives. That is still something that will, on occasion, come up so we can't forget about that.

We will discuss formulas for the following functions.

- Functions involving polynomials, roots and more generally, terms involving variables raised to a power.
- Trigonometric functions.
- Exponential and Logarithm functions.
- Inverse Trigonometric functions.
- Hyperbolic functions.

We'll also see very quickly that while the formulas for the functions above are nice they won't actually allow us to differentiate just any function that involved them. So, we will also discuss the Product and Quotient Rules allowing us to differentiate, oddly enough, products and quotients involving the functions listed above. We will also take a long look at something called the Chain Rule which will again greatly expand the number of functions we can differentiate. In fact, the Chain Rule may be the most important of the formulas we discuss as easily the majority of derivatives will be taking eventually will involve the Chain Rule at least partially.

In addition we will also take a look at implicit differentiation. This will, again, expend the number derivatives that we can find, including allowing us to find derivatives that we would not be able to find otherwise. Implicit differentiation will also allow us to look at the only application of derivatives that we will look at in this chapter, Related Rates. Related Rates problems will allow us to determine the rate of change of a quantity provided we know something about the rates of change for the other quantities in the problem.

We will also look at higher order derivatives. Or, in other words, we will take the derivative of a derivative and discuss an application of of at least one of the higher order derivatives.

We will then close out the chapter with a quick discussion of Logarithmic Differentiation. Logarithmic Differentiation is al alternative method of differentiation that can be used instead of the Product and Quotient Rule (sometimes easier sometimes not...). More importantly logarithmic differentiation will allow us to differentiate a class of functions that none of the formulas we will have discussed in this chapter up to this point would allow us differentiate.

The following sections are the practice problems, with solutions, for this material.

3.1 Definition of the Derivative

1. Use the definition of the derivative to find the derivative of,

$$f(x) = 6$$

Solution

There really isn't much to do for this problem other than to plug the function into the definition of the derivative and do a little algebra.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{6-6}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

So, the derivative for this function is,

$$\boxed{f'(x) = 0}$$

---

2. Use the definition of the derivative to find the derivative of,

$$V(t) = 3 - 14t$$

Step 1

First we need to plug the function into the definition of the derivative.

$$V'(t) = \lim_{h \to 0} \frac{V(t+h) - V(t)}{h} = \lim_{h \to 0} \frac{3 - 14(t+h) - (3 - 14t)}{h}$$

Make sure that you properly evaluate the first function evaluation. This is one of the more common errors that students make with these problems.

Also watch for the parenthesis on the second function evaluation. You are subtracting off the whole function and so you need to make sure that you deal with the minus sign properly. Either put in the parenthesis as we've done here or make sure the minus sign get distributed through properly. This is another very common error and one that if you make will often make the problem impossible to complete.

Step 2

Now all that we need to do is some quick algebra and we'll be done.

$$V'(t) = \lim_{h\to 0} \frac{3 - 14t - 14h - 3 + 14t}{h} = \lim_{h\to 0} \frac{-14h}{h} = \lim_{h\to 0} (-14) = -14$$

The derivative for this function is then,

$$\boxed{V'(t) = -14}$$

---

3. Use the definition of the derivative to find the derivative of,

$$g(x) = x^2$$

Step 1

First we need to plug the function into the definition of the derivative.

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

Make sure that you properly evaluate the first function evaluation. This is one of the more common errors that students make with these problems.

Step 2

Now all that we need to do is some quick algebra and we'll be done.

$$g'(x) = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x + h) = 2x$$

The derivative for this function is then,

$$\boxed{g'(x) = 2x}$$

---

4. Use the definition of the derivative to find the derivative of,

$$Q(t) = 10 + 5t - t^2$$

Step 1

First we need to plug the function into the definition of the derivative.

$$Q'(t) = \lim_{h \to 0} \frac{Q(t+h) - Q(t)}{h} = \lim_{h \to 0} \frac{10 + 5(t+h) - (t+h)^2 - \left(10 + 5t - t^2\right)}{h}$$

Make sure that you properly evaluate the first function evaluation. This is one of the more common errors that students make with these problems.

Also watch for the parenthesis on the second function evaluation. You are subtracting off the whole function and so you need to make sure that you deal with the minus sign properly. Either put in the parenthesis as we've done here or make sure the minus sign get distributed through properly. This is another very common error and one that if you make will often make the problem impossible to complete.

Step 2

Now all that we need to do is some algebra (and it might get a little messy here, but that is somewhat common with these types of problems) and we'll be done.

$$\begin{aligned} Q'(t) &= \lim_{h \to 0} \frac{10 + 5t + 5h - t^2 - 2th - h^2 - 10 - 5t + t^2}{h} \\ &= \lim_{h \to 0} \frac{h(5 - 2t - h)}{h} = \lim_{h \to 0} (5 - 2t - h) = 5 - 2t \end{aligned}$$

The derivative for this function is then,

$$\boxed{Q'(t) = 5 - 2t}$$

---

5. Use the definition of the derivative to find the derivative of,

$$W(z) = 4z^2 - 9z$$

Step 1

First we need to plug the function into the definition of the derivative.

$$W'(z) = \lim_{h \to 0} \frac{W(z+h) - W(z)}{h} = \lim_{h \to 0} \frac{4(z+h)^2 - 9(z+h) - (4z^2 - 9z)}{h}$$

Make sure that you properly evaluate the first function evaluation. This is one of the more common errors that students make with these problems.

Also watch for the parenthesis on the second function evaluation. You are subtracting off the whole function and so you need to make sure that you deal with the minus sign properly. Either put in the parenthesis as we've done here or make sure the minus sign get distributed through properly. This is another very common error and one that if you make will often make the problem impossible to complete.

Step 2

Now all that we need to do is some algebra (and it might get a little messy here, but that is somewhat common with these types of problems) and we'll be done.

$$\begin{aligned} W'(z) &= \lim_{h \to 0} \frac{4(z^2 + 2zh + h^2) - 9z - 9h - 4z^2 + 9z}{h} \\ &= \lim_{h \to 0} \frac{4z^2 + 8zh + 4h^2 - 9z - 9h - 4z^2 + 9z}{h} \\ &= \lim_{h \to 0} \frac{h(8z + 4h - 9)}{h} = \lim_{h \to 0} (8z + 4h - 9) = 8z - 9 \end{aligned}$$

The derivative for this function is then,

$$\boxed{W'(z) = 8z - 9}$$

6. Use the definition of the derivative to find the derivative of,

$$f(x) = 2x^3 - 1$$

Step 1

First we need to plug the function into the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x+h)^3 - 1 - \left(2x^3 - 1\right)}{h}$$

Make sure that you properly evaluate the first function evaluation. This is one of the more common errors that students make with these problems.

Also watch for the parenthesis on the second function evaluation. You are subtracting off the whole function and so you need to make sure that you deal with the minus sign properly. Either put in the parenthesis as we've done here or make sure the minus sign get distributed through properly. This is another very common error and one that if you make will often make the problem impossible to complete.

Step 2

Now all that we need to do is some algebra (and it might get a little messy here, but that is somewhat common with these types of problems) and we'll be done.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{2\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - 1 - 2x^3 + 1}{h} \\ &= \lim_{h \to 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 1 - 2x^3 + 1}{h} \\ &= \lim_{h \to 0} \frac{h\left(6x^2 + 6xh + 2h^2\right)}{h} = \lim_{h \to 0} \left(6x^2 + 6xh + 2h^2\right) = 6x^2 \end{aligned}$$

The derivative for this function is then,

$$\boxed{f'(x) = 6x^2}$$

7. Use the definition of the derivative to find the derivative of,

$$g(x) = x^3 - 2x^2 + x - 1$$

Step 1

First we need to plug the function into the definition of the derivative.

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^3 - 2(x+h)^2 + x + h - 1 - \left(x^3 - 2x^2 + x - 1\right)}{h} \end{aligned}$$

Make sure that you properly evaluate the first function evaluation. This is one of the more common errors that students make with these problems.

Also watch for the parenthesis on the second function evaluation. You are subtracting off the whole function and so you need to make sure that you deal with the minus sign properly. Either put in the parenthesis as we've done here or make sure the minus sign get distributed through properly. This is another very common error and one that if you make will often make the problem impossible to complete.

Step 2

Now all that we need to do is some algebra (and it will get a little messy here, but that is somewhat common with these types of problems) and we'll be done.

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 2\left(x^2 + 2xh + h^2\right) + x + h - 1 - \left(x^3 - 2x^2 + x - 1\right)}{h} \\ &= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 2x^2 - 4xh - 2h^2 + x + h - 1 - x^3 + 2x^2 - x + 1}{h} \\ &= \lim_{h \to 0} \frac{h\left(3x^2 + 3xh + h^2 - 4x - 2h + 1\right)}{h} \\ &= \lim_{h \to 0} \left(3x^2 + 3xh + h^2 - 4x - 2h + 1\right) = 3x^2 - 4x + 1 \end{aligned}$$

The derivative for this function is then,

$$\boxed{g'(x) = 3x^2 - 4x + 1}$$

8. Use the definition of the derivative to find the derivative of,

$$R(z) = \frac{5}{z}$$

Step 1

First we need to plug the function into the definition of the derivative.

$$R'(z) = \lim_{h \to 0} \frac{R(z+h) - R(z)}{h} = \lim_{h \to 0} \frac{1}{h}\left(\frac{5}{z+h} - \frac{5}{z}\right)$$

Make sure that you properly evaluate the first function evaluation. This is one of the more common errors that students make with these problems.

Also note that in order to make the problem a little easier to read rewrote the rational expression in the definition a little bit. This doesn't need to be done, but will make things a little nicer to look at.

Step 2

Next we need to combine the two rational expressions into a single rational expression.

$$R'(z) = \lim_{h \to 0} \frac{1}{h}\left(\frac{5z - 5(z+h)}{z(z+h)}\right)$$

Step 3

Now all that we need to do is some algebra and we'll be done.

$$R'(z) = \lim_{h \to 0} \frac{1}{h}\left(\frac{5z - 5z - 5h}{z(z+h)}\right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{-5h}{z(z+h)}\right) = \lim_{h \to 0} \frac{-5}{z(z+h)} = -\frac{5}{z^2}$$

The derivative for this function is then,

$$\boxed{R'(z) = -\frac{5}{z^2}}$$

9. Use the definition of the derivative to find the derivative of,

$$V(t) = \frac{t+1}{t+4}$$

Step 1

First we need to plug the function into the definition of the derivative.

$$V'(t) = \lim_{h\to 0} \frac{V(t+h) - V(t)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{t+h+1}{t+h+4} - \frac{t+1}{t+4}\right)$$

Make sure that you properly evaluate the first function evaluation. This is one of the more common errors that students make with these problems.

Also note that in order to make the problem a little easier to read rewrote the rational expression in the definition a little bit. This doesn't need to be done, but will make things a little nicer to look at.

Step 2

Next we need to combine the two rational expressions into a single rational expression.

$$V'(t) = \lim_{h\to 0} \frac{1}{h}\left(\frac{(t+h+1)(t+4) - (t+1)(t+h+4)}{(t+h+4)(t+4)}\right)$$

Step 3

Now all that we need to do is some algebra (and it will get a little messy here, but that is somewhat common with these types of problems) and we'll be done.

$$\begin{aligned} V'(t) &= \lim_{h\to 0} \frac{1}{h}\left(\frac{t^2 + th + 5t + 4h + 4 - \left(t^2 + th + 5t + h + 4\right)}{(t+h+4)(t+4)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{t^2 + th + 5t + 4h + 4 - t^2 - th - 5t - h - 4}{(t+h+4)(t+4)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{3h}{(t+h+4)(t+4)}\right) = \lim_{h\to 0} \frac{3}{(t+h+4)(t+4)} = \frac{3}{(t+4)^2} \end{aligned}$$

The derivative for this function is then,

$$\boxed{V'(t) = \frac{3}{(t+4)^2}}$$

10. Use the definition of the derivative to find the derivative of,

$$Z(t) = \sqrt{3t-4}$$

Step 1

First we need to plug the function into the definition of the derivative.

$$Z'(t) = \lim_{h \to 0} \frac{Z(t+h) - Z(t)}{h} = \lim_{h \to 0} \frac{\sqrt{3(t+h)-4} - \sqrt{3t-4}}{h}$$

Make sure that you properly evaluate the first function evaluation. This is one of the more common errors that students make with these problems.

Step 2

Next we need to rationalize the numerator.

$$Z'(t) = \lim_{h \to 0} \frac{\left(\sqrt{3(t+h)-4} - \sqrt{3t-4}\right)}{h} \frac{\left(\sqrt{3(t+h)-4} + \sqrt{3t-4}\right)}{\left(\sqrt{3(t+h)-4} + \sqrt{3t-4}\right)}$$

Step 3

Now all that we need to do is some algebra (and it will get a little messy here, but that is somewhat common with these types of problems) and we'll be done.

$$\begin{aligned} Z'(t) &= \lim_{h \to 0} \frac{3(t+h) - 4 - (3t-4)}{h\left(\sqrt{3(t+h)-4} + \sqrt{3t-4}\right)} \\ &= \lim_{h \to 0} \frac{3t + 3h - 4 - 3t + 4}{h\left(\sqrt{3(t+h)-4} + \sqrt{3t-4}\right)} \\ &= \lim_{h \to 0} \frac{3h}{h\left(\sqrt{3(t+h)-4} + \sqrt{3t-4}\right)} = \lim_{h \to 0} \frac{3}{\sqrt{3(t+h)-4} + \sqrt{3t-4}} \\ &= \frac{3}{2\sqrt{3t-4}} \end{aligned}$$

Be careful when multiplying out the numerator here. It is easy to lose track of the minus sign (or parenthesis for that matter) on the second term. This is a very common mistake that students make.

The derivative for this function is then,

$$\boxed{Z'(t) = \frac{3}{2\sqrt{3t-4}}}$$

---

11. Use the definition of the derivative to find the derivative of,

$$f(x) = \sqrt{1-9x}$$

Step 1

First we need to plug the function into the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{1-9(x+h)} - \sqrt{1-9x}}{h}$$

Make sure that you properly evaluate the first function evaluation. This is one of the more common errors that students make with these problems.

Step 2

Next we need to rationalize the numerator.

$$f'(x) = \lim_{h \to 0} \frac{\left(\sqrt{1-9(x+h)} - \sqrt{1-9x}\right)}{h} \frac{\left(\sqrt{1-9(x+h)} + \sqrt{1-9x}\right)}{\left(\sqrt{1-9(x+h)} + \sqrt{1-9x}\right)}$$

Step 3

Now all that we need to do is some algebra (and it will get a little messy here, but that is somewhat common with these types of problems) and we'll be done.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{1-9(x+h)-(1-9x)}{h\left(\sqrt{1-9(x+h)}+\sqrt{1-9x}\right)} \\ &= \lim_{h\to 0} \frac{1-9x-9h-1+9x}{h\left(\sqrt{1-9(x+h)}+\sqrt{1-9x}\right)} \\ &= \lim_{h\to 0} \frac{-9h}{h\left(\sqrt{1-9(x+h)}+\sqrt{1-9x}\right)} \\ &= \lim_{h\to 0} \frac{-9}{\sqrt{1-9(x+h)}+\sqrt{1-9x}} = \frac{-9}{2\sqrt{1-9x}} \end{aligned}$$

Be careful when multiplying out the numerator here. It is easy to lose track of the minus sign (or parenthesis for that matter) on the second term. This is a very common mistake that students make.

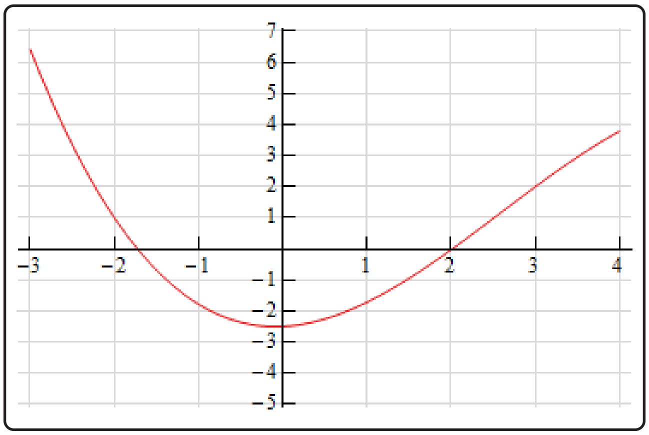
The derivative for this function is then,

$$\boxed{f'(x) = \frac{-9}{2\sqrt{1-9x}}}$$

3.2 Interpretation of the Derivative

1. Use the graph of the function, $f(x)$, estimate the value of $f'(a)$ for

(a) $a = -2$ **(b)** $a = 3$



Hint

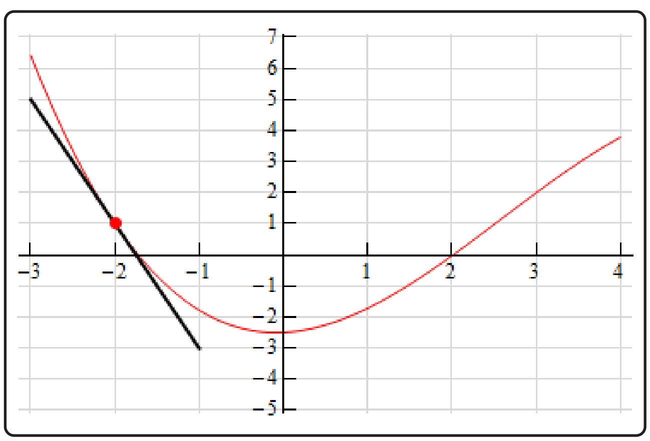
Remember that one of the interpretations of the derivative is the slope of the tangent line to the function.

Solutions

(a) $a = -2$

Step 1

Given that one of the interpretations of the derivative is that it is the slope of the tangent line to the function at a particular point let's first sketch in a tangent line at the point on the graph.



Step 2

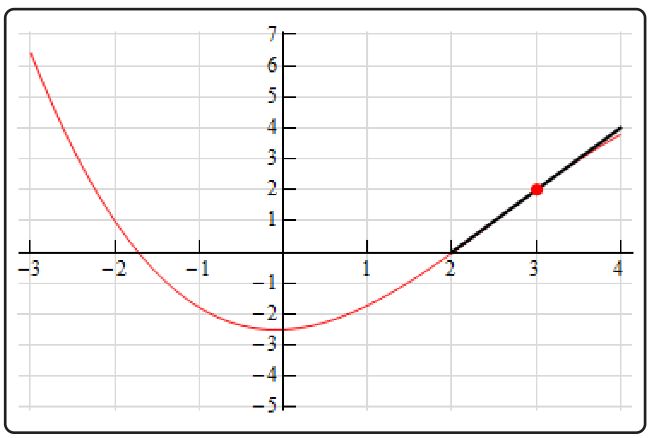
The function is clearly decreasing here and so we know that the derivative at this point will be negative. Now, from this sketch of the tangent line it looks like if we run over 1 we go down 4 and so we can estimate that,

$$\boxed{f'(-2) = -4}$$

(b) $a = 3$

Step 1

Given that one of the interpretations of the derivative is that it is the slope of the tangent line to the function at a particular point. Let's first sketch in a tangent line at the point.



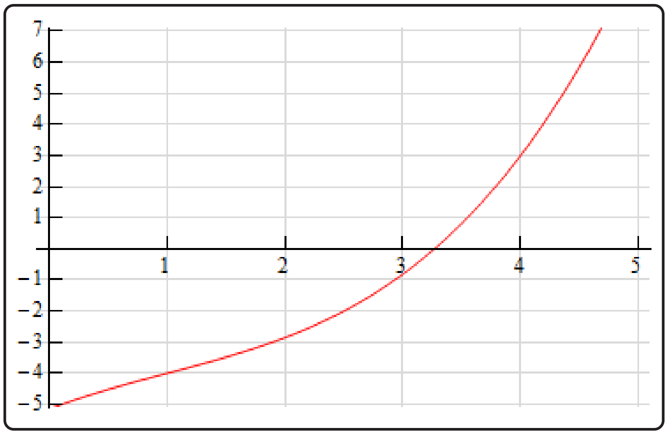
Step 2

The function is clearly increasing here and so we know that the derivative at this point will be positive. Now, from this sketch of the tangent line it looks like if we run over 1 we go up 2 and so we can estimate that,

$$\boxed{f'(3) = 2}$$

2. Use the graph of the function, $f(x)$, estimate the value of $f'(a)$ for

(a) $a = 1$ **(b)** $a = 4$



Hint

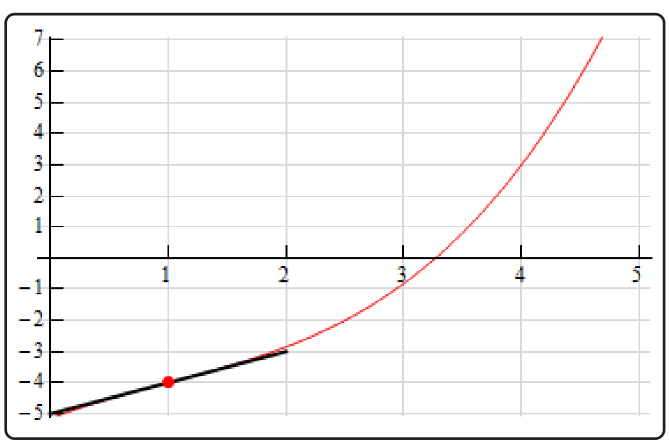
Remember that one of the interpretations of the derivative is the slope of the tangent line to the function.

Solutions

(a) $a = 1$

Step 1

Given that one of the interpretations of the derivative is that it is the slope of the tangent line to the function at a particular point let's first sketch in a tangent line at the point on the graph.



Step 2

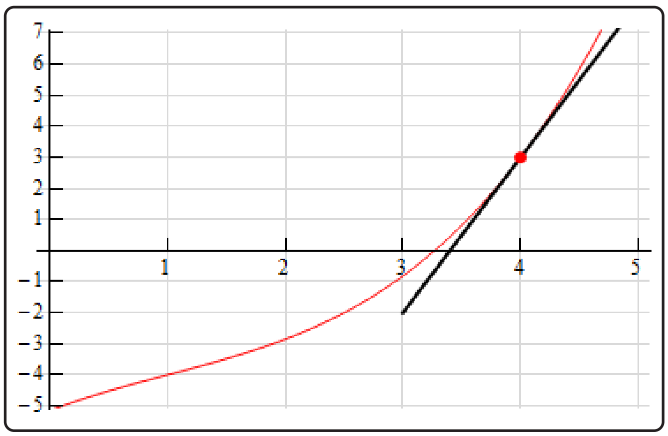
The function is clearly increasing here and so we know that the derivative at this point will be positive. Now, from this sketch of the tangent line it looks like if we run over 1 we go up 1 and so we can estimate that,

$$\boxed{f'(1) = 1}$$

(b) $a = 4$

Step 1

Given that one of the interpretations of the derivative is that it is the slope of the tangent line to the function at a particular point. Let's first sketch in a tangent line at the point.



Step 2

The function is clearly increasing here and so we know that the derivative at this point will be positive. Now, from this sketch of the tangent line it looks like if we run over 1 we go up 5 and so we can estimate that,

$$\boxed{f'(4) = 5}$$

3. Sketch the graph of a function that satisfies $f(1) = 3$, $f'(1) = 1$, $f(4) = 5$, $f'(4) = -2$.

Hint

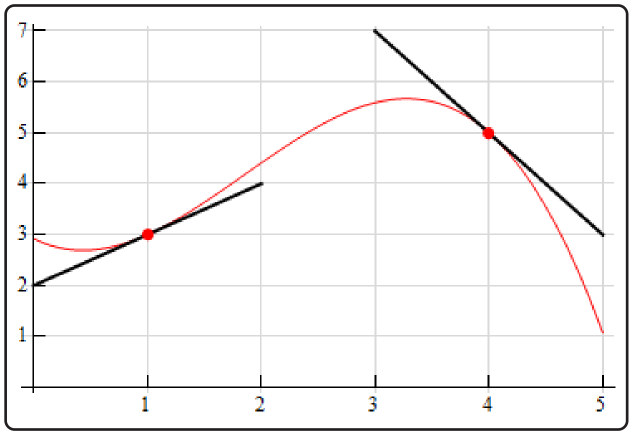
Remember that one of the interpretations of the derivative is the slope of the tangent line to the function.

Step 1

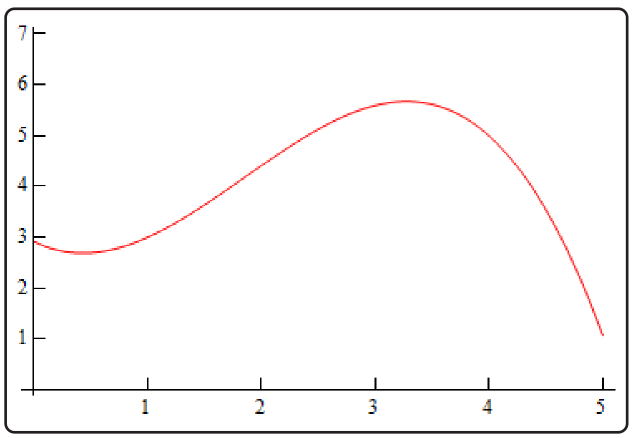
First, recall that one of the interpretations of the derivative is that it is the slope of the tangent line to the function at a particular point. So, let's start off with a graph that has the given points on it and a sketch of a tangent line at the points whose slope is the value of the derivative at the points.

Step 2

Now, all that we need to do is sketch in a graph that goes through the indicated points and at the same time it must be parallel to the tangents that we sketched. There are many possible sketches that we can make here and so don't worry if your sketch is not the same as the one here. This is just one possible sketch that meets the given conditions.



While, it's not really needed here is a sketch of the function without all the extra bits that we put in to help with the sketch.



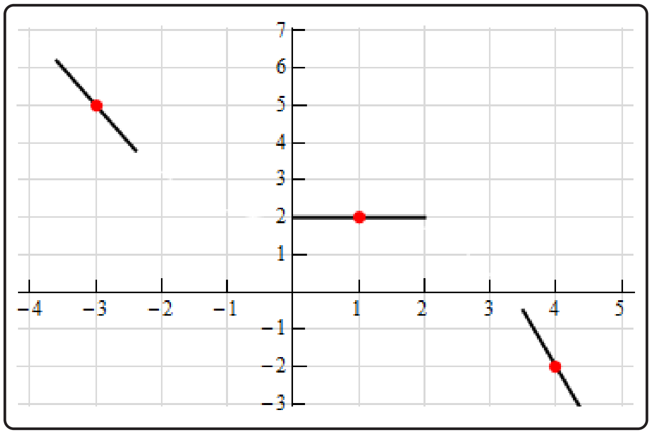
4. Sketch the graph of a function that satisfies $f(-3) = 5$, $f'(-3) = -2$, $f(1) = 2$, $f'(1) = 0$, $f(4) = -2$, $f'(4) = -3$.

Hint

Remember that one of the interpretations of the derivative is the slope of the tangent line to the function.

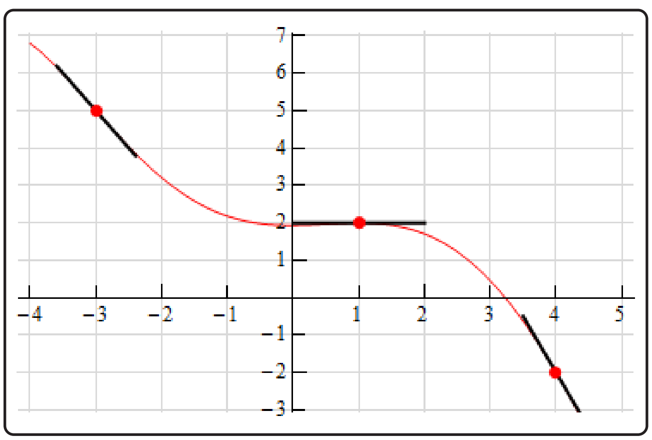
Step 1

First, recall that one of the interpretations of the derivative is that it is the slope of the tangent line to the function at a particular point. So, let's start off with a graph that has the given points on it and a sketch of a tangent line at the points whose slope is the value of the derivative at the points.

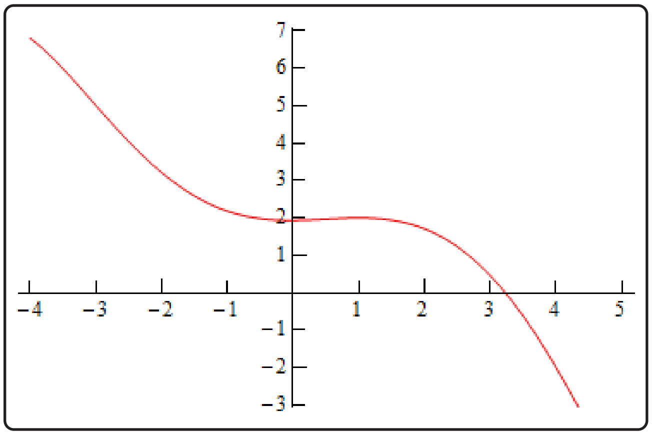


Step 2

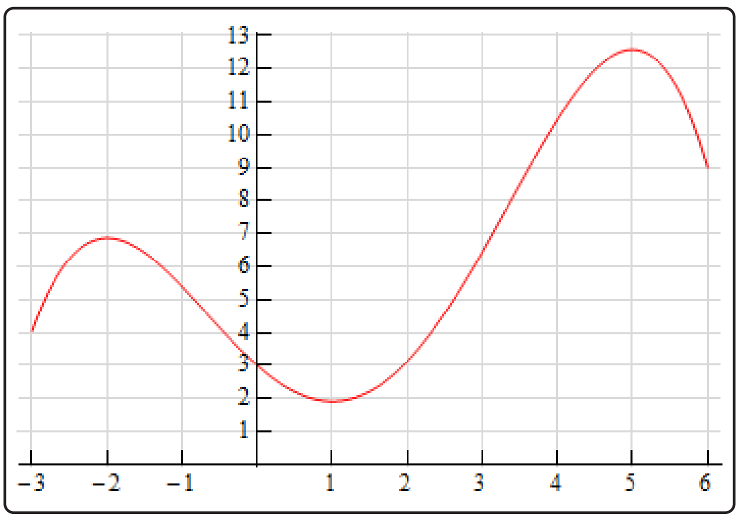
Now, all that we need to do is sketch in a graph that goes through the indicated points and at the same time it must be parallel to the tangents that we sketched. There are many possible sketches that we can make here and so don't worry if your sketch is not the same as the one here. This is just one possible sketch that meets the given conditions.



While, it's not really needed here is a sketch of the function without all the extra bits that we put in to help with the sketch.



5. Below is the graph of some function, $f(x)$. Use this to sketch the graph of the derivative, $f'(x)$.



Hint

Where will the derivative be zero? This gives us a couple of starting points for our sketch.

Step 1

From the graph of the function it is pretty clear that we will have horizontal tangent lines at $x = -2$, $x = 1$ and $x = 5$. Because we will have horizontal tangents here we also know that the derivative at these points must be zero. Therefore, we know the following derivative evaluations.

$$f'(-2) = 0 \qquad f'(1) = 0 \qquad f'(5) = 0$$

Hint

Recall that the derivative can also be used to tell us where the function is increasing and decreasing. Knowing this we can use the graph to determine where the derivative will be positive and where it will be negative.

Step 2

The points we found above break the x-axis up into regions where the function is increasing and decreasing. Recall that if the derivative is positive then the function is increasing and likewise if the derivative is negative then the function is decreasing. Using these ideas we can easily identify the sign of the derivative on each of the regions. Doing this gives,

$$\begin{aligned} x &< -2 & f'(x) &> 0 \\ -2 < x &< 1 & f'(x) &< 0 \\ 1 < x &< 5 & f'(x) &> 0 \\ x &> 5 & f'(x) &< 0 \end{aligned}$$

Hint

At this point all we have to do is try and put all this together and come up with a sketch of the derivative.

Step 3

In the range $x < -2$ we know the derivative must be positive and that it must be zero at $x = -2$ so it makes sense that just to the left of $x = -2$ the derivative must be decreasing.

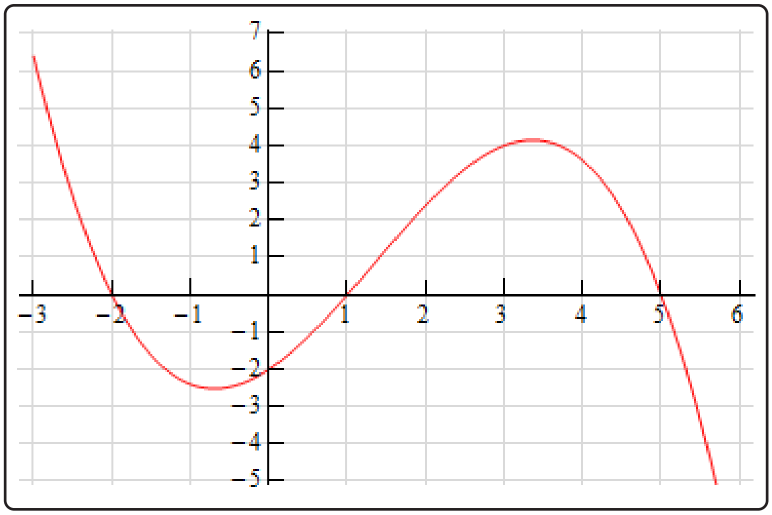
In the range $-2 < x < 1$ we know that the derivative will be negative and that it will be zero at the endpoints of the range. So, to the right of $x = -2$ the derivative will have to be decreasing (goes from zero to a negative number). Likewise, to the left of $x = 1$ the derivative will have to be increasing (goes from a negative number to zero).

Note that we don't really know just how the derivative will behave everywhere in this range, but we can use the general behavior near the endpoints and go with the simplest way to connect the two up to get an idea of what the derivative should look like.

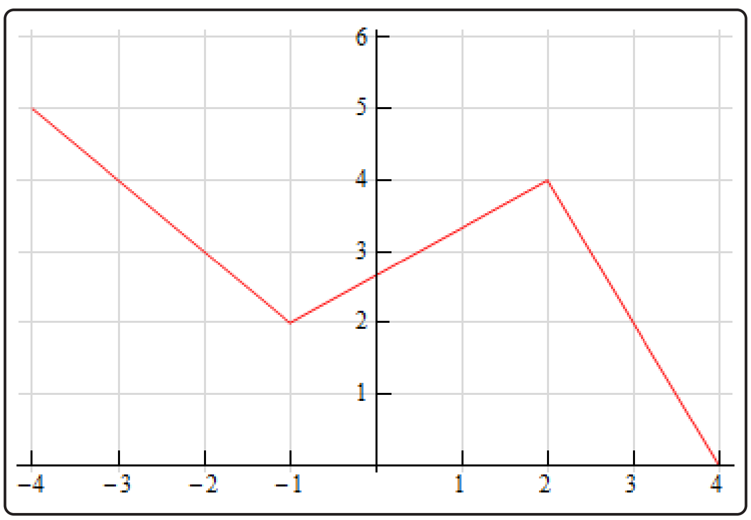
Following similar reasoning we can see that the derivative should be increasing just to the right of $x = 1$ (goes from zero to a positive number), decreasing just to the left of $x = 5$ (goes from a positive number to zero) and decreasing just to the right of $x = 5$ (goes from zero to a negative number).

Step 4

So, putting all of this together here is a sketch of the derivative. Note that we included the a scale on the vertical axis if you would like to try and estimate some specific values of the derivative as we did in Example 4 of this section.



6. Below is the graph of some function, $f(x)$. Use this to sketch the graph of the derivative, $f'(x)$.



Hint

Because the derivative of a function is also the slope of the tangent line. We can therefore determine actual values of the derivative at almost every spot.

Step 1

Because the three portions of the function are actually lines and the tangent line to a line would just be the line itself we can easily compute the derivative on each portion of the curve.

On each of the portions we can use the grid included on the graph to compute the slope of each part. Knowing the slope of the graph on each portion will in turn tell us the slope of the tangent line for each portion. This in turn tells us that the derivative on each of the three portions is then,

$$\begin{array}{rr} x < -1 & f'(x) = -1 \\ -1 < x < 2 & f'(x) = \dfrac{2}{3} \\ x > 2 & f'(x) = -2 \end{array}$$

Hint

What is the derivative at the "sharp points"?

Step 2

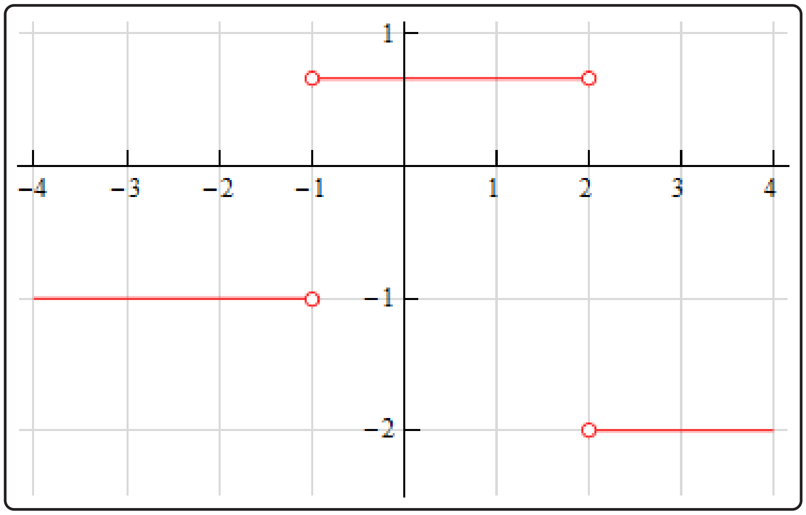
Recall Example 4 from the previous section. In that example we showed that the derivative of the absolute value function does not exist at $x = 0$. The limit on the left side of $x = 0$ (which gives the slope of the line on the left) and the limit on the right side of $x = 0$ (which gives the slope of the line on the right) were different and so the overall limit did not exist. This in turn tells us that the derivative doesn't exist at that point.

Here we have the same problem. We'll leave it to you to verify that the right and left-handed limits at $x = -1$ are not the same and so the derivative does not exist at $x = -1$. Likewise, the derivative does not exist at $x = 2$.

There will therefore be open dots on the graph at these two points.

Step 3

Here is the sketch of the derivative of this function.



7. Answer the following questions about the function $W(z) = 4z^2 - 9z$.

   (a) Is the function increasing or decreasing at $z = -1$?

   (b) Is the function increasing or decreasing at $z = 2$?

   (c) Does the function ever stop changing? If yes, at what value(s) of z does the function stop changing?

Solutions

(a) Is the function increasing or decreasing at $z = -1$?

Solution

We know that the derivative of a function gives us the rate of change of the function and so we'll first need the derivative of this function. We computed this derivative in Problem 5 from the previous section and so we won't show the work here. If you need the practice you should go back and redo that problem before proceeding.

So, from our previous work we know that the derivative is,

$$W'(z) = 8z - 9$$

Now all that we need to do is to compute : $W'(-1) = -17$. This is negative and so we know that the function must be **decreasing** at $z = -1$.

(b) Is the function increasing or decreasing at $z = 2$?

Solution

Again, all we need to do is compute a derivative and since we've got the derivative written down in the first part there's no reason to redo that here.

The evaluation is : $W'(2) = 7$. This is positive and so we know that the function must be **increasing** at $z = 2$.

(c) Does the function ever stop changing? If yes, at what value(s) of z does the function stop changing?

Solution

Here all that we're really asking is if the derivative is ever zero. So we need to solve,

$$W'(z) = 0 \qquad \rightarrow \qquad 8z - 9 = 0 \qquad \Rightarrow \qquad z = \frac{9}{8}$$

So, the function will stop changing at $z = \frac{9}{8}$.

---

8. What is the equation of the tangent line to $f(x) = 3 - 14x$ at $x = 8$.

Solution

We know that the derivative of a function gives us the slope of the tangent line and so we'll first need the derivative of this function. We computed this derivative in Problem 2 from the previous section and so we won't show the work here. If you need the practice you should go back and redo that problem before proceeding.

Note that we did use a different set of letters in the previous problem, but the work is identical. So, from our previous work (with a corresponding change of variables) we know that the derivative is,

$$f'(x) = -14$$

This tells us that the slope of the tangent line at $x = 8$ is then : $m = f'(8) = -14$. We also know that a point on the tangent line is : $(8, f(8)) = (8, -109)$.

The tangent line is then,

$$y = -109 - 14(x - 8) = 3 - 14x$$

Note that, in this case the tangent is the same as the function. This should not be surprising however as the function is a line and so any tangent line (*i.e.* parallel line) will in fact be the same as the line itself.

---

9. The position of an object at any time t is given by $s(t) = \dfrac{t+1}{t+4}$.

   (a) Determine the velocity of the object at any time t.

   (b) Does the object ever stop moving? If yes, at what time(s) does the object stop moving?

Solutions

(a) Determine the velocity of the object at any time t.

Solution

We know that the derivative of a function gives is the velocity of the object and so we'll first need the derivative of this function. We computed this derivative in Problem 9 from the previous section and so we won't show the work here. If you need the practice you should go back and redo that problem before proceeding.

Note that we did use a different letter for the function in the previous problem, but the work is identical. So, from our previous work we know that the derivative is,

$$s'(t) = \frac{3}{(t+4)^2}$$

(b) Does the object ever stop moving? If yes, at what time(s) does the object stop moving?

Solution

We know that the object will stop moving if the velocity (*i.e.* the derivative) is zero. In this case the derivative is a rational expression and clearly the numerator will never be zero. Therefore, the derivative will not be zero and therefore the object **never stops moving**.

---

10. What is the equation of the tangent line to $f(x) = \frac{5}{x}$ at $x = \frac{1}{2}$?

Solution

We know that the derivative of a function gives us the slope of the tangent line and so we'll first need the derivative of this function. We computed this derivative in Problem 8 from the previous section and so we won't show the work here. If you need the practice you should go back and redo that problem before proceeding.

Note that we did use a different set of letters in the previous problem, but the work is identical. So, from our previous work (with a corresponding change of variables) we know that the derivative is,

$$f'(x) = -\frac{5}{x^2}$$

This tells us that the slope of the tangent line at $x = \frac{1}{2}$ is then : $m = f'\left(\frac{1}{2}\right) = -20$. We also know that a point on the tangent line is : $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, 10\right)$.

The tangent line is then,

$$y = 10 - 20\left(x - \frac{1}{2}\right) = 20 - 20x$$

11. Determine where, if anywhere, the function $g(x) = x^3 - 2x^2 + x - 1$ stops changing.

Solution

We know that the derivative of a function gives us the rate of change of the function and so we'll first need the derivative of this function. We computed this derivative in Problem 7 from the previous section and so we won't show the work here. If you need the practice you should go back and redo that problem before proceeding.

From our previous work (with a corresponding change of variables) we know that the derivative is,

$$g'(x) = 3x^2 - 4x + 1$$

If the function stops changing at a point then the derivative will be zero at that point. So, to determine if we function stops changing we will need to solve,

$$\begin{aligned} g'(x) &= 0 \\ 3x^2 - 4x + 1 &= 0 \\ (3x - 1)(x - 1) &= 0 \qquad \Rightarrow \qquad x = \frac{1}{3},\ x = 1 \end{aligned}$$

So, the function will stop changing at $x = \frac{1}{3}$ and $x = 1$.

12. Determine if the function $Z(t) = \sqrt{3t-4}$ increasing or decreasing at the given points.

(a) $t = 5$

(b) $t = 10$

(c) $t = 300$

Solutions

(a) $t = 5$

Solution

We know that the derivative of a function gives us the rate of change of the function and so we'll first need the derivative of this function. We computed this derivative in Problem 10 from the previous section and so we won't show the work here. If you need the practice you should go back and redo that problem before proceeding.

So, from our previous work we know that the derivative is,

$$Z'(t) = \frac{3}{2\sqrt{3t-4}}$$

Now all that we need to do is to compute : $Z'(5) = \frac{3}{2\sqrt{11}}$. This is positive and so we know that the function must be **increasing** at $t = 5$.

(b) $t = 10$

Solution

Again, all we need to do is compute a derivative and since we've got the derivative written down in the first part there's no reason to redo that here. The evaluation is : $Z'(10) = \frac{3}{2\sqrt{26}}$. This is positive and so we know that the function must be **increasing** at $t = 10$.

(c) $t = 300$

Solution

Again, all we need to do is compute a derivative and since we've got the derivative written down in the first part there's no reason to redo that here. The evaluation is : $Z'(300) = \frac{3}{2\sqrt{896}}$. This is positive and so we know that the function must be **increasing** at $t = 300$.

Final Note

As a final note to all the parts of this problem let's notice that we did not really need to do any evaluations. Because we know that square roots will always be positive it is clear that the derivative will always be positive regardless of the value of t we plug in.

---

13. Suppose that the volume of water in a tank for $0 \le t \le 6$ is given by $Q(t) = 10 + 5t - t^2$.

(a) Is the volume of water increasing or decreasing at $t = 0$?

(b) Is the volume of water increasing or decreasing at $t = 6$?

(c) Does the volume of water ever stop changing? If yes, at what times(s) does the volume stop changing?

Solutions

(a) Is the volume of water increasing or decreasing at $t = 0$?

Solution

We know that the derivative of a function gives us the rate of change of the function and so we'll first need the derivative of this function. We computed this derivative in Problem 4 from the previous section and so we won't show the work here. If you need the practice you should go back and redo that problem before proceeding.

So, from our previous work we know that the derivative is,

$$Q'(t) = 5 - 2t$$

Now all that we need to do is to compute : $Q'(0) = 5$. This is positive and so we know that the volume of water in the tank must be **increasing** at $t = 0$.

(b) Is the volume of water increasing or decreasing at $t = 6$?

Solution

Again, all we need to do is compute a derivative and since we've got the derivative written down in the first part there's no reason to redo that here.

The evaluation is : $Q'(6) = -7$. This is negative and so we know that the volume of water in the tank must be **decreasing** at $t = 6$.

(c) Does the volume of water ever stop changing? If yes, at what times(s) does the volume stop changing?

Solution

Here all that we're really asking is if the derivative is ever zero. So we need to solve,

$$Q'(t) = 0 \qquad \to \qquad 5 - 2t = 0 \qquad \Rightarrow \qquad t = \frac{5}{2}$$

So, the volume of water will stop changing at $\frac{5}{2}$.

3.3 Derivative Formulas

1. Find the derivative of $f(x) = 6x^3 - 9x + 4$.

Solution

There isn't much to do here other than take the derivative using the rules we discussed in this section.

$$\boxed{f'(x) = 18x^2 - 9}$$

---

2. Find the derivative of $y = 2t^4 - 10t^2 + 13t$.

Solution

There isn't much to do here other than take the derivative using the rules we discussed in this section.

$$\boxed{\frac{dy}{dt} = 8t^3 - 20t + 13}$$

---

3. Find the derivative of $g(z) = 4z^7 - 3z^{-7} + 9z$.

Solution

There isn't much to do here other than take the derivative using the rules we discussed in this section.

$$\boxed{g'(z) = 28z^6 + 21z^{-8} + 9}$$

---

4. Find the derivative of $h(y) = y^{-4} - 9y^{-3} + 8y^{-2} + 12$.

Solution

There isn't much to do here other than take the derivative using the rules we discussed in this section.

$$\boxed{h'(y) = -4y^{-5} + 27y^{-4} - 16y^{-3}}$$

5. Find the derivative of $y = \sqrt{x} + 8\sqrt[3]{x} - 2\sqrt[4]{x}$.

Solution

There isn't much to do here other than take the derivative using the rules we discussed in this section.

Remember that you'll need to convert the roots to fractional exponents before you start taking the derivative. Here is the rewritten function.

$$y = x^{\frac{1}{2}} + 8x^{\frac{1}{3}} - 2x^{\frac{1}{4}}$$

The derivative is,

$$\boxed{\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{8}{3}x^{-\frac{2}{3}} - \frac{1}{2}x^{-\frac{3}{4}}}$$

6. Find the derivative of $f(x) = 10\sqrt[5]{x^3} - \sqrt{x^7} + 6\sqrt[3]{x^8} - 3$.

Solution

There isn't much to do here other than take the derivative using the rules we discussed in this section.

Remember that you'll need to convert the roots to fractional exponents before you start taking the derivative. Here is the rewritten function.

$$f(x) = 10\left(x^3\right)^{\frac{1}{5}} - \left(x^7\right)^{\frac{1}{2}} + 6\left(x^8\right)^{\frac{1}{3}} - 3 = 10x^{\frac{3}{5}} - x^{\frac{7}{2}} + 6x^{\frac{8}{3}} - 3$$

The derivative is,

$$f'(x) = 10\left(\frac{3}{5}\right)x^{-\frac{2}{5}} - \frac{7}{2}x^{\frac{5}{2}} + 6\left(\frac{8}{3}\right)x^{\frac{5}{3}} = \boxed{6x^{-\frac{2}{5}} - \frac{7}{2}x^{\frac{5}{2}} + 16x^{\frac{5}{3}}}$$

7. Find the derivative of $f\left(t\right) = \dfrac{4}{t} - \dfrac{1}{6t^3} + \dfrac{8}{t^5}$.

Solution

There isn't much to do here other than take the derivative using the rules we discussed in this section.

Remember that you'll need to rewrite the terms so that each of the t's are in the numerator with negative exponents before taking the derivative. Here is the rewritten function.

$$f\left(t\right) = 4t^{-1} - \frac{1}{6}t^{-3} + 8t^{-5}$$

The derivative is,

$$\boxed{f'\left(t\right) = -4t^{-2} + \frac{1}{2}t^{-4} - 40t^{-6}}$$

---

8. Find the derivative of $R\left(z\right) = \dfrac{6}{\sqrt{z^3}} + \dfrac{1}{8z^4} - \dfrac{1}{3z^{10}}$.

Solution

There isn't much to do here other than take the derivative using the rules we discussed in this section.

Remember that you'll need to rewrite the terms so that each of the z's are in the numerator with negative exponents and rewrite the root as a fractional exponent before taking the derivative. Here is the rewritten function.

$$R\left(z\right) = 6z^{-\frac{3}{2}} + \frac{1}{8}z^{-4} - \frac{1}{3}z^{-10}$$

The derivative is,

$$R'\left(z\right) = 6\left(-\frac{3}{2}\right)z^{-\frac{5}{2}} + \frac{1}{8}\left(-4\right)z^{-5} - \frac{1}{3}\left(-10\right)z^{-11} = \boxed{-9z^{-\frac{5}{2}} - \frac{1}{2}z^{-5} + \frac{10}{3}z^{-11}}$$

---

9. Find the derivative of $z = x\left(3x^2 - 9\right)$.

Solution

There isn't much to do here other than take the derivative using the rules we discussed in this section.

Remember that in order to do this derivative we'll first need to multiply the function out before we take the derivative. Here is the rewritten function.

$$z = 3x^3 - 9x$$

The derivative is,

$$\boxed{\frac{dz}{dx} = 9x^2 - 9}$$

---

10. Find the derivative of $g\left(y\right) = \left(y - 4\right)\left(2y + y^2\right)$.

Solution

There isn't much to do here other than take the derivative using the rules we discussed in this section.

Remember that in order to do this derivative we'll first need to multiply the function out before we take the derivative. Here is the rewritten function.

$$g\left(y\right) = y^3 - 2y^2 - 8y$$

The derivative is,

$$\boxed{g'\left(y\right) = 3y^2 - 4y - 8}$$

---

11. Find the derivative of $h\left(x\right) = \dfrac{4x^3 - 7x + 8}{x}$.

Solution

There isn't much to do here other than take the derivative using the rules we discussed in this section.

Remember that in order to do this derivative we'll first need to divide the function out and simplify before we take the derivative. Here is the rewritten function.

$$h(x) = \frac{4x^3}{x} - \frac{7x}{x} + \frac{8}{x} = 4x^2 - 7 + 8x^{-1}$$

The derivative is,

$$\boxed{h'(x) = 8x - 8x^{-2}}$$

---

12. Find the derivative of $f(y) = \dfrac{y^5 - 5y^3 + 2y}{y^3}$.

Solution

There isn't much to do here other than take the derivative using the rules we discussed in this section.

Remember that in order to do this derivative we'll first need to divide the function out and simplify before we take the derivative. Here is the rewritten function.

$$f(y) = \frac{y^5}{y^3} - \frac{5y^3}{y^3} + \frac{2y}{y^3} = y^2 - 5 + 2y^{-2}$$

The derivative is,

$$\boxed{f'(y) = 2y - 4y^{-3}}$$

---

13. Determine where, if anywhere, the function $f(x) = x^3 + 9x^2 - 48x + 2$ is not changing.

Hint

Recall the various interpretations of the derivative. One of them is exactly what we need to do this problem.

Step 1

Recall that one of the interpretations of the derivative is that it gives the rate of change of the function. So, the function won't be changing if its rate of change is zero and so all we need to do is find the derivative and set it equal to zero to determine where the rate of change is zero and hence the function will not be changing.

First the derivative, and we'll do a little factoring while we are at it.

$$f'(x) = 3x^2 + 18x - 48 = 3\left(x^2 + 6x - 16\right) = 3(x+8)(x-2)$$

Step 2

Now all that we need to do is set this equation to zero and solve.

$$f'(x) = 0 \qquad \Rightarrow \qquad 3(x+8)(x-2) = 0$$

We can easily see from this that the derivative will be zero at $x = -8$ and $x = 2$. The function therefore not be changing at,

$$\boxed{x = -8 \qquad \text{and} \qquad x = 2}$$

---

14. Determine where, if anywhere, the function $y = 2z^4 - z^3 - 3z^2$ is not changing.

Hint

Recall the various interpretations of the derivative. One of them is exactly what we need to do this problem.

Step 1

Recall that one of the interpretations of the derivative is that it gives the rate of change of the function. So, the function won't be changing if its rate of change is zero and so all we need to do is find the derivative and set it equal to zero to determine where the rate of change is zero and hence the function will not be changing.

First the derivative, and we'll do a little factoring while we are at it.

$$\frac{dy}{dz} = 8z^3 - 3z^2 - 6z = z\left(8z^2 - 3z - 6\right)$$

Step 2

Now all that we need to do is set this equation to zero and solve.

$$\frac{dy}{dz} = 0$$
$$z\left(8z^2 - 3z - 6\right) = 0 \qquad \rightarrow \qquad z = 0, \quad 8z^2 - 3z - 6 = 0$$

We can easily see from this that the derivative will be zero at $z = 0$, however, because the quadratic doesn't factor we'll need to use the quadratic formula to determine where, if anywhere, that will be zero.

$$z = \frac{3 \pm \sqrt{(-3)^2 - 4(8)(-6)}}{2(8)} = \frac{3 \pm \sqrt{201}}{16}$$

The function therefore not be changing at,

$$\boxed{z = 0 \qquad z = \frac{3 + \sqrt{201}}{16} = 1.07359 \qquad z = \frac{3 - \sqrt{201}}{16} = -0.69859}$$

15. Find the tangent line to $g(x) = \frac{16}{x} - 4\sqrt{x}$ at $x = 4$.

Hint

Recall the various interpretations of the derivative. One of them will help us do this problem.

Step 1

Recall that one of the interpretations of the derivative is that it gives slope of the tangent line to the graph of the function.

So, we'll need the derivative of the function. However before doing that we'll need to do a little rewrite. Here is that work as well as the derivative.

$$g(x) = 16x^{-1} - 4x^{\frac{1}{2}} \qquad \Rightarrow \qquad \boxed{g'(x) = -16x^{-2} - 2x^{-\frac{1}{2}} = -\frac{16}{x^2} - \frac{2}{\sqrt{x}}}$$

Note that we rewrote the derivative back into rational expressions with roots to help with the evaluation.

Step 2

Next we need to evaluate the function and derivative at $x = 4$.

$$g(4) = \frac{16}{4} - 4\sqrt{4} = -4 \qquad\qquad g'(4) = -\frac{16}{4^2} - \frac{2}{\sqrt{4}} = -2$$

Step 3

Now all that we need to do is write down the equation of the tangent line.

$$y = g(4) + g'(4)(x-4) = -4 - 2(x-4) \qquad \to \qquad \boxed{y = -2x + 4}$$

16. Find the tangent line to $f(x) = 7x^4 + 8x^{-6} + 2x$ at $x = -1$.

Hint

Recall the various interpretations of the derivative. One of them will help us do this problem.

Step 1

Recall that one of the interpretations of the derivative is that it gives slope of the tangent line to the graph of the function.

So, we'll need the derivative of the function.

$$\boxed{f'(x) = 28x^3 - 48x^{-7} + 2 = 28x^3 - \frac{48}{x^7} + 2}$$

Note that we rewrote the derivative back into rational expressions help a little with the evaluation.

Step 2

Next we need to evaluate the function and derivative at $x = -1$.

$$f(-1) = 7 + 8 - 2 = 13 \qquad f'(-1) = -28 + 48 + 2 = 22$$

Step 3

Now all that we need to do is write down the equation of the tangent line.

$$y = f(-1) + f'(-1)(x+1) = 13 + 22(x+1) \quad \rightarrow \quad \boxed{y = 22x + 35}$$

---

17. The position of an object at any time t is given by $s(t) = 3t^4 - 40t^3 + 126t^2 - 9$.

(a) Determine the velocity of the object at any time t.

(b) Does the object ever stop changing?

(c) When is the object moving to the right and when is the object moving to the left?

Solutions

Hint

Recall the various interpretations of the derivative. One of them is exactly what we need for this part.

(a) Determine the velocity of the object at any time t.

Solution

Recall that one of the interpretations of the derivative is that it gives the velocity of an object if we know the position function of the object.

We've been given the position function of the object and so all we need to do is find its derivative and we'll have the velocity of the object at any time t.

The velocity of the object is then,

$$\boxed{s'(t) = 12t^3 - 120t^2 + 252t = 12t(t-3)(t-7)}$$

Note that the derivative was factored for later parts and doesn't really need to be done in general.

Hint

If the object isn't moving what is the velocity?

(b) Does the object ever stop changing?

Solution

The object will not be moving if the velocity is ever zero and so all we need to do is set the derivative equal to zero and solve.

$$s'(t) = 0 \qquad \Rightarrow \qquad 12t\,(t-3)\,(t-7) = 0$$

From this it is pretty easy to see that the derivative will be zero, and hence the object will not be moving, at,

$$\boxed{t = 0 \qquad t = 3 \qquad t = 7}$$

Hint

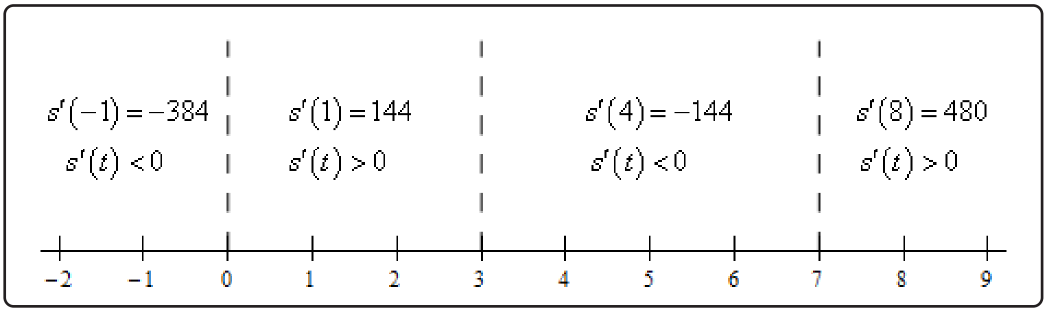
How does the direction (right vs. left) of movement relate to the sign (positive or negative) of the derivative?

(c) When is the object moving to the right and when is the object moving to the left?

Solution

To answer this part all we need to know is where the derivative is positive (and hence the object is moving to the right) or negative (and hence the object is moving to the left). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.



From this we get the following right/left movement information.

$$\begin{aligned} \text{Moving Right : } & 0 < t < 3, \quad 7 < t < \infty \\ \text{Moving Left : } & -\infty < t < 0, \quad 3 < t < 7 \end{aligned}$$

Note that depending upon your interpretation of t as time you may or may not have included the interval $-\infty < t < 0$ in the "Moving Left" portion.

18. Determine where the function $h(z) = 6 + 40z^3 - 5z^4 - 4z^5$ is increasing and decreasing.

Hint

Recall the various interpretations of the derivative. One of them is exactly what we need to get the problem started.

Step 1

Recall that one of the interpretations of the derivative is that it gives the rate of change of the function. Since we are talking about where the function is increasing and decreasing we are clearly talking about the rate of change of the function.

So, we'll need the derivative.

$$h'(z) = 120z^2 - 20z^3 - 20z^4 = -20z^2(z+3)(z-2)$$

Note that the derivative was factored for later steps and doesn't really need to be done in general.

Hint

Where is the function not changing?

Step 2

Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve.

$$h'(z) = 0 \qquad \Rightarrow \qquad -20z^2(z+3)(z-2) = 0$$

From this it is pretty easy to see that the derivative will be zero, and hence the function will not be moving, at,

$$\boxed{z = 0 \qquad z = -3 \qquad z = 2}$$

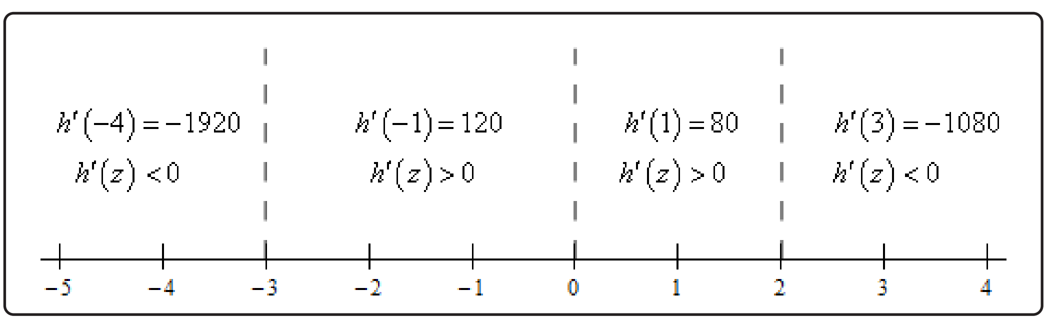
Hint

How does the increasing/decreasing behavior of the function relate to the sign (positive or negative) of the derivative?

Step 3

To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.



From this we get the following increasing/decreasing information.

$$\boxed{\begin{aligned} \text{Increasing :} &\quad -3 < z < 0, \quad 0 < z < 2 \\ \text{Decreasing :} &\quad -\infty < z < -3, \quad 2 < z < \infty \end{aligned}}$$

---

19. Determine where the function $R(x) = (x+1)(x-2)^2$ is increasing and decreasing.

Hint

Recall the various interpretations of the derivative. One of them is exactly what we need to get the problem started.

Step 1

Recall that one of the interpretations of the derivative is that it gives the rate of change of the function. Since we are talking about where the function is increasing and decreasing we are clearly talking about the rate of change of the function.

So, we'll need the derivative. First however we'll need to multiply out the function so we can actually take the derivative. Here is the rewritten function and the derivative.

$$R(x) = x^3 - 3x^2 + 4 \qquad \boxed{R'(x) = 3x^2 - 6x = 3x(x-2)}$$

Note that the derivative was factored for later steps and doesn't really need to be done in general.

Hint

Where is the function not changing?

Step 2

Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve.

$$R'(x) = 0 \qquad \Rightarrow \qquad 3x(x-2) = 0$$

From this it is pretty easy to see that the derivative will be zero, and hence the function will not be moving, at,

$$\boxed{x = 0 \qquad x = 2}$$

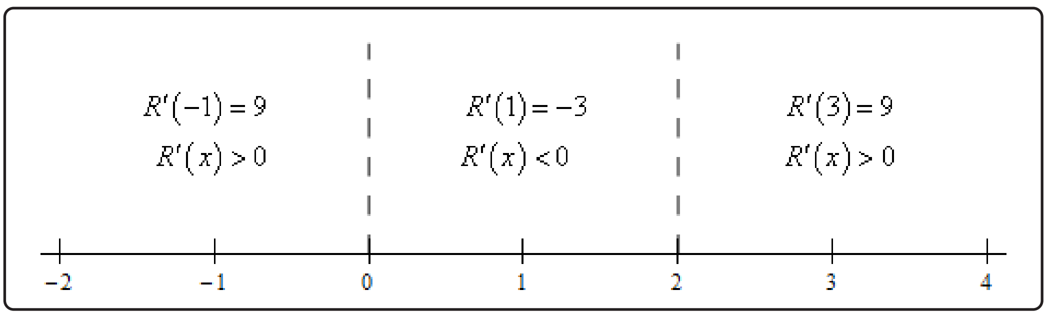
Hint

How does the increasing/decreasing behavior of the function relate to the sign (positive or negative) of the derivative?

Step 3

To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.



From this we get the following increasing/decreasing information.

Increasing : $-\infty < x < 0, \quad 2 < x < \infty$
Decreasing : $0 < x < 2$

---

20. Determine where, if anywhere, the tangent line to $f(x) = x^3 - 5x^2 + x$ is parallel to the line $y = 4x + 23$.

Step 1

The first thing that we'll need of course is the slope of the tangent line. So, all we need to do is take the derivative of the function.

$$\boxed{f'(x) = 3x^2 - 10x + 1}$$

Hint

What is the relationship between the slope of two parallel lines?

Step 2

Two lines that are parallel will have the same slope and so all we need to do is determine where the slope of the tangent line will be 4, the slope of the given line. In other words, we'll need to solve,

$$f'(x) = 4 \qquad \to \qquad 3x^2 - 10x + 1 = 4 \qquad \to \qquad 3x^2 - 10x - 3 = 0$$

This quadratic doesn't factor and so a quick use of the quadratic formula will solve this for us.

$$x = \frac{10 \pm \sqrt{136}}{6} = \frac{10 \pm 2\sqrt{34}}{6} = \frac{5 \pm \sqrt{34}}{3}$$

So, the tangent line will be parallel to $y = 4x + 23$ at,

$$\boxed{x = \frac{5 - \sqrt{34}}{3} = -0.276984 \qquad x = \frac{5 + \sqrt{34}}{3} = 3.61032}$$

3.4 Product & Quotient Rule

1. Use the Product Rule to find the derivative of $f(t) = \left(4t^2 - t\right)\left(t^3 - 8t^2 + 12\right)$.

Solution

There isn't much to do here other than take the derivative using the product rule.

$$\begin{aligned} f'(t) &= (8t-1)\left(t^3 - 8t^2 + 12\right) + \left(4t^2 - t\right)\left(3t^2 - 16t\right) \\ &= 20t^4 - 132t^3 + 24t^2 + 96t - 12 \end{aligned}$$

Note that we multiplied everything out to get a "simpler" answer.

---

2. Use the Product Rule to find the derivative of $y = \left(1 + \sqrt{x^3}\right)\left(x^{-3} - 2\sqrt[3]{x}\right)$.

Solution

There isn't much to do here other than take the derivative using the product rule. We'll also need to convert the roots to fractional exponents.

$$y = \left(1 + x^{\frac{3}{2}}\right)\left(x^{-3} - 2x^{\frac{1}{3}}\right)$$

The derivative is then,

$$\begin{aligned} \frac{dy}{dx} &= \left(\frac{3}{2}x^{\frac{1}{2}}\right)\left(x^{-3} - 2x^{\frac{1}{3}}\right) + \left(1 + x^{\frac{3}{2}}\right)\left(-3x^{-4} - \frac{2}{3}x^{-\frac{2}{3}}\right) \\ &= -3x^{-4} - \frac{3}{2}x^{-\frac{5}{2}} - \frac{2}{3}x^{-\frac{2}{3}} - \frac{11}{3}x^{\frac{5}{6}} \end{aligned}$$

Note that we multiplied everything out to get a "simpler" answer.

---

3. Use the Product Rule to find the derivative of $h(z) = \left(1 + 2z + 3z^2\right)\left(5z + 8z^2 - z^3\right)$.

Solution

There isn't much to do here other than take the derivative using the product rule.

$$\begin{aligned} h'(z) &= (2 + 6z)\left(5z + 8z^2 - z^3\right) + \left(1 + 2z + 3z^2\right)\left(5 + 16z - 3z^2\right) \\ &= 5 + 36z + 90z^2 + 88z^3 - 15z^4 \end{aligned}$$

Note that we multiplied everything out to get a "simpler" answer.

4. Use the Quotient Rule to find the derivative of $g\left(x\right)=\dfrac{6x^2}{2-x}$.

Solution

There isn't much to do here other than take the derivative using the quotient rule.

$$g'\left(x\right)=\frac{12x\left(2-x\right)-6x^2\left(-1\right)}{\left(2-x\right)^2}=\boxed{\frac{24x-6x^2}{\left(2-x\right)^2}}$$

5. Use the Quotient Rule to find the derivative of $R\left(w\right)=\dfrac{3w+w^4}{2w^2+1}$.

Solution

There isn't much to do here other than take the derivative using the quotient rule.

$$R'\left(w\right)=\frac{\left(3+4w^3\right)\left(2w^2+1\right)-\left(3w+w^4\right)\left(4w\right)}{\left(2w^2+1\right)^2}=\boxed{\frac{4w^5+4w^3-6w^2+3}{\left(2w^2+1\right)^2}}$$

6. Use the Quotient Rule to find the derivative of $f\left(x\right)=\dfrac{\sqrt{x}+2x}{7x-4x^2}$.

Solution

There isn't much to do here other than take the derivative using the quotient rule.

$$f'\left(x\right)=\frac{\left(\frac{1}{2}x^{-\frac{1}{2}}+2\right)\left(7x-4x^2\right)-\left(x^{\frac{1}{2}}+2x\right)\left(7-8x\right)}{\left(7x-4x^2\right)^2}$$

7. If $f\left(2\right)=-8$, $f'\left(2\right)=3$, $g\left(2\right)=17$ and $g'\left(2\right)=-4$ determine the value of $\left(f\,g\right)'\left(2\right)$.

Solution

We know that the product rule is,

$$\left(f\,g\right)'\left(x\right)=f'\left(x\right)g\left(x\right)+f\left(x\right)g'\left(x\right)$$

Now, we want to know the value of this at $x = 2$ and so all we need to do is plug this into the derivative. Doing this gives,

$$(f\,g)'(2) = f'(2)\,g(2) + f(2)\,g'(2)$$

Now, we were given values for all these quantities and so all we need to do is plug these into our "formula" above.

$$(f\,g)'(2) = (3)(17) + (-8)(-4) = \boxed{83}$$

---

8. If $f(x) = x^3 g(x)$, $g(-7) = 2$, $g'(-7) = -9$ determine the value of $f'(-7)$.

Hint

Even though we don't know what $g(x)$ is we can still use the product rule to take the derivative and then we can use the given information to get the value of $f'(-7)$.

Solution

Even though we don't know what $g(x)$ is we do have a product of two functions here and so we can use the product rule to determine the derivative of $f(x)$.

$$f'(x) = 3x^2 g(x) + x^3 g'(x)$$

Now all we need to do is plug $x = -7$ into this and use the given information to determine the value of $f'(-7)$.

$$f'(-7) = 3(-7)^2 g(-7) + (-7)^3 g'(-7) = 3(49)(2) + (-343)(-9) = \boxed{3381}$$

---

9. Find the equation of the tangent line to $f(x) = (1 + 12\sqrt{x})(4 - x^2)$ at $x = 9$.

Step 1

We know that the derivative of the function will give us the slope of the tangent line so we'll need the derivative of the function. We'll use the product rule to get the derivative.

$$f'(x) = \left(6x^{-\frac{1}{2}}\right)(4 - x^2) + (1 + 12\sqrt{x})(-2x) = \left(\frac{6}{\sqrt{x}}\right)(4 - x^2) - 2x(1 + 12\sqrt{x})$$

Step 2

Note that we didn't bother to "simplify" the derivative (other than converting the fractional exponent back to a root) because all we really need this for is a quick evaluation.

Speaking of which here are the evaluations that we'll need for this problem.

$$f(9) = (37)(-77) = -2849 \qquad f'(9) = (2)(-77) - 18(37) = -820$$

Step 3

Now all that we need to do is write down the equation of the tangent line.

$$y = f(9) + f'(9)(x-9) = -2849 - 820(x-9) \quad \rightarrow \quad \boxed{y = -820x + 4531}$$

10. Determine where $f(x) = \dfrac{x - x^2}{1 + 8x^2}$ is increasing and decreasing.

Step 1

We'll first need the derivative, which will require the quotient rule, because we know that the derivative will give us the rate of change of the function. Here is the derivative.

$$\boxed{f'(x) = \frac{(1-2x)\left(1+8x^2\right) - \left(x - x^2\right)(16x)}{\left(1+8x^2\right)^2} = \frac{1 - 2x - 8x^2}{\left(1+8x^2\right)^2}}$$

Step 2

Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve. In this case it is clear that the denominator will never be zero for any real number and so the derivative will only be zero where the numerator is zero. Therefore, setting the numerator equal to zero and solving gives,

$$1 - 2x - 8x^2 = -\left(8x^2 + 2x - 1\right) = -(4x-1)(2x+1) = 0$$

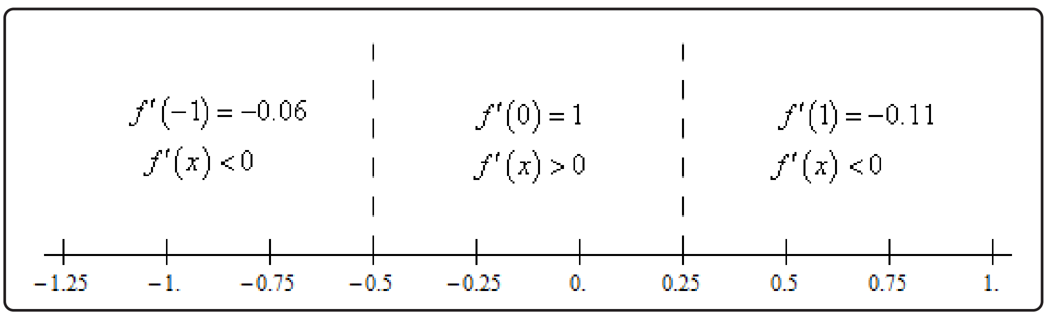
From this it is pretty easy to see that the derivative will be zero, and hence the function will not be changing, at,

$$x = -\frac{1}{2} \qquad x = \frac{1}{4}$$

Step 3

To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.



From this we get the following increasing/decreasing information.

$$\begin{aligned} \text{Increasing :} \quad & -\frac{1}{2} < x < \frac{1}{4} \\ \text{Decreasing :} \quad & -\infty < x < -\frac{1}{2}, \quad \frac{1}{4} < x < \infty \end{aligned}$$

11. Determine where $V(t) = (4 - t^2)(1 + 5t^2)$ is increasing and decreasing.

Step 1

We'll first need the derivative, for which we will use the product rule, because we know that the derivative will give us the rate of change of the function. Here is the derivative.

$$V'(t) = (-2t)(1 + 5t^2) + (4 - t^2)(10t) = 38t - 20t^3 = 2t(19 - 10t^2)$$

Step 2

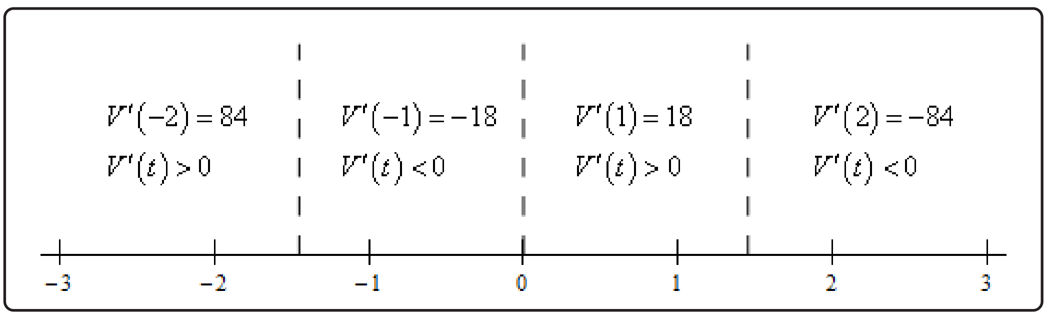
Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve. From the factored form of the derivative it is easy to see that the derivative will be zero at,

$$t = 0 \qquad t = \pm\sqrt{\frac{19}{10}} = \pm 1.3784$$

Step 3

To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.



From this we get the following increasing/decreasing information.

$$\text{Increasing : } -\infty < t < -\sqrt{\frac{19}{10}}, \quad 0 < t < \sqrt{\frac{19}{10}}$$

$$\text{Decreasing : } -\sqrt{\frac{19}{10}} < t < 0, \quad \sqrt{\frac{19}{10}} < t < \infty$$

3.5 Derivatives of Trig Functions

1. Evaluate $\displaystyle\lim_{z\to 0}\frac{\sin(10z)}{z}$.

Solution

All we need to do is set this up to allow us to use the fact from the notes in this section.

$$\lim_{z\to 0}\frac{\sin(10z)}{z}=\lim_{z\to 0}\frac{10\sin(10z)}{10z}=10\lim_{z\to 0}\frac{\sin(10z)}{10z}=10(1)=\boxed{10}$$

2. Evaluate $\displaystyle\lim_{\alpha\to 0}\frac{\sin(12\alpha)}{\sin(5\alpha)}$.

Solution

All we need to do is set this up to allow us to use the fact from the notes in this section.

$$\begin{aligned}\lim_{\alpha\to 0}\frac{\sin(12\alpha)}{\sin(5\alpha)}&=\lim_{\alpha\to 0}\left[\frac{12\alpha\sin(12\alpha)}{12\alpha}\frac{5\alpha}{5\alpha\sin(5\alpha)}\right]=\lim_{\alpha\to 0}\left[\frac{12\alpha}{5\alpha}\frac{\sin(12\alpha)}{12\alpha}\frac{5\alpha}{\sin(5\alpha)}\right]\\&=\lim_{\alpha\to 0}\left[\frac{12}{5}\frac{\sin(12\alpha)}{12\alpha}\frac{5\alpha}{\sin(5\alpha)}\right]=\frac{12}{5}\left[\lim_{\alpha\to 0}\frac{\sin(12\alpha)}{12\alpha}\right]\left[\lim_{\alpha\to 0}\frac{5\alpha}{\sin(5\alpha)}\right]\\&=\frac{12}{5}(1)(1)=\boxed{\frac{12}{5}}\end{aligned}$$

3. Evaluate $\displaystyle\lim_{x\to 0}\frac{\cos(4x)-1}{x}$.

Solution

All we need to do is set this up to allow us to use the fact from the notes in this section.

$$\lim_{x\to 0}\frac{\cos(4x)-1}{x}=\lim_{x\to 0}\frac{4(\cos(4x)-1)}{4x}=4\lim_{x\to 0}\frac{\cos(4x)-1}{4x}=4(0)=\boxed{0}$$

4. Differentiate $f(x) = 2\cos(x) - 6\sec(x) + 3$.

Solution

Not much to do here other than take the derivative.

$$f'(x) = -2\sin(x) - 6\sec(x)\tan(x)$$

---

5. Differentiate $g(z) = 10\tan(z) - 2\cot(z)$.

Solution

Not much to do here other than take the derivative.

$$g'(z) = 10\sec^2(z) + 2\csc^2(z)$$

---

6. Differentiate $f(w) = \tan(w)\sec(w)$.

Solution

Not much to do here other than take the derivative, which will require the product rule.

$$\begin{aligned} f'(w) &= \left[\sec^2(w)\right]\sec(w) + \tan(w)\left[\sec(w)\tan(w)\right] \\ &= \boxed{\sec^3(w) + \sec(w)\tan^2(w)} \end{aligned}$$

---

7. Differentiate $h(t) = t^3 - t^2\sin(t)$.

Solution

Not much to do here other than take the derivative, which will require the product rule for the second term.

You'll need to be careful with the minus sign on the second term. You can either use a set of parentheses around the derivative of the second term or you can think of the minus sign as part of the "first" function. We'll think of the minus sign as part of the first function for this problem.

$$\boxed{h'(t) = 3t^2 - 2t\sin(t) - t^2\cos(t)}$$

8. Differentiate $y = 6 + 4\sqrt{x}\,\csc(x)$.

Solution

Not much to do here other than take the derivative, which will require the product rule for the second term.

$$\begin{aligned} y' &= 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}}\csc(x) + 4\sqrt{x}\,(-\csc(x)\cot(x)) \\ &= \boxed{2x^{-\frac{1}{2}}\csc(x) - 4\sqrt{x}\,\csc(x)\cot(x)} \end{aligned}$$

9. Differentiate $R(t) = \dfrac{1}{2\sin(t) - 4\cos(t)}$.

Solution

Not much to do here other than take the derivative, which will require the quotient rule.

$$\begin{aligned} R'(t) &= \frac{(0)(2\sin(t) - 4\cos(t)) - (1)(2\cos(t) + 4\sin(t))}{(2\sin(t) - 4\cos(t))^2} \\ &= \boxed{\frac{-2\cos(t) - 4\sin(t)}{(2\sin(t) - 4\cos(t))^2}} \end{aligned}$$

10. Differentiate $Z(v) = \dfrac{v + \tan(v)}{1 + \csc(v)}$.

Solution

Not much to do here other than take the derivative, which will require the quotient rule.

$$\begin{aligned} Z'(v) &= \frac{(1 + \sec^2(v))(1 + \csc(v)) - (v + \tan(v))(-\csc(v)\cot(v))}{(1 + \csc(v))^2} \\ &= \boxed{\frac{(1 + \sec^2(v))(1 + \csc(v)) + \csc(v)\cot(v)(v + \tan(v))}{(1 + \csc(v))^2}} \end{aligned}$$

11. Find the tangent line to $f(x) = \tan(x) + 9\cos(x)$ at $x = \pi$.

Step 1

We know that the derivative of the function will give us the slope of the tangent line so we'll need the derivative of the function.

$$f'(x) = \sec^2(x) - 9\sin(x)$$

Step 2

Now all we need to do is evaluate the function and the derivative at the point in question.

$$f(\pi) = \tan(\pi) + 9\cos(\pi) = -9 \qquad f'(\pi) = \sec^2(\pi) - 9\sin(\pi) = 1$$

Step 3

Now all that we need to do is write down the equation of the tangent line.

$$y = f(\pi) + f'(\pi)(x - \pi) = -9 + (1)(x - \pi) \qquad \rightarrow \qquad \boxed{y = x - \pi - 9}$$

Don't get excited about the presence of the π in the answer. It is just a number like the 9 is and so is nothing to worry about.

---

12. The position of an object is given by $s(t) = 2 + 7\cos(t)$ determine all the points where the object is not moving.

Solution

We know that the object will not be moving if its velocity, which is simply the derivative of the position function, is zero. So, all we need to do is take the derivative, set it equal to zero and solve.

$$s'(t) = -7\sin(t) \qquad \Rightarrow \qquad -7\sin(t) = 0$$

So, for this problem the object will not be moving anywhere that sine is zero. From our recollection of the unit circle we know that will be at,

$$\boxed{t = 0 + 2\pi n = 2\pi n \qquad \text{and} \qquad t = \pi + 2\pi n \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots}$$

---

13. Where in the range $[-2, 7]$ is the function $f(x) = 4\cos(x) - x$ is increasing and decreasing.

Step 1

We'll first need the derivative because we know that the derivative will give us the rate of change of the function. Here is the derivative.

$$\boxed{f'(x) = -4\sin(x) - 1}$$

Step 2

Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve.

$$-4\sin(x) - 1 = 0 \qquad \Rightarrow \qquad \sin(x) = -\frac{1}{4}$$

A quick calculator computation tells us that,

$$x = \sin^{-1}\left(-\frac{1}{4}\right) = -0.2527$$

Recalling our work in the Review chapter on solving trig equations we know that a positive angle corresponding to this solution is : $x = 2\pi - 0.2527 = 6.0305$. Either can be used, but we will use the positive angle.

Also, from a quick check on a unit circle we can see that $x = \pi + 0.2527 = 3.3943$ will be a second solution.

Putting all of this together and we can see that the derivative will be zero at,

$$x = 6.0305 + 2\pi n \qquad \text{and} \qquad x = 3.3943 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Finally, all we need to so is plug in some n's to determine which solutions fall in the interval we are working on, $[-2, 7]$.

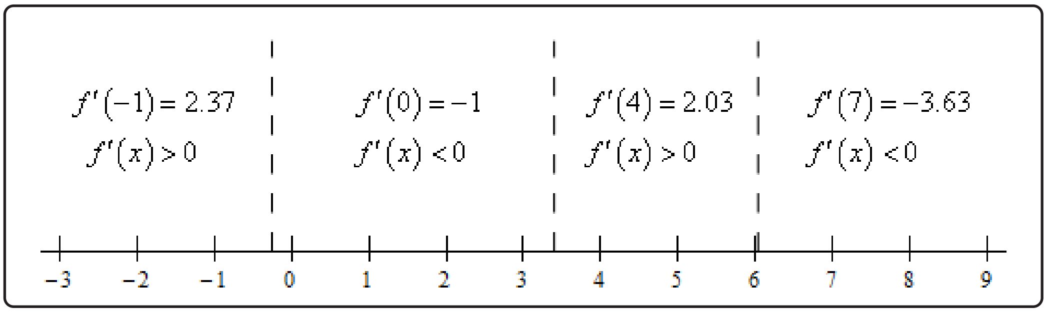
So, in the interval $[-2, 7]$ the function will stop changing at the following three points.

$$x = -0.2527, \quad 3.3943, \quad 6.0305$$

Step 3

To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.



From this we get the following increasing/decreasing information.

$$\begin{aligned} \text{Increasing : } & -2 \le x < -0.2527, \quad 3.3943 < x < 6.0305 \\ \text{Decreasing : } & -0.2527 < x < 3.3943, \quad 6.0305 < x \le 7 \end{aligned}$$

Note that because we've only looked at what is happening in the interval $[-2,7]$ we can't say anything about the increasing/decreasing nature of the function outside of this interval.

3.6 Derivatives of Exponential & Logarithm Functions

1. Differentiate $f(x) = 2\mathbf{e}^x - 8^x$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$\boxed{f'(x) = 2\mathbf{e}^x - 8^x \ln(8)}$$

---

2. Differentiate $g(t) = 4\log_3(t) - \ln(t)$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$\boxed{g'(t) = \frac{4}{t\ln(3)} - \frac{1}{t}}$$

---

3. Differentiate $R(w) = 3^w \log(w)$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$\boxed{R'(w) = 3^w \ln(3)\log(w) + \frac{3^w}{w\ln(10)}}$$

Recall that $\log(x)$ is the common logarithm and so is really $\log_{10}(x)$.

---

4. Differentiate $y = z^5 - \mathbf{e}^z \ln(z)$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$\boxed{y' = 5z^4 - \mathbf{e}^z \ln(z) - \frac{\mathbf{e}^z}{z}}$$

5. Differentiate $h(y) = \dfrac{y}{1 - \mathbf{e}^y}$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$h'(y) = \frac{(1)(1 - \mathbf{e}^y) - y(-\mathbf{e}^y)}{(1 - \mathbf{e}^y)^2} = \boxed{\frac{1 - \mathbf{e}^y + y\,\mathbf{e}^y}{(1 - \mathbf{e}^y)^2}}$$

6. Differentiate $f(t) = \dfrac{1 + 5t}{\ln(t)}$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$f(t) = \frac{1 + 5t}{\ln(t)} = \frac{5\ln(t) - (1 + 5t)\left(\frac{1}{t}\right)}{[\ln(t)]^2} = \boxed{\frac{5\ln(t) - \frac{1}{t} - 5}{[\ln(t)]^2}}$$

7. Find the tangent line to $f(x) = 7^x + 4\mathbf{e}^x$ at $x = 0$.

Step 1

We know that the derivative of the function will give us the slope of the tangent line so we'll need the derivative of the function.

$$f'(x) = 7^x \ln(7) + 4\mathbf{e}^x$$

Step 2

Now all we need to do is evaluate the function and the derivative at the point in question.

$$f(0) = 5 \qquad f'(0) = \ln(7) + 4 = 5.9459$$

Step 3

Now all that we need to do is write down the equation of the tangent line.

$$y = f(0) + f'(0)(x - 0) = \boxed{5 + (\ln(7) + 4)x = 5 + 5.9459x}$$

8. Find the tangent line to $f(x) = \ln(x)\log_2(x)$ at $x = 2$.

Step 1

We know that the derivative of the function will give us the slope of the tangent line so we'll need the derivative of the function.

$$f'(x) = \frac{\log_2(x)}{x} + \frac{\ln(x)}{x\ln(2)}$$

Step 2

Now all we need to do is evaluate the function and the derivative at the point in question.

$$f(2) = \ln(2)\log_2(2) = \ln(2) \qquad f'(2) = \frac{\log_2(2)}{2} + \frac{\ln(2)}{2\ln(2)} = 1$$

Step 3

Now all that we need to do is write down the equation of the tangent line.

$$y = f(2) + f'(2)(x - 2) = \ln(2) + (1)(x - 2) = \boxed{x - 2 + \ln(2)}$$

9. Determine if $V(t) = \dfrac{t}{\mathbf{e}^t}$ is increasing or decreasing at the following points.

(a) $t = -4$ **(b)** $t = 0$ **(c)** $t = 10$

Solutions

(a) $t = -4$

Solution

We know that the derivative of the function will give us the rate of change for the function and so we'll need that.

$$V'(t) = \frac{(1)\mathbf{e}^t - t(\mathbf{e}^t)}{(\mathbf{e}^t)^2} = \frac{\mathbf{e}^t - t\mathbf{e}^t}{(\mathbf{e}^t)^2} = \boxed{\frac{1-t}{\mathbf{e}^t}}$$

Now, all we need to do is evaluate the derivative at the point in question. So,

$$V'(-4) = \frac{5}{\mathbf{e}^{-4}} = 272.991 > 0$$

$V'(-4) > 0$ and so the function must be **increasing** at $t = -4$.

(b) $t = 0$

Solution

We found the derivative of the function in the first part so here all we need to do is the evaluation.

$$V'(0) = \frac{1}{\mathbf{e}^0} = 1 > 0$$

$V'(0) > 0$ and so the function must be **increasing** at $t = 0$.

(c) $t = 10$

Solution

We found the derivative of the function in the first part so here all we need to do is the evaluation.

$$V'(10) = \frac{-9}{\mathbf{e}^{10}} = -0.0004086 < 0$$

$V'(10) < 0$ and so the function must be **decreasing** at $t = 10$.

10. Determine if $G(z) = (z-6)\ln(z)$ is increasing or decreasing at the following points.

(a) $z = 1$ tasks $z = 5$ tasks
$z = 20$

Solutions

(a) $z = 1$

Solution

We know that the derivative of the function will give us the rate of change for the function and so we'll need that.

$$G'(z) = \ln(z) + \frac{z-6}{z}$$

Now, all we need to do is evaluate the derivative at the point in question. So,

$$G'(1) = \ln(1) - 5 = -5 < 0$$

$G'(1) < 0$ and so the function must be **decreasing** at $z = 1$.

(b) $z = 5$

Solution

We found the derivative of the function in the first part so here all we need to do is the evaluation.

$$G'(5) = \ln(5) - \frac{1}{5} = 1.40944 > 0$$

$G'(5) > 0$ and so the function must be **increasing** at $z = 5$.

(c) $z = 20$

Solution

We found the derivative of the function in the first part so here all we need to do is the evaluation.

$$G'(20) = \ln(20) + \frac{7}{10} = 3.69573$$

$G'(20) > 0$ and so the function must be **increasing** at $z = 20$.

3.7 Derivatives of Inverse Trig Functions

1. Differentiate $T(z) = 2\cos(z) + 6\cos^{-1}(z)$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$\boxed{T'(z) = -2\sin(z) - \frac{6}{\sqrt{1-z^2}}}$$

---

2. Differentiate $g(t) = \csc^{-1}(t) - 4\cot^{-1}(t)$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$\boxed{g'(t) = -\frac{1}{|t|\sqrt{t^2-1}} + \frac{4}{t^2+1}}$$

---

3. Differentiate $y = 5x^6 - \sec^{-1}(x)$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$\boxed{\frac{dy}{dx} = 30x^5 - \frac{1}{|x|\sqrt{x^2-1}}}$$

---

4. Differentiate $f(w) = \sin(w) + w^2\tan^{-1}(w)$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$\boxed{f'(w) = \cos(w) + 2w\tan^{-1}(w) + \frac{w^2}{1+w^2}}$$

5. Differentiate $h(x) = \dfrac{\sin^{-1}(x)}{1+x}$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$\boxed{h'(x) = \frac{\frac{1+x}{\sqrt{1-x^2}} - \sin^{-1}(x)}{(1+x)^2} = \frac{1+x-\sqrt{1-x^2}\sin^{-1}(x)}{\sqrt{1-x^2}(1+x)^2}}$$

3.8 Derivatives of Hyperbolic Functions

1. Differentiate $f(x) = \sinh(x) + 2\cosh(x) - \operatorname{sech}(x)$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$\boxed{f'(x) = \cosh(x) + 2\sinh(x) + \operatorname{sech}(x)\tanh(x)}$$

---

2. Differentiate $R(t) = \tan(t) + t^2\operatorname{csch}(t)$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$\boxed{R'(t) = \sec^2(t) + 2t\operatorname{csch}(t) - t^2\operatorname{csch}(t)\coth(t)}$$

---

3. Differentiate $g(z) = \dfrac{z+1}{\tanh(z)}$.

Solution

Not much to do here other than take the derivative using the formulas from class.

$$\boxed{g'(z) = \frac{\tanh(z) - (z+1)\operatorname{sech}^2(z)}{\tanh^2(z)}}$$

3.9 Chain Rule

1. Differentiate $f(x) = \left(6x^2 + 7x\right)^4$.

Hint

Recall that with Chain Rule problems you need to identify the "*inside*" and "*outside*" functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the exponent of 4 on the parenthesis while the inside function is the polynomial that is being raised to the power. The derivative is then,

$$f'(x) = 4\left(6x^2 + 7x\right)^3 (12x + 7) = \boxed{4(12x + 7)\left(6x^2 + 7x\right)^3}$$

2. Differentiate $g(t) = \left(4t^2 - 3t + 2\right)^{-2}$.

Hint

Recall that with Chain Rule problems you need to identify the "*inside*" and "*outside*" functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the exponent of -2 on the parenthesis while the inside function is the polynomial that is being raised to the power. The derivative is then,

$$g'(t) = -2\left(4t^2 - 3t + 2\right)^{-3} (8t - 3) = \boxed{-2(8t - 3)\left(4t^2 - 3t + 2\right)^{-3}}$$

3. Differentiate $y = \sqrt[3]{1 - 8z}$.

Hint

Recall that with Chain Rule problems you need to identify the "*inside*" and "*outside*" functions and then apply the chain rule.

Solution

For this problem, after converting the root to a fractional exponent, the outside function is (hopefully) clearly the exponent of $\frac{1}{3}$ while the inside function is the polynomial that is being raised to the power (or the polynomial inside the root - depending upon how you want to think about it). The derivative is then,

$$y = (1-8z)^{\frac{1}{3}} \qquad \Rightarrow \qquad \frac{dy}{dz} = \frac{1}{3}(1-8z)^{-\frac{2}{3}}(-8) = \boxed{-\frac{8}{3}(1-8z)^{-\frac{2}{3}}}$$

---

4. Differentiate $R(w) = \csc(7w)$.

Hint

Recall that with Chain Rule problems you need to identify the "*inside*" and "*outside*" functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the trig function and the inside function is the stuff inside of the trig function. The derivative is then,

$$\boxed{R'(w) = -7\csc(7w)\cot(7w)}$$

In dealing with functions like cosecant (or secant for that matter) be careful to make sure that the inside function gets substituted into both terms of the derivative of the outside function. One of the more common mistakes with this kind of problem is to only substitute the $7w$ into only the cosecant or only the cotangent instead of both as it should be.

---

5. Differentiate $G(x) = 2\sin(3x + \tan(x))$.

Hint

Recall that with Chain Rule problems you need to identify the "*inside*" and "*outside*" functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the sine function and the inside function is the stuff inside of the trig function. The derivative is then,

$$\boxed{G'(x) = 2\left(3 + \sec^2(x)\right)\cos\left(3x + \tan(x)\right)}$$

---

6. Differentiate $h(u) = \tan(4 + 10u)$.

Hint

Recall that with Chain Rule problems you need to identify the "*inside*" and "*outside*" functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the trig function and the inside function is the stuff inside of the trig function. The derivative is then,

$$\boxed{h'(u) = 10\sec^2(4 + 10u)}$$

---

7. Differentiate $f(t) = 5 + \mathbf{e}^{4t+t^7}$.

Hint

Recall that with Chain Rule problems you need to identify the "*inside*" and "*outside*" functions and then apply the chain rule.

Solution

Note that we only need to use the Chain Rule on the second term as we can differentiate the first term without the Chain Rule.

Now, recall that for exponential functions outside function is the exponential function itself and the inside function is the exponent. The derivative is then,

$$\boxed{f'(t) = \left(4 + 7t^6\right)\mathbf{e}^{4t+t^7}}$$

8. Differentiate $g(x) = \mathbf{e}^{1-\cos(x)}$.

Hint

Recall that with Chain Rule problems you need to identify the "*inside*" and "*outside*" functions and then apply the chain rule.

Solution

For exponential functions remember that the outside function is the exponential function itself and the inside function is the exponent. The derivative is then,

$$\boxed{g'(x) = \sin(x)\,\mathbf{e}^{1-\cos(x)}}$$

9. Differentiate $H(z) = 2^{1-6z}$.

Hint

Recall that with Chain Rule problems you need to identify the "*inside*" and "*outside*" functions and then apply the chain rule.

Solution

For exponential functions remember that the outside function is the exponential function itself and the inside function is the exponent. The derivative is then,

$$\boxed{H'(z) = -6\left(2^{1-6z}\right)\ln(2)}$$

10. Differentiate $u(t) = \tan^{-1}(3t-1)$.

Hint

Recall that with Chain Rule problems you need to identify the "*inside*" and "*outside*" functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the inverse tangent and the inside function is the stuff inside of the inverse tangent. The derivative is then,

$$\boxed{u'(t) = \frac{3}{(3t-1)^2 + 1}}$$

---

11. Differentiate $F(y) = \ln\left(1 - 5y^2 + y^3\right)$.

Hint

Recall that with Chain Rule problems you need to identify the "*inside*" and "*outside*" functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the logarithm and the inside function is the stuff inside of the logarithm. The derivative is then,

$$F(y) = \frac{1}{1 - 5y^2 + y^3}\left(-10y + 3y^2\right) = \boxed{\frac{-10y + 3y^2}{1 - 5y^2 + y^3}}$$

With logarithm problems remember that after differentiating the logarithm (*i.e.* the outside function) you need to substitute the inside function into the derivative. So, instead of getting just,

$$\frac{1}{y}$$

we get the following (*i.e.* we plugged the inside function into the derivative),

$$\frac{1}{1 - 5y^2 + y^3}$$

Then, we can't forget of course to multiply by the derivative of the inside function.

---

12. Differentiate $V(x) = \ln(\sin(x) - \cot(x))$.

Hint

Recall that with Chain Rule problems you need to identify the "*inside*" and "*outside*" functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the logarithm and the inside function is the stuff inside of the logarithm. The derivative is then,

$$V(x) = \frac{1}{\sin(x) - \cot(x)}\left(\cos(x) + \csc^2(x)\right) = \boxed{\frac{\cos(x) + \csc^2(x)}{\sin(x) - \cot(x)}}$$

With logarithm problems remember that after differentiating the logarithm (*i.e.* the outside function) you need to substitute the inside function into the derivative. So, instead of getting just,

$$\frac{1}{x}$$

we get the following (*i.e.* we plugged the inside function into the derivative),

$$\frac{1}{\sin(x) - \cot(x)}$$

Then, we can't forget of course to multiply by the derivative of the inside function.

13. Differentiate $h(z) = \sin\left(z^6\right) + \sin^6(z)$.

Hint

Don't get too locked into problems only requiring a single use of the Chain Rule. Sometimes separate terms will require different applications of the Chain Rule, or maybe only one of the terms will require the Chain Rule.

Solution

For this problem each term will require a separate application of the Chain Rule and don't forget that,

$$\sin^6(z) = [\sin(z)]^6$$

So, in the first term the outside function is the sine function, while the sine function is the inside function in the second term. The derivative is then,

$$\boxed{h'(z) = 6z^5 \cos\left(z^6\right) + 6\sin^5(z)\cos(z)}$$

14. Differentiate $S(w) = \sqrt{7w} + \mathbf{e}^{-w}$.

Hint

Don't get too locked into problems only requiring a single use of the Chain Rule. Sometimes separate terms will require different applications of the Chain Rule, or maybe only one of the terms will require the Chain Rule.

Solution

For this problem each term will require a separate application of the Chain Rule and make sure you are careful with parenthesis in dealing with the root in the first term.

The derivative is then,

$$S(w) = (7w)^{\frac{1}{2}} + \mathbf{e}^{-w} \qquad \Rightarrow \qquad S'(w) = \frac{1}{2}(7)(7w)^{-\frac{1}{2}} - \mathbf{e}^{-w} = \boxed{\frac{7}{2}(7w)^{-\frac{1}{2}} - \mathbf{e}^{-w}}$$

15. Differentiate $g(z) = 3z^7 - \sin\left(z^2 + 6\right)$.

Hint

Don't get too locked into problems only requiring a single use of the Chain Rule. Sometimes separate terms will require different applications of the Chain Rule, or maybe only one of the terms will require the Chain Rule.

Solution

For this problem the first term requires no Chain Rule and the second term will require the Chain Rule. The derivative is then,

$$\boxed{g'(z) = 21z^6 - 2z\cos\left(z^2 + 6\right)}$$

16. Differentiate $f(x) = \ln(\sin(x)) - (x^4 - 3x)^{10}$.

Hint

Don't get too locked into problems only requiring a single use of the Chain Rule. Sometimes separate terms will require different applications of the Chain Rule, or maybe only one of the terms will require the Chain Rule.

Solution

For this problem each term will require a separate application of the Chain Rule. The derivative is then,

$$f'(x) = \frac{\cos(x)}{\sin(x)} - 10(4x^3 - 3)(x^4 - 3x)^9 = \boxed{\cot(x) - 10(4x^3 - 3)(x^4 - 3x)^9}$$

17. Differentiate $h(t) = t^6\sqrt{5t^2 - t}$.

Hint

Don't forget the Product and Quotient Rule. Sometimes, in the process of doing the Product or Quotient Rule you'll need to use the Chain Rule when differentiating one or both of the terms in the product or quotient.

Solution

For this problem we'll need to do the Product Rule to start off the derivative. In the process we'll need to use the Chain Rule when we differentiate the second term.

The derivative is then,

$$\begin{aligned} h(t) &= t^6(5t^2 - t)^{\frac{1}{2}} \\ h'(t) &= 6t^5(5t^2 - t)^{\frac{1}{2}} + t^6\left(\frac{1}{2}\right)(5t^2 - t)^{-\frac{1}{2}}(10t - 1) \\ &= \boxed{6t^5(5t^2 - t)^{\frac{1}{2}} + \frac{1}{2}t^6(10t - 1)(5t^2 - t)^{-\frac{1}{2}}} \end{aligned}$$

18. Differentiate $q(t) = t^2\ln(t^5)$.

Hint

Don't forget the Product and Quotient Rule. Sometimes, in the process of doing the Product or Quotient Rule you'll need to use the Chain Rule when differentiating one or both of the terms in the product or quotient.

Solution

For this problem we'll need to do the Product Rule to start off the derivative. In the process we'll need to use the Chain Rule when we differentiate the second term.

The derivative is then,

$$q'(t) = 2t\ln\left(t^5\right) + t^2\left(\frac{5t^4}{t^5}\right) = \boxed{2t\ln\left(t^5\right) + 5t}$$

---

19. Differentiate $g(w) = \cos(3w)\sec(1-w)$.

Hint

Don't forget the Product and Quotient Rule. Sometimes, in the process of doing the Product or Quotient Rule you'll need to use the Chain Rule when differentiating one or both of the terms in the product or quotient.

Solution

For this problem we'll need to do the Product Rule to start off the derivative. In the process we'll need to use the Chain Rule when we differentiate each term.

The derivative is then,

$$\begin{aligned} g'(w) &= -\sin(3w)(3)\sec(1-w) + \cos(3w)\sec(1-w)\tan(1-w)(-1) \\ &= \boxed{-3\sin(3w)\sec(1-w) - \cos(3w)\sec(1-w)\tan(1-w)} \end{aligned}$$

---

20. Differentiate $y = \dfrac{\sin(3t)}{1+t^2}$.

Hint

Don't forget the Product and Quotient Rule. Sometimes, in the process of doing the Product or Quotient Rule you'll need to use the Chain Rule when differentiating one or both of the terms in the product or quotient.

Solution

For this problem we'll need to do the Quotient Rule to start off the derivative. In the process we'll need to use the Chain Rule when we differentiate the numerator.

The derivative is then,

$$\frac{dy}{dt} = \frac{3\cos(3t)\left(1+t^2\right) - \sin(3t)(2t)}{\left(1+t^2\right)^2} = \boxed{\frac{3\cos(3t)\left(1+t^2\right) - 2t\sin(3t)}{\left(1+t^2\right)^2}}$$

---

21. Differentiate $K(x) = \dfrac{1+\mathbf{e}^{-2x}}{x+\tan(12x)}$.

Hint

Don't forget the Product and Quotient Rule. Sometimes, in the process of doing the Product or Quotient Rule you'll need to use the Chain Rule when differentiating one or both of the terms in the product or quotient.

Solution

For this problem we'll need to do the Quotient Rule to start off the derivative. In the process we'll need to use the Chain Rule when we differentiate both the numerator and the denominator.

The derivative is then,

$$\boxed{K'(x) = \frac{-2\mathbf{e}^{-2x}\left(x+\tan(12x)\right) - \left(1+\mathbf{e}^{-2x}\right)\left(1+12\sec^2(12x)\right)}{\left(x+\tan(12x)\right)^2}}$$

---

22. Differentiate $f(x) = \cos(x^2\mathbf{e}^x)$.

Hint

Don't forget the Product and Quotient Rule. Sometimes, in the process of using the Chain Rule, you'll also need the Product and/or Quotient Rule.

Solution

For this problem we'll start off using the Chain Rule, however when we differentiate the inside function we'll need to do the Product Rule.

The derivative is then,

$$f'(x) = -\left(2x\mathbf{e}^x + x^2\mathbf{e}^x\right)\sin\left(x^2\mathbf{e}^x\right)$$

---

23. Differentiate $z = \sqrt{5x + \tan(4x)}$.

Hint

Sometimes the Chain Rule will need to be done multiple times before we finish taking the derivative.

Step 1

This problem will require multiple uses of the Chain Rule and so we'll step though the derivative process to make each use clear.

Here is the first step of the derivative and we'll need to use the Chain Rule in this step.

$$z = (5x + \tan(4x))^{\frac{1}{2}}$$
$$\frac{dz}{dx} = \frac{1}{2}(5x + \tan(4x))^{-\frac{1}{2}}\frac{d}{dx}(5x + \tan(4x))$$

Step 2

In this step we can see that we'll need to use the Chain Rule on the second term.

The derivative is then,

$$\boxed{\frac{dz}{dx} = \frac{1}{2}(5x + \tan(4x))^{-\frac{1}{2}}\left(5 + 4\sec^2(4x)\right)}$$

In this step we were using the Chain Rule on the second term and so when multiplying by the derivative of the inside function we only multiply the second term by the derivative of the inside function and not both terms.

---

24. Differentiate $f(t) = \left(\mathbf{e}^{-6t} + \sin(2-t)\right)^3$.

Hint

Sometimes the Chain Rule will need to be done multiple times before we finish taking the derivative.

Step 1

This problem will require multiple uses of the Chain Rule and so we'll step though the derivative process to make each use clear.

Here is the first step of the derivative and we'll need to use the Chain Rule in this step.

$$f'(t) = 3\left(\mathbf{e}^{-6t} + \sin(2-t)\right)^2 \frac{d}{dt}\left(\mathbf{e}^{-6t} + \sin(2-t)\right)$$

Step 2

In this step we can see that we'll need to use the Chain Rule on each of the terms.

The derivative is then,

$$\boxed{f'(t) = 3\left(\mathbf{e}^{-6t} + \sin(2-t)\right)^2\left(-6\mathbf{e}^{-6t} - \cos(2-t)\right)}$$

---

25. Differentiate $g(x) = \left(\ln\left(x^2+1\right) - \tan^{-1}(6x)\right)^{10}$.

Hint

Sometimes the Chain Rule will need to be done multiple times before we finish taking the derivative.

Step 1

This problem will require multiple uses of the Chain Rule and so we'll step though the derivative process to make each use clear.

Here is the first step of the derivative and we'll need to use the Chain Rule in this step.

$$g'(x) = 10\left(\ln\left(x^2+1\right) - \tan^{-1}(6x)\right)^9 \frac{d}{dx}\left(\ln\left(x^2+1\right) - \tan^{-1}(6x)\right)$$

Step 2

In this step we can see that we'll need to use the Chain Rule on each of the terms.

The derivative is then,

$$\boxed{g'(x) = 10\left(\ln\left(x^2+1\right) - \tan^{-1}(6x)\right)^9 \left(\frac{2x}{x^2+1} - \frac{6}{36x^2+1}\right)}$$

---

26. Differentiate $h(z) = \tan^4\left(z^2+1\right)$.

Hint

Sometimes the Chain Rule will need to be done multiple times before we finish taking the derivative.

Step 1

This problem will require multiple uses of the Chain Rule and so we'll step though the derivative process to make each use clear. Also, recall that,

$$\tan^4(x) = [\tan(x)]^4$$

Here is the first step of the derivative and we'll need to use the Chain Rule in this step.

$$h'(z) = 4\tan^3(z^2+1)\frac{d}{dz}\left[\tan(z^2+1)\right]$$

Step 2

As we can see the derivative from the previous step will also require the Chain Rule.

The derivative is then,

$$h'(z) = 4\tan^3(z^2+1)\sec^2(z^2+1)(2z) = \boxed{8z\tan^3(z^2+1)\sec^2(z^2+1)}$$

---

27. Differentiate $f(x) = \left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-1}$.

Hint

Sometimes the Chain Rule will need to be done multiple times before we finish taking the derivative.

Step 1

This problem will require multiple uses of the Chain Rule and so we'll step though the derivative process to make each use clear.

Here is the first step of the derivative and we'll need to use the Chain Rule in this step.

$$f'(x) = -\left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-2}\frac{d}{dx}\left((12x)^{\frac{1}{3}} + \sin^2(3x)\right)$$

Step 2

As we can see the derivative from the previous step will also require the Chain Rule on each of the terms.

The derivative from this step is,

$$f'(x) = -\left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-2}\left(\frac{1}{3}(12x)^{-\frac{2}{3}}(12) + 2\sin(3x)\frac{d}{dx}\left(\sin(3x)\right)\right)$$

Step 3

The second term will again use the Chain Rule as we can see.

The derivative is then,

$$\boxed{f'(x) = -\left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-2}\left(4(12x)^{-\frac{2}{3}} + 6\sin(3x)\cos(3x)\right)}$$

---

28. Find the tangent line to $f(x) = 4\sqrt{2x} - 6\mathbf{e}^{2-x}$ at $x = 2$.

Step 1

We know that the derivative of the function will give us the slope of the tangent line so we'll need the derivative of the function. Differentiating each term will require the Chain Rule as well.

$$f(x) = 4(2x)^{\frac{1}{2}} - 6\mathbf{e}^{2-x}$$

$$f'(x) = 4\left(\frac{1}{2}\right)(2x)^{-\frac{1}{2}}(2) - 6\mathbf{e}^{2-x}(-1) = 4(2x)^{-\frac{1}{2}} + 6\mathbf{e}^{2-x} = \boxed{\frac{4}{\sqrt{2x}} + 6\mathbf{e}^{2-x}}$$

Step 2

Now all we need to do is evaluate the function and the derivative at the point in question.

$$f(2) = 4(2) - 6\mathbf{e}^{0} = 2 \qquad f'(2) = \frac{4}{2} + 6\mathbf{e}^{0} = 8$$

Step 3

Now all that we need to do is write down the equation of the tangent line.

$$y = f(2) + f'(2)(x-2) = 2 + 8(x-2) \quad \rightarrow \quad \boxed{y = 8x - 14}$$

---

29. Determine where $V(z) = z^4(2z-8)^3$ is increasing and decreasing.

Step 1

We'll first need the derivative because we know that the derivative will give us the rate of change of the function. Here is the derivative.

$$V'(z) = 4z^3(2z-8)^3 + z^4(3)(2z-8)^2(2)$$
$$= 2z^3(2z-8)^2\left[2(2z-8)+3z\right] = \boxed{2z^3(2z-8)^2(7z-16)}$$

Note that we factored the derivative to help with the next step. In general, we don't need to do this.

Step 2

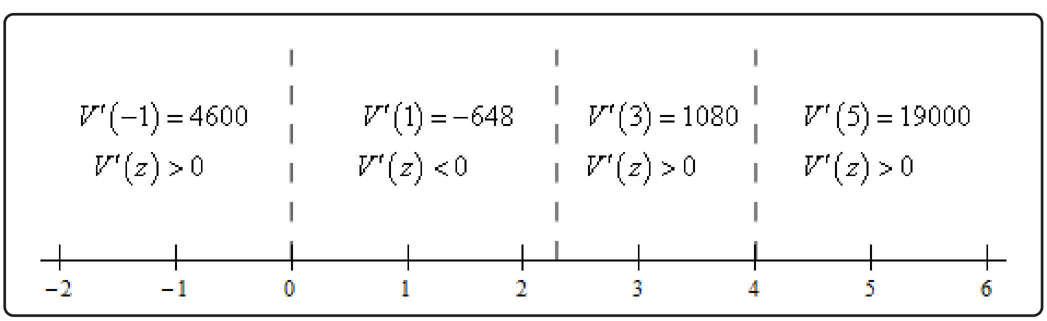
Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve. In this case it's pretty easy to spot where the derivative will be zero.

$$2z^3(2z-8)^2(7z-16) = 0 \qquad \Rightarrow \qquad z=0, \quad z=4, \quad z=\frac{16}{7}=2.2857$$

Step 3

To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.



From this we get the following increasing/decreasing information.

$$\text{Increasing}: \ -\infty < z < 0, \quad \frac{16}{7} < z < 4, \quad 4 < z < \infty$$
$$\text{Decreasing}: \ 0 < z < \frac{16}{7}$$

---

30. The position of an object is given by $s(t) = \sin(3t) - 2t + 4$. Determine where in the interval $[0, 3]$ the object is moving to the right and moving to the left.

Step 1

We'll first need the derivative because we know that the derivative will give us the velocity of the object. Here is the derivative.

$$\boxed{s'(t) = 3\cos(3t) - 2}$$

Step 2

Next, we need to know where the object stops moving and so all we need to do is set the derivative equal to zero and solve.

$$3\cos(3t) - 2 = 0 \qquad \Rightarrow \qquad \cos(3t) = \frac{2}{3}$$

A quick calculator computation tells us that,

$$3t = \cos^{-1}\left(\frac{2}{3}\right) = 0.8411$$

Recalling our work in the Review chapter and a quick check on a unit circle we can see that either $3t = -0.8411$ or $3t = 2\pi - 0.8411 = 5.4421$ can be used for the second angle. Note that either will work, but we'll use the second simply because it is the positive angle.

Putting all of this together and dividing by 3 we can see that the derivative will be zero at,

$$3t = 0.8411 + 2\pi n \quad \text{and} \quad 3t = 5.4421 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$
$$t = 0.2804 + \frac{2\pi n}{3} \quad \text{and} \quad t = 1.8140 + \frac{2\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Finally, all we need to do is plug in some n's to determine which solutions fall in the interval we are working on, $[0,3]$.

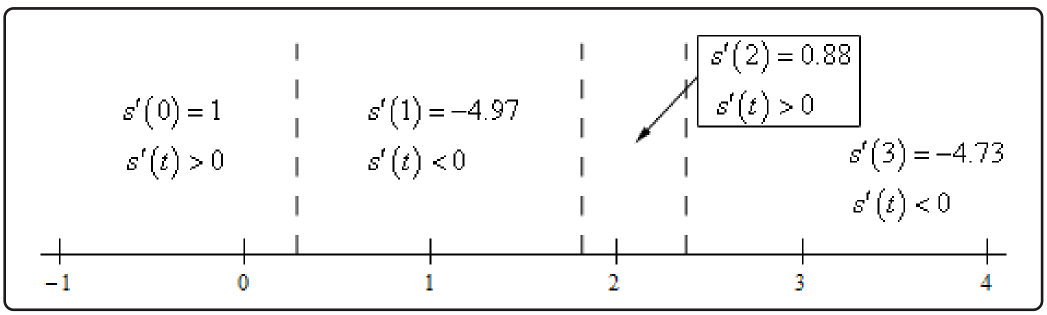
So, in the interval $[0,3]$, the object stops moving at the following three points.

$$t = 0.2804, \quad 1.8140, \quad 2.3748$$

Step 3

To get the answer to this problem all we need to know is where the derivative is positive (and hence the object is moving to the right) or negative (and hence the object is moving to the left). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.



From this we get the following moving right/moving left information.

Moving Right :	$0 \le t < 0.2804, \quad 1.8140 < t < 2.3748$
Moving Left :	$0.2804 < t < 1.8140, \quad 2.3748 < t \le 3$

Note that because we've only looked at what is happening in the interval $[0,3]$ we can't say anything about the moving right/moving left behavior of the object outside of this interval.

31. Determine where $A(t) = t^2 \mathbf{e}^{5-t}$ is increasing and decreasing.

Step 1

We'll first need the derivative because we know that the derivative will give us the rate of change of the function. Here is the derivative.

$$A'(t) = 2t\,\mathbf{e}^{5-t} - t^2\mathbf{e}^{5-t} = t\,\mathbf{e}^{5-t}(2-t)$$

Note that we factored the derivative to help with the next step. In general, we don't need to do this.

Step 2

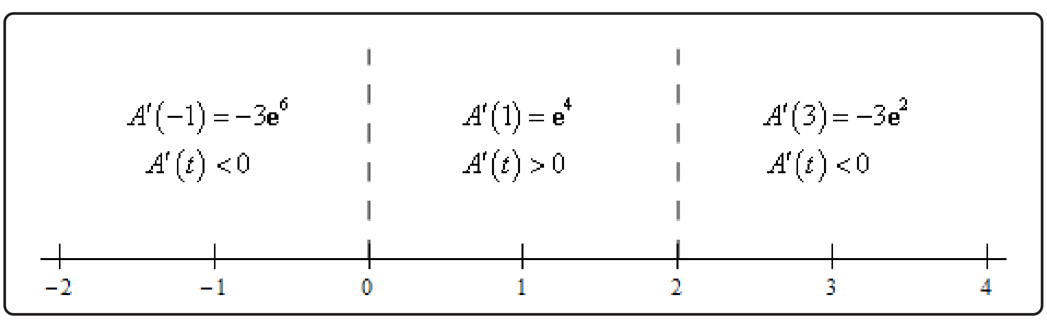
Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve. In this case it's pretty easy to spot where the derivative will be zero.

$$t\,\mathbf{e}^{5-t}(2-t) = 0 \qquad \Rightarrow \qquad t = 0, \quad t = 2$$

Step 3

To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.



From this we get the following increasing/decreasing information.

$$\boxed{\begin{array}{l}\text{Increasing}: 0 < t < 2 \\ \text{Decreasing}: -\infty < t < 0, \quad 2 < t < \infty\end{array}}$$

32. Determine where in the interval $[-1, 20]$ the function $f(x) = \ln\left(x^4 + 20x^3 + 100\right)$ is increasing and decreasing.

Step 1

We'll first need the derivative because we know that the derivative will give us the rate of change of the function. Here is the derivative.

$$f(x) = \frac{4x^3 + 60x^2}{x^4 + 20x^3 + 100} = \frac{4x^2(x + 15)}{x^4 + 20x^3 + 100}$$

Note that we factored the numerator to help with the next step. In general, we don't need to do this.

Step 2

Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve.

$$\frac{4x^2(x + 15)}{x^4 + 20x^3 + 100} = 0 \quad \rightarrow \quad 4x^2(x + 15) = 0 \quad \Rightarrow \quad x = 0, \quad x = -15$$

Note, that in general, we also need to be concerned with where the derivative is not defined as well. Functions can (and often do) change sign where they are not defined. In this case however we've restricted the interval down to a range where the function exists and is continuous on the given interval and so this is something we need to worry about for this problem.

In the next Chapter we will start also looking at what happens if the derivative is also not defined at particular points.

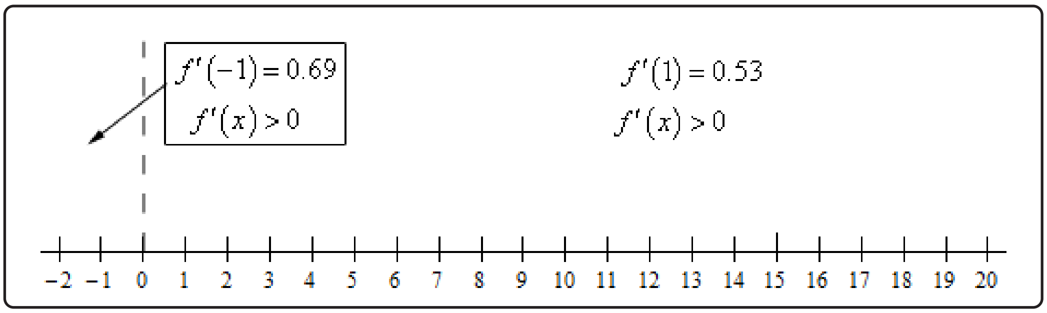
Note as well that we really should at this point step back and recall that we are working on the interval $[-1, 20]$. We are only interested in what is happening on this interval and so we should make sure that the points found above are inside the interval.

In this case only $x = 0$ is in the interval and so we'll need to exclude $x = -15$ from our work for the rest of this problem.

Step 3

To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.



So, we can see that, in this case function is increasing everywhere in the interval $[-1, 20]$ except $x = 0$. Recall that at this point the derivative was zero and hence the function is not changing (and therefore can't be increasing).

So, the formal answer for this problem is,

$$\boxed{\text{Increasing} : -1 \leq x < 0, \quad 0 < x \leq 20}$$

Note that because we've only looked at what is happening in the interval $[-1, 20]$ we can't say anything about the increasing/decreasing nature of the function outside of this interval.

3.10 Implicit Differentiation

1. For $\dfrac{x}{y^3} = 1$ do each of the following.

   (a) Find y' by solving the equation for y and differentiating directly.

   (b) Find y' by implicit differentiation.

   (c) Check that the derivatives in (a) and (b) are the same.

Solutions

(a) Find y' by solving the equation for y and differentiating directly.

Step 1

First, we just need to solve the equation for y.

$$y^3 = x \qquad \Rightarrow \qquad y = x^{\frac{1}{3}}$$

Step 2

Now differentiate with respect to x.

$$\boxed{y' = \frac{1}{3}x^{-\frac{2}{3}}}$$

Hint

Don't forget that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y! Also, don't forget that because y is really $y(x)$ we may well have a Product and/or a Quotient Rule buried in the problem.

(b) Find y' by implicit differentiation.

Step 1

First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y.

Also, prior to taking the derivative a little rewrite might make this a little easier.

$$x\,y^{-3} = 1$$

Now take the derivative and don't forget that we actually have a product of functions of x here and so we'll need to use the Product Rule when differentiating the left side.

$$y^{-3} - 3x\,y^{-4}y' = 0$$

Step 2

Finally, all we need to do is solve this for y'.

$$y' = \frac{y^{-3}}{3xy^{-4}} = \boxed{\frac{y}{3x}}$$

Hint

To show they are the same all we need is to plug the formula for y (which we already have....) into the derivative we found in **(b)** and, potentially with a little work, show that we get the same derivative as we got in **(a)**.

(c) Check that the derivatives in (a) and (b) are the same.

Solution

From **(a)** we have a formula for y written explicitly as a function of x so plug that into the derivative we found in **(b)** and, with a little simplification/work, show that we get the same derivative as we got in **(a)**.

$$y' = \frac{y}{3x} = \frac{x^{\frac{1}{3}}}{3x} = \frac{1}{3}x^{-\frac{2}{3}}$$

So, we got the same derivative as we should.

2. For $x^2 + y^3 = 4$ do each of the following.

(a) Find y' by solving the equation for y and differentiating directly.

(b) Find y' by implicit differentiation.

(c) Check that the derivatives in (a) and (b) are the same.

Solutions

(a) Find y' by solving the equation for y and differentiating directly.

Step 1

First, we just need to solve the equation for y.

$$y^3 = 4 - x^2 \qquad \Rightarrow \qquad y = \left(4 - x^2\right)^{\frac{1}{3}}$$

Step 2

Now differentiate with respect to x.

$$\boxed{y' = -\frac{2}{3}x\left(4 - x^2\right)^{-\frac{2}{3}}}$$

Hint

Don't forget that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y!

(b) Find y' by implicit differentiation.

Step 1

First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y.

Taking the derivative gives,

$$2x + 3y^2y' = 0$$

Step 2

Finally, all we need to do is solve this for y'.

$$y' = \boxed{-\frac{2x}{3y^2}}$$

Hint

To show they are the same all we need is to plug the formula for y (which we already have....) into the derivative we found in **(b)** and, potentially with a little work, show that we get the same derivative as we got in **(a)**.

(c) Check that the derivatives in (a) and (b) are the same.

Solution

From **(a)** we have a formula for y written explicitly as a function of x so plug that into the derivative we found in **(b)** and, with a little simplification/work, show that we get the same derivative as we got in **(a)**.

$$y' = -\frac{2x}{3y^2} = -\frac{2x}{3(4-x^2)^{\frac{2}{3}}} = -\frac{2}{3}x\left(4-x^2\right)^{-\frac{2}{3}}$$

So, we got the same derivative as we should.

3. For $x^2 + y^2 = 2$ do each of the following.

   (a) Find y' by solving the equation for y and differentiating directly.

   (b) Find y' by implicit differentiation.

   (c) Check that the derivatives in (a) and (b) are the same.

Solutions

(a) Find y' by solving the equation for y and differentiating directly.

Step 1

First, we just need to solve the equation for y.

$$y^2 = 2 - x^2 \qquad \Rightarrow \qquad y = \pm\left(2 - x^2\right)^{\frac{1}{2}}$$

Note that because we have no restriction on y (*i.e.* we don't know if y is positive or negative) we really do need to have the "$\pm$" there and that does lead to issues when taking the derivative.

Hint

Two formulas for y and so two derivatives.

Step 2

Now, because there are two formulas for y we will also have two formulas for the derivative, one for each formula for y.

The derivatives are then,

$$y = \left(2 - x^2\right)^{\frac{1}{2}} \qquad \Rightarrow \qquad \boxed{y' = -x\left(2 - x^2\right)^{-\frac{1}{2}}} \qquad (y > 0)$$

$$y = -\left(2 - x^2\right)^{\frac{1}{2}} \qquad \Rightarrow \qquad \boxed{y' = x\left(2 - x^2\right)^{-\frac{1}{2}}} \qquad (y < 0)$$

As noted above the first derivative will hold for $y > 0$ while the second will hold for $y < 0$ and we can use either for $y = 0$ as the plus/minus won't affect that case.

Hint

Don't forget that y is really $y\left(x\right)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y!

(b) Find y' by implicit differentiation.

Step 1

First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y\left(x\right)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y.

Taking the derivative gives,

$$2x + 2y\,y' = 0$$

Step 2

Finally, all we need to do is solve this for y'.

$$y' = \boxed{-\frac{x}{y}}$$

Hint

To show they are the same all we need is to plug the formula for y (which we already have....) into the derivative we found in **(b)** and, potentially with a little work, show that we get the same derivative as we got in **(a)**. Again, two formulas for y so two derivatives...

(c) Check that the derivatives in (a) and (b) are the same.

Solution

From **(a)** we have a formula for y written explicitly as a function of x so plug that into the derivative we found in **(b)** and, with a little simplification/work, show that we get the same derivative as we got in **(a)**.

Also, because we have two formulas for y we will have two formulas for the derivative.

First, if $y > 0$ we will have,

$$y = \left(2 - x^2\right)^{\frac{1}{2}} \qquad \Rightarrow \qquad y' = -\frac{x}{y} = -\frac{x}{\left(2 - x^2\right)^{\frac{1}{2}}} = -x\left(2 - x^2\right)^{-\frac{1}{2}}$$

Next, if $y < 0$ we will have,

$$y = -\left(2 - x^2\right)^{\frac{1}{2}} \qquad \Rightarrow \qquad y' = -\frac{x}{y} = -\frac{x}{-\left(2 - x^2\right)^{\frac{1}{2}}} = x\left(2 - x^2\right)^{-\frac{1}{2}}$$

So, in both cases, we got the same derivative as we should.

---

4. Find y' by implicit differentiation for $2y^3 + 4x^2 - y = x^6$.

Hint

Don't forget that y is really $y\left(x\right)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y!

Step 1

First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y\left(x\right)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y.

Differentiating with respect to x gives,

$$6y^2\,y' + 8x - y' = 6x^5$$

Step 2

Finally, all we need to do is solve this for y'.

$$\left(6y^2 - 1\right)y' = 6x^5 - 8x \qquad \Rightarrow \qquad \boxed{y' = \frac{6x^5 - 8x}{6y^2 - 1}}$$

---

5. Find y' by implicit differentiation for $7y^2 + \sin\left(3x\right) = 12 - y^4$.

Hint

Don't forget that y is really $y\left(x\right)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y!

Step 1

First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y.

Differentiating with respect to x gives,

$$14y\,y' + 3\cos(3x) = -4y^3y'$$

Step 2

Finally, all we need to do is solve this for y'.

$$\left(14y + 4y^3\right)y' = -3\cos(3x) \qquad \Rightarrow \qquad \boxed{y' = \frac{-3\cos(3x)}{14y + 4y^3}}$$

6. Find y' by implicit differentiation for $\mathbf{e}^x - \sin(y) = x$.

Hint

Don't forget that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y!

Step 1

First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y.

Differentiating with respect to x gives,

$$\mathbf{e}^x - \cos(y)\,y' = 1$$

Don't forget the y' on the cosine after differentiating. Again, y is really $y(x)$ and so when differentiating $\sin(y)$ we really differentiating $\sin[y(x)]$ and so we are differentiating using the Chain Rule!

Step 2

Finally, all we need to do is solve this for y'.

$$\boxed{y' = \frac{1 - \mathbf{e}^x}{-\cos(y)} = (\mathbf{e}^x - 1)\sec(y)}$$

---

7. Find y' by implicit differentiation for $4x^2y^7 - 2x = x^5 + 4y^3$.

Hint

Don't forget that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y! Also, don't forget that because y is really $y(x)$ we may well have a Product and/or a Quotient Rule buried in the problem.

Step 1

First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y. This also means that the first term on the left side is really a product of functions of x and hence we will need to use the Product Rule when differentiating that term.

Differentiating with respect to x gives,

$$8xy^7 + 28x^2y^6y' - 2 = 5x^4 + 12y^2y'$$

Step 2

Finally, all we need to do is solve this for y'.

$$8xy^7 - 5x^4 - 2 = \left(12y^2 - 28x^2y^6\right)y' \qquad \Rightarrow \qquad \boxed{y' = \frac{8xy^7 - 5x^4 - 2}{12y^2 - 28x^2y^6}}$$

---

8. Find y' by implicit differentiation for $\cos\left(x^2+2y\right)+x\,\mathbf{e}^{y^2}=1$.

Hint

Don't forget that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y! Also, don't forget that because y is really $y(x)$ we may well have a Product and/or a Quotient Rule buried in the problem.

Step 1

First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y. This also means that the second term on the left side is really a product of functions of x and hence we will need to use the Product Rule when differentiating that term.

Differentiating with respect to x gives,

$$-\left(2x+2y'\right)\sin\left(x^2+2y\right)+\mathbf{e}^{y^2}+2y\,y'x\,\mathbf{e}^{y^2}=0$$

Step 2

Finally, all we need to do is solve this for y' (with some potentially messy algebra).

$$\begin{aligned}-2x\sin\left(x^2+2y\right)-2y'\sin\left(x^2+2y\right)+\mathbf{e}^{y^2}+2y\,y'x\,\mathbf{e}^{y^2}&=0\\ \left(2yx\,\mathbf{e}^{y^2}-2\sin\left(x^2+2y\right)\right)y'&=0+2x\sin\left(x^2+2y\right)-\mathbf{e}^{y^2}\\ y'&=\boxed{\frac{2x\sin\left(x^2+2y\right)-\mathbf{e}^{y^2}}{2yx\,\mathbf{e}^{y^2}-2\sin\left(x^2+2y\right)}}\end{aligned}$$

9. Find y' by implicit differentiation for $\tan\left(x^2y^4\right)=3x+y^2$.

Hint

Don't forget that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y! Also, don't forget that because y is really $y(x)$ we may well have a Product and/or a Quotient Rule buried in the problem.

Step 1

First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y. This also means that the when doing Chain Rule on the first tangent on the left side we will need to do Product Rule when differentiating the "inside term".

Differentiating with respect to x gives,

$$\left(2x\,y^4 + 4x^2y^3y'\right)\sec^2\left(x^2y^4\right) = 3 + 2y\,y'$$

Step 2

Finally, all we need to do is solve this for y' (with some potentially messy algebra).

$$\begin{aligned}2x\,y^4\sec^2\left(x^2y^4\right) + 4x^2y^3y'\sec^2\left(x^2y^4\right) &= 3 + 2y\,y' \\ \left(4x^2y^3\sec^2\left(x^2y^4\right) - 2y\right)y' &= 3 - 2x\,y^4\sec^2\left(x^2y^4\right) \\ y' &= \boxed{\frac{3 - 2x\,y^4\sec^2\left(x^2y^4\right)}{4x^2y^3\sec^2\left(x^2y^4\right) - 2y}}\end{aligned}$$

---

10. Find the equation of the tangent line to $x^4 + y^2 = 3$ at $\left(1, -\sqrt{2}\right)$.

Hint

We know how to compute the slope of tangent lines and with implicit differentiation that shouldn't be too hard at this point.

Step 1

The first thing to do is use implicit differentiation to find y' for this function.

$$4x^3 + 2y\,y' = 0 \qquad \Rightarrow \qquad \underline{y' = -\frac{2x^3}{y}}$$

Step 2

Evaluating the derivative at the point in question to get the slope of the tangent line gives,

$$m = y'\big|_{x=1,\ y=-\sqrt{2}} = -\frac{2}{-\sqrt{2}} = \sqrt{2}$$

Step 3

Now, we just need to write down the equation of the tangent line.

$$y - \left(-\sqrt{2}\right) = \sqrt{2}\left(x-1\right) \quad \Rightarrow \quad y = \sqrt{2}\left(x-1\right) - \sqrt{2} = \boxed{\sqrt{2}\left(x-2\right)}$$

---

11. Find the equation of the tangent line to $y^2\mathbf{e}^{2x} = 3y + x^2$ at $(0,3)$.

Hint

We know how to compute the slope of tangent lines and with implicit differentiation that shouldn't be too hard at this point.

Step 1

The first thing to do is use implicit differentiation to find y' for this function.

$$2yy'\mathbf{e}^{2x} + 2y^2\mathbf{e}^{2x} = 3y' + 2x \quad \Rightarrow \quad \underline{y' = \frac{2x - 2y^2\mathbf{e}^{2x}}{2y\mathbf{e}^{2x} - 3}}$$

Step 2

Evaluating the derivative at the point in question to get the slope of the tangent line gives,

$$m = y'\big|_{x=0,\ y=3} = \frac{-18}{3} = -6$$

Step 3

Now, we just need to write down the equation of the tangent line.

$$y - 3 = -6(x - 0) \quad \Rightarrow \quad \boxed{y = -6x + 3}$$

---

12. Assume that $x = x(t)$, $y = y(t)$ and $z = z(t)$ then differentiate $x^2 - y^3 + z^4 = 1$ with respect to t.

Hint

This is just implicit differentiation like we've been doing to this point. The only difference is that now all the functions are functions of some fourth variable, t. Outside of that there is nothing different between this and the previous problems.

Solution

Differentiating with respect to t gives,

$$\boxed{2x\,x' - 3y^2\,y' + 4z^3\,z' = 0}$$

Note that because we were not asked to give the formula for a specific derivative we don't need to go any farther. We could however, if asked, solved this for any of the three derivatives that are present.

---

13. Assume that $x = x(t)$, $y = y(t)$ and $z = z(t)$ and differentiate $x^2 \cos(y) = \sin(y^3 + 4z)$ with respect to t.

Hint

This is just implicit differentiation like we've been doing to this point. The only difference is that now all the functions are functions of some fourth variable, t. Outside of that there is nothing different between this and the previous problems.

Solution

Differentiating with respect to t gives,

$$\boxed{2x\,x'\cos\left(y\right) - x^2\sin\left(y\right)y' = \left(3y^2y' + 4z'\right)\cos\left(y^3 + 4z\right)}$$

Note that because we were not asked to give the formula for a specific derivative we don't need to go any farther. We could however, if asked, solved this for any of the three derivatives that are present.

3.11 Related Rates

1. In the following assume that x and y are both functions of t. Given $x = -2$, $y = 1$ and $x' = -4$ determine y' for the following equation.

$$6y^2 + x^2 = 2 - x^3\mathbf{e}^{4-4y}$$

Hint

This is just like the problems worked in the section notes. The only difference is that you've been given the equation and all the needed information and so you won't have to worry about finding that.

Step 1

The first thing that we need to do here is use implicit differentiation to differentiate the equation with respect to t.

$$12y\,y' + 2x\,x' = -3x^2x'\mathbf{e}^{4-4y} + 4x^3\mathbf{e}^{4-4y}y'$$

Step 2

All we need to do now is plug in the given information and solve for y'.

$$12y' + 16 = 48 - 32y' \qquad \Rightarrow \qquad \boxed{y' = \frac{8}{11}}$$

2. In the following assume that x, y and z are all functions of t. Given $x = 4$, $y = -2$, $z = 1$, $x' = 9$ and $y' = -3$ determine z' for the following equation.

$$x\left(1 - y\right) + 5z^3 = y^2z^2 + x^2 - 3$$

Hint

This is just like the problems worked in the section notes. The only difference is that you've been given the equation and all the needed information and so you won't have to worry about finding that.

Step 1

The first thing that we need to do here is use implicit differentiation to differentiate the equation with respect to t.

$$x'(1-y) - x\,y' + 15z^2z' = 2y\,y'z^2 + 2y^2z\,z' + 2x\,x'$$

Step 2

All we need to do now is plug in the given information and solve for z'.

$$27 + 12 + 15z' = 12 + 8\,z' + 72 \qquad \Rightarrow \qquad \boxed{z' = \frac{45}{7}}$$

---

3. For a certain rectangle the length of one side is always three times the length of the other side.

   (a) If the shorter side is decreasing at a rate of 2 inches/minute at what rate is the longer side decreasing?

   (b) At what rate is the enclosed area decreasing when the shorter side is 6 inches long and is decreasing at a rate of 2 inches/minute?

Solutions

Hint

The equation needed here is a really simple equation. In fact, so simple it might be easy to miss...

(a) If the shorter side is decreasing at a rate of 2 inches/minute at what rate is the longer side decreasing?

Step 1

Let's call the shorter side x and the longer side y. We know that $x' = -2$ and want to find y'.

Now all we need is an equation that relates these two quantities and from the problem statement we know the longer side is three times shorter side and so the equation is,

$$y = 3x$$

Step 2

Next step is to simply differentiate the equation with respect to t.

$$y' = 3x'$$

Step 3

Finally, plug in the known quantity and solve for what we want : $\boxed{y' = -6}$

Hint

Once we have the equation for the area we can either simplify the equation as we did in this section or we can use the result from the previous step and the equation directly.

(b) At what rate is the enclosed area decreasing when the shorter side is 6 inches long and is decreasing at a rate of 2 inches/minute?

Step 1

Again, we'll call the shorter side x and the longer side y as with the last part. We know that $x = 6$, $x' = -2$ and want to find A'.

The equation we'll need is just the area formula for a rectangle : $A = xy$ At this point we can either leave the equation as is and differentiate it or we can plug in $y = 3x$ to simplify the equation down to a single variable then differentiate. Doing this gives,

$$A(x) = 3x^2$$

Step 2

Now we need to differentiate with respect to t.

If we use the equation in terms of only x, which is probably the easiest to use we get,

$$A' = 6x\,x'$$

If we use the equation in terms of both x and y we get,

$$A' = x\,y' + x'y$$

Step 3

Now all we need to do is plug in the known quantities and solve for A'.

Using the equation in terms of only x is the "easiest" because we already have all the known quantities from the problem statement itself. Doing this gives,

$$\boxed{A' = 6\,(6)\,(-2) = -72}$$

Now let's use the equation in terms of x and y. We know that $x = 6$ and $x' = -2$ from the problem statement. From part **(a)** we have $y' = -6$ and we also know that $y = 3\,(6) = 18$. Using these gives,

$$\boxed{A' = (6)\,(-6) + (-2)\,(18) = -72}$$

So, as we can see both gives the same result, but the second method is slightly more work, although not much more.

---

4. A thin sheet of ice is in the form of a circle. If the ice is melting in such a way that the area of the sheet is decreasing at a rate of 0.5 m^2/sec at what rate is the radius decreasing when the area of the sheet is 12 m^2?

Step 1

We'll call the area of the sheet A and the radius r and we know that the area of a circle is given by,

$$A = \pi r^2$$

We know that $A' = -0.5$ and want to determine r' when $A = 12$.

Step 2

Next step is to simply differentiate the equation with respect to t.

$$A' = 2\pi r\, r'$$

Step 3

Now, to finish this problem off we'll first need to go back to the equation of the area and use the fact that we know the area at the point we are interested in and determine the radius at that time.

$$12 = \pi r^2 \qquad \Rightarrow \qquad r = \sqrt{\frac{12}{\pi}} = 1.9544$$

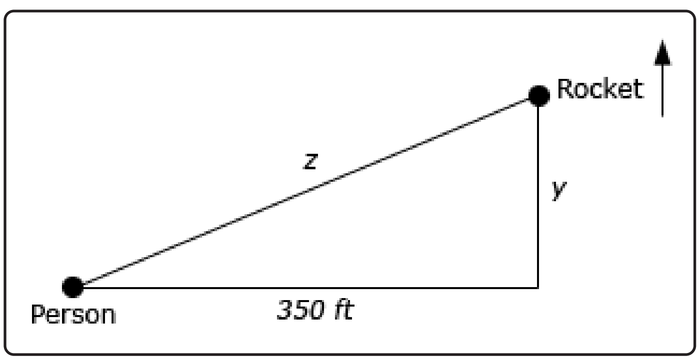
The rate of change of the radius is then,

$$-0.5 = 2\pi\,(1.9544)\; r' \qquad \Rightarrow \qquad \boxed{r' = -0.040717}$$

5. A person is standing 350 feet away from a model rocket that is fired straight up into the air at a rate of 15 ft/sec. At what rate is the distance between the person and the rocket increasing **(a)** 20 seconds after liftoff? **(b)** 1 minute after liftoff?

Step 1

Here is a sketch for this situation that will work for both parts so we'll put it here.



Step 2

In both parts we know that $y' = 15$ and want to determine z' for each given time. Using the Pythagorean Theorem we get the following equation to relate y and z.

$$z^2 = y^2 + 350^2 = y^2 + 122500$$

Step 3

Finally, let's differentiate this with respect to t and we can even solve it for z' so the actual solution will be quick and simple to find.

$$2z\,z' = 2y\,y' \qquad \Rightarrow \qquad z' = \frac{y\,y'}{z}$$

We have now reached a point where the process will differ for each part.

(a) At what rate is the distance between the person and the rocket increasing 20 seconds after liftoff?

Solution

To finish off this problem all we need to do is determine y (from the speed of the rocket and given time) and z (reusing the Pythagorean Theorem).

$$y = (15)(20) = 300 \qquad z = \sqrt{300^2 + 350^2} = \sqrt{212500} = 50\sqrt{85} = 460.9772$$

The rate of change of the distance between the two is then,

$$z' = \frac{(300)(15)}{460.9772} = \boxed{9.76187}$$

(b) At what rate is the distance between the person and the rocket increasing 1 minute after liftoff?

Solution

This part is nearly identical to the first part with the exception that the time is now 60 seconds (and note that we MUST be in seconds because the speeds are in time of seconds).

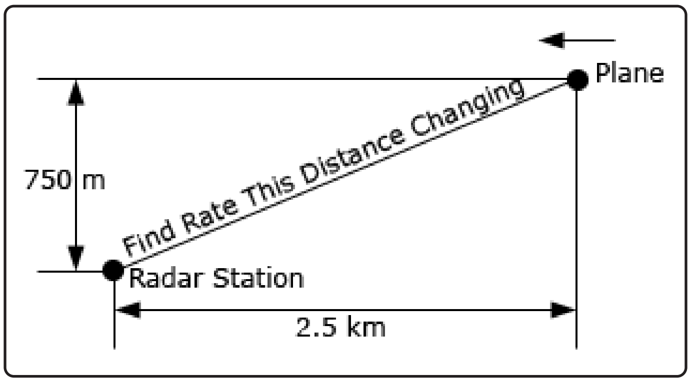
Here is the work for this problem.

$$y = (15)(60) = 900 \qquad z = \sqrt{900^2 + 350^2} = \sqrt{932500} = 50\sqrt{373} = 965.6604$$

$$z' = \frac{(900)(15)}{965.6604} = \boxed{13.98007}$$

---

6. A plane is 750 meters in the air flying parallel to the ground at a speed of 100 m/s and is initially 2.5 kilometers away from a radar station. At what rate is the distance between the plane and the radar station changing **(a)** initially and **(b)** 30 seconds after it passes over the radar station?



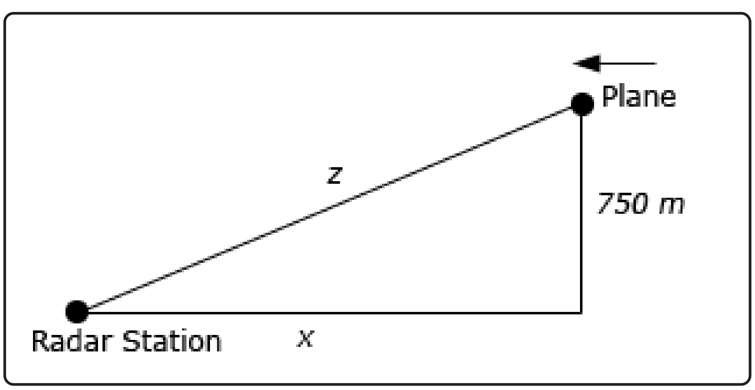
Solutions

(a) At what rate is the distance between the plane and the radar station changing initially?

Step 1

For this part we know that $x' = -100$ when $x = 2500$. In this case note that x' must be negative because x will be decreasing in this part. Also note that we converted x to meters since all the other quantities are in meters.

Here is a sketch for this part.



Step 2

We want to determine z' in this part so using the Pythagorean Theorem we get the following equation to relate x and z.

$$z^2 = x^2 + 750^2 = x^2 + 562500$$

Step 3

Finally, let's differentiate this with respect to t and we can even solve it for z' so the actual solution will be quick and simple to find.

$$2z\,z' = 2x\,x' \qquad \Rightarrow \qquad z' = \frac{x\,x'}{z}$$

Step 4

To finish off this problem all we need to do is determine z (reusing the Pythagorean Theorem) and then plug into the equation from Step 3 above.

$$z = \sqrt{2500^2 + 750^2} = \sqrt{6812500} = 250\sqrt{109} = 2610.0766$$

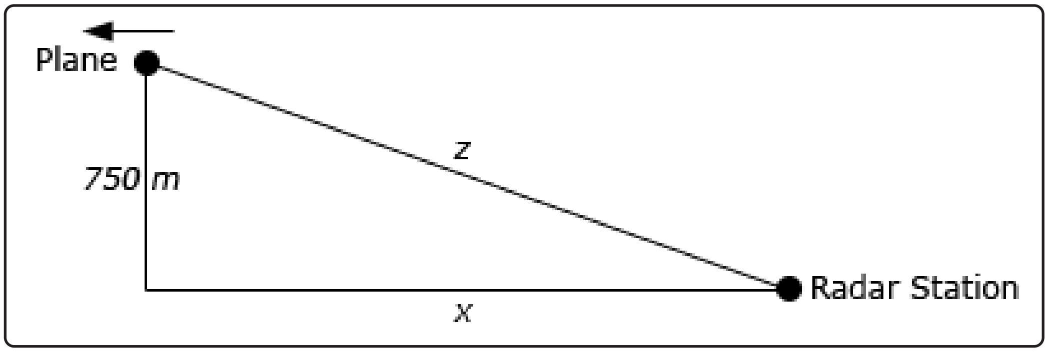
The rate of change of the distance between the two for this part is,

$$z' = \frac{(2500)(-100)}{2610.0766} = \boxed{-95.7826}$$

(b) At what rate is the distance between the plane and the radar station changing 30 seconds after it passes over the radar station?

Step 1

For this part we know that $x' = 100$ and it will be positive in this case because x will now be increasing as we can see in the sketch below.



Step 2

As with the previous part we want to determine z' and equation we'll need is identical to the previous part so we'll just rewrite both it and its derivative here.

$$z^2 = x^2 + 750^2 = x^2 + 562500$$
$$2z\,z' = 2x\,x' \qquad \Rightarrow \qquad z' = \frac{x\,x'}{z}$$

Step 3

To finish off this problem all we need to do is determine both x and z. For x we know the speed of the plane and the fact that it has flown for 30 seconds after passing over the radar station. So x is,

$$x = (100)(30) = 3000$$

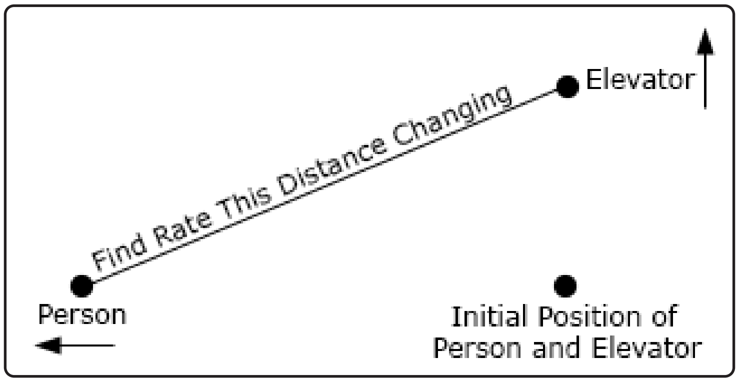
For z we just need to reuse the Pythagorean Theorem.

$$z = \sqrt{3000^2 + 750^2} = \sqrt{9562500} = 750\sqrt{17} = 3092.3292$$

The rate of change of the distance between the two for this part is then,

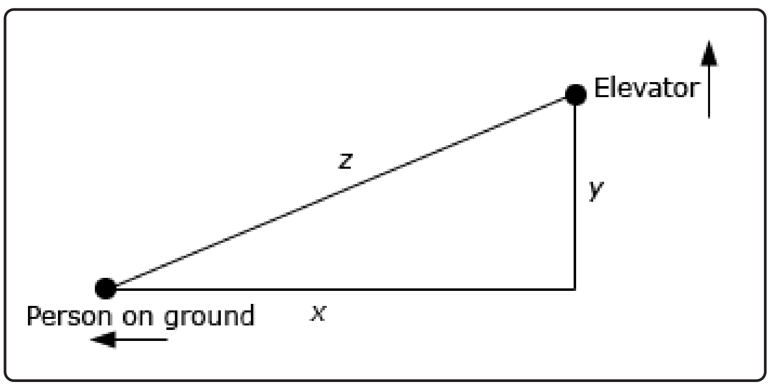
$$z' = \frac{(3000)(100)}{3092.3292} = \boxed{97.0143}$$

7. Two people are at an elevator. At the same time one person starts to walk away from the elevator at a rate of 2 ft/sec and the other person starts going up in the elevator at a rate of 7 ft/sec. What rate is the distance between the two people changing 15 seconds later?



Step 1

Here is a sketch for this part.



We want to determine z' after 15 seconds given that $x' = 2$, $y' = 7$ and assuming that they start at the same point.

Step 2

Hopefully it's clear that we'll need the Pythagorean Theorem to solve this problem so here is that.

$$z^2 = x^2 + y^2$$

Step 3

Finally, let's differentiate this with respect to t and we can even solve it for z' so the actual solution will be quick and simple to find.

$$2z\,z' = 2x\,x' + 2y\,y' \qquad \Rightarrow \qquad z' = \frac{x\,x' + y\,y'}{z}$$

Step 4

To finish off this problem all we need to do is determine all three lengths of the triangle in the sketch above. We can find x and y using their speeds and time while we can find z by reusing the Pythagorean Theorem.

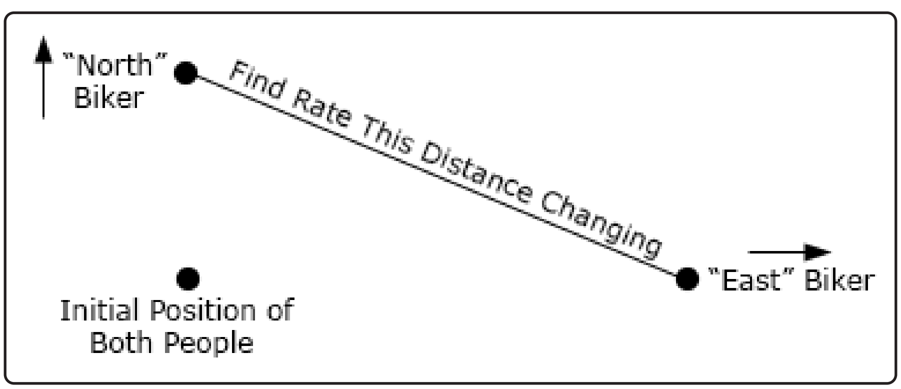
$$x = (2)(15) = 30 \qquad\qquad y = (7)(15) = 105$$
$$z = \sqrt{30^2 + 105^2} = \sqrt{11925} = 15\sqrt{53} = 109.2016$$

The rate of change of the distance between the two people is then,

$$z' = \frac{(30)(2) + (105)(7)}{109.2016} = \boxed{7.2801}$$

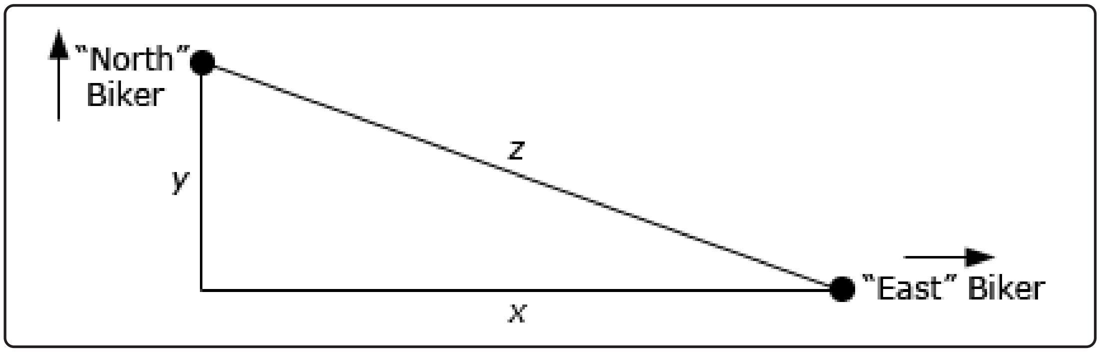
---

8. Two people on bikes are at the same place. One of the bikers starts riding directly north at a rate of 8 m/sec. Five seconds after the first biker started riding north the second starts to ride directly east at a rate of 5 m/sec. At what rate is the distance between the two riders increasing 20 seconds after the second person started riding?



Step 1

Here is a sketch of this situation.



We want to determine z' after 20 seconds after the second biker starts riding east given that $x' = 5$, $y' = 8$ and assuming that they start at the same point.

Step 2

Hopefully it's clear that we'll need the Pythagorean Theorem to solve this problem so here is that.

$$z^2 = x^2 + y^2$$

Step 3

Finally, let's differentiate this with respect to t and we can even solve it for z' so the actual solution will be quick and simple to find.

$$2z\,z' = 2x\,x' + 2y\,y' \qquad \Rightarrow \qquad z' = \frac{x\,x' + y\,y'}{z}$$

Step 4

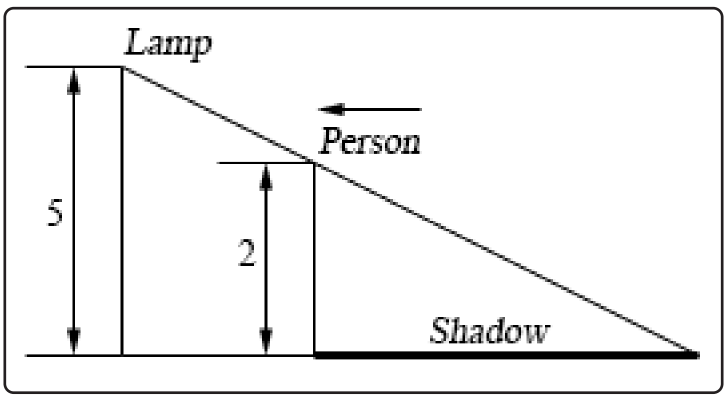
To finish off this problem all we need to do is determine all three lengths of the triangle in the sketch above. We can find x and y using their speeds and time while we can find z by reusing the Pythagorean Theorem. Note that the biker riding east will be riding for 20 seconds and the biker riding north will be riding for 25 seconds (this biker started 5 seconds earlier...).

$$x = (5)(20) = 100 \qquad y = (8)(25) = 200$$
$$z = \sqrt{100^2 + 200^2} = \sqrt{50000} = 100\sqrt{5} = 223.6068$$

The rate of change of the distance between the two people is then,

$$z' = \frac{(100)(5) + (200)(8)}{223.6068} = \boxed{9.3915}$$

9. A light is mounted on a wall 5 meters above the ground. A 2 meter tall person is initially 10 meters from the wall and is moving towards the wall at a rate of 0.5 m/sec. After 4 seconds of moving is the tip of the shadow moving **(a)** towards or away from the person and **(b)** towards or away from the wall?

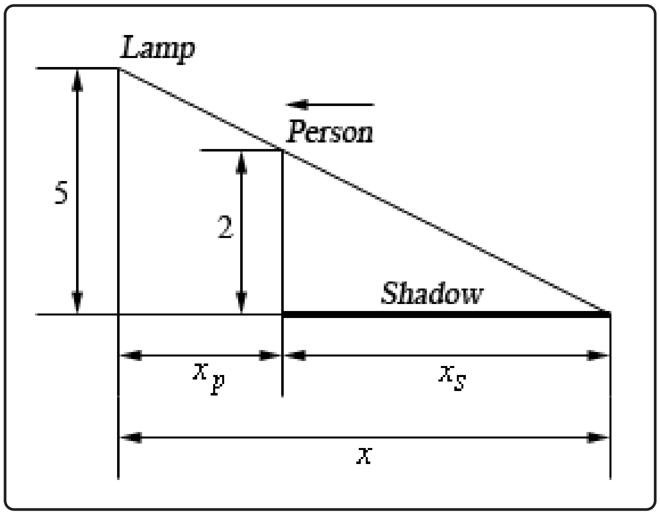


Solutions

(a) After 4 seconds of moving is the tip of the shadow towards or away from the person?

Step 1

Here is a sketch for this situation that will work for both parts so we'll put it here. Also note that we know that $x_p' = -0.5$ for both parts.



Step 2

In this case we want to determine x_s' when $x_p = 10 - 4(0.5) = 8$ (although it will turn out that we simply don't need this piece of information for this problem....).

We can use the idea of similar triangles to get the following equation.

$$\frac{2}{5} = \frac{x_s}{x} = \frac{x_s}{x_p + x_s}$$

If we solve this for x_s we arrive at,

$$\begin{aligned} \frac{2}{5}(x_p + x_s) &= x_s \\ \frac{2}{5}x_p + \frac{2}{5}x_s &= x_s \qquad \Rightarrow \qquad x_s = \frac{2}{3}x_p \end{aligned}$$

This equation will work perfectly for us.

Step 3

Differentiation with respect to t will give us,

$$x_s' = \frac{2}{3}x_p'$$

Step 4

Finishing off this problem is very simple as all we need to do is plug in the known speed.

$$x_s' = \frac{2}{3}(-0.5) = -\frac{1}{3}$$

Because this rate is negative we can see that the tip of the shadow is moving towards the person at a rate of $\frac{1}{3}$ m/s.

(b) After 4 seconds of moving is the tip of the shadow towards or away from the wall?

Step 1

In this case we want to determine x' and the equation is really simple. All we need is,

$$x = x_p + x_s$$

Step 2

Differentiation with respect to t will give us,

$$x' = x_p' + x_s'$$

Step 3

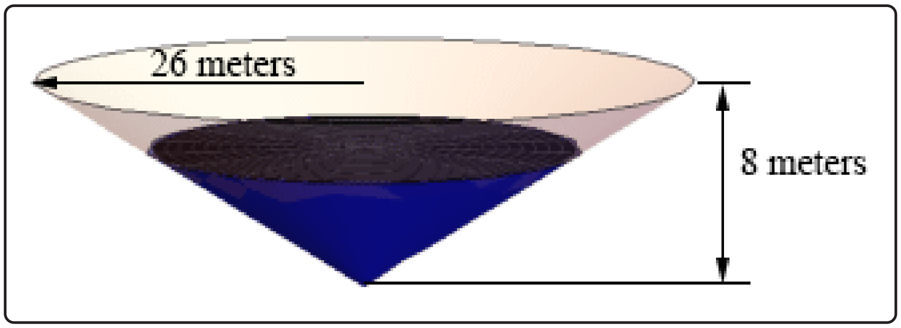
Finishing off this problem is very simple as all we need to do is plug in the known speeds and note that we will need to result from the first part here. So we have $x_p' = -\frac{1}{2}$ from the problem statement and $x_s' = -\frac{1}{3}$ from the previous part.

$$x' = -\frac{1}{2} + \left(-\frac{1}{3}\right) = -\frac{5}{6}$$

Because this rate is negative we can see that the tip of the shadow is moving towards the wall at a rate of $\frac{5}{6}$ m/s.

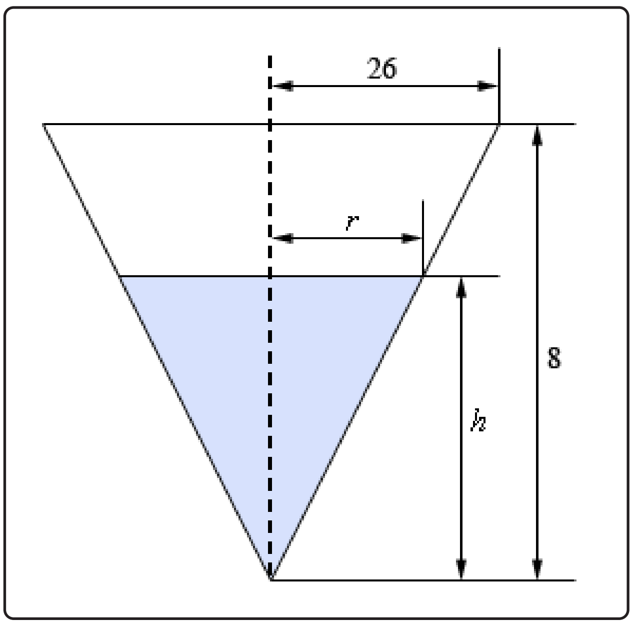
---

10. A tank of water in the shape of a cone is being filled with water at a rate of 12 m^3/sec. The base radius of the tank is 26 meters and the height of the tank is 8 meters. At what rate is the depth of the water in the tank changing when the radius of the top of the water is 10 meters?



Step 1

Here is a sketch of the cross section of the tank and it is not even remotely to scale as I found it easier to reuse an old image that I had lying around. I can be a little lazy sometimes. At least I was less lazy with the image in the problem statement....



We want to determine h' when $r = 10$ and we know that $V' = 12$.

Step 2

We'll need the equation for the volume of a cone.

$$V = \frac{1}{3}\pi r^2 h$$

This is a problem however as it has both r and h in it and it would be best to have only h since we need h'. We can use similar triangles to fix this up. Based on similar triangles we get the following equation which can be solved for r.

$$\frac{r}{h} = \frac{26}{8} \qquad \Rightarrow \qquad r = \frac{13}{4}h$$

Plugging this into the volume equation gives,

$$V = \frac{169}{48}\pi h^3$$

Step 3

Next, let's differentiate this with respect to t.

$$V' = \frac{169}{16}\pi h^2 h'$$

Step 4

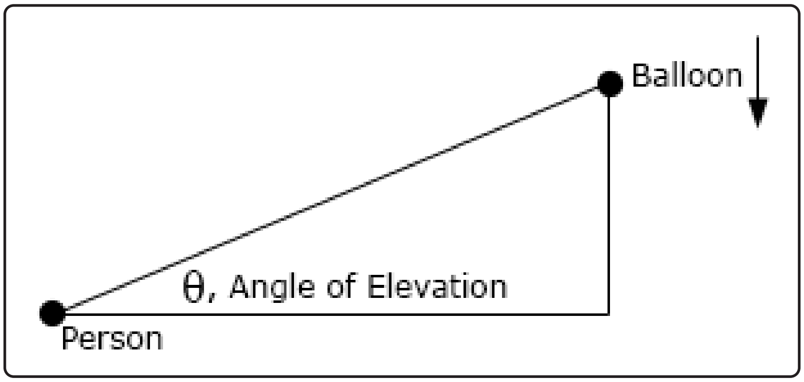
To finish off this problem all we need to do is determine the value of h for the time we are interested in. This can easily be done from the similar triangle equation and the fact that we know $r = 10$.

$$h = \frac{4}{13}r = \frac{4}{13}(10) = \frac{40}{13}$$

The rate of change of the height of the water is then,

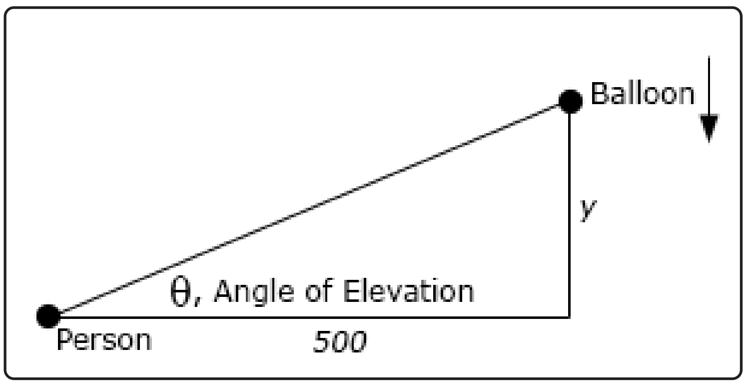
$$12 = \frac{169}{16}\pi\left(\frac{40}{13}\right)^2 h' = 100\pi h' \qquad \Rightarrow \qquad \boxed{h' = \frac{3}{25\pi}}$$

11. The angle of elevation is the angle formed by a horizontal line and a line joining the observer's eye to an object above the horizontal line. A person is 500 feet way from the launch point of a hot air balloon. The hot air balloon is starting to come back down at a rate of 15 ft/sec. At what rate is the angle of elevation, θ, changing when the hot air balloon is 200 feet above the ground.



Step 1

Putting variables and known quantities on the sketch from the problem statement gives,



We want to determine θ' when $y = 200$ and we know that $y' = -15$.

Step 2

There are a variety of equations that we could use here but probably the best one that involves all of the known and needed quantities is,

$$\tan(\theta) = \frac{y}{500}$$

Step 3

Differentiating with respect to t gives,

$$\sec^2(\theta)\ \theta' = \frac{y'}{500} \qquad \Rightarrow \qquad \theta' = \frac{y'}{500}\cos^2(\theta)$$

Step 4

To finish off this problem all we need to do is determine the value of θ for the time in question. We can either use the original equation to do this or we could acknowledge that all we really need is $\cos(\theta)$ and we could do a little right triangle trig to determine that.

For this problem we'll just use the original equation to find the value of θ.

$$\tan(\theta) = \frac{200}{500} \qquad \Rightarrow \qquad \theta = \tan^{-1}\left(\frac{2}{5}\right) = 0.38051 \text{ radians}$$

The rate of change of the angle of elevation is then,

$$\theta' = \frac{-15}{500}\cos^2(0.38051) = \boxed{-0.02586}$$

3.12 Higher Order Derivatives

1. Determine the fourth derivative of $h(t) = 3t^7 - 6t^4 + 8t^3 - 12t + 18$

Step 1

Not much to this problem other than to take four derivatives so each step will show each successive derivative until we get to the fourth. The first derivative is then,

$$h'(t) = 21t^6 - 24t^3 + 24t^2 - 12$$

Step 2

The second derivative is,

$$h''(t) = 126t^5 - 72t^2 + 48t$$

Step 3

The third derivative is,

$$h'''(t) = 630t^4 - 144t + 48$$

Step 4

The fourth, and final derivative for this problem, is,

$$\boxed{h^{(4)}(t) = 2520t^3 - 144}$$

---

2. Determine the fourth derivative of $V(x) = x^3 - x^2 + x - 1$

Step 1

Not much to this problem other than to take four derivatives so each step will show each successive derivative until we get to the fourth. The first derivative is then,

$$V'(x) = 3x^2 - 2x + 1$$

Step 2

The second derivative is,

$$V''(x) = 6x - 2$$

Step 3

The third derivative is,

$$V'''(x) = 6$$

Step 4

The fourth, and final derivative for this problem, is,

$$\boxed{V^{(4)}(x) = 0}$$

Note that we could have just as easily used the Fact from the notes to arrive at this answer in one step.

---

3. Determine the fourth derivative of $f(x) = 4\sqrt[5]{x^3} - \dfrac{1}{8x^2} - \sqrt{x}$

Step 1

Not much to this problem other than to take four derivatives so each step will show each successive derivative until we get to the fourth. After a quick rewrite of the function to help with the differentiation the first derivative is,

$$f(x) = 4x^{\frac{3}{5}} - \frac{1}{8}x^{-2} - x^{\frac{1}{2}} \qquad \Rightarrow \qquad f'(x) = \frac{12}{5}x^{-\frac{2}{5}} + \frac{1}{4}x^{-3} - \frac{1}{2}x^{-\frac{1}{2}}$$

Step 2

The second derivative is,

$$f''(x) = -\frac{24}{25}x^{-\frac{7}{5}} - \frac{3}{4}x^{-4} + \frac{1}{4}x^{-\frac{3}{2}}$$

Step 3

The third derivative is,

$$f'''(x) = \frac{168}{125}x^{-\frac{12}{5}} + 3x^{-5} - \frac{3}{8}x^{-\frac{5}{2}}$$

Step 4

The fourth, and final derivative for this problem, is,

$$\boxed{f^{(4)}(x) = -\frac{2016}{625}x^{-\frac{17}{5}} - 15x^{-6} + \frac{15}{16}x^{-\frac{7}{2}}}$$

---

4. Determine the fourth derivative of $f(w) = 7\sin\left(\frac{w}{3}\right) + \cos(1-2w)$

Step 1

Not much to this problem other than to take four derivatives so each step will show each successive derivative until we get to the fourth. The first derivative is then,

$$f'(w) = \frac{7}{3}\cos\left(\frac{w}{3}\right) + 2\sin(1-2w)$$

Step 2

The second derivative is,

$$f''(w) = -\frac{7}{9}\sin\left(\frac{w}{3}\right) - 4\cos(1-2w)$$

Step 3

The third derivative is,

$$f'''(w) = -\frac{7}{27}\cos\left(\frac{w}{3}\right) - 8\sin(1-2w)$$

Step 4

The fourth, and final derivative for this problem, is,

$$\boxed{f^{(4)}(w) = \frac{7}{81}\sin\left(\frac{w}{3}\right) + 16\cos(1-2w)}$$

5. Determine the fourth derivative of $y = \mathbf{e}^{-5z} + 8\ln\left(2z^4\right)$

Step 1

Not much to this problem other than to take four derivatives so each step will show each successive derivative until we get to the fourth. The first derivative is then,

$$\frac{dy}{dz} = -5\mathbf{e}^{-5z} + 8\left(\frac{8z^3}{2z^4}\right) = -5\mathbf{e}^{-5z} + 32z^{-1}$$

Step 2

The second derivative is,

$$\frac{d^2y}{dz^2} = 25\mathbf{e}^{-5z} - 32z^{-2}$$

Step 3

The third derivative is,

$$\frac{d^3y}{dz^3} = -125\mathbf{e}^{-5z} + 64z^{-3}$$

Step 4

The fourth, and final derivative for this problem, is,

$$\boxed{\frac{d^4y}{dz^4} = 625\mathbf{e}^{-5z} - 192z^{-4}}$$

6. Determine the second derivative of $g(x) = \sin(2x^3 - 9x)$

Step 1

Not much to this problem other than to take two derivatives so each step will show each successive derivative until we get to the second. The first derivative is then,

$$g'(x) = (6x^2 - 9)\cos(2x^3 - 9x)$$

Step 2

Do not forget that often we will end up needing to do a product rule in the second derivative even though we did not need to do that in the first derivative. The second derivative is then,

$$\boxed{g''(x) = 12x\cos(2x^3 - 9x) - (6x^2 - 9)^2\sin(2x^3 - 9x)}$$

7. Determine the second derivative of $z = \ln(7 - x^3)$

Step 1

Not much to this problem other than to take two derivatives so each step will show each successive derivative until we get to the second. The first derivative is then,

$$\frac{dz}{dx} = \frac{-3x^2}{7 - x^3}$$

Step 2

Do not forget that often we will end up needing to do a quotient rule in the second derivative even though we did not need to do that in the first derivative. The second derivative is then,

$$\frac{d^2z}{dx^2} = \frac{-6x\left(7-x^3\right)-\left(-3x^2\right)\left(-3x^2\right)}{\left(7-x^3\right)^2} = \boxed{\frac{-42x-3x^4}{\left(7-x^3\right)^2}}$$

---

8. Determine the second derivative of $Q\left(v\right) = \dfrac{2}{\left(6+2v-v^2\right)^4}$

Step 1

Not much to this problem other than to take two derivatives so each step will show each successive derivative until we get to the second. We'll do a quick rewrite of the function to help with the derivatives and then the first derivative is,

$$Q\left(v\right) = 2\left(6+2v-v^2\right)^{-4}$$
$$Q'\left(v\right) = -8\left(2-2v\right)\left(6+2v-v^2\right)^{-5}$$

Step 2

Do not forget that often we will end up needing to do a product rule in the second derivative even though we did not need to do that in the first derivative. The second derivative is then,

$$\boxed{Q''\left(v\right) = 16\left(6+2v-v^2\right)^{-5} + 40(2-2v)^2\left(6+2v-v^2\right)^{-6}}$$

---

9. Determine the second derivative of $H\left(t\right) = \cos^2\left(7t\right)$

Step 1

Not much to this problem other than to take two derivatives so each step will show each successive derivative until we get to the second. The first derivative is then,

$$H'\left(t\right) = -14\cos\left(7t\right)\sin\left(7t\right)$$

Step 2

Do not forget that often we will end up needing to do a product rule in the second derivative even though we did not need to do that in the first derivative. The second derivative is then,

$$H''(t) = 98\sin(7t)\sin(7t) - 98\cos(7t)\cos(7t) = \boxed{98\sin^2(7t) - 98\cos^2(7t)}$$

Note that, in this case, if we recall our trig formulas we could have reduced the product in the first derivative to a single trig function which would have then allowed us to avoid the product rule for the second derivative. Can you figure out what the formula is?

---

10. Determine the second derivative of $2x^3 + y^2 = 1 - 4y$

Step 1

Not much to this problem other than to take two derivatives so each step will show each successive derivative until we get to the second. Note however that we are going to have to do implicit differentiation to do each derivative.

Here is the work for the first derivative. If you need a refresher on implicit differentiation go back to that section and check some of the problems in that section.

$$6x^2 + 2y\,y' = -4y'$$

$$(2y+4)\,y' = -6x^2 \qquad \Rightarrow \qquad y' = \frac{-6x^2}{2y+4} = \frac{-3x^2}{y+2}$$

Step 2

Now, the second derivative will also need implicit differentiation. Note as well that we can work with the first derivative in its present form which will require the quotient rule or we can rewrite it as,

$$y' = -3x^2(y+2)^{-1}$$

and use the product rule.

These get messy enough as it is so we'll go with the product rule to try and keep the "mess" down a little. Using implicit differentiation to take the derivative of first derivative gives,

$$y'' = \frac{d}{dx}\left(y'\right) = -6x(y+2)^{-1} + 3x^2(y+2)^{-2}y'$$

Step 3

Finally, recall that we don't want a y' in the second derivative so to finish this out we need to plug in the formula for y' (which we know...) and do a little simplifying to get the final answer.

$$\begin{aligned} y'' &= -6x(y+2)^{-1} + 3x^2(y+2)^{-2}\left(-3x^2(y+2)^{-1}\right) \\ &= \boxed{-6x(y+2)^{-1} - 9x^4(y+2)^{-3}} \end{aligned}$$

11. Determine the second derivative of $6y - xy^2 = 1$

Step 1

Not much to this problem other than to take two derivatives so each step will show each successive derivative until we get to the second. Note however that we are going to have to do implicit differentiation to do each derivative.

Here is the work for the first derivative. If you need a refresher on implicit differentiation go back to that section and check some of the problems in that section.

$$\begin{aligned} 6y' - y^2 - 2xy\,y' &= 0 \\ (6-2xy)\,y' &= y^2 \qquad \Rightarrow \qquad y' = \frac{y^2}{6-2xy} \end{aligned}$$

Step 2

Now, the second derivative will also need implicit differentiation. Note as well that we can work with the first derivative in its present form which will require the quotient rule or we can rewrite it as,

$$y' = y^2(6-2xy)^{-1}$$

and use the product rule.

These get messy enough as it is so we'll go with the product rule to try and keep the "mess" down a little. Using implicit differentiation to take the derivative of first derivative gives,

$$y'' = \frac{d}{dx}\left(y'\right) = 2y\,y'(6-2xy)^{-1} - y^2(6-2xy)^{-2}\left(-2y-2xy'\right)$$

Step 3

Finally, recall that we don't want a y' in the second derivative. So, to finish this out let's do a little "simplifying" of the to make it "easier" to plug in the formula for y'.

$$\begin{aligned} y'' &= 2y\,y'(6-2xy)^{-1} + 2y^3(6-2xy)^{-2} + 2xy^2\,y'(6-2xy)^{-2} \\ &= 2y\,y'(6-2xy)^{-1}\left(1+xy(6-2xy)^{-1}\right) + 2y^3(6-2xy)^{-2} \end{aligned}$$

The point of all of this was to get down to a single y' in the formula for the second derivative, which won't always be possible to do, and a little factoring to try and make things a little easier to deal with.

Finally, all we need to do is plug in the formula for y' to get the final answer.

$$\begin{aligned} y'' &= 2y\left[y^2(6-2xy)^{-1}\right](6-2xy)^{-1}\left(1+xy(6-2xy)^{-1}\right) + 2y^3(6-2xy)^{-2} \\ &= \boxed{2y^3(6-2xy)^{-2}\left(1+xy(6-2xy)^{-1}\right) + 2y^3(6-2xy)^{-2}} \end{aligned}$$

Note that for a further simplification step, if we wanted to go further, we could factor a

$$2y^3(6-2xy)^{-2}$$

out of both terms to get,

$$\boxed{y'' = 2y^3(6-2xy)^{-2}\left(2+xy(6-2xy)^{-1}\right)}$$

3.13 Logarithmic Differentiation

1. Use logarithmic differentiation to find the first derivative of $f(x) = \left(5 - 3x^2\right)^7 \sqrt{6x^2 + 8x - 12}$.

Step 1

Take the logarithm of both sides and do a little simplifying.

$$\begin{aligned} \ln\left[f(x)\right] &= \ln\left[\left(5 - 3x^2\right)^7 \sqrt{6x^2 + 8x - 12}\right] \\ &= \ln\left[\left(5 - 3x^2\right)^7\right] + \ln\left[\left(6x^2 + 8x - 12\right)^{\frac{1}{2}}\right] \\ &= 7\ln\left(5 - 3x^2\right) + \frac{1}{2}\ln\left(6x^2 + 8x - 12\right) \end{aligned}$$

Step 2

Use implicit differentiation to differentiate both sides with respect to x.

$$\frac{f'(x)}{f(x)} = 7\frac{-6x}{5 - 3x^2} + \frac{1}{2}\frac{12x + 8}{6x^2 + 8x - 12} = \frac{-42x}{5 - 3x^2} + \frac{6x + 4}{6x^2 + 8x - 12}$$

Step 3

Finally, solve for the derivative and plug in the equation for $f(x)$.

$$\begin{aligned} f'(x) &= f(x)\left[\frac{-42x}{5 - 3x^2} + \frac{6x + 4}{6x^2 + 8x - 12}\right] \\ &= \boxed{\left(5 - 3x^2\right)^7 \sqrt{6x^2 + 8x - 12}\left[\frac{-42x}{5 - 3x^2} + \frac{6x + 4}{6x^2 + 8x - 12}\right]} \end{aligned}$$

2. Use logarithmic differentiation to find the first derivative of $y = \dfrac{\sin\left(3z + z^2\right)}{\left(6 - z^4\right)^3}$.

Step 1

Take the logarithm of both sides and do a little simplifying.

$$\begin{aligned}\ln(y) &= \ln\left[\frac{\sin\left(3z+z^2\right)}{\left(6-z^4\right)^3}\right] = \ln\left[\sin\left(3z+z^2\right)\right] - \ln\left[\left(6-z^4\right)^3\right] \\ &= \ln\left[\sin\left(3z+z^2\right)\right] - 3\ln\left[6-z^4\right]\end{aligned}$$

Step 2

Use implicit differentiation to differentiate both sides with respect to z.

$$\frac{y'}{y} = \frac{(3+2z)\cos\left(3z+z^2\right)}{\sin\left(3z+z^2\right)} - 3\left[\frac{-4z^3}{6-z^4}\right] = (3+2z)\cot\left(3z+z^2\right) + \frac{12z^3}{6-z^4}$$

Step 3

Finally, solve for the derivative and plug in the equation for y .

$$\begin{aligned}y' &= y\left[(3+2z)\cot\left(3z+z^2\right) + \frac{12z^3}{6-z^4}\right] \\ &= \boxed{\frac{\sin\left(3z+z^2\right)}{\left(6-z^4\right)^3}\left[(3+2z)\cot\left(3z+z^2\right) + \frac{12z^3}{6-z^4}\right]}\end{aligned}$$

3. Use logarithmic differentiation to find the first derivative of $h(t) = \dfrac{\sqrt{5t+8}\;\sqrt[3]{1-9\cos(4t)}}{\sqrt[4]{t^2+10t}}$.

Step 1

Take the logarithm of both sides and do a little simplifying.

$$\begin{aligned}
\ln[h(t)] &= \ln\left[\frac{\sqrt{5t+8}\;\sqrt[3]{1-9\cos(4t)}}{\sqrt[4]{t^2+10t}}\right]\\
&= \ln\left[\sqrt{5t+8}\;\sqrt[3]{1-9\cos(4t)}\right] - \ln\left[\sqrt[4]{t^2+10t}\right]\\
&= \ln\left[(5t+8)^{\frac{1}{2}}\right] + \ln\left[(1-9\cos(4t))^{\frac{1}{3}}\right] - \ln\left[\left(t^2+10t\right)^{\frac{1}{4}}\right]\\
&= \frac{1}{2}\ln(5t+8) + \frac{1}{3}\ln(1-9\cos(4t)) - \frac{1}{4}\ln\left(t^2+10t\right)
\end{aligned}$$

Note that the logarithm simplification work was a little complicated for this problem, but if you know your logarithm properties you should be okay with that.

Step 2

Use implicit differentiation to differentiate both sides with respect to t.

$$\frac{h'(t)}{h(t)} = \frac{1}{2}\frac{5}{5t+8} + \frac{1}{3}\frac{36\sin(4t)}{1-9\cos(4t)} - \frac{1}{4}\frac{2t+10}{t^2+10t}$$

Step 3

Finally, solve for the derivative and plug in the equation for $h(t)$.

$$\begin{aligned}
h'(t) &= h(t)\left[\frac{\frac{5}{2}}{5t+8} + \frac{12\sin(4t)}{1-9\cos(4t)} - \frac{\frac{1}{2}t+\frac{5}{2}}{t^2+10t}\right]\\
&= \boxed{\frac{\sqrt{5t+8}\;\sqrt[3]{1-9\cos(4t)}}{\sqrt[4]{t^2+10t}}\left[\frac{\frac{5}{2}}{5t+8} + \frac{12\sin(4t)}{1-9\cos(4t)} - \frac{\frac{1}{2}t+\frac{5}{2}}{t^2+10t}\right]}
\end{aligned}$$

4. Find the first derivative of $g\left(w\right) = \left(3w - 7\right)^{4w}$.

Step 1

We just need to do some logarithmic differentiation so take the logarithm of both sides and do a little simplifying.

$$\ln\left[g\left(w\right)\right] = \ln\left[\left(3w-7\right)^{4w}\right] = 4w\ \ln\left(3w-7\right)$$

Step 2

Use implicit differentiation to differentiate both sides with respect to w. Don't forget to product rule the right side.

$$\frac{g'\left(w\right)}{g\left(w\right)} = 4\ln\left(3w-7\right) + 4w\frac{3}{3w-7} = 4\ln\left(3w-7\right) + \frac{12w}{3w-7}$$

Step 3

Finally, solve for the derivative and plug in the equation for $g\left(w\right)$.

$$g'\left(w\right) = g\left(w\right)\left[4\ln\left(3w-7\right) + \frac{12w}{3w-7}\right]$$
$$= \boxed{\left(3w-7\right)^{4w}\left[4\ln\left(3w-7\right) + \frac{12w}{3w-7}\right]}$$

5. Find the first derivative of $f\left(x\right) = \left(2x - \mathbf{e}^{8x}\right)^{\sin\left(2x\right)}$.

Step 1

We just need to do some logarithmic differentiation so take the logarithm of both sides and do a little simplifying.

$$\ln\left[f\left(x\right)\right] = \ln\left[\left(2x-\mathbf{e}^{8x}\right)^{\sin\left(2x\right)}\right] = \sin\left(2x\right)\ln\left(2x-\mathbf{e}^{8x}\right)$$

Step 2

Use implicit differentiation to differentiate both sides with respect to x. Don't forget to product rule the right side.

$$\frac{f'(x)}{f(x)} = 2\cos(2x)\ln\left(2x - \mathbf{e}^{8x}\right) + \sin(2x)\frac{2 - 8\mathbf{e}^{8x}}{2x - \mathbf{e}^{8x}}$$
$$= 2\cos(2x)\ln\left(2x - \mathbf{e}^{8x}\right) + \sin(2x)\frac{2 - 8\mathbf{e}^{8x}}{2x - \mathbf{e}^{8x}}$$

Step 3

Finally, solve for the derivative and plug in the equation for $f(x)$.

$$f'(x) = f(x)\left[2\cos(2x)\ln\left(2x - \mathbf{e}^{8x}\right) + \sin(2x)\frac{2 - 8\mathbf{e}^{8x}}{2x - \mathbf{e}^{8x}}\right]$$
$$= \boxed{\left(2x - \mathbf{e}^{8x}\right)^{\sin(2x)}\left[2\cos(2x)\ln\left(2x - \mathbf{e}^{8x}\right) + \sin(2x)\frac{2 - 8\mathbf{e}^{8x}}{2x - \mathbf{e}^{8x}}\right]}$$

4 Derivative Applications

In the previous chapter we focused almost exclusively, with the exception of Related Rates, on the computation and interpretation of derivatives. In this chapter will focus on applications of derivatives and it is important to always remember that we didn't spend a whole chapter talking about how to compute derivatives just to be talking about them. Each of the applications here will require us to compute at least one derivative and that computation will often be the very first step in the problem. So, if you are rusty with your differentiation skills you will need to go back to the previous chapter and scrap some of that rust off, so to speak, or you will find yourself struggling a lot in this chapter.

There are quite a few important applications to derivatives but almost all of them require the use of something called a critical point. So, we will first need to define just what a critical point is and make sure we are comfortable with finding critical points.

Once we have a good understanding of critical points we will turn our focus to the first application that we'll be spending quite bit of time on in this chapter. Namely, how we can use derivatives to find some important information about a function. We already know how to determine if a function is increasing or decreasing as we discussed that and worked a few problems on that in the last chapter. We will work some more problems involving increasing and decreasing functions to make sure we are clear on how that works. As we know from the last chapter we use the first derivative to determine where a function is increasing and decreasing. So we will then move on to see what the second derivative can tell us about a function. As we will see the second derivative can be used to determine the concavity of a function. The concavity of a function gives, in some way, the "curvature" of a function.

Once we have discussed all the information that derivatives can tell us about a function we'll use that information to get a sketch of the graph of a function without any kind of computational aid outside of occasionally needing a calculator to compute the value of the function at a few points. As we'll see we will often get a fairly good sketch of the graph from just this information.

The other topic that we will focus on in this chapter will be optimizing functions. By optimizing a function we mean finding the minimum and maximum value that a function can take. In addition, we will, on occasion, include a constraint on the function we are trying to optimize. The constraint will be an additional equation that the variable(s) in the function we are optimizing must also satisfy.

We will also take a quick look at a couple of other applications. These will include linear approximations (*i.e.* find a linear function that can approximate the function for at least a range of variables), Newton's Method (*i.e.* approximating the solution to an equation) as well as a couple of applications of derivatives to some business applications.

We will also briefly revisit limits to discuss L'Hospital's Rule. This is a method of computing some limits of functions that are of "indeterminate form" (defined later) that we cannot, at this point, computer. A valid question is why did we not discuss L'Hospital's Rule back in the Limits chapter? That is a valid question and it has a simple answer. We couldn't discuss L'Hospitals's Rule until this point because it involved taking some derivatives which we (clearly) did not yet know when we first looked at limits.

The following sections are the practice problems, with solutions, for this material.

4.1 Rates of Change

As noted in the text for this section the purpose of this section is only to remind you of certain types of applications that were discussed in the previous chapter. As such there aren't any problems written for this section. Instead here is a list of links (note that these will only be active links in the web version and not the pdf version) to problems from the relevant sections from the previous chapter.

Each of the following sections has a selection of increasing/decreasing problems towards the bottom of the problem set.

Differentiation Formulas

Product & Quotient Rules

Derivatives of Trig Functions

Derivatives of Exponential and Logarithm Functions

Chain Rule

Related Rates problems are in the Related Rates section.

4.2 Critical Points

1. Determine the critical points of $f(x) = 8x^3 + 81x^2 - 42x - 8$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$f'(x) = 24x^2 + 162x - 42 = 6(x+7)(4x-1)$$

Factoring the derivative as much as possible will help with the next step.

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is just a polynomial and we know that exists everywhere and so we don't need to worry about that. So, all we need to do is set the derivative equal to zero and solve for the critical points.

$$6(x+7)(4x-1) = 0 \qquad \Rightarrow \qquad \boxed{x = -7, \quad x = \frac{1}{4}}$$

2. Determine the critical points of $R(t) = 1 + 80t^3 + 5t^4 - 2t^5$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$R'(t) = 240t^2 + 20t^3 - 10t^4 = -10t^2(t+4)(t-6)$$

Factoring the derivative as much as possible will help with the next step.

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is just a polynomial and we know that exists everywhere and so we don't need to worry about that. So, all we need to do is set the derivative equal to zero and solve for the critical points.

$$-10t^2(t+4)(t-6)=0 \qquad \Rightarrow \qquad \boxed{t=0, \quad t=-4, \quad t=6}$$

3. Determine the critical points of $g(w)=2w^3-7w^2-3w-2$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$g'(w)=6w^2-14w-3$$

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is just a polynomial and we know that exists everywhere and so we don't need to worry about that. So, all we need to do is set the derivative equal to zero and solve for the critical points.

$$6w^2-14w-3=0 \qquad \Rightarrow \qquad \boxed{w=\frac{14\pm\sqrt{268}}{12}=\frac{7\pm\sqrt{67}}{6}}$$

As we can see in this case we needed to use the quadratic formula to find the critical points. Not all quadratics will factor so don't forget about the quadratic formula!

4. Determine the critical points of $g(x)=x^6-2x^5+8x^4$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$g'(x)=6x^5-10x^4+32x^3=2x^3\left(3x^2-5x+16\right)$$

Factoring the derivative as much as possible will help with the next step.

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is just a polynomial and we know that exists everywhere and so we don't need to worry about that. So, all we need to do is set the derivative equal to zero and solve for the critical points.

$$2x^3\left(3x^2 - 5x + 16\right) = 0 \qquad \Rightarrow \qquad 2x^3 = 0 \qquad \text{OR} \qquad 3x^2 - 5x + 16 = 0$$

From the first term we clearly see that $x = 0$ is a critical point. The second term does not factor and we we'll need to use the quadratic formula to solve this equation.

$$x = \frac{5 \pm \sqrt{-167}}{6} = \frac{5 \pm \sqrt{167}\,i}{6}$$

Remember that not all quadratics will factor so don't forget about the quadratic formula!

Step 3

Now, recall that we don't use complex numbers in this class and so the solutions from the second term are not critical points. Therefore, the only critical point of this function is,

$$\boxed{x = 0}$$

5. Determine the critical points of $h\left(z\right) = 4z^3 - 3z^2 + 9z + 12$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$h'\left(z\right) = 12z^2 - 6z + 9$$

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is just a polynomial and we know that exists everywhere and so we don't need to worry about that. So, all we need to do is set the derivative equal to zero and solve for the critical points.

$$12z^2 - 6z + 9 = 0 \qquad \Rightarrow \qquad z = \frac{6 \pm \sqrt{-396}}{24} = \frac{1 \pm \sqrt{11}\,i}{4}$$

As we can see in this case we needed to use the quadratic formula to solve the quadratic. Remember that not all quadratics will factor so don't forget about the quadratic formula!

Step 3

Now, recall that we don't use complex numbers in this class and so the solutions are not critical points. Therefore, there are **no critical points** for this function.

Do not get excited about there being no critical points for a function. There is no rule that says that every function has to have critical points!

6. Determine the critical points of $Q\left(x\right) = (2 - 8x)^4\left(x^2 - 9\right)^3$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$\begin{aligned} Q'\left(x\right) &= 4(2-8x)^3\left(-8\right)\left(x^2-9\right)^3 + (2-8x)^4\left(3\right)\left(x^2-9\right)^2\left(2x\right) \\ &= 2(2-8x)^3\left(x^2-9\right)^2\left[-16\left(x^2-9\right) + 3x\left(2-8x\right)\right] \\ &= 2(2-8x)^3\left(x^2-9\right)^2\left[-40x^2 + 6x + 144\right] \\ &= -4(2-8x)^3\left(x^2-9\right)^2\left[20x^2 - 3x - 72\right] \end{aligned}$$

Factoring the derivative as much as possible will help with the next step. For this problem (unlike some of the previous problems) this extra factoring is all but required to make this easier to finish.

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is just a polynomial, (admittedly a somewhat messy polynomial) and we know that exists everywhere and so we don't need to worry about that. So, all we need to do is set the derivative equal to zero and solve for the critical points.

$$-4(2-8x)^3\left(x^2-9\right)^2\left[20x^2-3x-72\right]=0$$

From this we get the following three equations that we need to solve.

$$\begin{aligned}(2-8x)^3&=0\\ \left(x^2-9\right)^2&=0\\ 20x^2-3x-72&=0\end{aligned}$$

For the first two equations all we really need to do is set the quantity inside the parenthesis to zero (the exponent on the parenthesis won't affect the solution) and the third requires the quadratic formula.

$$2-8x=0 \qquad \Rightarrow \qquad \boxed{x=\frac{1}{4}}$$

$$x^2-9=0 \qquad \Rightarrow \qquad \boxed{x=\pm 3}$$

$$20x^2-3x-72=0 \qquad \Rightarrow \qquad x=\frac{3\pm\sqrt{3^2-4\,(20)\,(-72)}}{2\,(20)}=\boxed{\frac{3\pm\sqrt{5769}}{40}}$$

So, we get the 5 critical points boxed in above.

---

7. Determine the critical points of $f\left(z\right)=\dfrac{z+4}{2z^2+z+8}$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$f'\left(z\right)=\frac{(1)\left(2z^2+z+8\right)-(z+4)\,(4z+1)}{\left(2z^2+z+8\right)^2}=\frac{-2z^2-16z+4}{\left(2z^2+z+8\right)^2}=\frac{-2\left(z^2+8z-2\right)}{\left(2z^2+z+8\right)^2}$$

The "-2" was factored out of the numerator only to make it a little nicer for the next step and doesn't really need to be done.

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is a rational expression. Therefore, we know that the derivative will be zero if the numerator is zero (and the denominator is also not zero for the same values of course). We also know that the derivative won't exist if we get division by zero.

So, all we need to do is set the numerator and denominator equal to zero and solve. Note as well that the "-2" we factored out of the numerator will not affect where it is zero and so can be ignored. Likewise, the exponent on the whole denominator will not affect where it is zero and so can also be ignored. This means we need to solve the following two equations.

$$z^2 + 8z - 2 = 0 \qquad \Rightarrow \qquad z = \frac{-8 \pm \sqrt{72}}{2} = -4 \pm 3\sqrt{2}$$

$$2z^2 + z + 8 = 0 \qquad \Rightarrow \qquad z = \frac{-1 \pm \sqrt{-63}}{4} = \frac{-1 \pm \sqrt{63}\,i}{4}$$

As we can see in this case we needed to use the quadratic formula both of the quadratic equations. Remember that not all quadratics will factor so don't forget about the quadratic formula!

Step 3

Now, recall that we don't use complex numbers in this class and so the solutions from where the denominator is zero (*i.e.* the derivative doesn't exist) won't be critical points. Therefore, the only critical points of this function are,

$$\boxed{x = -4 \pm 3\sqrt{2}}$$

---

8. Determine the critical points of $R(x) = \dfrac{1-x}{x^2 + 2x - 15}$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$R'(x) = \frac{(-1)\left(x^2 + 2x - 15\right) - (1-x)(2x+2)}{\left(x^2 + 2x - 15\right)^2} = \frac{x^2 - 2x + 13}{\left(x^2 + 2x - 15\right)^2}$$

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is a rational expression. Therefore, we know that the derivative will be zero if the numerator is zero (and the denominator is also not zero for the same values of course). We also know that the derivative won't exist if we get division by zero.

So, all we need to do is set the numerator and denominator equal to zero and solve. Note that the exponent on the whole denominator will not affect where it is zero and so can be ignored. This means we need to solve the following two equations.

$$x^2 - 2x + 13 = 0 \qquad \Rightarrow \qquad x = \frac{2 \pm \sqrt{-48}}{2} = 1 \pm 2\sqrt{3}\,i$$

$$x^2 + 2x - 15 = (x+5)(x-3) = 0 \qquad \Rightarrow \qquad x = -5,\ 3$$

As we can see in this case we needed to use the quadratic formula on the first quadratic equation. Remember that not all quadratics will factor so don't forget about the quadratic formula!

Step 3

Now, recall that we don't use complex numbers in this class and so the solutions from where the numerator is zero won't be critical points.

Also recall that a point will only be a critical point if the function (not the derivative, but the original function) exists at the point. For this problem we found two values where the derivative doesn't exist, however the function also doesn't exist at these points and so neither of these will be critical points either.

Therefore, this function has **no critical points**. Do not get excited about this when it happens. Not all functions will have critical points!

9. Determine the critical points of $r(y) = \sqrt[5]{y^2 - 6y}$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$r'(y) = \frac{1}{5}(2y-6)\left(y^2-6y\right)^{-\frac{4}{5}} = \frac{2y-6}{5(y^2-6y)^{\frac{4}{5}}}$$

We took the term with the negative exponent to the denominator for the discussion in the next step. While it doesn't really need to be done this will make sure that there are no inadvertent mistakes down the road.

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is a rational expression. Therefore, we know that the derivative will be zero if the numerator is zero (and the denominator is also not zero for the same values of course). We also know that the derivative won't exist if we get division by zero.

So, all we need to do is set the numerator and denominator equal to zero and solve. Note that the exponent on the whole denominator will not affect where it is zero and so can be ignored. This means we need to solve the following two equations.

$$2y - 6 = 0 \qquad \Rightarrow \qquad y = 3$$

$$y^2 - 6y = y\left(y - 6\right) = 0 \qquad \Rightarrow \qquad y = 0, 6$$

Step 3

Note as well that the reason for moving the term to the denominator as we did in the first step is to make it clear that the last two critical points are critical points because the derivative does not exist at those points and not because the derivative is zero at those points. Also note that they are critical points because the function does exist at these points.

Therefore, along with the first critical point (where the derivative is zero), we get the following critical points for this function.

$$\boxed{y = 0,\ 3,\ 6}$$

---

10. Determine the critical points of $h(t) = 15 - (3-t)\left[t^2 - 8t + 7\right]^{\frac{1}{3}}$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$\begin{aligned}
h'(t) &= \left[t^2 - 8t + 7\right]^{\frac{1}{3}} - (3-t)\left(\frac{1}{3}\right)(2t-8)\left[t^2 - 8t + 7\right]^{-\frac{2}{3}} \\
&= \left[t^2 - 8t + 7\right]^{\frac{1}{3}} - \frac{(3-t)(2t-8)}{3\left[t^2 - 8t + 7\right]^{\frac{2}{3}}} \\
&= \frac{3\left(t^2 - 8t + 7\right) - (3-t)(2t-8)}{3\left(t^2 - 8t + 7\right)^{\frac{2}{3}}} \\
&= \frac{5t^2 - 38t + 45}{3\left(t^2 - 8t + 7\right)^{\frac{2}{3}}}
\end{aligned}$$

After differentiating we moved the term with the negative exponent to the denominator and then combined everything into a single term. This will help with the next step considerably.

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist.

Because we moved the term with the negative exponent to the denominator and then combined everything into a single term we now have written the derivative as a rational expression. Therefore, we know that the derivative will be zero if the numerator is zero (and the denominator is also not zero for the same values of course). We also know that the derivative won't exist if we get division by zero.

So, all we need to do is set the numerator and denominator equal to zero and solve. Note that the exponent on the whole denominator will not affect where it is zero and so can be ignored. This means we need to solve the following two equations.

$$5t^2 - 38t + 45 = 0 \qquad \Rightarrow \qquad t = \frac{38 \pm \sqrt{544}}{10} = \frac{19 \pm 2\sqrt{34}}{5}$$

$$t^2 - 8t + 7 = (t-7)(t-1) = 0 \qquad \Rightarrow \qquad t = 1,\ 7$$

Step 3

Note that because we combined all the terms in the derivative into a single term it was much easier to determine the critical points for this function. If we had not combined the terms the solving work would have been more complicated, although not impossible.

Doing this also makes it clear that the last two critical points are critical points because the derivative does not exist at those points and not because the derivative is zero at those points. Also note that they are critical points because the function does exist at these points.

Therefore, along with the first two critical points (where the derivative is zero), we get the following critical points for this function.

$$\boxed{t = 1,\ 7,\ \frac{19 \pm 2\sqrt{34}}{5}}$$

---

11. Determine the critical points of $s(z) = 4\cos(z) - z$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$s'(z) = -4\sin(z) - 1$$

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist.

This derivative exists everywhere and so we don't need to worry about that. Therefore, all we need to do is determine where the derivative is zero. So, all we need to do is solve the equation,

$$-4\sin(z) - 1 = 0 \qquad \to \qquad \sin(z) = -\frac{1}{4} \qquad \to \qquad z = \sin^{-1}\left(-\frac{1}{4}\right) = -0.2527$$

This is the answer we got from a calculator and we could use this or we could use the equivalent positive angle : $2\pi - 0.2527 = 6.0305$. Either can be used, but we'll use the positive one for this problem.

Now, a quick look at a unit circle gives us a second solution of $\pi + 0.2527 = 3.3943$.

Finally, all possible solutions to this equation, and hence, all the critical points of the original function are,

$$\boxed{\begin{array}{ll} z = 6.0305 + 2\pi n & \\ & n = 0, \pm 1, \pm 2, \pm, \ldots \\ z = 3.3943 + 2\pi n & \end{array}}$$

If you don't remember how to solve trig equations you should go back and review those sections in the Review Chapter of the notes.

---

12. Determine the critical points of $f(y) = \sin\left(\frac{y}{3}\right) + \frac{2y}{9}$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$f'(y) = \frac{1}{3}\cos\left(\frac{y}{3}\right) + \frac{2}{9}$$

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist.

This derivative exists everywhere and so we don't need to worry about that. Therefore, all we need to do is determine where the derivative is zero. So, all we need to do is solve the equation,

$$\frac{1}{3}\cos\left(\frac{y}{3}\right) + \frac{2}{9} = 0 \quad \to \quad \cos\left(\frac{y}{3}\right) = -\frac{2}{3} \quad \to \quad \frac{y}{3} = \cos^{-1}\left(-\frac{2}{3}\right) = 2.3005$$

This is the answer we got from a calculator and a quick look at a unit circle gives us a second solution of either -2.3005 or if you want the positive equivalent we could use $2\pi - 2.3005 = 3.9827$. For this problem we'll use the positive one, although the negative one could just as easily be used if you wanted to.

All possible solutions to $\cos\left(\frac{y}{3}\right) = -\frac{2}{3}$ are then,

$$\begin{array}{ll} \frac{y}{3} = 2.3005 + 2\pi n & \\ & n = 0, \pm 1, \pm 2, \pm, \ldots \\ \frac{y}{3} = 3.9827 + 2\pi n & \end{array}$$

Finally solving for y gives all the critical points of the function.

$$\boxed{\begin{array}{ll} y = 6.9015 + 6\pi n & \\ y = 11.9481 + 6\pi n & \quad n = 0, \pm 1, \pm 2, \pm, \ldots \end{array}}$$

If you don't remember how to solve trig equations you should go back and review those sections in the Review Chapter of the notes.

---

13. Determine the critical points of $V(t) = \sin^2(3t) + 1$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$V'(t) = 6\sin(3t)\cos(3t)$$

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist.

This derivative exists everywhere and so we don't need to worry about that. Therefore, all we need to do is determine where the derivative is zero. So, all we need to do is solve the equation,

$$6\sin(3t)\cos(3t) = 0 \qquad \to \qquad \sin(3t) = 0 \quad \text{or} \quad \cos(3t) = 0$$

Step 3

So, we now need to solve these two trig equations.

From a quick look at a unit circle we can see that sine is zero at 0 and π and so all solutions to $\sin(3t) = 0$ are then,

$$\begin{array}{l} 3t = 0 + 2\pi n \\ 3t = \pi + 2\pi n \end{array} \qquad \to \qquad \begin{array}{l} t = \frac{2}{3}\pi n \\ t = \frac{1}{3}\pi + \frac{2}{3}\pi n \end{array} \qquad n = 0, \pm 1, \pm 2, \pm, \ldots$$

Another look at a unit circle and we can see that cosine is zero at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ and so all solutions to cos $(3t) = 0$ are then,

$$3t = \frac{\pi}{2} + 2\pi n \qquad\qquad t = \frac{\pi}{6} + \frac{2}{3}\pi n$$
$$\rightarrow \qquad\qquad n = 0, \pm 1, \pm 2, \pm, \ldots$$
$$3t = \frac{3\pi}{2} + 2\pi n \qquad\qquad t = \frac{\pi}{2} + \frac{2}{3}\pi n$$

Therefore, critical points of the function are,

$$\boxed{t = \frac{2}{3}\pi n, \; t = \frac{1}{3}\pi + \frac{2}{3}\pi n, \; t = \frac{\pi}{6} + \frac{2}{3}\pi n, \; t = \frac{\pi}{2} + \frac{2}{3}\pi n \qquad n = 0, \pm 1, \pm 2, \pm, \ldots}$$

If you don't remember how to solve trig equations you should go back and review those sections in the Review Chapter of the notes.

14. Determine the critical points of $f(x) = 5x\,\mathbf{e}^{9-2x}$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$f'(x) = 5\mathbf{e}^{9-2x} + 5x(-2)\,\mathbf{e}^{9-2x} = 5\mathbf{e}^{9-2x}(1-2x)$$

We did some quick factoring to help with the next step and while it doesn't technically need to be done it will significantly reduce the amount work required in the next step.

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist.

This derivative exists everywhere and so we don't need to worry about that. Therefore, all we need to do is determine where the derivative is zero.

Notice as well that because we know that exponential functions are never zero and so the derivative will only be zero if,

$$1 - 2x = 0 \qquad \rightarrow \qquad x = \frac{1}{2}$$

So, we have a single critical point, $\boxed{x = \frac{1}{2}}$, for this function.

15. Determine the critical points of $g(w) = \mathbf{e}^{w^3 - 2w^2 - 7w}$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$g'(w) = \left(3w^2 - 4w - 7\right)\mathbf{e}^{w^3 - 2w^2 - 7w} = (3w - 7)(w + 1)\,\mathbf{e}^{w^3 - 2w^2 - 7w}$$

We did some quick factoring to help with the next step.

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist.

This derivative exists everywhere and so we don't need to worry about that. Therefore, all we need to do is determine where the derivative is zero.

Notice as well that because we know that exponential functions are never zero and so the derivative will only be zero if,

$$(3w - 7)(w + 1) = 0 \qquad \rightarrow \qquad w = \frac{7}{3},\ -1$$

So, we have a two critical points, $\boxed{w = \frac{7}{3}}$ and $\boxed{w = -1}$ for this function.

16. Determine the critical points of $R(x) = \ln\left(x^2 + 4x + 14\right)$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$R'(x) = \frac{2x + 4}{x^2 + 4x + 14}$$

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is a rational expression.

So, we know that the derivative will be zero if the numerator is zero (and the denominator is also not zero for the same values of course).

We also know that the derivative won't exist if we get division by zero. However, in this case note that the denominator is also the polynomial that is inside the logarithm and so any values of x for which the denominator is zero will not be in the domain of the original function (*i.e.* the function, $R(x)$, won't exist at those values of x because we can't take the logarithm of zero). Therefore, these points will not be critical points and we don't need to bother determining where the derivative will be zero.

So, setting the numerator equal to zero gives,

$$2x + 4 = 0 \qquad \Rightarrow \qquad x = -2$$

Step 3

As a final step we really should check that $R(-2)$ exists since there is always a chance that it won't since we are dealing with a logarithm. It does exist ($R(-2) = \ln(10)$) and so the only critical point for this function is,

$$\boxed{x = -2}$$

---

17. Determine the critical points of $A(t) = 3t - 7\ln(8t + 2)$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$A'(t) = 3 - 7\left(\frac{8}{8t + 2}\right) = 3 - \frac{56}{8t + 2} = \frac{24t - 50}{8t + 2}$$

We did quite a bit of simplification of the derivative to help with the next step. While not technically required it will mean the next step will be a fair amount simpler to do.

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is a rational expression.

So, we know that the derivative will be zero if the numerator is zero (and the denominator is also not zero for the same values of course).

We also know that the derivative won't exist if we get division by zero. However, in this case note that the denominator is also the polyomial that is inside the logarithm and so any values of t for which the denominator is zero (*i.e.* $t = -\frac{1}{4}$ since it's easy to see that point) will not be in the domain of the original function (*i.e.* the function, $A\left(-\frac{1}{4}\right)$, won't exist because we can't take the logarithm of zero). Therefore, this point will not be a critical point.

So, setting the numerator equal to zero gives,

$$24t - 50 = 0 \qquad \Rightarrow \qquad t = \frac{25}{12}$$

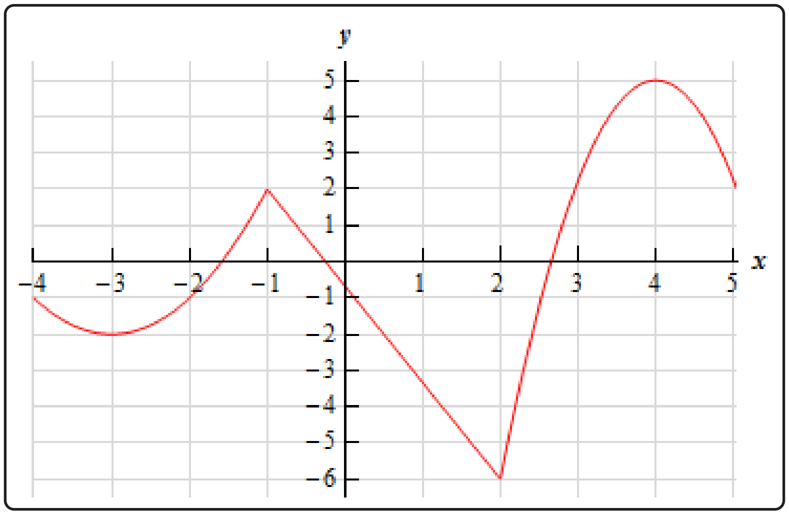
Step 3

As a final step we really should check that $A\left(\frac{25}{12}\right)$ exists since there is always a chance that it won't since we are dealing with a logarithm. It does exist ($A\left(\frac{25}{12}\right) = \frac{75}{12} - 7\ln\left(\frac{65}{3}\right)$) and so the only critical point for this function is,

$$\boxed{t = \frac{25}{12}}$$

4.3 Minimum and Maximum Values

1. Below is the graph of some function, $f(x)$. Identify all of the relative extrema and absolute extrema of the function.



Solution

There really isn't all that much to this problem. We know that absolute extrema are the highest/lowest point on the graph and that they may occur at the endpoints or in the interior of the graph. Relative extrema on the other hand, are "humps" or "bumps" in the graph where in the region around that point the "bump" is a maximum or minimum. Also recall that relative extrema only occur in the interior of the graph and not at the end points of the interval.

Also recall that relative extrema can also be absolute extrema.

So, we have the following absolute/relative extrema.

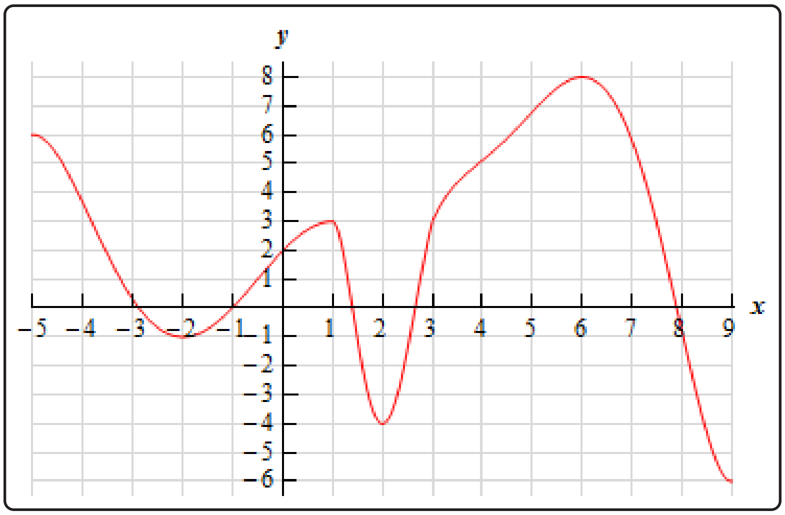
Absolute Maximum : $(4, 5)$

Absolute Minimum : $(2, -6)$

Relative Maximums : $(-1, 2)$ and $(4, 5)$

Relative Minimums : $(-3, -2)$ and $(2, -6)$

2. Below is the graph of some function, $f(x)$. Identify all of the relative extrema and absolute extrema of the function.



Solution

There really isn't all that much to this problem. We know that absolute extrema are the highest/lowest point on the graph and that they may occur at the endpoints or in the interior of the graph. Relative extrema on the other hand, are "humps" or "bumps" in the graph where in the region around that point the "bump" is a maximum or minimum. Also recall that relative extrema only occur in the interior of the graph and not at the end points of the interval.

Also recall that relative extrema can also be absolute extrema.

So, we have the following absolute/relative extrema.

Absolute Maximum : $(6, 8)$

Absolute Minimum : $(9, -6)$

Relative Maximums : $(1, 3)$ and $(6, 8)$

Relative Minimums : $(-2, -1)$ and $(2, -4)$

3. Sketch the graph of $g(x) = x^2 - 4x$ and identify all the relative extrema and absolute extrema of the function on each of the following intervals.

   (a) $(-\infty, \infty)$

   (b) $[-1, 4]$

   (c) $[1, 3]$

   (d) $[3, 5]$

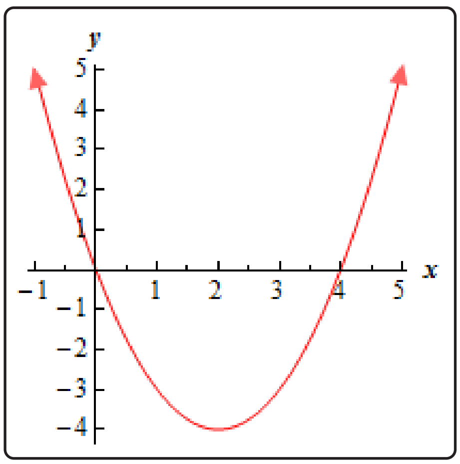
   (e) $(-1, 5]$

Solutions

(a) $(-\infty, \infty)$

Solution

Here's a graph of the function on the interval.



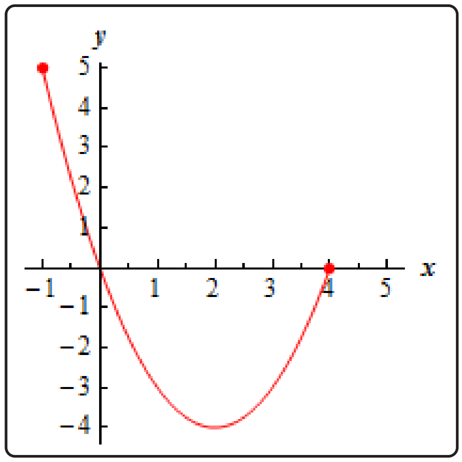
If you don't recall how to graph parabolas you should check out the section on graphing parabolas in the Algebra notes.

So, on the interval $(-\infty, \infty)$, we can clearly see that there are **no absolute maximums** (the graph increases without bounds on both the left and right side of the graph.). There are also **no relative maximums** (there are no "bumps" in which the graph is a maximum in the region around the point). The point $(2, -4)$ is both a **relative minimum** and an **absolute minimum**.

(b) $[-1, 4]$

Solution

Here's a graph of the function on this interval.

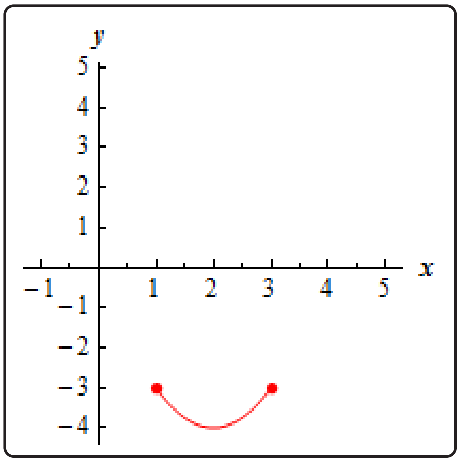


The point $(2, -4)$ is still both a **relative minimum** and an **absolute minimum**. There are still **no relative maximums**. However, because we are now working on a closed interval (*i.e.* we are working on an interval with finite endpoints and we are including the endpoints) we can see that we have an **absolute maximum** at the point $(-1, 5)$.

(c) $[1, 3]$

Solution

Here's a graph of the function on this interval.



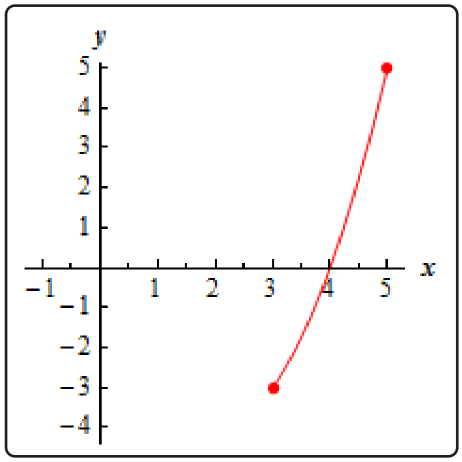
The point $(2, -4)$ is still both a **relative minimum** and an **absolute minimum**. There are still **no relative maximums** of the function on this interval. However, because we are now working on a closed interval (*i.e.* we are working on an interval with finite endpoints and we are including the endpoints) we can see that we have an **absolute maximum** that occurs at the points $(1, -3)$ and $(3, -3)$.

Recall that while there can only be one absolute maximum value of a function (or minimum value if that is the case) it can occur at more than one point.

(d) $[3,5]$

Solution

Here's a graph of the function on this interval.

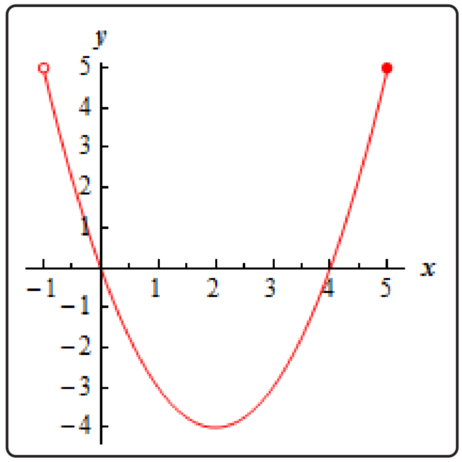


On this interval we clearly do not have any "bumps" in the interior of the interval and so, for this interval, there are **no relative extrema** of the function on this interval. However, we are working on a closed interval and so we can clearly see that there is an **absolute maximum** at the point $(5,5)$ and an **absolute minimum** at the point $(3,-3)$.

(e) $(-1, 5]$

Solution

Here's a graph of the function on this interval.



The point $(2, -4)$ is both a **relative minimum** and an **absolute minimum**. There are **no relative maximums** of the function on this interval.

For the absolute maximum we need to be a little careful however. In this case we are including the right endpoint of the interval, but not the left endpoint. Therefore, there is an **absolute maximum** at the point $(5, 5)$. There is not, however, an absolute maximum at the left point because that point is not being included in the interval.

Because we are not including the left endpoint in the interval and so x will get closer and closer to $x = -1$ without actually reaching $x = -1$. This means that while the graph will get closer and closer to $y = 5$ it will never actually reach $y = 5$ and so there will not be an absolute maximum at the left end point.

4. Sketch the graph of $h(x) = -(x+4)^3$ and identify all the relative extrema and absolute extrema of the function on each of the following intervals.

(a) $(-\infty, \infty)$

(b) $[-5.5, -2]$

(c) $[-4, -3)$

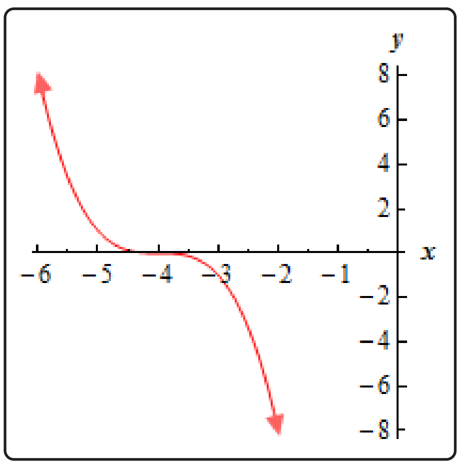
(d) $[-4, -3]$

Solutions

(a) $(-\infty, \infty)$

Solution

Here's a graph of the function on the interval.



To graph this recall the transformations of graphs. The "-" in front simply reflects the graph of x^3 about the x-axis and the "+4" shifts that graph 4 units to the left.

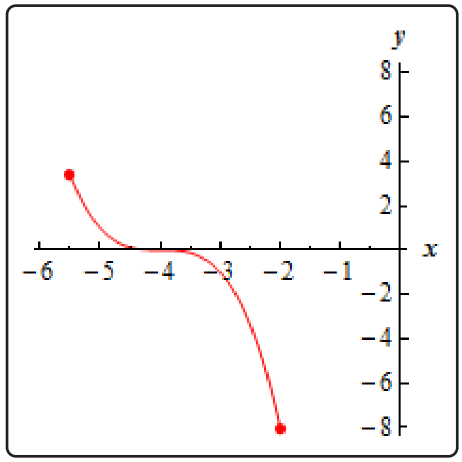
So, on the interval $(-\infty, \infty)$, we can clearly see that there are **no absolute extrema** (the graph increases/decreases without bounds on both the left/right side of the graph.). There are also **no relative extrema** (there are no "bumps" in which the graph is a maximum or minimum in the region around the point).

Don't get so locked into functions having to have extrema of some kind. There are all sorts of graphs that do not have absolute or relative extrema. This is one of those.

(b) $[-5.5, -2]$

Solution

Here's a graph of the function on this interval.



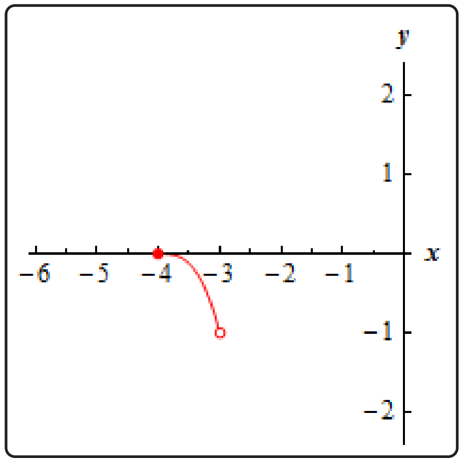
As with the first part we still have no **relative extrema**. However, because we are now working on a closed interval (*i.e.* we are working on an interval with finite end-points and we are including the endpoints) we can see that we will have **absolute extrema** in the interval.

We will have an **absolute maximum** at the point $(-5.5,\ 3.375)$ and an **absolute minimum** at the point $(-2, -8)$.

(c) $[-4, -3)$

Solution

Here's a graph of the function on this interval.



We still have **no relative extrema** for this function.

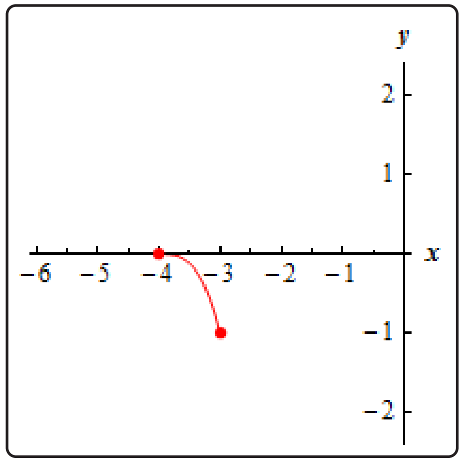
Because we are including the left endpoint in the interval we can see that we have an **absolute maximum** at the point $(-4, 0)$.

We need to be careful with the right endpoint however. It may look like we have an absolute minimum at that point, but we don't. We are not including $x = -3$ in our interval. What this means is that we are going to continue to take values of x that are closer and closer to $x = -3$ and graphing them, but we aren't going to ever reach $x = -3$. Therefore, technically, the graph will continually decreases without ever actually reaching a final value. It will get closer and closer to -1, but will never actually reach that point. What this means for us is that there will be **no absolute minimum** of the function on the given interval.

(d) $[-4, -3]$

Solution

Here is a graph of the function on this interval.



Note that the only difference between this part and the previous part is that we are now including the right endpoint in the interval. Because of that most of the answers here are identical to part **(c)**.

There are **no relative extrema** of the function on the interval and there is an **absolute maximum** at the point $(-4, 0)$.

Now, unlike part **(c)** we are including $x = -3$ in the interval and so the graph will reach a final point, so to speak, as we move to the right. Therefore, for this interval, we have an **absolute minimum** at the point $(-3, -1)$.

---

5. Sketch the graph of some function on the interval $[1, 6]$ that has an absolute maximum at $x = 6$ and an absolute minimum at $x = 3$.

Hint

Do not let the apparent difficulty of this problem fool you. It's not asking us to find an actual function that meets these conditions. It's only asking for a graph that meets the conditions and we know what absolute extrema look like so just start sketching and keep in mind what the conditions are.

Step 1

So, we need a graph of some function (not the function itself, only the graph). The graph must be on the interval $[1, 6]$ and must have absolute extrema at the specified points.

By this point we should have seen enough sketches of graphs to have a pretty good idea of what absolute minimums that are not at the endpoints of an interval should look like on a graph. Therefore, we should know basically what the graph should look like at

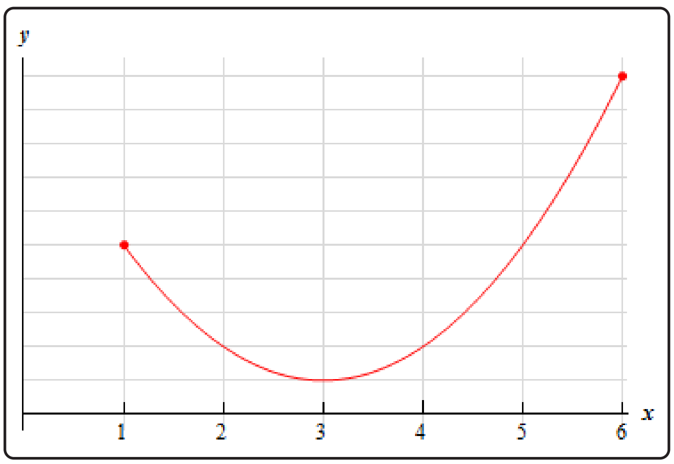
$x = 3$.

Next, we know that the absolute maximum must occur at the right end point of the interval and so all we need to do is sketch a curve from the absolute minimum up to the right endpoint and make sure that the graph at the right endpoint is simply higher than every other point on the graph.

For the graph to the left of the absolute minimum we can sketch in pretty much anything until we reach the left end point, we just need to make sure that no portion of it goes below the absolute minimum or above the absolute maximum.

Step 2

There are literally an infinite number of graphs that we could do here. Some will be more complicated that others, but here is probably one of the simpler graphs that we could use here.



6. Sketch the graph of some function on the interval $[-4, 3]$ that has an absolute maximum at $x = -3$ and an absolute minimum at $x = 2$.

Hint

Do not let the apparent difficulty of this problem fool you. It's not asking us to find an actual function that meets these conditions. It's only asking for a graph that meets the conditions and we know what absolute extrema look like so just start sketching and keep in mind what the conditions are.

Step 1

So, we need a graph of some function (not the function itself, only the graph). The graph must be on the interval $[-4, 3]$ and must have absolute extrema at the specified points.

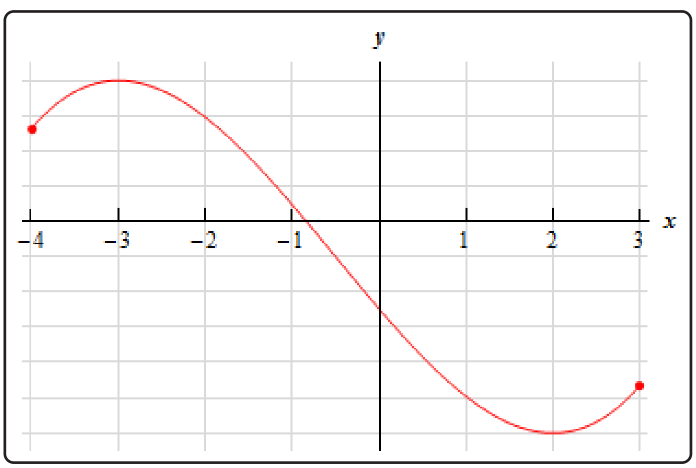
By this point we should have seen enough sketches of graphs to have a pretty good idea of what absolute maximums/minimums that are not at the endpoints of an interval should look like on a graph. Therefore, we should know basically what the graph should look like at $x = -3$ and $x = 2$. There are many ways we could sketch the graph between these two points, but there is no reason to overly complicate the graph so the best thing to do is probably just sketch in a short smooth curve connecting the two points.

Also, because the absolute extrema occur interior to the interval we know that the graph at the endpoints of the interval must fall somewhere between the maximum/minimum values of the graph. This means that as we sketch the graph from the absolute maximum to the left end point we can sketch anything we just need to make sure it never rises above the highest point on the graph or below the lowest point on the graph.

Similarly, as we sketch the graph from the absolute minimum to the right endpoint we just need to make sure it stays between the highest and lowest point on the graph.

Step 2

There are literally an infinite number of graphs that we could do here. Some will be more complicated that others, but here is probably one of the simpler graphs that we could use here.



7. Sketch the graph of some function that meets the following conditions :

(a) The function is continuous.

(b) Has two relative minimums.

(c) One of relative minimums is also an absolute minimum and the other relative minimum is not an absolute minimum.

(d) Has one relative maximum.

(e) Has no absolute maximum.

Hint

Do not let the apparent difficulty of this problem fool you. It's not asking us to find an actual function that meets these conditions. It's only asking for a graph that meets the conditions and we know what absolute and relative extrema look like so just start sketching and keep in mind what the conditions are.

Step 1

So, we need a graph of some function (not the function itself, only the graph) that meets the given conditions. We were not given an interval as one of the conditions so it's okay to assume that the interval is $(-\infty, \infty)$ for this problem.

From the first condition we know that we can't have any holes or breaks in the graph in order for the function to be continuous.

Now let's take care of the next two conditions as they are related to each other. By this point we've seen enough sketches of graphs to have a pretty good idea of what absolute and relative minimums looks like. So, we're going to need two downwards pointing "bumps" in the graph to give use the two relative minimums. Also, one of them must be the lowest point on the graph and other must be higher so it is not also an absolute minimum.

Next, we want to think about how to connect the two relative minimums. This is also where the fourth condition comes in. As we'll see because we have a continuous function we'll need that to connect the two relative minimums.

Let's start with the leftmost relative minimum. In order for it to be a minimum the graph must be increasing as we move to the right. However, if we also want to get the minimum to the right of this the graph will have to, at some point, start decreasing again. If you think about it that is exactly what a relative maximum will look like. So, in moving from the leftmost relative minimum to the rightmost relative minimum we must have a relative maximum between them and so the fourth condition is automatically met.

Note that if we don't insist on a continuous function it is possible to get from one to the other without having a relative maximum. All it would take is to have a division by zero discontinuity somewhere between the two relative minimums in which the graph goes to positive infinity on both sides of the discontinuity.

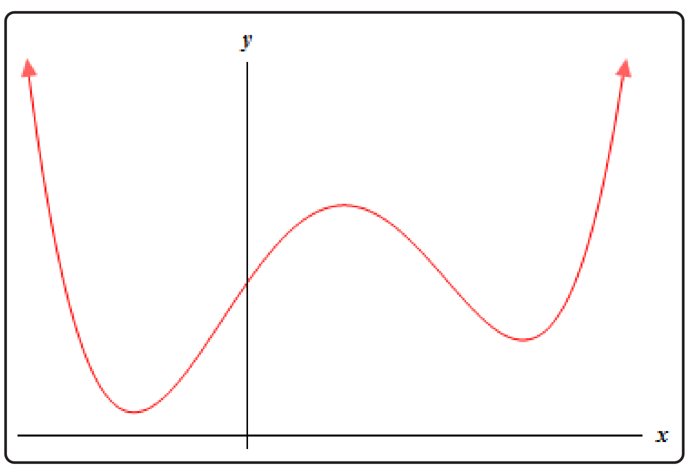
This would maintain the relative minimums and at the same time would not be a relative maximum.

Now let's deal with the final condition. In order for the graph to have no absolute maximum all we really need to do is make sure that the graph increases without bound as we move to the right and left of the graph. This will also match up nicely with the relative minimums that we are required to have.

To the left of the leftmost relative minimum the graph must be increasing and so we may as well just let it increase forever on that side. Likewise, on the right side of the rightmost relative minimum the graph will need to be increasing. So, again let's just let the graph increase forever on that side.

Step 2

There are literally an infinite number of graphs that we could do here. Some will be more complicated that others, but here is probably one of the simpler graphs that we could use here.



4.4 Finding Absolute Extrema

1. Determine the absolute extrema of $f(x) = 8x^3 + 81x^2 - 42x - 8$ on $[-8, 2]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with a polynomial and this is continuous everywhere and so will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$f'(x) = 24x^2 + 162x - 42 = 6(4x-1)(x+7) = 0 \qquad \Rightarrow \qquad x = -7, \;\; x = \frac{1}{4}$$

Step 2

Now, recall that we actually are only interested in the critical points that are in the given interval and so, in this case, the critical points that we need are,

$$x = -7, \quad x = \frac{1}{4}$$

Step 3

The next step is to evaluate the function at the critical points from the second step and at the end points of the given interval. Here are those function evaluations.

$$f(-8) = 1416 \qquad f(-7) = 1511 \qquad f\left(\frac{1}{4}\right) = -13.3125 \qquad f(2) = 296$$

Do not forget to evaluate the function at the end points! This is one of the biggest mistakes that people tend to make with this type of problem.

Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

$$\begin{array}{c}\text{Absolute Maximum : } 1511 \text{ at } x = -7 \\ \text{Absolute Minimum : } -13.3125 \text{ at } x = \dfrac{1}{4}\end{array}$$

---

2. Determine the absolute extrema of $f(x) = 8x^3 + 81x^2 - 42x - 8$ on $[-4, 2]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with a polynomial and this is continuous everywhere and so will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$f'(x) = 24x^2 + 162x - 42 = 6(4x - 1)(x + 7) = 0 \qquad \Rightarrow \qquad x = -7, \;\; x = \frac{1}{4}$$

Step 2

Now, recall that we actually are only interested in the critical points that are in the given interval and so, in this case, the only critical point that we need is,

$$x = \frac{1}{4}$$

Step 3

The next step is to evaluate the function at the critical point from the second step and at the end points of the given interval. Here are those function evaluations.

$$f(-4) = 944 \qquad f\left(\frac{1}{4}\right) = -13.3125 \qquad f(2) = 296$$

Do not forget to evaluate the function at the end points! This is one of the biggest mistakes that people tend to make with this type of problem.

Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

Absolute Maximum : 944 at $x = -4$

Absolute Minimum : -13.3125 at $x = \frac{1}{4}$

Note the importance of paying attention to the interval with this problem. Had we neglected to exclude $x = -7$ we would have gotten the wrong answer for the absolute maximum (check out the previous problem to see this....).

---

3. Determine the absolute extrema of $R(t) = 1 + 80t^3 + 5t^4 - 2t^5$ on $[-4.5,\ 4]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with a polynomial and this is continuous everywhere and so will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$R'(t) = 240t^2 + 20t^3 - 10t^4 = -10t^2(t-6)(t+4) = 0 \qquad \Rightarrow \qquad t=-4, \quad t=0, \quad t=6$$

Step 2

Now, recall that we actually are only interested in the critical points that are in the given interval and so, in this case, the critical points that we need are,

$$t=-4, \quad t=0$$

Step 3

The next step is to evaluate the function at the critical points from the second step and at the end points of the given interval. Here are those function evaluations.

$$R(-4.5) = -1548.13 \qquad R(-4) = -1791 \qquad R(0) = 1 \qquad R(4) = 4353$$

Do not forget to evaluate the function at the end points! This is one of the biggest mistakes that people tend to make with this type of problem.

Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

$$\boxed{\begin{array}{r}\text{Absolute Maximum}: 4353 \text{ at } t = 4\\ \text{Absolute Minimum}: -1791 \text{ at } t = -4\end{array}}$$

Note the importance of paying attention to the interval with this problem. Had we neglected to exclude $t = 6$ we would have gotten the wrong answer for the absolute maximum. Also note that if we'd neglected to check the endpoints at all we also would have gotten the wrong absolute maximum.

---

4. Determine the absolute extrema of $R(t) = 1 + 80t^3 + 5t^4 - 2t^5$ on $[0, 7]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with a polynomial and this is continuous everywhere and so will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$R'(t) = 240t^2 + 20t^3 - 10t^4 = -10t^2(t-6)(t+4) = 0 \qquad \Rightarrow \qquad t = -4, \quad t = 0, \quad t = 6$$

Step 2

Now, recall that we actually are only interested in the critical points that are in the given interval and so, in this case, the critical points that we need are,

$$t = 0, \quad t = 6$$

Do not get excited about the fact that one of the critical points also happens to be one of the end points of the interval. This happens on occasion.

Step 3

The next step is to evaluate the function at the critical points from the second step and at the end points of the given interval. Here are those function evaluations.

$$R(0) = 1 \qquad R(6) = 8209 \qquad R(7) = 5832$$

Do not forget to evaluate the function at the end points! This is one of the biggest mistakes that people tend to make with this type of problem.

Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

Absolute Maximum : 8209 at $t = 6$
Absolute Minimum : 1 at $t = 0$

Note the importance of paying attention to the interval with this problem. Had we neglected to exclude $t = -4$ we would have gotten the wrong answer for the absolute minimum.

---

5. Determine the absolute extrema of $h(z) = 4z^3 - 3z^2 + 9z + 12$ on $[-2, 1]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with a polynomial and this is continuous everywhere and so will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$h'(z) = 12z^2 - 6z + 9 = 0 \qquad \Rightarrow \qquad z = \frac{6 \pm \sqrt{-396}}{24} = \frac{1 \pm \sqrt{11}\,i}{4}$$

Now, recall that we only work with real numbers here and so we ignore complex roots. Therefore, this function has no critical points.

Step 2

Technically the next step is to determine all the critical points that are in the given interval. However, there are no critical points for this function and so there are also no critical points in the given interval.

Step 3

The next step is to evaluate the function at the critical points from the second step and at the end points of the given interval. However, since there are no critical points for this function all we need to do is evaluate the function at the end points of the interval.

Here are those function evaluations.

$$h(-2) = -50 \qquad\qquad h(1) = 22$$

Do not forget to evaluate the function at the end points! This is one of the biggest mistakes that people tend to make with this type of problem. That is especially true for this problem as there would be no points to evaluate at without the end points.

Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

$$\boxed{\begin{array}{r}\text{Absolute Maximum : } 22 \text{ at } z = 1\\ \text{Absolute Minimum : } -50 \text{ at } z = -2\end{array}}$$

Note that if we hadn't remembered to evaluate the function at the end points of the interval we would not have had an answer for this problem!

---

6. Determine the absolute extrema of $g(x) = 3x^4 - 26x^3 + 60x^2 - 11$ on $[1, 5]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with a polynomial and this is continuous everywhere and so will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$g'(x) = 12x^3 - 78x^2 + 120x = 6x(x-4)(2x-5) = 0 \qquad \Rightarrow \qquad x = 0, \quad x = \frac{5}{2}, \quad x = 4$$

Step 2

Now, recall that we actually are only interested in the critical points that are in the given interval and so, in this case, the critical points that we need are,

$$x = \frac{5}{2}, \quad x = 4$$

Step 3

The next step is to evaluate the function at the critical points from the second step and at the end points of the given interval. Here are those function evaluations.

$$g(1) = 26 \qquad g\left(\frac{5}{2}\right) = 74.9375 \qquad g(4) = 53 \qquad g(5) = 114$$

Do not forget to evaluate the function at the end points! This is one of the biggest mistakes that people tend to make with this type of problem.

Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

$$\boxed{\begin{array}{c}\text{Absolute Maximum : } 114 \text{ at } x = 5 \\ \text{Absolute Minimum : } 26 \text{ at } x = 1\end{array}}$$

Note that if we hadn't remembered to evaluate the function at the end points of the interval we would have gotten both of the answers incorrect!

---

7. Determine the absolute extrema of $Q(x) = (2-8x)^4 (x^2-9)^3$ on $[-3, 3]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with a polynomial and this is continuous everywhere and so will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$\begin{aligned} Q'(x) &= 4(-8)(2-8x)^3\left(x^2-9\right)^3 + 3(2x)(2-8x)^4\left(x^2-9\right)^2 \\ &= -4(2-8x)^3\left(x^2-9\right)^2\left(20x^2-3x-72\right) \\ &= 0 \qquad\qquad \Rightarrow \qquad x=\frac{1}{4}, \quad x=\pm 3, \quad x=\frac{3\pm\sqrt{5769}}{40} = -1.8239,\ 1.9739 \end{aligned}$$

Step 2

Now, recall that we actually are only interested in the critical points that are in the given interval and so, in this case, we need all the critical points from the first step.

$$x=\frac{1}{4}, \quad x=\pm 3, \quad x=\frac{3\pm\sqrt{5769}}{40} = -1.8239,\ 1.9739$$

Do not get excited about the fact that both end points of the interval are also critical points. It happens sometimes and in this case it will reduce the number of computations required in the next step by 2 and that's not a bad thing.

Step 3

The next step is to evaluate the function at the critical points from the second step and at the end points of the given interval. Here are those function evaluations.

$$\begin{aligned} &Q(-3)=0 \quad Q(-1.8239) = -1.38\times 10^7 \\ &Q\left(\frac{1}{4}\right)=0 \quad Q(1.9739) = -4.81\times 10^6 \quad Q(3)=0 \end{aligned}$$

Do not get excited about the large numbers for the two non-zero function values. This is something that is going to happen on occasion and we shouldn't worry about it when it does happen.

Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

$$\boxed{\begin{array}{l}\text{Absolute Maximum : } 0 \text{ at } x=-3,\;\; x=\frac{1}{4},\;\; x=3\\ \text{Absolute Minimum : } -1.38\times10^{7} \text{ at } x=-1.8239\end{array}}$$

Recall that while we can only have one largest possible value (*i.e.* only one absolute maximum) it is completely possible for it to occur at more than one point (3 points in this case).

---

8. Determine the absolute extrema of $h(w) = 2w^3(w+2)^5$ on $\left[-\frac{5}{2}, \frac{1}{2}\right]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with a polynomial and this is continuous everywhere and so will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$\begin{aligned} h'(w) &= 6w^2(w+2)^5 + 10w^3(w+2)^4 \\ &= 4w^2(w+2)^4(4w+3) = 0 \qquad \Rightarrow \qquad w=0,\;\; w=-\frac{3}{4},\;\; w=-2 \end{aligned}$$

Step 2

Now, recall that we actually are only interested in the critical points that are in the given interval and so, in this case, we need all the critical points from the first step.

$$w = 0, \quad w = -\frac{3}{4}, \quad w = -2$$

Step 3

The next step is to evaluate the function at the critical points from the second step and at the end points of the given interval. Here are those function evaluations.

$$h\left(-\frac{5}{2}\right) = 0.9766 \quad h(-2) = 0 \quad h\left(-\frac{3}{4}\right) = -2.5749 \quad h(0) = 0 \quad h\left(\frac{1}{2}\right) = 24.4141$$

Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

Absolute Maximum : 24.4141 at $w = \frac{1}{2}$

Absolute Minimum : -2.5749 at $w = -\frac{3}{4}$

---

9. Determine the absolute extrema of $f(z) = \dfrac{z+4}{2z^2+z+8}$ on $[-10, 0]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with a rational expression in which both the numerator and denominator are continuous everywhere. Also notice that the rational expression exists at all points in the interval and so will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$f'(z) = \frac{(1)\left(2z^2 + z + 8\right) - (z+4)(4z+1)}{\left(2z^2 + z + 8\right)^2}$$
$$= \frac{-2\left(z^2 + 8z - 2\right)}{\left(2z^2 + z + 8\right)^2} = 0 \quad \Rightarrow \quad z = \frac{-8 \pm \sqrt{72}}{2} = -4 \pm 3\sqrt{2} = -8.2426,\ \ 0.2426$$

Step 2

Now, recall that we actually are only interested in the critical points that are in the given interval and so, in this case, the only critical point that we need is,

$$z = -4 - 3\sqrt{2} = -8.2426$$

Step 3

The next step is to evaluate the function at the critical point from the second step and at the end points of the given interval. Here are those function evaluations.

$$f(-10) = -\frac{1}{33} = -0.0303 \qquad f(-8.2426) = -0.03128 \qquad f(0) = \frac{1}{2}$$

Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

$$\boxed{\begin{array}{l}\text{Absolute Maximum : } \dfrac{1}{2} \text{ at } z = 0 \\ \text{Absolute Minimum : } -0.03128 \text{ at } z = -4 - 3\sqrt{2}\end{array}}$$

Note the importance of paying attention to the interval with this problem. Had we neglected to exclude $z = -4 + 3\sqrt{2} = 0.2426$ we would have gotten the wrong answer for the absolute maximum.

This problem also shows that we need to be very careful with doing too much rounding of our answers. Had we rounded down to say 2 decimal places we would have been tempted to say that the absolute minimum occurred at two places when in fact one of the points was lower than the other.

---

10. Determine the absolute extrema of $A(t) = t^2 (10 - t)^{\frac{2}{3}}$ on $[2,\ 10.5]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with a product of a polynomial and a cube root function. Both are continuous everywhere and so the product will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$\begin{aligned} A'(t) &= 2t\,(10-t)^{\frac{2}{3}} + t^2\left(\frac{2}{3}\right)(-1)\,(10-t)^{-\frac{1}{3}} = 2t\,(10-t)^{\frac{2}{3}} - \frac{2t^2}{3(10-t)^{\frac{1}{3}}} \\ &= \frac{6t\,(10-t) - 2t^2}{3(10-t)^{\frac{1}{3}}} = \frac{60t - 8t^2}{3(10-t)^{\frac{1}{3}}} = \frac{4t\,(15-2t)}{3(10-t)^{\frac{1}{3}}} \\ &= 0 \qquad\qquad t = 0, \quad t = \frac{15}{2}, \quad t = 10 \end{aligned}$$

Don't forget about critical points where the derivative doesn't exist!

Step 2

Now, recall that we actually are only interested in the critical points that are in the given interval and so, in this case, the critical points that we need are,

$$t = \frac{15}{2}, \quad t = 10$$

Step 3

The next step is to evaluate the function at the critical points from the second step and at the end points of the given interval. Here are those function evaluations.

$$A(2) = 16 \quad A\left(\frac{15}{2}\right) = 103.613 \quad A(10) = 0 \quad A(10.5) = 69.4531$$

Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

$$\boxed{\begin{array}{l} \text{Absolute Maximum : } 103.613 \text{ at } t = \frac{15}{2} \\ \text{Absolute Minimum : } 0 \text{ at } t = 10 \end{array}}$$

Note the importance of paying attention to the interval with this problem. Had we neglected to exclude $t = 0$ we would have had the absolute minimum showing up at two places instead of only the one place inside the given interval.

11. Determine the absolute extrema of $f(y) = \sin\left(\frac{y}{3}\right) + \frac{2y}{9}$ on $[-10, 15]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with the sine function and this is continuous everywhere and so will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$f'(y) = \frac{1}{3}\cos\left(\frac{y}{3}\right) + \frac{2}{9} = 0 \quad \rightarrow \quad \cos\left(\frac{y}{3}\right) = -\frac{2}{3} \quad \rightarrow \quad \frac{y}{3} = \cos^{-1}\left(-\frac{2}{3}\right) = 2.3005$$

$$\begin{aligned} \frac{y}{3} &= 2.3005 + 2\pi n \\ \frac{y}{3} &= 3.9827 + 2\pi n \end{aligned} \quad \Rightarrow \quad \begin{aligned} y &= 6.9016 + 6\pi n \\ y &= 11.9481 + 6\pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \pm, \ldots$$

If you need some review on solving trig equations please go back to the Review chapter and work some of the problems the Solving Trig Equations sections.

Step 2

Now, recall that we actually are only interested in the critical points that are in the given interval and so, in this case, the critical points that we need are,

$$y = -6.9016, \quad y = 6.9016, \quad y = 11.9481$$

Note that we got these values by plugging in values of n into the solutions above and checking the results against the given interval.

Step 3

The next step is to evaluate the function at the critical points from the second step and at the end points of the given interval. Here are those function evaluations.

$$f(-10) = -2.0317 \quad f(-6.9016) = -2.2790 \quad f(6.9016) = 2.2790$$
$$f(11.9481) = 1.9098 \quad f(15) = 2.3744$$

Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

$$\boxed{\begin{array}{l}\text{Absolute Maximum : } 2.3744 \text{ at } y = 15 \\ \text{Absolute Minimum : } -2.2790 \text{ at } y = -6.9016\end{array}}$$

Note the importance of paying attention to the interval with this problem. Without an interval we would have had (literally) an infinite number of critical points to check. Also, without an interval (as a quick graph of the function would show) there would be no absolute extrema for this function.

12. Determine the absolute extrema of $g(w) = \mathbf{e}^{w^3 - 2w^2 - 7w}$ on $\left[-\frac{1}{2}, \frac{5}{2}\right]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with an exponential function with a polynomial in the exponent. The exponent is continuous everywhere and so we can see that the exponential function will also be continuous everywhere. Therefore, the function will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$g'(w) = \left(3w^2 - 4w - 7\right)\mathbf{e}^{w^3 - 2w^2 - 7w}$$
$$= (w+1)(3w-7)\mathbf{e}^{w^3 - 2w^2 - 7w} = 0 \qquad \Rightarrow \qquad w = -1, \quad w = \frac{7}{3}$$

Step 2

Now, recall that we actually are only interested in the critical points that are in the given interval and so, in this case, the only critical point that we need is,

$$w = \frac{7}{3}$$

Step 3

The next step is to evaluate the function at the critical point from the second step and at the end points of the given interval. Here are those function evaluations.

$$g\left(-\frac{1}{2}\right) = \mathbf{e}^{\frac{23}{8}} \qquad g\left(\frac{7}{3}\right) = \mathbf{e}^{-\frac{392}{27}} \qquad g\left(\frac{5}{2}\right) = \mathbf{e}^{-\frac{115}{8}}$$

Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

$$\boxed{\begin{array}{c}\text{Absolute Maximum : } \mathbf{e}^{\frac{23}{8}} \text{ at } w = -\frac{1}{2} \\ \text{Absolute Minimum : } \mathbf{e}^{-\frac{392}{27}} \text{ at } w = \frac{7}{3}\end{array}}$$

Note the importance of paying attention to the interval with this problem. Had we neglected to exclude $w = -1$ we would have gotten the absolute maximum wrong.

Also note that we need to be careful with rounding with this problem. Both of the exponentials with negative exponents are very small and rounding could cause some real issues here. However, we don't need to actually do any calculator work for this anyway. Recall that the more negative the exponent is the smaller the exponential will be.

So, because $\frac{392}{27} > \frac{115}{8}$ we must have $\mathbf{e}^{-\frac{392}{27}} < \mathbf{e}^{-\frac{115}{8}}$.

13. Determine the absolute extrema of $R(x) = \ln(x^2 + 4x + 14)$ on $[-4, 2]$.

Hint

Just recall the process for finding absolute extrema outlined in the notes for this section and you'll be able to do this problem!

Step 1

First, notice that we are working with a logarithm whose argument is a polynomial (which is continuous everywhere) that is always positive in the interval. Because of this we can see that the function will be continuous on the given interval. Recall that this is important because we now know that absolute extrema will in fact exist by the Extreme Value Theorem!

Now that we know that absolute extrema will in fact exist on the given interval we'll need to find the critical points of the function.

Given that the purpose of this section is to find absolute extrema we'll not be putting much work/explanation into the critical point steps. If you need practice finding critical points please go back and work some problems from that section.

Here are the critical points for this function.

$$R'(x) = \frac{2x + 4}{x^2 + 4x + 14} \qquad \Rightarrow \qquad x = -2$$

Step 2

Now, recall that we actually are only interested in the critical points that are in the given interval and so, in this case, the critical point that we need is,

$$x = -2$$

Step 3

The next step is to evaluate the function at the critical points from the second step and at the end points of the given interval. Here are those function evaluations.

$$R(-4) = 2.6391 \qquad R(-2) = 2.3026 \qquad R(2) = 3.2581$$

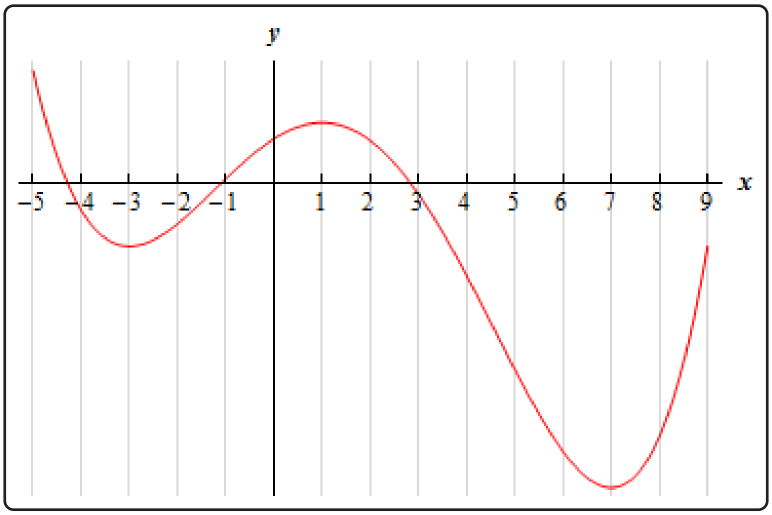
Step 4

The final step is to identify the absolute extrema. So, the answers for this problem are then,

$$\boxed{\begin{array}{c}\text{Absolute Maximum : } 3.2581 \text{ at } x = 2 \\ \text{Absolute Minimum : } 2.3026 \text{ at } x = -2\end{array}}$$

4.5 The Shape of a Graph, Part I

1. The graph of a function is given below. Determine the intervals on which the function increases and decreases.



Solution

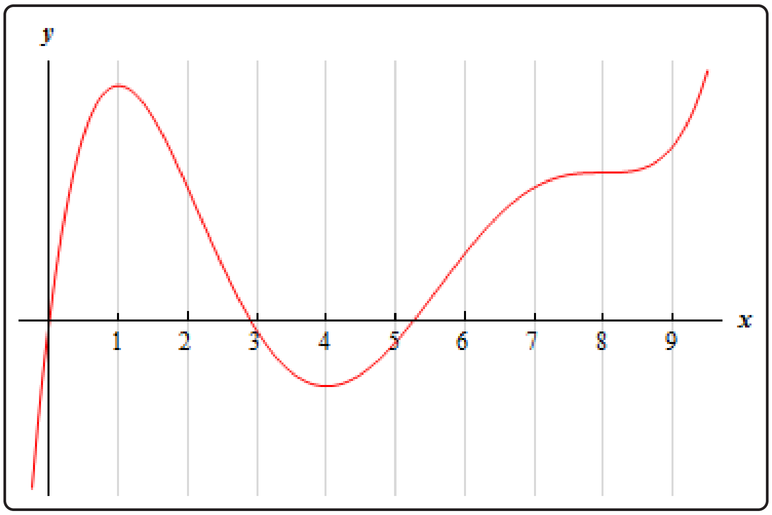
Solution

There really isn't too much to this problem. We can easily see from the graph where the function in increasing/decreasing and so all we need to do is write down the intervals.

$$\boxed{\text{Increasing : } (-3,1) \quad \& \quad (7,\infty) \qquad\qquad \text{Decreasing : } (-\infty,-3) \quad \& \quad (1,7)}$$

Note as well that we don't include the end points in the interval. For this problem that is important because at the end points we are at infinity or the function is either not increasing or decreasing.

2. The graph of a function is given below. Determine the intervals on which the function increases and decreases.



Solution

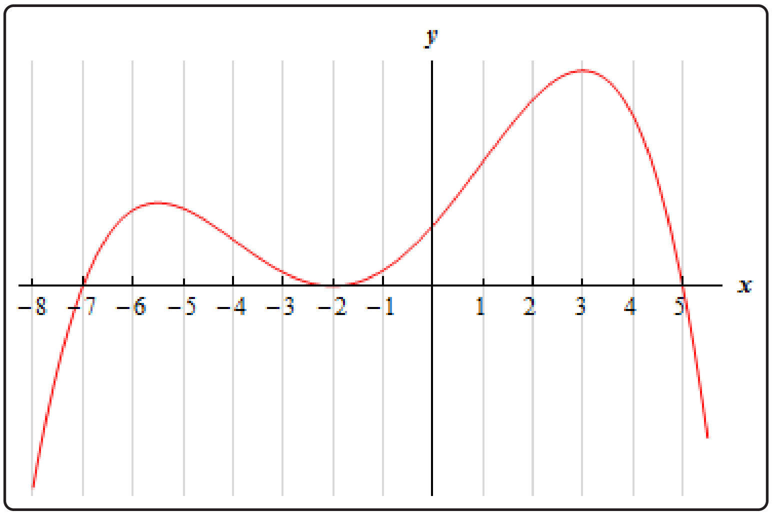
Solution

There really isn't too much to this problem. We can easily see from the graph where the function in increasing/decreasing and so all we need to do is write down the intervals.

Increasing : $(-\infty, 1)$, $(4, 8)$ & $(8, \infty)$ Decreasing : $(1, 4)$

Note as well that we don't include the end points in the interval. For this problem that is important because at the end points we are at infinity or the function is either not increasing or decreasing.

3. Below is the graph of the **derivative** of a function. From this graph determine the intervals in which the **function** increases and decreases.



Hint

Be careful with this problem. The graph is of the **derivative** of the function and so we don't just write down intervals where the graph is increasing and decreasing. Recall how the derivative tells us where the function is increasing and decreasing and this problem is not too bad.

Solution

We have to be careful and not do this problem as we did the first two practice problems. The graph given is the graph of the **derivative** and not the graph of the function. So, the answer is not just where the graph is increasing or decreasing.

Instead we need to recall that the sign of the derivative tells us where the function is increasing and decreasing. If the derivative is positive (*i.e.* its graph is above the x-axis) then the function is increasing and if the derivative is negative (*i.e.* its graph is below the x-axis) then the function is decreasing.

So, it is fairly clear where the graph is above/below the x-axis and so we have the following intervals of increase/decrease.

Increasing : $(-7,-2)$ & $(-2,5)$ Decreasing : $(-\infty,-7)$ & $(5,\infty)$

4. This problem is about some function. All we know about the function is that it exists everywhere and we also know the information given below about the derivative of the function. Answer each of the following questions about this function.

   (a) Identify the critical points of the function.

   (b) Determine the intervals on which the function increases and decreases.

   (c) Classify the critical points as relative maximums, relative minimums or neither.

$$f'(-5) = 0 \qquad f'(-2) = 0 \qquad f'(4) = 0 \qquad f'(8) = 0$$
$$f'(x) < 0 \quad \text{on} \quad (-5,-2)\,,\ \ (-2,4)\,,\ \ (8,\infty) \qquad\qquad f'(x) > 0 \quad \text{on} \quad (-\infty,-5)\,,\ \ (4,8)$$

Hint

This problem is actually quite simple. Just keep in mind how critical points are defined and how we can answer the last two parts from the derivative of the function.

Solutions

(a) Identify the critical points of the function.

Solution

Okay, let's recall the definition of a critical point. A critical point is any point in which the function exists and the derivative is either zero or doesn't exist.

We are given that the function exists everywhere (and in fact this part is why that in there at all....) and so we don't really need to worry about that part of the definition for this problem.

Also, from the given information about the derivative we can see that at every point the derivative is either zero, positive or negative. In other words, the derivative will exist at every point.

So, all this means that the critical points of the function are those points were the derivative is zero and we are given those in the information.

Therefore, the critical points of the function are,

$$\boxed{x = -5, \qquad x = -2, \qquad x = 4, \qquad x = 8}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

There is really not a lot to this part. We know that the function will increase where the derivative is positive and it will decrease where the derivative is negative. This positive and negative information is clearly listed above in the given information so here are the increasing/decreasing intervals for this function.

$$\begin{array}{l} \text{Increasing : } (-\infty,-5) \ \ \& \ \ (4,8) \\ \text{Decreasing : } (-5,-2)\,,\ \ (-2,4) \ \ \& \ \ (8,\infty) \end{array}$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

Okay, there isn't a lot that we need to do here. We know that relative maximums are increasing on the left and decreasing on the right and relative minimums are decreasing on the left and increasing on the right. We have all the increasing/decreasing information from the second part so here is the answer to this part.

$$\begin{array}{lll} x=-5 & : & \text{Relative Maximum} \\ x=-2 & : & \text{Neither} \\ x=4 & : & \text{Relative Minimum} \\ x=8 & : & \text{Relative Maximum} \end{array}$$

---

5. For $f(x) = 2x^3 - 9x^2 - 60x$ answer each of the following questions.

(a) Identify the critical points of the function.

(b) Determine the intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solutions

(a) Identify the critical points of the function.

Solution

We need the 1^{st} derivative to get the critical points so here it is.

$$f'(x) = 6x^2 - 18x - 60 = 6\left(x^2 - 3x - 10\right) = 6(x-5)(x+2)$$

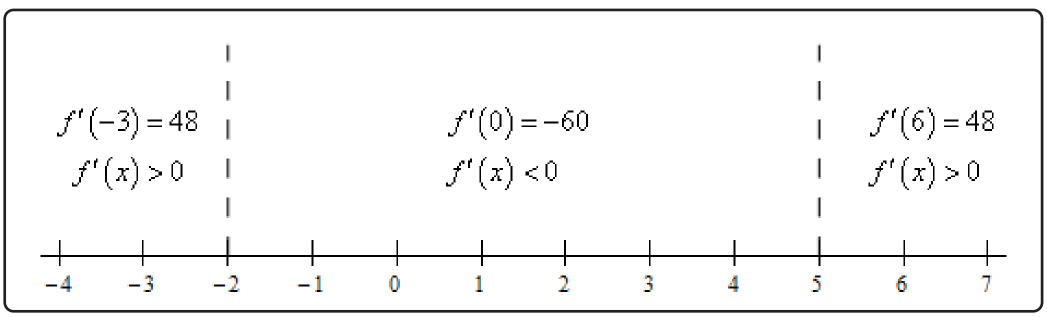
Now, recall that critical points are where the derivative doesn't exist or is zero. Clearly this derivative exists everywhere (it's a polynomial....) and because we factored the derivative we can easily identify where the derivative is zero. The critical points of the function are,

$$\boxed{x = -2, \qquad x = 5}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To determine the increase/decrease information for the function all we need is a quick number line for the derivative. Here is the number line.



From this we get the following increasing/decreasing information for the function.

$$\boxed{\text{Increasing : } (-\infty, -2) \ \ \& \ \ (5, \infty) \qquad \text{Decreasing : } (-2, 5)}$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

With the increasing/decreasing information from the previous step we can easily classify the critical points using the 1^{st} derivative test. Here is classification of the functions critical points.

$$\boxed{\begin{array}{l} x = -2 : \text{ Relative Maximum} \\ x = 5 : \text{ Relative Minimum} \end{array}}$$

---

6. For $h(t) = 50 + 40t^3 - 5t^4 - 4t^5$ answer each of the following questions.

(a) Identify the critical points of the function.

(b) Determine the intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solutions

(a) Identify the critical points of the function.

Solution

We need the 1^{st} derivative to get the critical points so here it is.

$$h'(t) = 120t^2 - 20t^3 - 20t^4 = -20t^2\left(t^2 + t - 6\right) = -20t^2(t+3)(t-2)$$

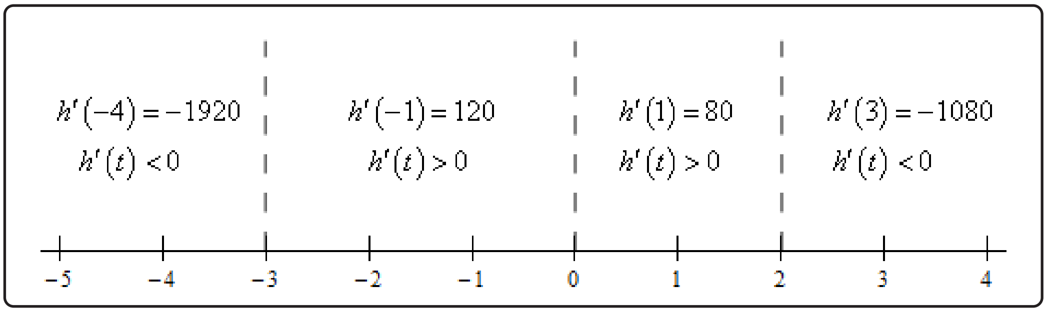
Now, recall that critical points are where the derivative doesn't exist or is zero. Clearly this derivative exists everywhere (it's a polynomial....) and because we factored the derivative we can easily identify where the derivative is zero. The critical points of the function are,

$$\boxed{t = -3, \quad t = 0, \quad t = 2}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To determine the increase/decrease information for the function all we need is a quick number line for the derivative. Here is the number line.



From this we get the following increasing/decreasing information for the function.

Increasing : $(-3, 0)$ & $(0, 2)$ Decreasing : $(-\infty, -3)$ & $(2, \infty)$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

With the increasing/decreasing information from the previous step we can easily classify the critical points using the 1^{st} derivative test. Here is classification of the functions critical points.

$t = -3$: Relative Minimum
$t = 0$: Neither
$t = 2$: Relative Maximum

---

7. For $y = 2x^3 - 10x^2 + 12x - 12$ answer each of the following questions.

   (a) Identify the critical points of the function.

   (b) Determine the intervals on which the function increases and decreases.

   (c) Classify the critical points as relative maximums, relative minimums or neither.

Solutions

(a) Identify the critical points of the function.

Solution

We need the 1^{st} derivative to get the critical points so here it is.

$$\frac{dy}{dx} = 6x^2 - 20x + 12$$

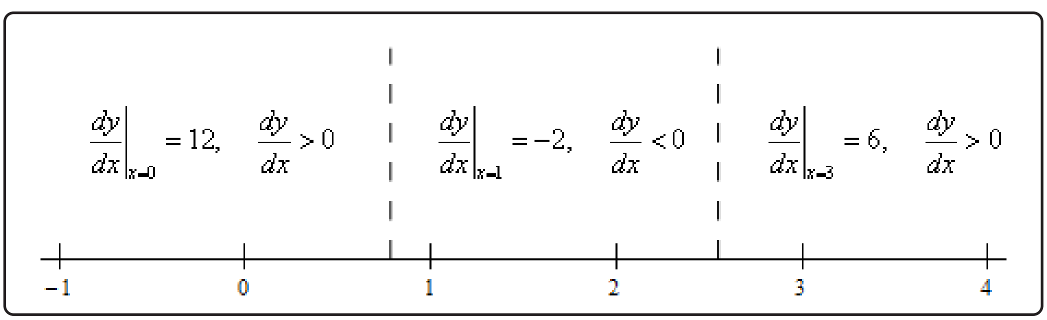
Now, recall that critical points are where the derivative doesn't exist or is zero. Clearly this derivative exists everywhere (it's a polynomial....) and because the derivative can't be factored in this case we'll need to do a quick quadratic formula to find where the derivative is zero. The critical points of the function are,

$$\boxed{x = \frac{5 \pm \sqrt{7}}{3} = 0.78475,\ 2.54858}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To determine the increase/decrease information for the function all we need is a quick number line for the derivative. Here is the number line.



From this we get the following increasing/decreasing information for the function.

$$\text{Increasing : } \left(-\infty, \frac{5-\sqrt{7}}{3}\right) \quad \& \quad \left(\frac{5+\sqrt{7}}{3}, \infty\right)$$
$$\text{Decreasing : } \left(\frac{5-\sqrt{7}}{3}, \frac{5+\sqrt{7}}{3}\right)$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

With the increasing/decreasing information from the previous step we can easily classify the critical points using the 1^{st} derivative test. Here is classification of the functions critical points.

$$x = \frac{5-\sqrt{7}}{3} = 0.78475 \quad : \quad \text{Relative Maximum}$$
$$x = \frac{5+\sqrt{7}}{3} = 2.54858 \quad : \quad \text{Relative Minimum}$$

8. For $p(x) = \cos(3x) + 2x$ answer each of the following questions on $\left[-\frac{3}{2}, 2\right]$.

(a) Identify the critical points of the function.

(b) Determine the intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solutions

(a) Identify the critical points of the function.

Solution

We need the 1^{st} derivative to get the critical points so here it is.

$$p'(x) = -3\sin(3x) + 2$$

Now, recall that critical points are where the derivative doesn't exist or is zero. Clearly this derivative exists everywhere (the sine function exists everywhere....) and so all we need to do is set the derivative equal to zero and solve. We're not going to show all of those details so if you need to do some review of the process go back to the Solving Trig Equations sections for some examples.

Here are all the critical points.

$$\begin{aligned} x &= 0.2432 + \frac{2\pi}{3}n \\ x &= 0.8040 + \frac{2\pi}{3}n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

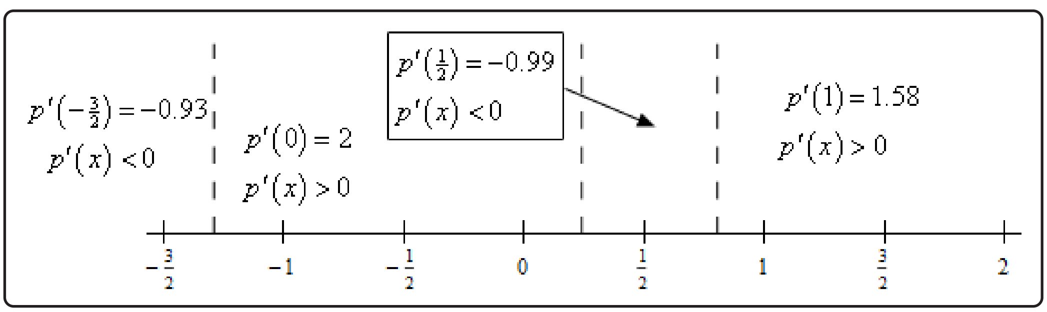
Plugging in some n's gives the following critical points in the interval $\left[-\frac{3}{2}, 2\right]$.

$$\boxed{x = -1.2904, \qquad x = 0.2432, \qquad x = 0.8040}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To determine the increase/decrease information for the function all we need is a quick number line for the derivative. Here is the number line.



From this we get the following increasing/decreasing information for the function.

$$\boxed{\begin{aligned} \text{Increasing : } & (-1.2904, 0.2432) \ \ \& \ \ (0.8040, 2] \\ \text{Decreasing : } & \left[-\frac{3}{2}, -1.2904\right) \ \ \& \ \ (0.2432, 0.8040) \end{aligned}}$$

Be careful with the end points of these intervals! We are working on the interval $\left[-\frac{3}{2}, 2\right]$ and we've done no work for increasing and decreasing outside of this interval and so we can't say anything about what happens outside of the interval.

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

With the increasing/decreasing information from the previous step we can easily classify the critical points using the 1^{st} derivative test. Here is classification of the functions critical points.

$$\boxed{\begin{array}{lcl} x = -1.2904 & : & \text{Relative Minimum} \\ x = 0.2432 & : & \text{Relative Maximum} \\ x = 0.8040 & : & \text{Relative Minimum} \end{array}}$$

As with the last step, we need to again recall that we are only working on the interval $\left[-\frac{3}{2}, 2\right]$ and the classifications given here are only for those critical points in the interval. There are, of course, an infinite number of critical points outside of this interval and they can all be classified as relative minimums or relative maximums provided we do the work to justify the classifications.

---

9. For $R(z) = 2 - 5z - 14\sin\left(\frac{z}{2}\right)$ answer each of the following questions on $[-10, 7]$.

(a) Identify the critical points of the function.

(b) Determine the intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solutions

(a) Identify the critical points of the function.

Solution

We need the 1^{st} derivative to get the critical points so here it is.

$$R'(z) = -5 - 7\cos\left(\frac{z}{2}\right)$$

Now, recall that critical points are where the derivative doesn't exist or is zero.

Clearly this derivative exists everywhere (the cosine function exists everywhere....) and so all we need to do is set the derivative equal to zero and solve. We're not going to show all of those details so if you need to do some review of the process go back to the Solving Trig Equations sections for some examples.

Here are all the critical points.

$$\begin{aligned} z &= 4.7328 + 4\pi n \\ z &= 7.8336 + 4\pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

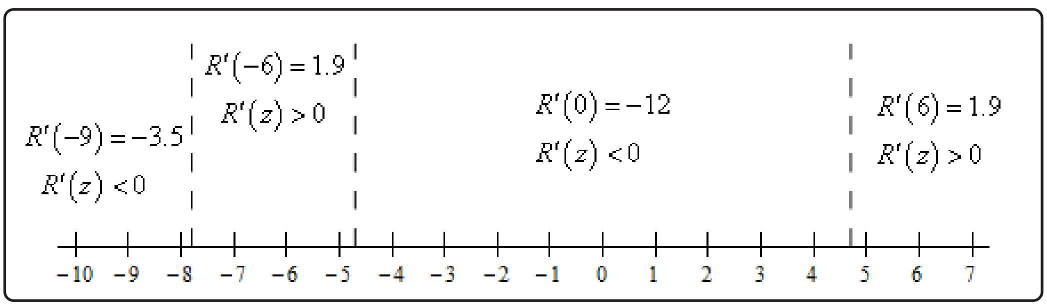
Plugging in some n's gives the following critical points in the interval $[-10, 7]$.

$$\boxed{z = -7.8336, \qquad z = -4.7328, \qquad z = 4.7328}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To determine the increase/decrease information for the function all we need is a quick number line for the derivative. Here is the number line.



From this we get the following increasing/decreasing information for the function.

$$\boxed{\begin{aligned} \text{Increasing : } & (-7.8336, -4.7328) \ \ \& \ \ (4.7328, 7] \\ \text{Decreasing : } & [-10, -7.8336) \ \ \& \ \ (-4.7328, 4.7328) \end{aligned}}$$

Be careful with the end points of these intervals! We are working on the interval $[-10, 7]$ and we've done no work for increasing and decreasing outside of this interval and so we can't say anything about what happens outside of the interval.

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

With the increasing/decreasing information from the previous step we can easily classify the critical points using the 1^{st} derivative test. Here is classification of the functions critical points.

$$\boxed{\begin{aligned} z &= -7.8336 &&: \text{ Relative Minimum} \\ z &= -4.7328 &&: \text{ Relative Maximum} \\ z &= 4.7328 &&: \text{ Relative Minimum} \end{aligned}}$$

As with the last step, we need to again recall that we are only working on the interval $[-10, 7]$ and the classifications given here are only for those critical points in the interval. There are, of course, an infinite number of critical points outside of this interval and they can all be classified as relative minimums or relative maximums provided we do the work to justify the classifications.

10. For $h(t) = t^2 \sqrt[3]{t-7}$ answer each of the following questions.

(a) Identify the critical points of the function.

(b) Determine the intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solutions

(a) Identify the critical points of the function.

Solution

We need the 1^{st} derivative to get the critical points so here it is.

$$h'(t) = 2t(t-7)^{\frac{1}{3}} + t^2\left(\frac{1}{3}\right)(t-7)^{-\frac{2}{3}} = 2t(t-7)^{\frac{1}{3}} + \frac{t^2}{3(t-7)^{\frac{2}{3}}}$$
$$= \frac{6t(t-7)+t^2}{3(t-7)^{\frac{2}{3}}} = \frac{7t^2-42t}{3(t-7)^{\frac{2}{3}}} = \frac{7t(t-6)}{3(t-7)^{\frac{2}{3}}}$$

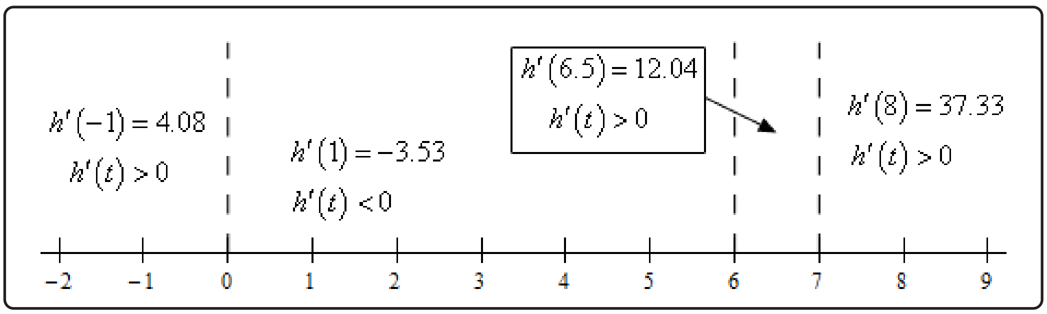
Now, recall that critical points are where the derivative doesn't exist or is zero. Because we simplified and factored the derivative as much as possible we can clearly see that the derivative does not exist at $t = 7$ (and the function exists here...) and that the derivative is zero at $t = 0$ and $t = 6$. The critical points of this function are then,

$$\boxed{t = 0, \quad t = 6, \quad t = 7}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To determine the increase/decrease information for the function all we need is a quick number line for the derivative. Here is the number line.



From this we get the following increasing/decreasing information for the function.

Increasing : $(-\infty, 0)$, $(6, 7)$ & $(7, \infty)$	Decreasing : $(0, 6)$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

With the increasing/decreasing information from the previous step we can easily classify the critical points using the 1^{st} derivative test. Here is classification of the functions critical points.

$t = 0$	:	Relative Maximum
$t = 6$	:	Relative Minimum
$t = 7$	:	Neither

11. For $f(w) = w\mathbf{e}^{2 - \frac{1}{2}w^2}$ answer each of the following questions.

   (a) Identify the critical points of the function.

   (b) Determine the intervals on which the function increases and decreases.

   (c) Classify the critical points as relative maximums, relative minimums or neither.

Solutions

(a) Identify the critical points of the function.

Solution

We need the 1^{st} derivative to get the critical points so here it is.

$$f'(w) = \mathbf{e}^{2 - \frac{1}{2}w^2} - w^2\mathbf{e}^{2 - \frac{1}{2}w^2} = \mathbf{e}^{2 - \frac{1}{2}w^2}\left(1 - w^2\right) = \mathbf{e}^{2 - \frac{1}{2}w^2}(1 - w)(1 + w)$$

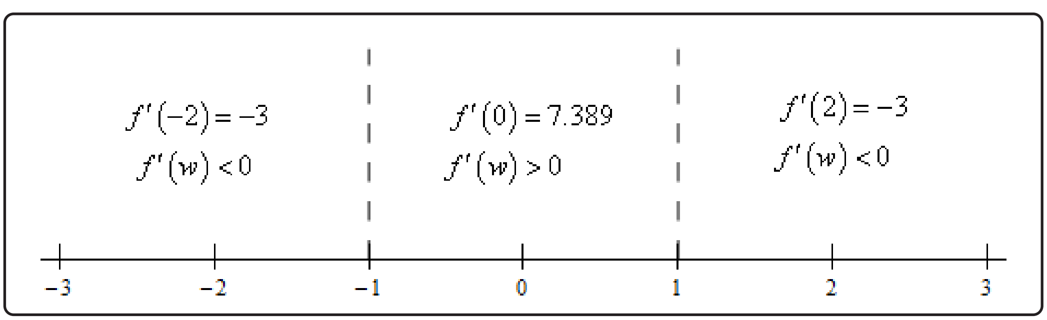
Now, recall that critical points are where the derivative doesn't exist or is zero. Because we simplified and factored the derivative as much as possible we can clearly see that the derivative will exist everywhere (it's the product of functions that exist everywhere). We can also easily see where the derivative is zero. The critical points of this function are then,

$$\boxed{w = -1, \quad w = 1}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To determine the increase/decrease information for the function all we need is a quick number line for the derivative. Here is the number line.



From this we get the following increasing/decreasing information for the function.

Increasing : $(-1, 1)$ Decreasing : $(-\infty, -1)$ & $(1, \infty)$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

With the increasing/decreasing information from the previous step we can easily classify the critical points using the 1^{st} derivative test. Here is classification of the functions critical points.

$w = -1$: Relative Minimum
$w = 1$: Relative Maximum

---

12. For $g(x) = x - 2\ln\left(1 + x^2\right)$ answer each of the following questions.

(a) Identify the critical points of the function.

(b) Determine the intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solutions

(a) Identify the critical points of the function.

Solution

We need the 1^{st} derivative to get the critical points so here it is.

$$g'(x) = 1 - 2\frac{2x}{1+x^2} = \frac{1 - 4x + x^2}{1+x^2}$$

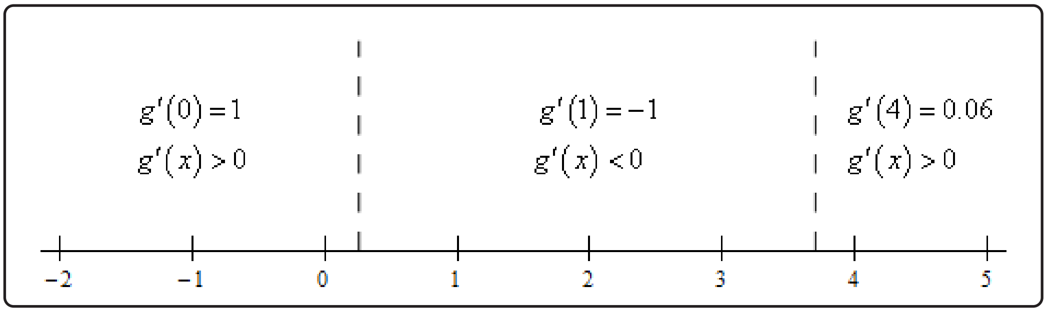
Now, recall that critical points are where the derivative doesn't exist or is zero. Because we simplified and factored the derivative as much as possible we can clearly see that the derivative will exist everywhere (or at least the denominator will not be zero for any real numbers...). We'll also need the quadratic formula to determine where the numerator, and hence the derivative, is zero. The critical points of this function are then,

$$x = 2 \pm \sqrt{3} = 0.2679, \quad 3.7321$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To determine the increase/decrease information for the function all we need is a quick number line for the derivative. Here is the number line.



From this we get the following increasing/decreasing information for the function.

Increasing : $(-\infty, 0.2679)$ & $(3.7321, \infty)$
Decreasing : $(0.2679, 3.7321)$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

With the increasing/decreasing information from the previous step we can easily classify the critical points using the 1^{st} derivative test. Here is classification of the functions critical points.

$x = 0.2679$: Relative Maximum
$x = 3.7321$: Relative Minimum

13. For some function, $f(x)$, it is known that there is a relative maximum at $x = 4$. Answer each of the following questions about this function.

(a) What is the simplest form for the derivative of this function?

Note : There really are many possible forms of the derivative so to make the rest of this problem as simple as possible you will want to use the simplest form of the derivative that you can come up with.

(b) Using your answer from (a) determine the most general form of the function.

(c) Given that $f(4) = 1$ find a function that will have a relative maximum at $x = 4$.

Note : You should be able to use your answer from (b) to determine an answer to this part.

Solutions

Hint

As noted in the problem there are many possible forms that the derivative can take. However, if we want things to remain simple just keep in mind what it takes for a point to be a critical point (why a critical point?). With that in mind it should be pretty simple to figure out a really simple form for the derivative to take to make sure we get a relative maximum at the point.

(a) What is the simplest form for the derivative of this function?

Solution

Okay, let's get started with this problem.

The first thing that we'll do is assume that the derivative exists everywhere. Making assumptions in a math class is generally a bad thing. However, in this case, because we are being asked to come up the form of the derivative all we are really doing here is starting that process. If we can't find a derivative that will have a relative maximum at the point that also exists everywhere we can come back and change things up. If we can find a derivative that will exist everywhere (which we can as we'll see) this assumption will help with keeping the derivative as simple as possible.

Now, given that we are assuming that the derivative exists everywhere and we know that if we have a relative maximum at $x = 4$ then $x = 4$ must also be a critical point (recall Fermat's Theorem from a couple of sections ago...). This is another reason for the assumption we made above. Fermat's theorem requires that the derivative exist at the point in order to know that it is also a critical point.

Next, because we assumed that the derivative exists everywhere (and in particular it exists at $x = 4$) we know that in order for it to be a critical point we must also have $f'(4) = 0$. There are lots and lots of functions that will be zero at $x = 4$ but probably one of the simplest is use,

$$f'(x) = x - 4$$

This does give $f'(4) = 0$ as we need, however we have a problem. We can clearly see that if $x < 4$ we would have $f'(x) < 0$ and if $x > 4$ we would have $f'(x) > 0$. This says that $x = 4$ would have to be a relative minimum and not the relative maximum that we wanted it to be.

Luckily enough for us this is easy to fix. The only problem with our original guess is that the signs of the derivative to the right and left of $x = 4$ are opposite what we need them to be. Therefore, all we need to do is change them and that can easily be done by multiplying by a negative or,

$$\boxed{f'(x) = 4 - x}$$

With this choice we still have $f'(4) = 0$ and now the derivative is positive if $x < 4$ and negative if $x > 4$ which means that $x = 4$ will be a relative maximum.

As noted in the problem statement there are many possible answers to this part. We will be working with the one given above. However, just to make the point here are a sampling of other derivatives all of which come from functions that have a relative maximum at $x = 4$.

$$f'(x) = 16 - x^2 \quad f'(x) = 24 + 18x - 6x^2 \quad f'(x) = \mathbf{e}^{4-x} - 1 \quad f'(x) = \sin(2\pi - \pi x)$$

After working the rest of this problem with $f'(x) = 4 - x$ you might want to come back and see if you can repeat the problem with one or more of these to see what you get.

Hint

This is where the problem gets a little tricky (as if the previous part wasn't tricky huh?). What we need to do here is "undo" the derivative. However, if you kept the answer from the previous part simple and you understand how differentiation works then it shouldn't be too hard to "undo" the differentiation and determine a function that gives your derivative. Keep in mind however that the problem statement asks for the most general possible function and that is where most will run into problems with this part.

(b) Using your answer from **a)** determine the most general form of the function.

Solution

Okay, from the previous part we have assumed that the derivative of our function is $f'(x) = 4 - x$ and we need to "undo" the differentiation to determine the most general possible function that we could have had.

So, before doing this let's recall what we know about differentiation. First, let's recall the following formula,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

When we differentiate a power of x the power goes down by one. So, what would we have to differentiate to get x? We'll, since the exponent on the x is 1 we would have had to differentiate an x^2. However, because we know that that the derivative of x^2 is 2 x and we want just an x that would mean that in fact we would have had to differentiate $\frac{1}{2}x^2$ to get x.

Next, also recall,

$$\frac{d}{dx}(kx) = k$$

So, using this as a guide is should be pretty simple to see that we would need to differentiate $4x$ to get 4.

So, if we put these two parts together it looks like we could use the following function.

$$f(x) = 4x - \frac{1}{2}x^2$$

The derivative of this function is clearly $f'(x) = 4 - x$. However, it is not the most general possible function that gives this derivative.

Do not forget that the derivative of a constant is zero and so we could add any constant onto our function and get the same derivative.

This is one of the biggest mistakes that students make with learning to "undo" differentiation. Any time we undo differentiation there is always the possibility that there was a constant on the original function and so we need to acknowledge that. We usually do this by adding a "$+c$" onto the end of our function. We use a general c because we have no way of knowing that the constant would be and this allows for all possible constants.

Therefore, the most general function that we could use to get $f'(x) = 4 - x$ is,

$$\boxed{f(x) = 4x - \frac{1}{2}x^2 + c}$$

Hint

This is the "easy" part of this problem. All we are really being asked to do is determine the specific value of c that we would need in order have the function have the value of 1 at $x = 4$. If you've reached this point of a Calculus course you should have the required Algebra knowledge (and yes it really is just Algebra) to do this part.

(c) Given that $f(4) = 1$ find a function that will have a relative maximum at $x = 4$.

Solution

To do this part all we really need to do is plug $x = 4$ into our answer from the previous part and set the result equal to 1. This will result in an equation with a single unknown value, c. So all we need to do then is solve for c.

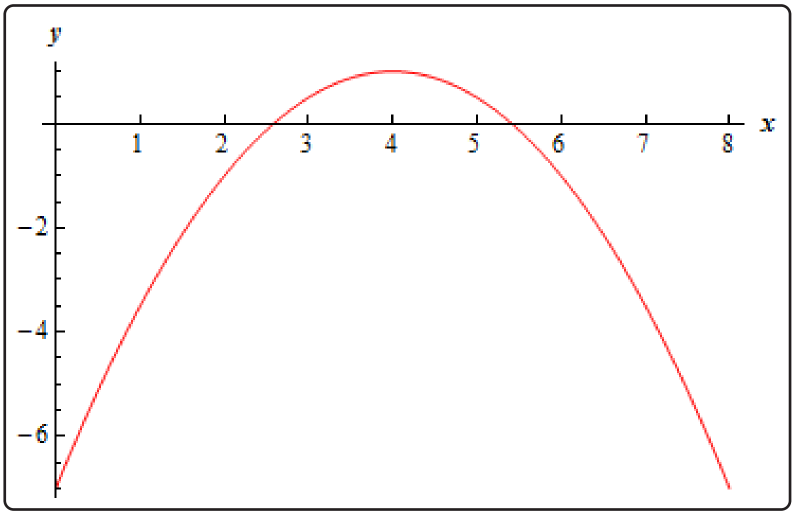
Here is the work for this part.

$$1 = f(4) = 4(4) - \frac{1}{2}(4)^2 + c = 8 + c \quad \rightarrow \quad 1 = 8 + c \quad \rightarrow \quad c = -7$$

So, it looks like one possible function that will have a relative maximum at $x = 4$ is,

$$\boxed{f(x) = 4x - \frac{1}{2}x^2 - 7}$$

As a final part to this problem, here is a quick graph of this function to verify that it does in fact have a relative maximum at $x = 4$.



14. Given that $f(x)$ and $g(x)$ are increasing functions. If we define $h(x) = f(x) + g(x)$ show that $h(x)$ is an increasing function.

Hint

At first glance this problem may seem quite difficult. However, just keep in mind how we have been determining whether functions where increasing to this point and that should suggest a first step.

Step 1

To this point we've always used the derivative to determine if a function was increasing so let's do that here as well.

Note that this may not seem all that useful because we don't actually know what any of the functions are. However, just because we don't know what the functions are doesn't mean that we can't at least write down a formula for $h'(x)$. Here is that formula.

$$h'(x) = f'(x) + g'(x)$$

Hint

We were told that $f(x)$ and $g(x)$ are increasing functions so what does that tell us about their derivatives?

Step 2

We are told that both $f(x)$ and $g(x)$ are increasing functions so this means that we know that both of their derivatives must be positive. Or,

$$f'(x) > 0 \qquad\qquad g'(x) > 0$$

Hint

Given the formula for $h'(x)$ we found in Step 1 and what we noticed about the signs of $f'(x)$ and $g'(x)$ that we noted in Step 2 what can we say about the sign of $h(x)$?

Step 3

Okay, we are pretty much done at this point. We know from Step 1 that,

$$h'(x) = f'(x) + g'(x)$$

Also from Step 2 we know that both $f'(x)$ and $g'(x)$ are positive. So, $h'(x)$ is the sum of two positive functions and in turn means that we must have,

$$h'(x) > 0$$

Therefore we can see that $h(x)$ must be an increasing function.

Final Thoughts / Strategy Discussion

Figuring out how to do these kinds of problems can definitely seem quite daunting at times. That is especially true when the statement we are being asked to prove seems to be fairly “obvious” as is the case here. The sum of two increasing functions intuitively should also be increasing. The problem is that we are being asked to actually prove that and not just say “well it makes sense so it should be true”.

What we want to discuss here is not the proof of this fact (that is given above after all...). Instead let’s take a look at the thought process that went into constructing the proof above.

The first step is to really look at what we are being asked to prove. This means not just reading the statement, but reading the statement and trying to relate what we are being asked to prove to something we already know.

In this case, we’re being asked to show that a general function is increasing given a set of assumptions. By this point we know how to prove that specific functions are increasing. So, let’s start with that.

We know that in order to use Calculus to prove that a function is increasing we need to look at the derivative of the function. We also know that, at least symbolically, we write down the formula for the derivative of the function we are interested in for this problem.

Now for the next step in the thought process. We've got a formula and we know that we need to show that it is positive. At this point we need to think about the assumptions that we were given. Don't forget the assumptions. They were given for a reason and we'll need to use them. What do the assumptions tell us? How can we relate them to what we are being asked to prove?

In this case, we know from the assumptions that the two derivatives were positive.

For this problem this wasn't a particularly difficult step, but for other problems this can be a little tricky.

Finally, we need to put the two previous thought process steps together. This can also be a fairly tricky step. If you haven't had a lot of exposure to "mathematical logic/proofs" it can be daunting to put all the information together. Often times you will need to try various ways of putting the information together before something "clicks" and you can see how to proceed. You may even need to go back to the previous step and see if there is something about the assumptions that you may have missed.

In this case we could see that the derivative was the sum of two other derivatives and from our assumptions we knew that the two individual derivatives had to be positive. We also know from basic Algebra knowledge that the sum of two positive quantities has to be positive and so we are done.

The key part of this whole process is that you will have to persevere. Try not to get discouraged and if something doesn't work out move on and try something else. Also, do not get so wrapped up in the process that you don't take breaks occasionally. If you keep running into road blocks then step away for a while and come back at a later point. Sometimes that is all it takes to get a fresh idea.

Another thing that students often initially have difficulty with is trying to mathematically write this kind of thing down. In your mind you may have been able to see all the "logic" involved in the proof, but just couldn't see how to put it all together and write it down. If you are having that problem the best thing to do is just start writing things down.

For instance, you know you need the derivative of the given function so write that down. If you don't have a specific function to differentiate can you at least symbolically write down the derivative as we did here?

Once you have that written down look at the pieces and start writing down what you know about them. Actually write down what you know (*i.e.* things like $f'(x) > 0$). This seems silly at times, but it really can help with the process.

Once you have everything written down you might be able to see how to string everything together with words/explanations to prove what you want to prove.

15. Given that $f(x)$ is an increasing function and define $h(x) = [f(x)]^2$. Will $h(x)$ be an increasing function? If yes, prove that $h(x)$ is an increasing function. If not, can you determine any other conditions needed on the function $f(x)$ that will guarantee that $h(x)$ will also increase?

Hint

If you have trouble with these kinds of "proof" problems you might want to check out the discussion at the end of the previous problem for a "strategy" that might be useful here. This isn't quite the same kind of problem but the strategy given there should help here as well.

Hint

How do we use Calculus to determine if a function is increasing?

Step 1

We know that the derivative can be used to tell us if a function is increasing so let's find the derivative of $h(x)$. Do not get excited about the fact that we don't know what $f(x)$ is. We can symbolically take the derivative with a quick application of the chain rule.

Here is the derivative of $h(x)$.

$$h'(x) = 2[f(x)]^1 f'(x) = 2f(x) f'(x)$$

Hint

How do we use the derivative to determine if a function is increasing?

Step 2

We know that a function will be increasing if its derivative is positive. So, the question we need to answer is can we guarantee that $h'(x) > 0$ if we only take into account the assumption that $f(x)$ is an increasing function.

From our assumption that $f(x)$ is an increasing function and so we know that $f'(x) > 0$.

Now, let's see what $h'(x)$ tells us. We can see that $h'(x)$ is a product of a number and two functions. The "2" is positive and so the sign of the derivative will come from the sign of the product of $f(x)$ and $f'(x)$.

Hint

So, will $h'(x)$ be positive?

Step 3

Okay, from our assumption we know that $f'(x)$ is positive. However, the product isn't guaranteed to be positive.

For example, consider $f(x) = x$. Clearly, $f'(x) = 1 > 0$, and so this is an increasing function. However, $f(x) f'(x) = (x)(1) = x$. Therefore, we can see that the product will not always be positive. This shouldn't be too surprising given that,

$$h(x) = [f(x)]^2 = [x]^2 = x^2$$

In this case we can clearly see that $h(x)$ will not always be an increasing function.

On the other had if we let $f(x) = \mathbf{e}^x$ we can see that $f'(x) = \mathbf{e}^x > 0$ and we can also see that $f(x) f'(x) = (\mathbf{e}^x)(\mathbf{e}^x) = \mathbf{e}^{2x} > 0$.

So, from these two examples we can see that we can find increasing functions, $f(x)$, for which $[f(x)]^2$ may or may not always be increasing.

Hint

Can you use the examples above to determine a condition on $f(x)$ that will guarantee that $h(x)$ will be increasing?

Step 4

So, just what was the difference between the two examples above?

The problem with the first example, $f(x) = x$ was that it wasn't always positive and so the product of $f(x)$ and $f'(x)$ would not always be positive as we needed it to be.

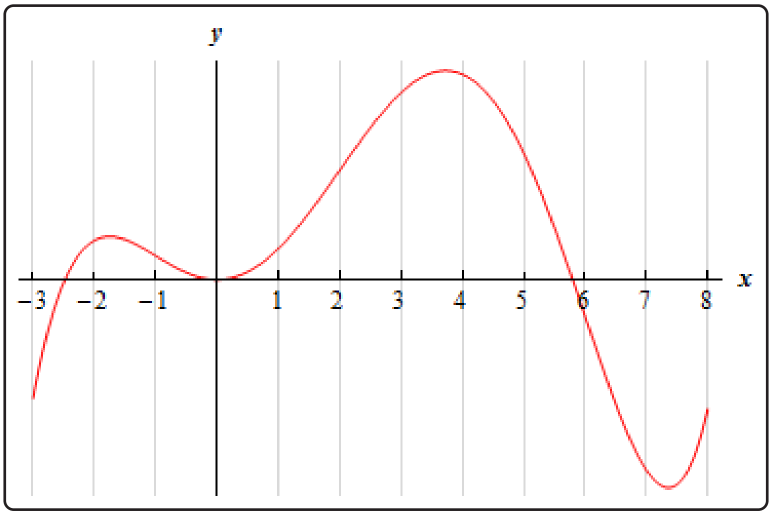
This was not a problem with the second example however, $f(x) = \mathbf{e}^x$. In this case the function is always positive and so the product of the function and its derivative will also be positive.

That is also the added condition that we need to guarantee that $h(x)$ will be positive.

If we start with the assumption that $f(x)$ in an increasing function we need to further assume that $f(x)$ is a positive function in order to guarantee that $h(x) = [f(x)]^2$ will be an increasing function.

4.6 The Shape of a Graph, Part II

1. The graph of a function is given below. Determine the intervals on which the function is concave up and concave down.



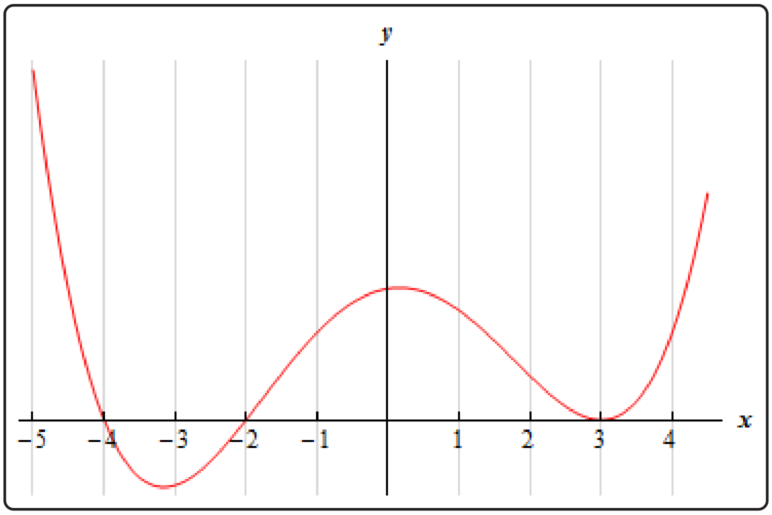
Solution

There really isn't too much to this problem. We can easily see from the graph where the function in concave up/concave down and so all we need to do is estimate where the concavity changes (and this really will be an estimate as it won't always be clear) and write down the intervals.

$$\boxed{\text{Concave Up : } (-1,2) \;\&\; (6,\infty) \qquad \text{Concave Down : } (-\infty,-1) \;\&\; (2,6)}$$

Again, the endpoints of these intervals are, at best, estimates as it won't always be clear just where the concavity changes.

2. Below is the graph the 2^{nd} **derivative** of a function. From this graph determine the intervals in which the **function** is concave up and concave down.



Hint

Be careful with this problem. The graph is of the 2^{nd} **derivative** of the function and so we don't just write down intervals where the graph is concave up and concave down. Recall how the 2^{nd} derivative tells us where the function is concave up and concave down and this problem is not too bad.

Solution

We need to be careful and not do this problem as we did the first practice problem. The graph given is the graph of the 2^{nd} **derivative** and not the graph of the function. Therefore, the answer is not just where the graph is concave up or concave down.

What we need to do here is to recall that if the 2^{nd} derivative is positive (*i.e.* the graph is above the x-axis) then the function in concave up and if the 2^{nd} derivative is negative (*i.e.* the graph is below the x-axis) then the function is concave down.

So, it is fairly clear where the graph is above/below the x-axis and so we have the following intervals of concave up/concave down.

Concave Up : $(-\infty, -4), (-2, 3)$ & $(3, \infty)$ Concave Down : $(-4, -2)$

Even though the problem didn't ask for it we can also identify that $x = -4$ and $x = -2$ are inflection points because at these points the concavity changes. Note that $x = 3$ is **not** an inflection point. Clearly the 2^{nd} derivative is zero here, but the concavity doesn't change at this point and so it is not an inflection point. Keep in mind inflection points are only where the concavity changes and not simply where the 2^{nd} derivative is zero.

3. For $f(x) = 12 + 6x^2 - x^3$ answer each of the following questions.

   (a) Determine a list of possible inflection points for the function.

   (b) Determine the intervals on which the function is concave up and concave down.

   (c) Determine the inflection points of the function.

Solutions

(a) Determine a list of possible inflection points for the function.

Solution

To get the list of possible inflection points for the function we'll need the 2^{nd} derivative of the function so here that is.

$$f'(x) = 12x - 3x^2 \qquad\qquad \underline{f''(x) = 12 - 6x}$$

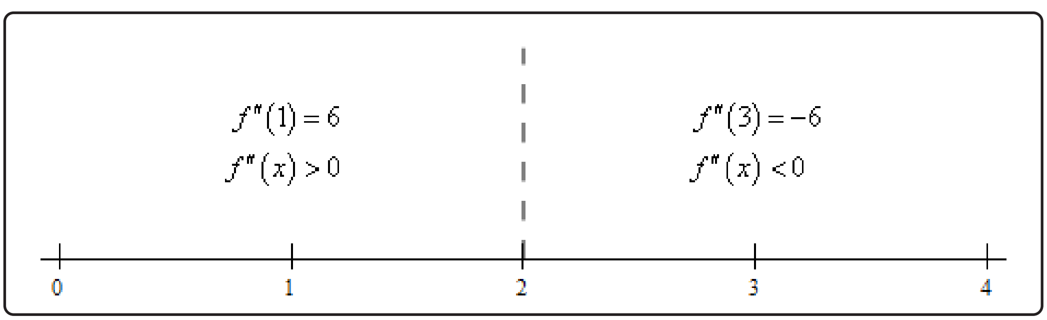
Now, recall that possible inflection points are where the 2^{nd} derivative either doesn't exist or is zero. Clearly the 2^{nd} derivative exists everywhere (it's a polynomial....) and, in this case, it should be fairly clear where the 2^{nd} derivative is zero. The only possible inflection critical point of the function in this case is,

$$\underline{x = 2}$$

(b) Determine the intervals on which the function is concave up and concave down.

Solution

There isn't much to this part. All we really need here is a number line for the 2^{nd} derivative. Here that is,



From this we get the following concave up/concave down information for the function.

$$\boxed{\text{Concave Up : } (-\infty, 2) \qquad \text{Concave Down : } (2, \infty)}$$

(c) Determine the inflection points of the function.

Solution

For this part all we need to do is interpret the results from the previous step. Recall that inflection points are points where the concavity changes (as opposed to simply the points where the 2^{nd} derivative is zero or doesn't exist). Therefore, the single inflection point for this function is,

$$\boxed{x = 2}$$

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4. For $g(z) = z^4 - 12z^3 + 84z + 4$ answer each of the following questions.

(a) Determine a list of possible inflection points for the function.

(b) Determine the intervals on which the function is concave up and concave down.

(c) Determine the inflection points of the function.

Solutions

(a) Determine a list of possible inflection points for the function.

Solution

To get the list of possible inflection points for the function we'll need the 2^{nd} derivative of the function so here that is.

$$g'(z) = 4z^3 - 36z^2 + 84 \qquad \underline{g''(z) = 12z^2 - 72z = 12z(z-6)}$$

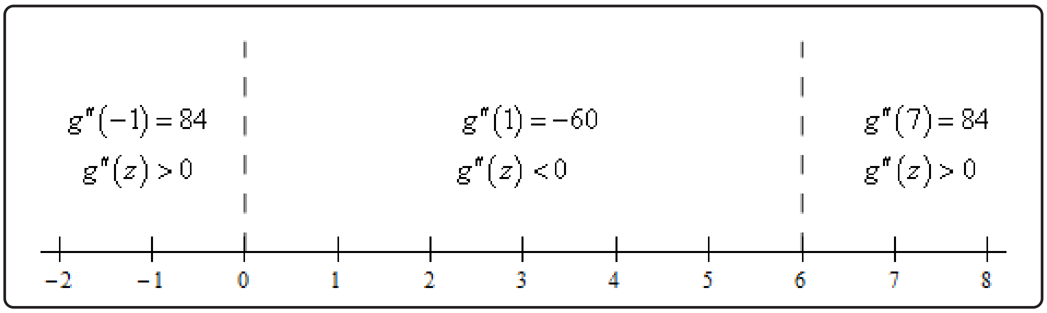
Now, recall that possible inflection points are where the 2^{nd} derivative either doesn't exist or is zero. Clearly the 2^{nd} derivative exists everywhere (it's a polynomial....) and, because we factored the 2^{nd} derivative, it should be fairly clear where the 2^{nd} derivative is zero. The possible inflection critical points of this function are,

$$\underline{z = 0 \quad \& \quad z = 6}$$

(b) Determine the intervals on which the function is concave up and concave down.

Solution

There isn't much to this part. All we really need here is a number line for the 2^{nd} derivative. Here that is,



From this we get the following concave up/concave down information for the function.

$$\boxed{\text{Concave Up : } (-\infty, 0) \ \ \& \ \ (6, \infty) \qquad \text{Concave Down : } (0, 6)}$$

(c) Determine the inflection points of the function.

Solution

For this part all we need to do is interpret the results from the previous step. Recall that inflection points are points where the concavity changes (as opposed to simply the points where the 2^{nd} derivative is zero or doesn't exist). Therefore, the inflection points for this function are,

$$\boxed{z = 0 \ \ \& \ \ z = 6}$$

5. For $h(t) = t^4 + 12t^3 + 6t^2 - 36t + 2$ answer each of the following questions.

   (a) Determine a list of possible inflection points for the function.

   (b) Determine the intervals on which the function is concave up and concave down.

   (c) Determine the inflection points of the function.

Solutions

(a) Determine a list of possible inflection points for the function.

Solution

To get the list of possible inflection points for the function we'll need the 2^{nd} derivative of the function so here that is.

$$h'(t) = 4t^3 + 36t^2 + 12t - 36 \qquad \underline{h''(t) = 12t^2 + 72t + 12 = 12\left(t^2 + 6t + 1\right)}$$

Now, recall that possible inflection points are where the 2^{nd} derivative either doesn't exist or is zero. Clearly the 2^{nd} derivative exists everywhere (it's a polynomial....). In this case the 2^{nd} derivative doesn't factor and so we'll need to use the quadratic formula to determine where the 2^{nd} derivative is zero.

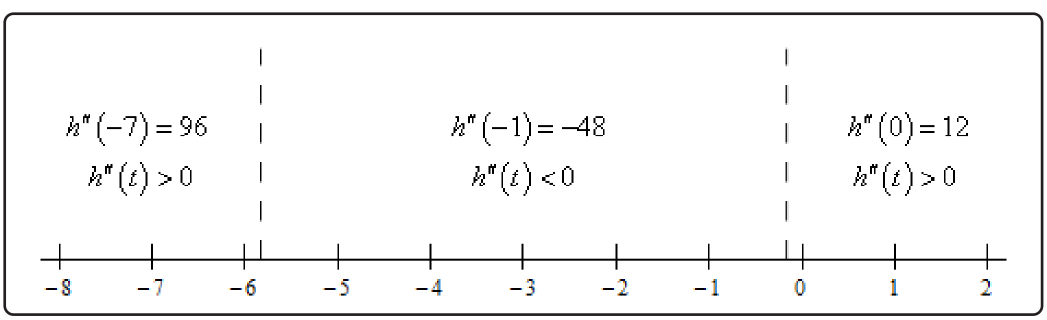
The possible inflection critical points of this function are,

$$\underline{t = -3 \pm 2\sqrt{2} = -5.8284, \quad -0.1716}$$

(b) Determine the intervals on which the function is concave up and concave down.

Solution

There isn't much to this part. All we really need here is a number line for the 2^{nd} derivative. Here that is,



From this we get the following concave up/concave down information for the function.

$$\begin{aligned} \text{Concave Up : } & \left(-\infty, -3-2\sqrt{2}\right) \quad \& \quad \left(-3+2\sqrt{2}, \infty\right) \\ \text{Concave Down : } & \left(-3-2\sqrt{2}, -3+2\sqrt{2}\right) \end{aligned}$$

(c) Determine the inflection points of the function.

Solution

For this part all we need to do is interpret the results from the previous step. Recall that inflection points are points where the concavity changes (as opposed to simply the points where the 2^{nd} derivative is zero or doesn't exist). Therefore, the inflection points for this function are,

$$\boxed{t = -3-2\sqrt{2} \quad \& \quad t = -3+2\sqrt{2}}$$

---

6. For $h(w) = 8 - 5w + 2w^2 - \cos(3w)$ on $[-1, 2]$ answer each of the following questions.

(a) Determine a list of possible inflection points for the function.

(b) Determine the intervals on which the function is concave up and concave down.

(c) Determine the inflection points of the function.

Solutions

(a) Determine a list of possible inflection points for the function.

Solution

To get the list of possible inflection points for the function we'll need the 2^{nd} derivative of the function so here that is.

$$h'(w) = -5 + 4w + 3\sin(3w) \qquad \underline{h''(w) = 4 + 9\cos(3w)}$$

Now, recall that possible inflection points are where the 2^{nd} derivative either doesn't exist or is zero. Clearly the 2^{nd} derivative exists everywhere (the cosine function exists everywhere...) and so all we need to do is set the 2^{nd} derivative equal to zero and solve. We're not going to show all of those details so if you need to do some review of the process go back to the Solving Trig Equations sections for some examples.

The possible inflection critical points of this function are,

$$\begin{aligned} w &= 0.6771 + \tfrac{2\pi}{3}n \\ w &= 1.4173 + \tfrac{2\pi}{3}n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

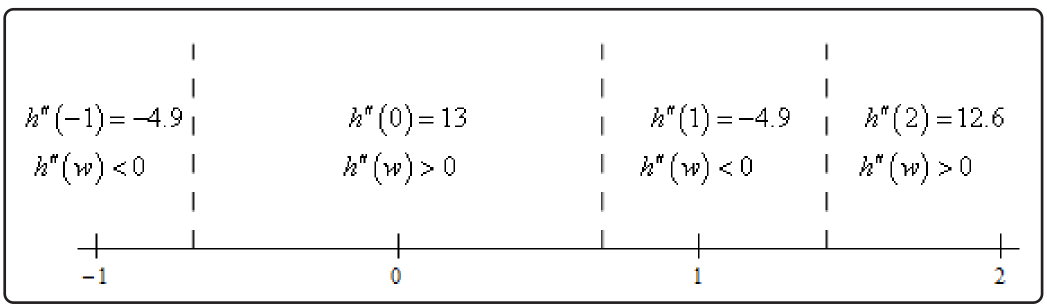
Plugging in some n's gives the following possible inflection points in the interval $[-1, 2]$.

$$\underline{w = -0.6771 \qquad w = 0.6771 \qquad w = 1.4173}$$

(b) Determine the intervals on which the function is concave up and concave down.

Solution

There isn't much to this part. All we really need here is a number line for the 2^{nd} derivative. Here that is,



From this we get the following concave up/concave down information for the function.

$$\begin{aligned} \text{Concave Up}: &\ (-0.6771,\ 0.6771) \quad \& \quad (1.4173,\ 2] \\ \text{Concave Down}: &\ [-1,\ -0.6771) \quad \& \quad (0.6771,\ 1.4173) \end{aligned}$$

Be careful with the end points of these intervals! We are working on the interval $[-1, 2]$ and we've done no work for concavity outside of this interval and so we can't say anything about what happens outside of the interval.

(c) Determine the inflection points of the function.

Solution

For this part all we need to do is interpret the results from the previous step. Recall that inflection points are points where the concavity changes (as opposed to simply the points where the 2^{nd} derivative is zero or doesn't exist). Therefore, the inflection points for this function are,

$$\boxed{w = -0.6771 \qquad w = 0.6771 \qquad w = 1.4173}$$

As with the previous step we have to be careful and recall that we are working on the interval $[-1, 2]$. There are infinitely many more possible inflection points and we've done no work outside of the interval to determine if they are in fact inflection points!

---

7. For $R(z) = z(z+4)^{\frac{2}{3}}$ answer each of the following questions.

(a) Determine a list of possible inflection points for the function.

(b) Determine the intervals on which the function is concave up and concave down.

(c) Determine the inflection points of the function.

Solutions

(a) Determine a list of possible inflection points for the function.

Solution

To get the list of possible inflection points for the function we'll need the 2^{nd} derivative of the function so here that is.

$$R'(z) = (z+4)^{\frac{2}{3}} + z\left(\frac{2}{3}\right)(z+4)^{-\frac{1}{3}} = \frac{5z+12}{3(z+4)^{\frac{1}{3}}}$$

$$R''(z) = \frac{5\left(3(z+4)^{\frac{1}{3}}\right) - (5z+12)(z+4)^{-\frac{2}{3}}}{\left[3(z+4)^{\frac{1}{3}}\right]^2}$$

$$= \frac{\left[15(z+4) - (5z+12)\right](z+4)^{-\frac{2}{3}}}{9(z+4)^{\frac{2}{3}}} = \underline{\frac{10z+48}{9(z+4)^{\frac{4}{3}}}}$$

Note that we simplified the derivatives at each step to help with the next step. You don't technically need to do this but having the 2^{nd} derivative in its "simplest" form will definitely help with getting the answer to this part.

Now, recall that possible inflection points are where the 2^{nd} derivative either doesn't exist or is zero. Because we simplified the 2^{nd} derivative as much as possible it is clear that the 2^{nd} derivative won't exist at $z = -4$ (and the function exists at this point as well!). It should also be clear that the 2^{nd} derivative is zero at $z = -\frac{48}{10} = -\frac{24}{5}$.

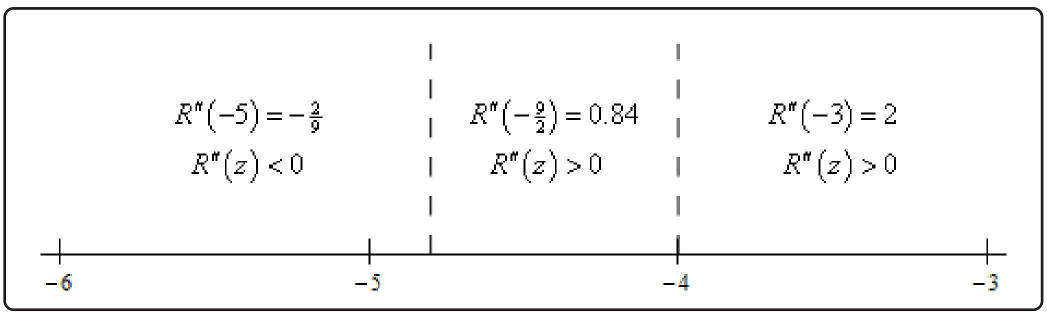
The possible inflection critical points of this function are then,

$$\underline{z = -\frac{24}{5} = -4.8 \quad \& \quad z = -4}$$

(b) Determine the intervals on which the function is concave up and concave down.

Solution

There isn't much to this part. All we really need here is a number line for the 2^{nd} derivative. Here that is,



From this we get the following concave up/concave down information for the function.

$$\text{Concave Up : } \left(-\frac{24}{5}, -4\right) \quad \& \quad (-4, \infty)$$

$$\text{Concave Down : } \left(-\infty, -\frac{24}{5}\right)$$

(c) Determine the inflection points of the function.

Solution

For this part all we need to do is interpret the results from the previous step. Recall that inflection points are points where the concavity changes (as opposed to simply the points where the 2^{nd} derivative is zero or doesn't exist). Therefore, the only inflection point for this function is,

$$\boxed{z = -\frac{24}{5}}$$

---

8. For $h(x) = \mathbf{e}^{4-x^2}$ answer each of the following questions.

(a) Determine a list of possible inflection points for the function.

(b) Determine the intervals on which the function is concave up and concave down.

(c) Determine the inflection points of the function.

Solutions

(a) Determine a list of possible inflection points for the function.

Solution

To get the list of possible inflection points for the function we'll need the 2^{nd} derivative of the function so here that is.

$$h'(x) = -2x\mathbf{e}^{4-x^2} \qquad h''(x) = -2\mathbf{e}^{4-x^2} + 4x^2\mathbf{e}^{4-x^2} = \underline{2\mathbf{e}^{4-x^2}\left(2x^2 - 1\right)}$$

Don't forget to product rule for the 2^{nd} derivative and factoring the exponential out will help a little with the next step.

Now, recall that possible inflection points are where the 2^{nd} derivative either doesn't exist or is zero. It should be fairly clear that the 2^{nd} derivative exists everywhere (it is a product of two functions that exist everywhere...). We also know that exponentials are never zero and so the 2^{nd} derivative will be zero at the solutions to $2x^2 - 1 = 0$

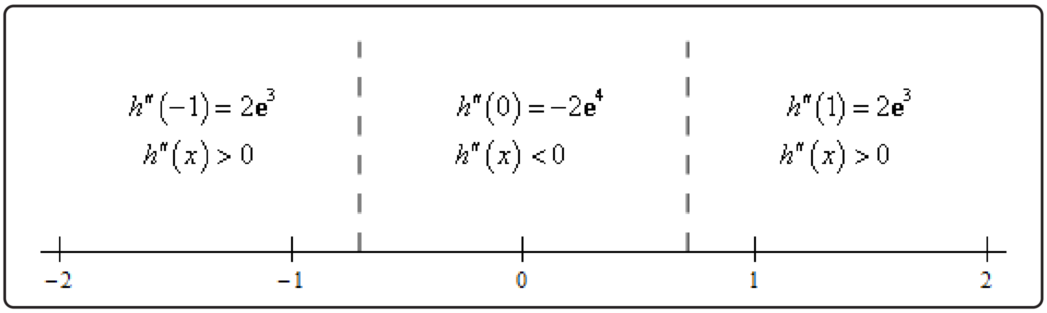
The possible inflection critical points of this function are then,

$$\underline{x = \pm\frac{1}{\sqrt{2}} = \pm 0.7071}$$

(b) Determine the intervals on which the function is concave up and concave down.

Solution

There isn't much to this part. All we really need here is a number line for the 2^{nd} derivative. Here that is,



From this we get the following concave up/concave down information for the function.

$$\boxed{\begin{array}{l} \text{Concave Up : } \left(-\infty, -\frac{1}{\sqrt{2}}\right) \quad \& \quad \left(\frac{1}{\sqrt{2}}, \infty\right) \\ \text{Concave Down : } \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \end{array}}$$

(c) Determine the inflection points of the function.

Solution

For this part all we need to do is interpret the results from the previous step. Recall that inflection points are points where the concavity changes (as opposed to simply the points where the 2^{nd} derivative is zero or doesn't exist). Therefore, the only inflection point for this function is,

$$\boxed{x = -\frac{1}{\sqrt{2}} = -0.7071 \qquad x = \frac{1}{\sqrt{2}} = 0.7071}$$

9. For $g(t) = t^5 - 5t^4 + 8$ answer each of the following questions.

(a) Identify the critical points of the function.

(b) Determine the intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

(d) Determine the intervals on which the function is concave up and concave down.

(e) Determine the inflection points of the function.

(f) Use the information from steps **(a)** - **(e)** to sketch the graph of the function.

Solutions

(a) Identify the critical points of the function.

Solution

The parts to this problem (with the exception of the last part) are just like the basic increasing/decreasing problems from the previous section and the basic concavity problems from earlier in this section. Because of that we will not be putting in quite as much detail here. If you are still unsure how to work the parts of this problem you should go back and work a few of the basic problems from the previous section and/or earlier in this section before proceeding.

We will need the 1^{st} derivative to start things off.

$$\underline{g'(t) = 5t^4 - 20t^3 = 5t^3(t-4)}$$

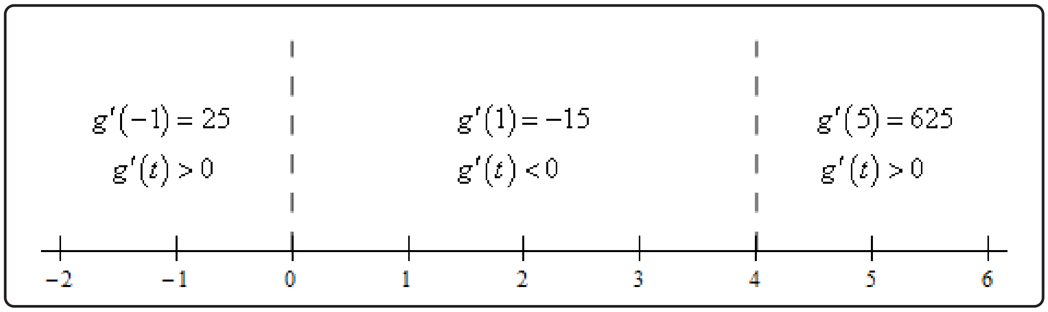
From the 1^{st} derivative we can see that the critical points of this function are then,

$$\underline{t = 0 \quad \& \quad t = 4}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To answer this part all we need is the number line for the 1^{st} derivative.



From this we get the following increasing/decreasing information for the function.

$$\boxed{\text{Increasing : } (-\infty, 0) \ \ \& \ \ (4, \infty) \qquad \text{Decreasing : } (0, 4)}$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

From the number line in the previous step we get the following classifications of the critical points.

$$\boxed{t = 0 : \text{ Relative Maximum} \qquad t = 4 : \text{ Relative Minimum}}$$

(d) Determine the intervals on which the function is concave up and concave down.

Solution

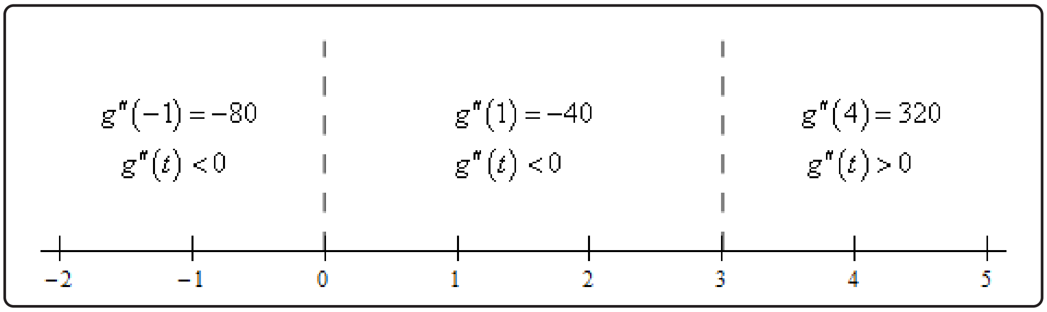
We'll need the 2^{nd} derivative to find the list of possible inflection points.

$$\underline{g''(t) = 20t^3 - 60t^2 = 20t^2(t - 3)}$$

The possible inflection points for this problem are,

$$\underline{t = 0 \qquad \& \qquad t = 3}$$

To get the intervals of concavity we'll need the number line for the 2^{nd} derivative.



From this we get the following concavity information for the function.

$$\boxed{\text{Concave Up : } (3,\infty) \qquad \text{Concave Down : } (-\infty,0) \quad \& \quad (0,3)}$$

(e) Determine the inflection points of the function.

Solution

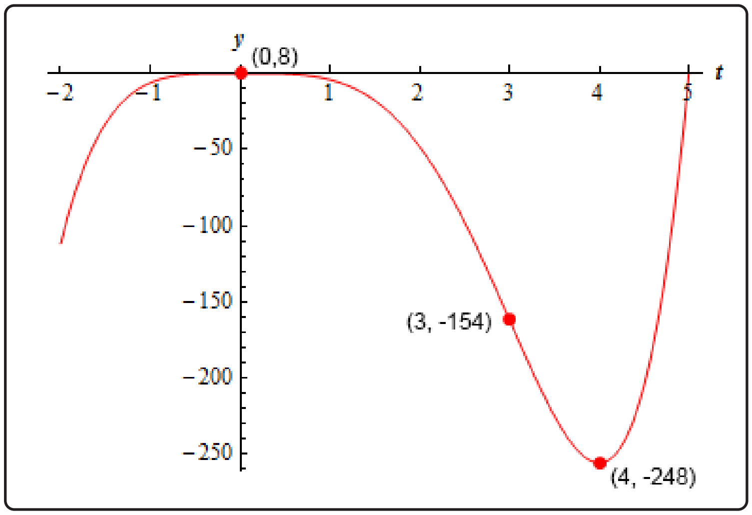
From the concavity information in the previous step we can see that the single inflection point for the function is,

$$\boxed{t = 3}$$

(f) Use the information from steps **(a)** - **(e)** to sketch the graph of the function.

Solution

Here is a sketch of the graph of this function using the information above. As we did in problems in this section we can start at the left and work our way to the right on the graph. As we do this we first pay attention to the increasing/decreasing information and then make sure that the curve has the correct concavity as we sketch it in.



Note that because we used a computer to generate the sketch it is possible that your sketch won't be quite the same. It should however, have the same points listed on the graph above, the same basic increasing/decreasing nature and the same basic concavity.

---

10. For $f(x) = 5 - 8x^3 - x^4$ answer each of the following questions.

(a) Identify the critical points of the function.

(b) Determine the intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

(d) Determine the intervals on which the function is concave up and concave down.

(e) Determine the inflection points of the function.

(f) Use the information from steps **(a)** - **(e)** to sketch the graph of the function.

Solutions

(a) Identify the critical points of the function.

Solution

The parts to this problem (with the exception of the last part) are just like the basic increasing/decreasing problems from the previous section and the basic concavity problems from earlier in this section. Because of that we will not be putting in quite as much detail here. If you are still unsure how to work the parts of this problem you should go back and work a few of the basic problems from the previous section and/or earlier in this section before proceeding.

We will need the 1^{st} derivative to start things off.

$$\underline{f'(x) = -24x^2 - 4x^3 = -4x^2(x+6)}$$

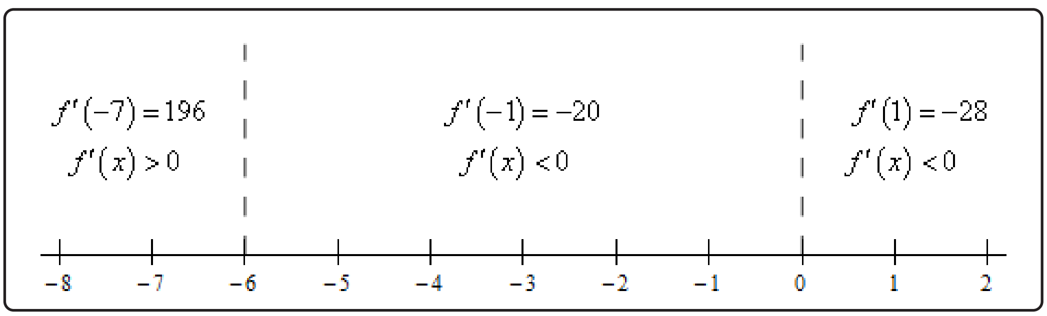
From the 1^{st} derivative we can see that the critical points of this function are then,

$$\underline{x = -6 \quad \& \quad x = 0}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To answer this part all we need is the number line for the 1^{st} derivative.



From this we get the following increasing/decreasing information for the function.

Increasing : $(-\infty, -6)$ Decreasing : $(-6, 0)$ & $(0, \infty)$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

From the number line in the previous step we get the following classifications of the critical points.

$$\boxed{x = -6: \text{ Relative Maximum} \qquad x = 0: \text{ Neither}}$$

(d) Determine the intervals on which the function is concave up and concave down.

Solution

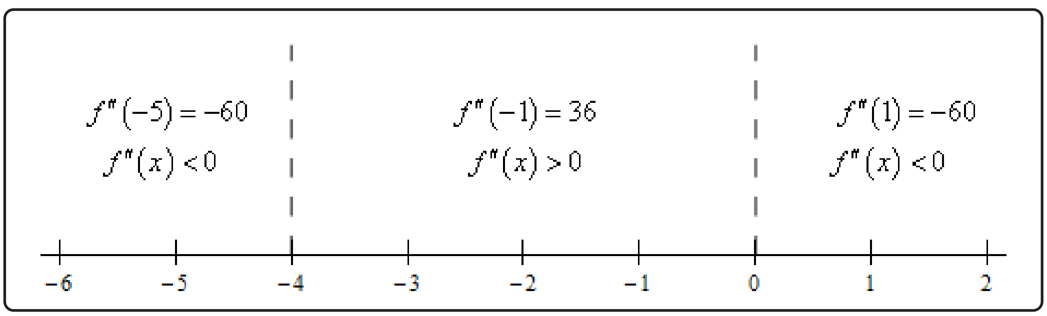
We'll need the 2^{nd} derivative to find the list of possible inflection points.

$$\underline{f''(x) = -48x - 12x^2 = -12x(x+4)}$$

The possible inflection points for this function are,

$$\underline{x = -4 \qquad \& \qquad x = 0}$$

To get the intervals of concavity we'll need the number line for the 2^{nd} derivative.



From this we get the following concavity information for the function.

$$\boxed{\text{Concave Up}: (-4, 0) \qquad \text{Concave Down}: (-\infty, -4) \ \& \ (0, \infty)}$$

(e) Determine the inflection points of the function.

Solution

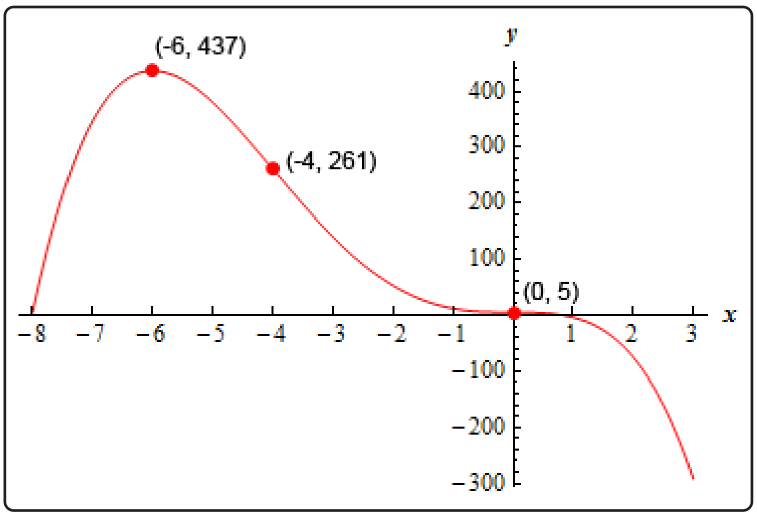
From the concavity information in the previous step we can see that the inflection points for the function are,

$$\boxed{x = -4 \qquad \& \qquad x = 0}$$

(f) Use the information from steps **(a)** - **(e)** to sketch the graph of the function.

Solution

Here is a sketch of the graph of this function using the information above. As we did in problems in this section we can start at the left and work our way to the right on the graph. As we do this we first pay attention to the increasing/decreasing information and then make sure that the curve has the correct concavity as we sketch it in.



Note that because we used a computer to generate the sketch it is possible that your sketch won't be quite the same. It should however, have the same points listed on the graph above, the same basic increasing/decreasing nature and the same basic concavity.

11. For $h(z) = z^4 - 2z^3 - 12z^2$ answer each of the following questions.

(a) Identify the critical points of the function.

(b) Determine the intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

(d) Determine the intervals on which the function is concave up and concave down.

(e) Determine the inflection points of the function.

(f) Use the information from steps **(a)** - **(e)** to sketch the graph of the function.

Solutions

(a) Identify the critical points of the function.

Solution

The parts to this problem (with the exception of the last part) are just like the basic increasing/decreasing problems from the previous section and the basic concavity problems from earlier in this section. Because of that we will not be putting in quite as much detail here. If you are still unsure how to work the parts of this problem you should go back and work a few of the basic problems from the previous section and/or earlier in this section before proceeding.

We will need the 1^{st} derivative to start things off.

$$\underline{h'(z) = 4z^3 - 6z^2 - 24z = 2z\left(2z^2 - 3z - 12\right)}$$

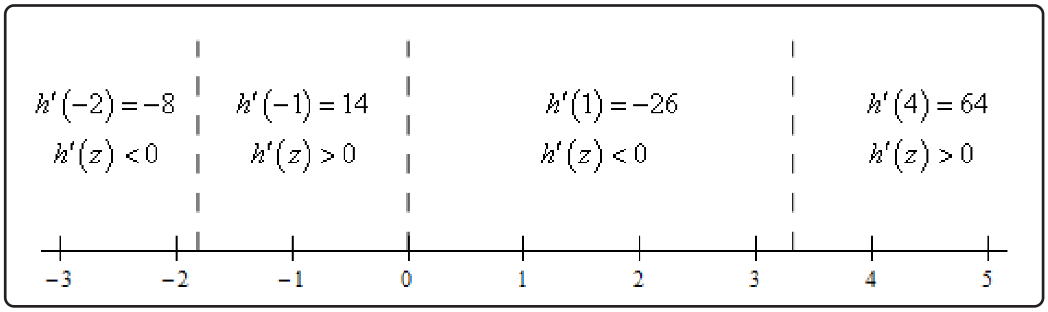
From the 1^{st} derivative we can see that the critical points of this function are then,

$$\underline{x = 0 \quad \& \quad x = \frac{3 \pm \sqrt{105}}{4} = -1.8117,\ 3.3117}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To answer this part all we need is the number line for the 1^{st} derivative.



From this we get the following increasing/decreasing information for the function.

$$\text{Increasing : } \left(\frac{3-\sqrt{105}}{4},0\right) \quad \& \quad \left(\frac{3+\sqrt{105}}{4},\infty\right)$$
$$\text{Decreasing : } \left(-\infty,\frac{3-\sqrt{105}}{4}\right) \quad \& \quad \left(0,\frac{3+\sqrt{105}}{4}\right)$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

From the number line in the previous step we get the following classifications of the critical points.

$$z = \frac{3 \pm \sqrt{105}}{4} : \text{ Relative Minimum} \qquad z = 0 : \text{ Relative Maximum}$$

(d) Determine the intervals on which the function is concave up and concave down.

Solution

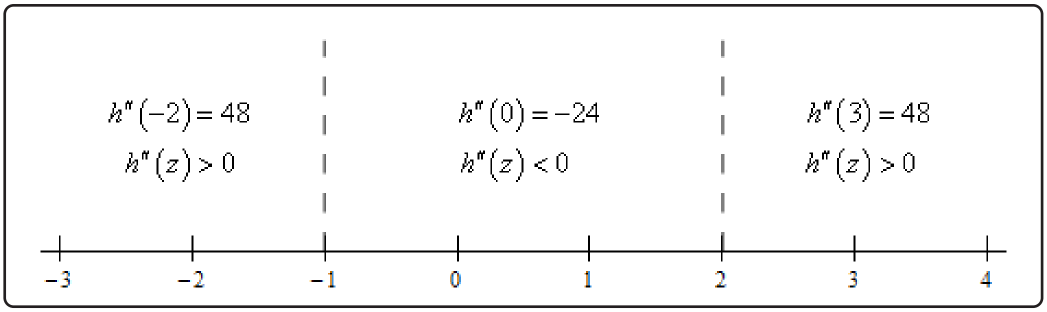
We'll need the 2^{nd} derivative to find the list of possible inflection points.

$$\underline{h''(z) = 12z^2 - 12z - 24 = 12(z-2)(z+1)}$$

The possible inflection points for this function are,

$$\underline{z = -1 \qquad \& \qquad z = 2}$$

To get the intervals of concavity we'll need the number line for the 2^{nd} derivative.



From this we get the following concavity information for the function.

$$\boxed{\text{Concave Up : } (-\infty,-1) \quad \& \quad (2,\infty) \qquad \text{Concave Down : } (-1,2)}$$

(e) Determine the inflection points of the function.

Solution

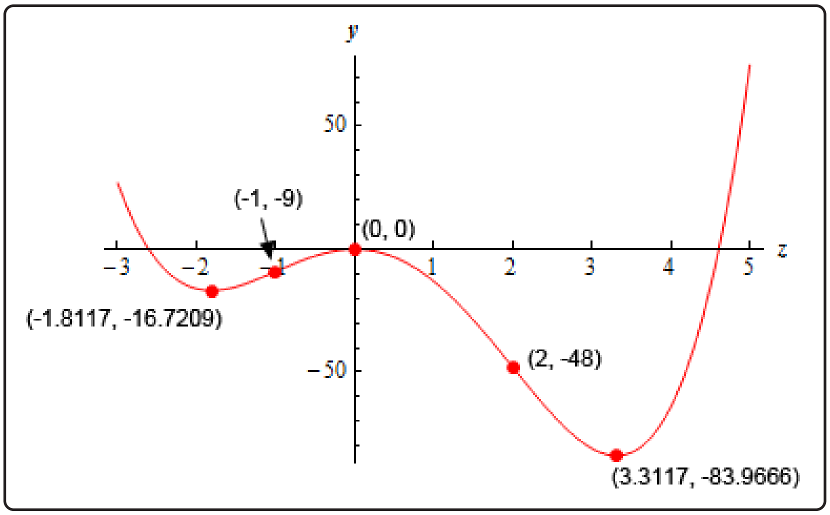
From the concavity information in the previous step we can see that the inflection points for the function are,

$$\boxed{z = -1 \qquad \& \qquad z = 2}$$

(f) Use the information from steps **(a)** - **(e)** to sketch the graph of the function.

Solution

Here is a sketch of the graph of this function using the information above. As we did in problems in this section we can start at the left and work our way to the right on the graph. As we do this we first pay attention to the increasing/decreasing information and then make sure that the curve has the correct concavity as we sketch it in.



Note that because we used a computer to generate the sketch it is possible that your sketch won't be quite the same. It should however, have the same points listed on the graph above, the same basic increasing/decreasing nature and the same basic concavity.

12. For $Q(t) = 3t - 8\sin\left(\frac{t}{2}\right)$ on $[-7, 4]$ answer each of the following questions.

(a) Identify the critical points of the function.

(b) Determine the intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

(d) Determine the intervals on which the function is concave up and concave down.

(e) Determine the inflection points of the function.

(f) Use the information from steps **(a)** - **(e)** to sketch the graph of the function.

Solutions

(a) Identify the critical points of the function.

Solution

The parts to this problem (with the exception of the last part) are just like the basic increasing/decreasing problems from the previous section and the basic concavity problems from earlier in this section. Because of that we will not be putting in quite as much detail here. If you are still unsure how to work the parts of this problem you should go back and work a few of the basic problems from the previous section and/or earlier in this section before proceeding.

We will need the 1^{st} derivative to start things off.

$$\underline{Q'(t) = 3 - 4\cos\left(\frac{t}{2}\right)}$$

From the 1^{st} derivative all of the critical points are,

$$\begin{aligned} t &= 1.4454 + 4\pi n \\ t &= 11.1210 + 4\pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

If you need some review of the solving trig equation process go back to the Solving Trig Equations sections for some examples.

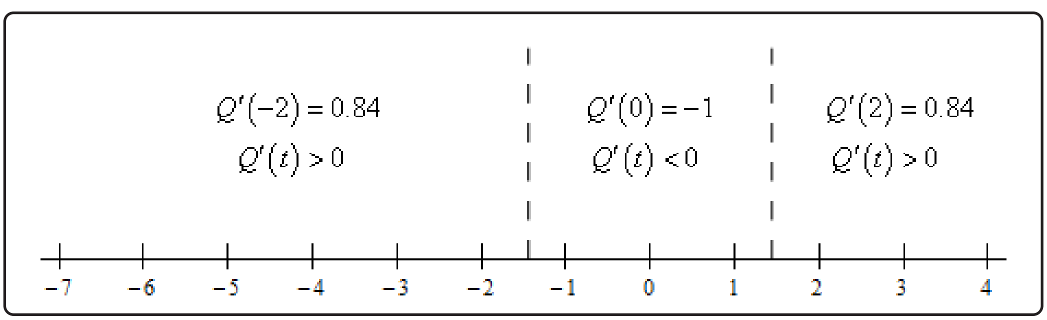
Plugging in some values of n we see that the critical points in the interval $[-7, 4]$ are,

$$\underline{t = -1.4454 \quad \& \quad t = 1.4454}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To answer this part all we need is the number line for the 1^{st} derivative.



From this we get the following increasing/decreasing information for the function.

Increasing : $[-7, -1.4454)$ & $(1.4454, 4]$
Decreasing : $(-1.4454, 1.4454)$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

From the number line in the previous step we get the following classifications of the critical points.

$t = -1.4454$: Relative Maximum $t = 1.4454$: Relative Minimum

---

(d) Determine the intervals on which the function is concave up and concave down.

Solution

We'll need the 2^{nd} derivative to find the list of possible inflection points.

$$\underline{Q''(t) = 2\sin\left(\frac{t}{2}\right)}$$

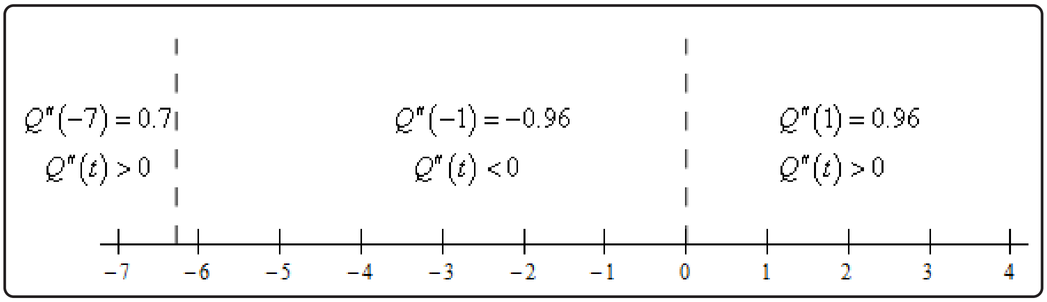
All possible inflection points of the function are,

$$\begin{aligned} t &= 4\pi n \\ t &= 2\pi + 4\pi n \end{aligned} \qquad n = 0, \pm1, \pm2, \pm3, \ldots$$

Plugging in some values of n we see that the possible inflection points in the interval $[-7, 4]$ are,

$$\underline{t = -6.2832 \qquad \& \qquad t = 0}$$

To get the intervals of concavity we'll need the number line for the 2^{nd} derivative.



From this we get the following concavity information for the function.

$$\text{Concave Up : } [-7, -6.2832) \text{ \& } (0, 4] \qquad \text{Concave Down : } (-6.2832, 0)$$

(e) Determine the inflection points of the function.

Solution

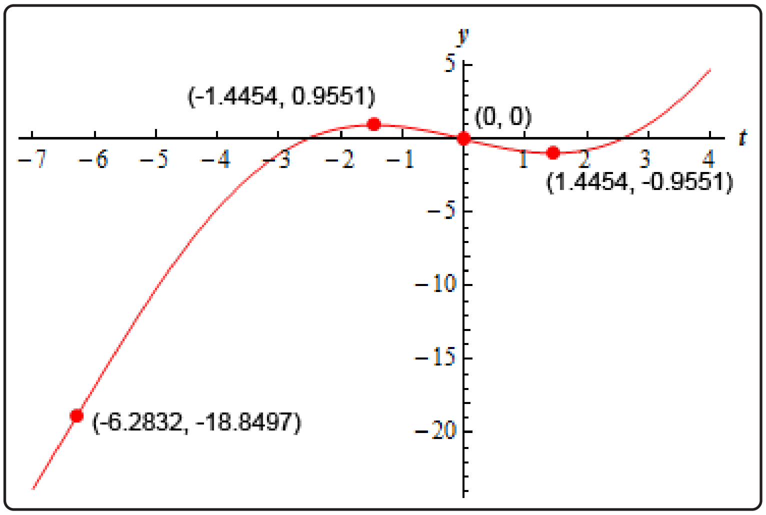
From the concavity information in the previous step we can see that the inflection points for the function are,

$$t = -6.2832 \qquad \& \qquad t = 0$$

(f) Use the information from steps **(a)** - **(e)** to sketch the graph of the function.

Solution

Here is a sketch of the graph of this function using the information above. As we did in problems in this section we can start at the left and work our way to the right on the graph. As we do this we first pay attention to the increasing/decreasing information and then make sure that the curve has the correct concavity as we sketch it in.



Note that because we used a computer to generate the sketch it is possible that your sketch won't be quite the same. It should however, have the same points listed on the graph above, the same basic increasing/decreasing nature and the same basic concavity.

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13. For $f(x) = x^{\frac{4}{3}}(x-2)$ answer each of the following questions.

(a) Identify the critical points of the function.

(b) Determine the intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

(d) Determine the intervals on which the function is concave up and concave down.

(e) Determine the inflection points of the function.

(f) Use the information from steps **(a)** - **(e)** to sketch the graph of the function.

Solutions

(a) Identify the critical points of the function.

Solution

The parts to this problem (with the exception of the last part) are just like the basic increasing/decreasing problems from the previous section and the basic concavity problems from earlier in this section. Because of that we will not be putting in quite as much detail here. If you are still unsure how to work the parts of this problem you should go back and work a few of the basic problems from the previous section and/or earlier in this section before proceeding.

We will need the 1^{st} derivative to start things off.

$$f(x) = x^{\frac{7}{3}} - 2x^{\frac{4}{3}} \qquad \rightarrow \qquad \underline{f'(x) = \frac{7}{3}x^{\frac{4}{3}} - \frac{8}{3}x^{\frac{1}{3}} = \frac{1}{3}x^{\frac{1}{3}}(7x-8)}$$

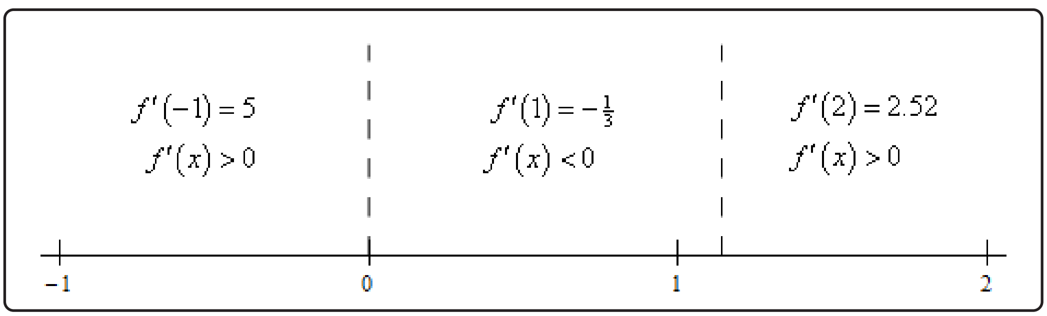
Note that by factoring the $x^{\frac{1}{3}}$ out we made it a little easier to quickly see that the critical points are,

$$\underline{x = 0 \quad \& \quad x = \frac{8}{7} = 1.1429}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To answer this part all we need is the number line for the 1^{st} derivative.



From this we get the following increasing/decreasing information for the function.

$$\boxed{\text{Increasing : } (-\infty, 0) \quad \& \quad \left(\frac{8}{7}, \infty\right) \qquad \text{Decreasing : } \left(0, \frac{8}{7}\right)}$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

From the number line in the previous step we get the following classifications of the critical points.

$$\boxed{x = 0: \text{ Relative Maximum} \qquad x = \frac{8}{7}: \text{ Relative Minimum}}$$

(d) Determine the intervals on which the function is concave up and concave down.

Solution

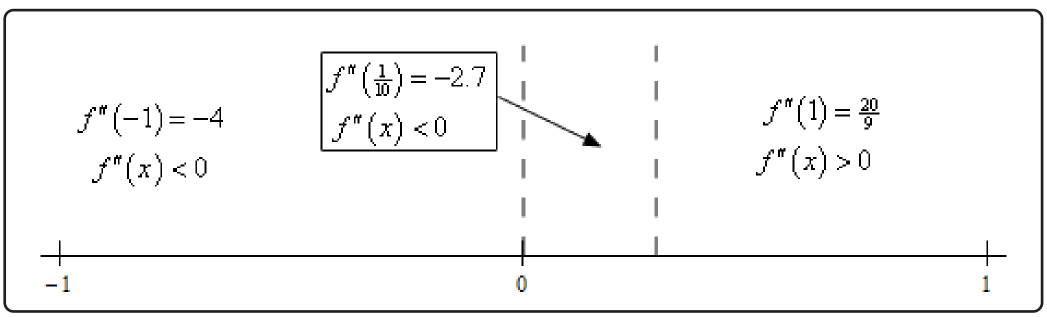
We'll need the 2^{nd} derivative to find the list of possible inflection points.

$$f''(x) = \frac{28}{9}x^{\frac{1}{3}} - \frac{8}{9}x^{-\frac{2}{3}} = \frac{28x - 8}{9x^{\frac{2}{3}}}$$

The possible inflection points for this function are,

$$x = 0 \qquad \& \qquad x = \frac{2}{7} = 0.2857$$

To get the intervals of concavity we'll need the number line for the 2^{nd} derivative.



From this we get the following concavity information for the function.

$$\boxed{\text{Concave Up}: \left(\frac{2}{7}, \infty\right) \qquad \text{Concave Down}: (-\infty, 0) \quad \& \quad \left(0, \frac{2}{7}\right)}$$

(e) Determine the inflection points of the function.

Solution

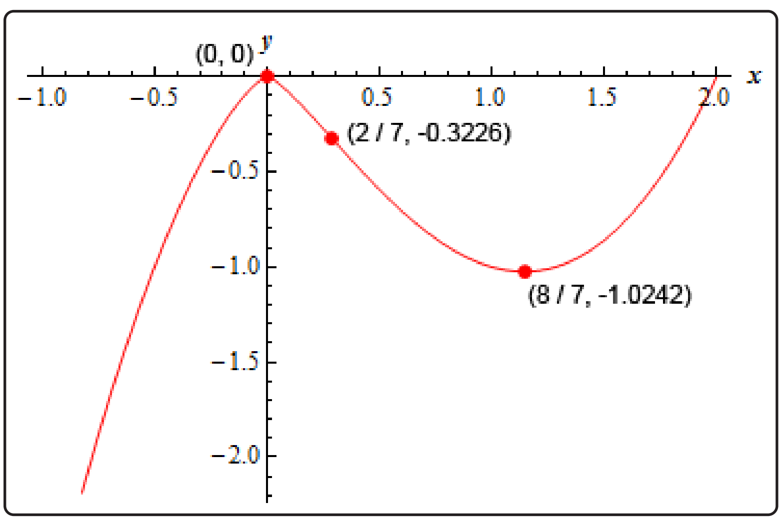
From the concavity information in the previous step we can see that the single inflection point for the function is,

$$\boxed{x = \frac{2}{7}}$$

(f) Use the information from steps **(a)** - **(e)** to sketch the graph of the function.

Solution

Here is a sketch of the graph of this function using the information above. As we did in problems in this section we can start at the left and work our way to the right on the graph. As we do this we first pay attention to the increasing/decreasing information and then make sure that the curve has the correct concavity as we sketch it in.



Note that because we used a computer to generate the sketch it is possible that your sketch won't be quite the same. It should however, have the same points listed on the graph above, the same basic increasing/decreasing nature and the same basic concavity.

14. For $P(w) = w\,\mathbf{e}^{4w}$ answer each of the following questions.

(a) Identify the critical points of the function.

(b) Determine the intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

(d) Determine the intervals on which the function is concave up and concave down.

(e) Determine the inflection points of the function.

(f) Use the information from steps **(a)** - **(e)** to sketch the graph of the function.

Solutions

(a) Identify the critical points of the function.

Solution

The parts to this problem (with the exception of the last part) are just like the basic increasing/decreasing problems from the previous section and the basic concavity problems from earlier in this section. Because of that we will not be putting in quite as much detail here. If you are still unsure how to work the parts of this problem you should go back and work a few of the basic problems from the previous section and/or earlier in this section before proceeding.

Also note that we haven't discussed L'Hospital's Rule yet (that comes in a few sections...) and that makes the behavior of the graph as $w \to \pm\infty$ a little trickier. Once we cover that section you might want to come back see if you agree with the behavior of the graph as $w \to \pm\infty$.

We will need the 1^{st} derivative to start things off.

$$\underline{P'(w) = \mathbf{e}^{4w} + 4w\,\mathbf{e}^{4w} = \mathbf{e}^{4w}(1+4w)}$$

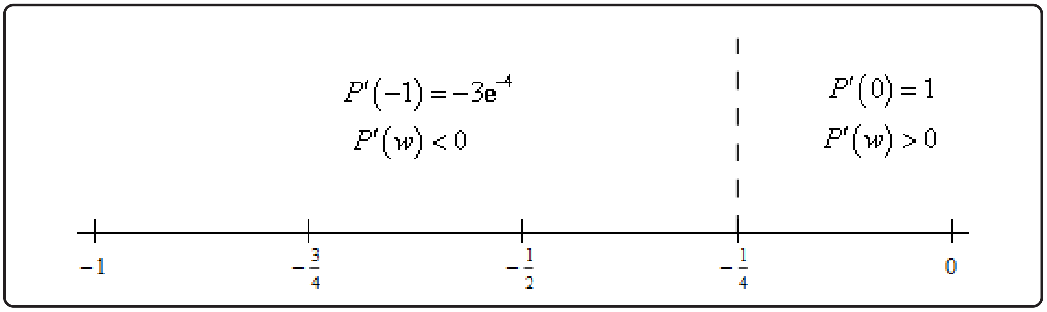
From the 1^{st} derivative we can see that the only critical points of this function is,

$$\underline{w = -\frac{1}{4}}$$

(b) Determine the intervals on which the function increases and decreases.

Solution

To answer this part all we need is the number line for the 1^{st} derivative.



From this we get the following increasing/decreasing information for the function.

$$\text{Increasing : } \left(-\frac{1}{4}, \infty\right) \qquad \text{Decreasing : } \left(-\infty, -\frac{1}{4}\right)$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

Solution

From the number line in the previous step we get the following classification of the critical point.

$$w = -\frac{1}{4} : \text{ Relative Minimum}$$

(d) Determine the intervals on which the function is concave up and concave down.

Solution

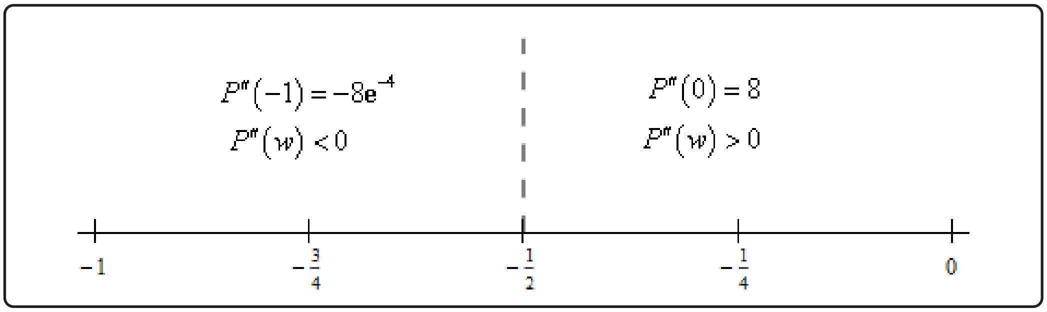
We'll need the 2^{nd} derivative to find the list of possible inflection points.

$$P''(w) = \mathbf{e}^{4w}(4) + 4\mathbf{e}^{4w}(1 + 4w) = 8\mathbf{e}^{4w}(1 + 2w)$$

The only possible inflection point for this function is,

$$\underline{w = -\frac{1}{2}}$$

To get the intervals of concavity we'll need the number line for the 2^{nd} derivative.



From this we get the following concavity information for the function.

$$\boxed{\text{Concave Up : } \left(-\frac{1}{2}, \infty\right) \qquad \text{Concave Down : } \left(-\infty, -\frac{1}{2}\right)}$$

(e) Determine the inflection points of the function.

Solution

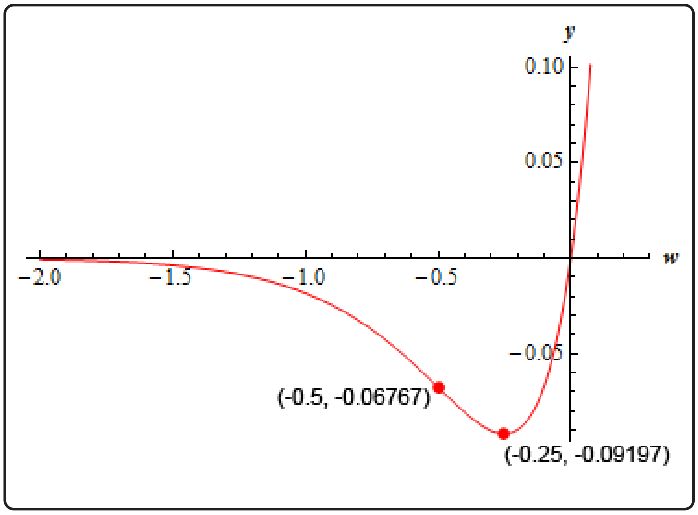
From the concavity information in the previous step we can see that the single inflection point for the function is,

$$\boxed{w = -\frac{1}{2}}$$

(f) Use the information from steps **(a)** - **(e)** to sketch the graph of the function.

Solution

Here is a sketch of the graph of this function using the information above. As we did in problems in this section we can start at the left and work our way to the right on the graph. As we do this we first pay attention to the increasing/decreasing information and then make sure that the curve has the correct concavity as we sketch it in.



Note that because we used a computer to generate the sketch it is possible that your sketch won't be quite the same. It should however, have the same points listed on the graph above, the same basic increasing/decreasing nature and the same basic concavity.

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15. Determine the minimum degree of a polynomial that has exactly one inflection point.

Hint

What is the simplest possible form of the 2^{nd} derivative that we can have that will guarantee that we have a single inflection point?

Step 1

First, let's suppose that the single inflection point occurs at $x = a$ for some number a. The value of a is not important, this only allows us to discuss the problem.

Now, if we start with a polynomial, call it $p(x)$, then the 2^{nd} derivative must also be a polynomial and we have to have $p''(a) = 0$. In addition, we know that the 2^{nd} derivative must change signs at $x = a$.

The simplest polynomial that we can have that will do this is,

$$\underline{p''(x) = x - a}$$

This clearly has $p''(a) = 0$ and it will change sign at $x = a$. Note as well that we don't really care which side is concave up and which side is concave down. We only care that the 2^{nd} derivative changes sign at $x = a$ as it does here.

Hint

We saw how to "undo" differentiation in the practice problems in the previous section. Here we simply need to do that twice and note that we don't actually have to undo the derivatives here, just think about what they would have to look like.

Step 2

Okay, saw how to "undo" differentiation in the practice problems of the previous section. We don't actually need to do that here, but we do need to think about what undoing differentiation will give here.

The 2^{nd} derivative is a 1^{st} degree polynomial and that means the 1^{st} derivative had to be a 2^{nd} degree polynomial. This should make sense to you if you understand how differentiation works.

We know that we have to differentiate the 1^{st} derivative to get the 2^{nd} derivative. Therefore, because the highest power of x in the 2^{nd} derivative is 1 and we know that differentiation lowers the power by 1 the highest power of x in the 1^{st} derivative must have been 2.

Okay, we've figured out that the 1^{st} derivative must have been a 2^{nd} degree polynomial. This in turn means that the original function must have been a 3^{rd} degree polynomial. Again, differentiation lowers the power of x by 1 and if the highest power of x in the 1^{st} derivative is 2 then the highest power of *x* in the original function must have been 3.

So, the minimum degree of a polynomial that has exactly one inflection point must be **three** (*i.e.* a cubic polynomial).

Note that we can have higher degree polynomials with exactly one inflection point. This is simply the minimal degree that will give exactly one inflection point.

16. Suppose that we know that $f(x)$ is a polynomial with critical points $x = -1$, $x = 2$ and $x = 6$. If we also know that the 2^{nd} derivative is $f''(x) = -3x^2 + 14x - 4$. If possible, classify each of the critical points as relative minimums, relative maximums. If it is not possible to classify the critical points clearly explain why they cannot be classified.

Hint

We do **NOT** need the 1^{st} derivative to answer this question. We are in the 2^{nd} derivative section and we did see a way in the notes on how to use the 2^{nd} derivative (which we have nicely been given...) to classify most critical points.

Solution

This problem is not as difficult as many students originally make it out to be. We've been given the 2^{nd} derivative and we saw how the 2^{nd} derivative test can be used to classify most critical points so let's use that.

First, we should note that because we have been told that $f(x)$ is a polynomial it should be fairly clear that, regardless of what the 1^{st} derivative actually is, we should have,

$$f'(-1) = 0 \qquad f'(2) = 0 \qquad f'(6) = 0$$

What this means is that we can use the 2^{nd} derivative test as it only works for these kinds of critical points.

All we need to do then is plug the critical points into the 2^{nd} derivative and use the 2^{nd} derivative test to classify the critical points.

$$\boxed{\begin{array}{rl} f''(-1) = -21 < 0 & \text{Relative Maximum} \\ f''(2) = 12 > 0 & \text{Relative Minimum} \\ f''(6) = -28 < 0 & \text{Relative Maximum} \end{array}}$$

So, in this case it was possible to classify all of the given critical points. Recall that if the 2^{nd} derivative had been zero for any of them we would not have been able to classify that critical point without the 1^{st} derivative which we don't have for this case.

4.7 Mean Value Theorem

1. Determine all the number(s) c which satisfy the conclusion of Rolle's Theorem for $f(x) = x^2 - 2x - 8$ on $[-1, 3]$.

Solution

The first thing we should do is actually verify that Rolle's Theorem can be used here.

The function is a polynomial which is continuous and differentiable everywhere and so will be continuous on $[-1, 3]$ and differentiable on $(-1, 3)$.

Next, a couple of quick function evaluations shows that $f(-1) = f(3) = -5$.

Therefore, the conditions for Rolle's Theorem are met and so we can actually do the problem.

Note that this may seem to be a little silly to check the conditions but it is a really good idea to get into the habit of doing this stuff. Since we are in this section it is pretty clear that the conditions will be met or we wouldn't be asking the problem. However, once we get out of this section and you want to use the Theorem the conditions may not be met. If you are in the habit of not checking you could inadvertently use the Theorem on a problem that can't be used and then get an incorrect answer.

Now that we know that Rolle's Theorem can be used there really isn't much to do. All we need to do is take the derivative,

$$f'(x) = 2x - 2$$

and then solve $f'(c) = 0$.

$$2c - 2 = 0 \qquad \Rightarrow \qquad c = 1$$

So, we found a single value and it is in the interval so the value we want is,

$$\boxed{c = 1}$$

2. Determine all the number(s) c which satisfy the conclusion of Rolle's Theorem for $g(t) = 2t - t^2 - t^3$ on $[-2, 1]$.

Solution

The first thing we should do is actually verify that Rolle's Theorem can be used here.

The function is a polynomial which is continuous and differentiable everywhere and so will be continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

Next, a couple of quick function evaluations shows that $g(-2) = g(1) = 0$.

Therefore, the conditions for Rolle's Theorem are met and so we can actually do the problem.

Note that this may seem to be a little silly to check the conditions but it is a really good idea to get into the habit of doing this stuff. Since we are in this section it is pretty clear that the conditions will be met or we wouldn't be asking the problem. However, once we get out of this section and you want to use the Theorem the conditions may not be met. If you are in the habit of not checking you could inadvertently use the Theorem on a problem that can't be used and then get an incorrect answer.

Now that we know that Rolle's Theorem can be used there really isn't much to do. All we need to do is take the derivative,

$$g'(t) = 2 - 2t - 3t^2$$

and then solve $g'(c) = 0$.

$$-3c^2 - 2c + 2 = 0 \qquad \Rightarrow \qquad c = \frac{1 \pm \sqrt{7}}{-3} = -1.2153, \;\; 0.5486$$

So, we found two values and, in this case, they are both in the interval so the values we want are,

$$\boxed{c = \frac{1 \pm \sqrt{7}}{-3} = -1.2153, \;\; 0.5486}$$

3. Determine all the number(s) c which satisfy the conclusion of Mean Value Theorem for $h(z) = 4z^3 - 8z^2 + 7z - 2$ on $[2, 5]$.

Solution

The first thing we should do is actually verify that the Mean Value Theorem can be used here.

The function is a polynomial which is continuous and differentiable everywhere and so will be continuous on $[2, 5]$ and differentiable on $(2, 5)$.

Therefore, the conditions for the Mean Value Theorem are met and so we can actually do the problem.

Note that this may seem to be a little silly to check the conditions but it is a really good idea to get into the habit of doing this stuff. Since we are in this section it is pretty clear that the conditions will be met or we wouldn't be asking the problem. However, once we get out of this section and you want to use the Theorem the conditions may not be met. If you are in the habit of not checking you could inadvertently use the Theorem on a problem that can't be used and then get an incorrect answer.

Now that we know that the Mean Value Theorem can be used there really isn't much to do. All we need to do is do some function evaluations and take the derivative.

$$h(2) = 12 \qquad h(5) = 333 \qquad h'(z) = 12z^2 - 16z + 7$$

The final step is to then plug into the formula from the Mean Value Theorem and solve for c.

$$12c^2 - 16c + 7 = \frac{333 - 12}{5 - 2} = 107 \qquad \to \qquad 12c^2 - 16c - 100 = 0$$

$$c = \frac{2 \pm \sqrt{79}}{3} = -2.2961, \quad 3.6294$$

So, we found two values and, in this case, only the second is in the interval and so the value we want is,

$$\boxed{c = \frac{2 + \sqrt{79}}{3} = 3.6294}$$

4. Determine all the number(s) c which satisfy the conclusion of Mean Value Theorem for $A(t) = 8t + \mathbf{e}^{-3t}$ on $[-2, 3]$.

Solution

The first thing we should do is actually verify that the Mean Value Theorem can be used here.

The function is a sum of a polynomial and an exponential function both of which are continuous and differentiable everywhere. This in turn means that the sum is also continuous and differentiable everywhere and so the function will be continuous on $[-2, 3]$ and differentiable on $(-2, 3)$.

Therefore, the conditions for the Mean Value Theorem are met and so we can actually do the problem.

Note that this may seem to be a little silly to check the conditions but it is a really good idea to get into the habit of doing this stuff. Since we are in this section it is pretty clear that the conditions will be met or we wouldn't be asking the problem. However, once we get out of this section and you want to use the Theorem the conditions may not be met. If you are in the habit of not checking you could inadvertently use the Theorem on a problem that can't be used and then get an incorrect answer.

Now that we know that the Mean Value Theorem can be used there really isn't much to do. All we need to do is do some function evaluations and take the derivative.

$$A(-2) = -16 + \mathbf{e}^{6} \qquad A(3) = 24 + \mathbf{e}^{-9} \qquad A'(t) = 8 - 3\mathbf{e}^{-3t}$$

The final step is to then plug into the formula from the Mean Value Theorem and solve for c.

$$\begin{aligned} 8 - 3\mathbf{e}^{-3c} &= \frac{24 + \mathbf{e}^{-9} - \left(-16 + \mathbf{e}^{6}\right)}{3 - (-2)} = -72.6857 \\ 3\mathbf{e}^{-3c} &= 80.6857 \\ \mathbf{e}^{-3c} &= 26.8952 \\ -3c &= \ln(26.8952) = 3.29195 \qquad \Rightarrow \qquad c = -1.0973 \end{aligned}$$

So, we found a single value and it is in the interval and so the value we want is,

$$\boxed{c = -1.0973}$$

5. Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7,0]$, that $f(-7) = -3$ and that $f'(x) \le 2$. What is the largest possible value for $f(0)$?

Step 1

We were told in the problem statement that the function (whatever it is) satisfies the conditions of the Mean Value Theorem so let's start out this that and plug in the known values.

$$f(0) - f(-7) = f'(c)(0 - (-7)) \qquad \to \qquad f(0) + 3 = 7f'(c)$$

Step 2

Next, let's solve for $f(0)$.

$$f(0) = 7f'(c) - 3$$

Step 3

Finally, let's take care of what we know about the derivative. We are told that the maximum value of the derivative is 2. So, plugging the maximum possible value of the derivative into $f'(c)$ above will, in this case, give us the maximum value of $f(0)$. Doing this gives,

$$f(0) = 7f'(c) - 3 \le 7(2) - 3 = 11$$

So, the largest possible value for $f(0)$ is 11. Or, written as an inequality this would be written as,

$$\boxed{f(0) \le 11}$$

---

6. Show that $f(x) = x^3 - 7x^2 + 25x + 8$ has exactly one real root.

Hint

Can you use the Intermediate Value Theorem to prove that it has at least one real root?

Step 1

First let's note that $f(0) = 8$. If we could find a function value that was negative the Intermediate Value Theorem (which can be used here because the function is continuous everywhere) would tell us that the function would have to be zero somewhere. In other words, there would have to be at least one real root.

Because the largest power of x is 3 it looks like if we let x be large enough and negative the function should also be negative. All we need to do is start plugging in negative x's until we find one that works. In fact, we don't even need to do much : $f(-1) = -25$.

So, we can see that $-25 = f(-1) < 0 < f(0) = 8$ and so by the Intermediate Value Theorem the function must be zero somewhere in the interval $(-1, 0)$. The interval itself is not important. What is important is that we have at least one real root.

Hint

What would happen if there were more than one real root?

Step 2

Next, let's assume that there is more than one real root. Assuming this means that there must be two numbers, say a and b, so that,

$$f(a) = f(b) = 0$$

Next, because $f(x)$ is a polynomial it is continuous and differentiable everywhere and so we could use Rolle's Theorem to see that there must be a real value, c, so that,

$$f'(c) = 0$$

Note that Rolle's Theorem tells us that c must be between a and b. Since both of these are real values then c must also be real.

Hint

Is that possible?

Step 3

Because $f(x)$ is a polynomial it is easy enough to see if such a c exists.

$$f'(x) = 3x^2 - 14x + 25 \qquad \to \qquad 3c^2 - 14c + 25 = 0 \qquad \to \qquad c = \frac{7 \pm \sqrt{26}\, i}{3}$$

So, we can see that in fact the only two places where the derivative is zero are complex numbers and so are not real numbers. Therefore, it is not possible for there to be more than one real root.

From Step 1 we know that there is at least one real root and we've just proven that we can't have more than one real root. Therefore, there must be **exactly one real root**.

4.8 Optimization

1. Find two positive numbers whose sum is 300 and whose product is a maximum.

Step 1

The first step is to write down equations describing this situation.

Let's call the two numbers x and y and we are told that the sum is 300 (this is the constraint for the problem) or,

$$x + y = 300$$

We are being asked to maximize the product,

$$A = xy$$

Step 2

We now need to solve the constraint for x or y (and it really doesn't matter which variable we solve for in this case) and plug this into the product equation.

$$y = 300 - x \qquad \Rightarrow \qquad A\left(x\right) = x\left(300 - x\right) = 300x - x^2$$

Step 3

The next step is to determine the critical points for this equation.

$$A'\left(x\right) = 300 - 2x \qquad \to \qquad 300 - 2x = 0 \qquad \to \qquad x = 150$$

Step 4

Now for the step many neglect as unnecessary. Just because we got a single value we can't just assume that this will give a maximum product. We need to do a quick check to see if it does give a maximum.

As discussed in notes there are several methods for doing this, but in this case we can quickly see that,

$$A''\left(x\right) = -2$$

Step 5

From this we can see that the second derivative is always negative and so $A(x)$ will always be concave down and so the single critical point we got in Step 3 must be a relative maximum and hence must be the value that gives a maximum product.

Step 5

Finally, let's actually answer the question. We need to give both values. We already have x so we need to determine y and that is easy to do from the constraint.

$$y = 300 - 150 = 150$$

The final answer is then,

$$\boxed{x = 150 \qquad y = 150}$$

2. Find two positive numbers whose product is 750 and for which the sum of one and 10 times the other is a minimum.

Step 1

The first step is to write down equations describing this situation.

Let's call the two numbers x and y and we are told that the product is 750 (this is the constraint for the problem) or,

$$xy = 750$$

We are then being asked to minimize the sum of one and 10 times the other,

$$S = x + 10y$$

Note that it really doesn't worry which is x and which is y in the sum so we simply chose the y to be multiplied by 10.

Step 2

We now need to solve the constraint for x or y (and it really doesn't matter which variable we solve for in this case) and plug this into the product equation.

$$x = \frac{750}{y} \qquad \Rightarrow \qquad S(y) = \frac{750}{y} + 10y$$

Step 3

The next step is to determine the critical points for this equation.

$$S'(y) = -\frac{750}{y^2} + 10 \qquad \to \qquad -\frac{750}{y^2} + 10 = 0 \qquad \to \qquad y = \pm\sqrt{75} = 5\sqrt{3}$$

Because we are told that y must be positive we can eliminate the negative value and so the only value we really get out of this step is : $y = \sqrt{75} = 5\sqrt{3}$.

Step 4

Now for the step many neglect as unnecessary. Just because we got a single value we can't just assume that this will give a minimum sum. We need to do a quick check to see if it does give a minimum.

As discussed in notes there are several methods for doing this, but in this case we can quickly see that,

$$S''(y) = \frac{1500}{y^3}$$

From this we can see that, provided we recall that y is positive, then the second derivative will always be positive. Therefore, $S(y)$ will always be concave up and so the single critical point from Step 3 that we can use must be a relative minimum and hence must be the value that gives a minimum sum.

Step 5

Finally, let's actually answer the question. We need to give both values. We already have y so we need to determine x and that is easy to do from the constraint.

$$x = \frac{750}{5\sqrt{3}} = 50\sqrt{3}$$

The final answer is then,

$$\boxed{x = 50\sqrt{3} \qquad y = 5\sqrt{3}}$$

---

3. Let x and y be two positive numbers such that $x + 2y = 50$ and $(x+1)(y+2)$ is a maximum.

Step 1

In this case we were given the constraint in the problem,

$$x + 2y = 50$$

We are also told the equation to maximize,

$$f = (x+1)(y+2)$$

So, let's just solve the constraint for x or y (we'll solve for x to avoid fractions...) and plug this into the product equation.

$$x = 50 - 2y \qquad \Rightarrow \qquad f(y) = (50 - 2y + 1)(y+2) = (51 - 2y)(y+2) = 102 + 47y - 2y^2$$

Step 2

The next step is to determine the critical points for this equation.

$$f'(y) = 47 - 4y \qquad \to \qquad 47 - 4y = 0 \qquad \to \qquad y = \frac{47}{4}$$

Step 3

Now for the step many neglect as unnecessary. Just because we got a single value we can't just assume that this will give a maximum product. We need to do a quick check to see if it does give a maximum.

As discussed in notes there are several methods for doing this, but in this case we can quickly see that,

$$f''(y) = -4$$

From this we can see that the second derivative is always negative and so $f(y)$ will always be concave down and so the single critical point we got in Step 2 must be a relative maximum and hence must be the value that gives a maximum.

Step 4

Finally, let's actually answer the question. We need to give both values. We already have y so we need to determine x and that is easy to do from the constraint.

$$x = 50 - 2\left(\frac{47}{4}\right) = \frac{53}{2}$$

The final answer is then,

$$\boxed{x = \frac{53}{2} \qquad y = \frac{47}{4}}$$

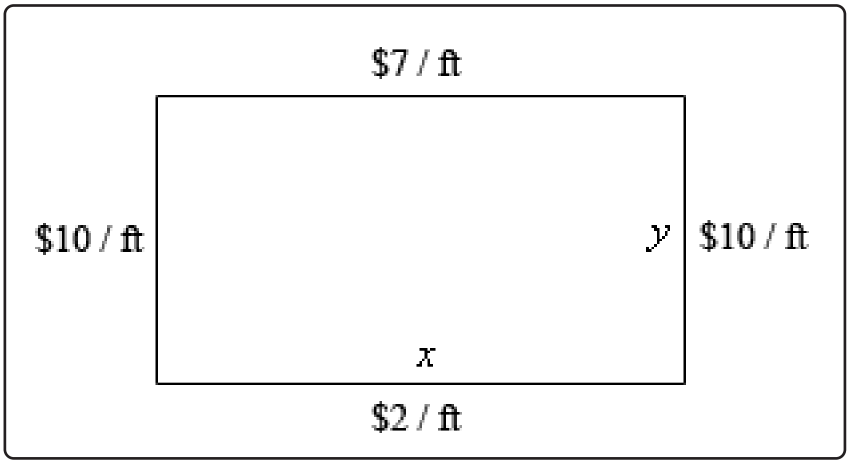
---

4. We are going to fence in a rectangular field. If we look at the field from above the cost of the vertical sides are \$10/ft, the cost of the bottom is \$2/ft and the cost of the top is \$7/ft. If we have \$700 determine the dimensions of the field that will maximize the enclosed area.

Step 1

The first step is to do a quick sketch of the problem. We could probably skip the sketch in this case, but that is a really bad habit to get into. For many of these problems a sketch is really convenient and it can be used to help us keep track of some of the important information in the problem and to "define" variables for the problem.

Here is the sketch for this problem.



Step 2

Next, we need to set up the constraint and equation that we are being asked to optimize.

We are told that we have $700 to spend and so the cost of the material will be the constraint for this problem. The cost for the material is then,

$$700 = 10y + 2x + 10y + 7x = 20y + 9x$$

We are being asked to maximize the area so that equation is,

$$A = xy$$

Step 3

Now, let's solve the constraint for y (that looks like it will only have one fraction in it and so may be "easier"...).

$$y = 35 - \frac{9}{20}x$$

Plugging this into the area formula gives,

$$A\left(x\right) = x\left(35 - \frac{9}{20}x\right) = 35x - \frac{9}{20}x^2$$

Step 4

Finding the critical point(s) for this shouldn't be too difficult at this point so here is that work.

$$A'(x) = 35 - \frac{9}{10}x \qquad \rightarrow \qquad 35 - \frac{9}{10}x = 0 \qquad \rightarrow \qquad x = \frac{350}{9}$$

Step 5

The second derivative of the area function is,

$$A''(x) = -\frac{9}{10}$$

From this we can see that the second derivative is always negative and so $A(x)$ will always be concave down and so the single critical point we got in Step 4 must be a relative maximum and hence must be the value that gives a maximum area.

Step 6

Now, let's finish the problem by getting the second dimension.

$$y = 35 - \frac{9}{20}\left(\frac{350}{9}\right) = \frac{35}{2}$$

The final dimensions are then,

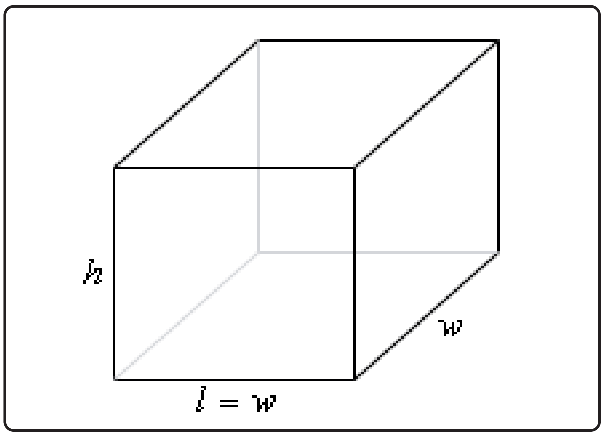
$$\boxed{x = \frac{350}{9} \qquad y = \frac{35}{2}}$$

5. We have 45 m^2 of material to build a box with a square base and no top. Determine the dimensions of the box that will maximize the enclosed volume.

Step 1

The first step is to do a quick sketch of the problem. We could probably skip the sketch in this case, but that is a really bad habit to get into. For many of these problems a sketch is really convenient and it can be used to help us keep track of some of the important information in the problem and to "define" variables for the problem.

Here is the sketch for this problem.



Step 2

Next, we need to set up the constraint and equation that we are being asked to optimize.

We are told that we have 45 m^2 of material to build the box and so that is the constraint. The amount of material that we need to build the box is then,

$$45 = lw + 2\left(lh\right) + 2\left(wh\right) = w^2 + 2wh + 2wh = w^2 + 4wh$$

Note that because there is no top the first term won't have the 2 that the second and third term have. Be careful with this kind of thing it is easy to miss if you aren't paying attention.

We are being asked to maximize the volume so that equation is,

$$V = lwh = w^2 h$$

Note as well that we went ahead and used fact that $l = w$ in both of these equations to reduce the three variables in the equation down to two variables.

Step 3

Now, let's solve the constraint for h (that will allow us to avoid dealing with roots, plus there is only one h in the constraint so it will simply be easier to deal with).

$$h = \frac{45 - w^2}{4w}$$

Plugging this into the volume formula gives,

$$V\left(w\right) = w^2\left(\frac{45 - w^2}{4w}\right) = \frac{1}{4}w\left(45 - w^2\right) = \frac{1}{4}\left(45w - w^3\right)$$

Step 4

Finding the critical point(s) for this shouldn't be too difficult at this point so here is that work.

$$V'\left(w\right) = \frac{1}{4}\left(45 - 3w^2\right) \qquad \to \qquad \frac{1}{4}\left(45 - 3w^2\right) = 0 \qquad \to \qquad w = \pm\sqrt{\frac{45}{3}} = \pm\sqrt{15}$$

Because we are dealing with the dimensions of a box the negative width doesn't make any sense and so the only critical point that we can use here is : $w = \sqrt{15}$.

Be careful here and do not get into the habit of just eliminating the negative values. The only reason for eliminating it in this case is for physical reasons. If we had just given the equations without any physical reasoning it would have to be included in the rest of the work!

Step 5

The second derivative of the volume function is,

$$V''\left(w\right) = -\frac{3}{2}w$$

From this we can see that the second derivative is always negative for positive w (which we will always have for this case since w is the width of a box). Therefore, provided w is positive, $V\left(w\right)$ will always be concave down and so the single critical point we got in Step 4 must be a relative maximum and hence must be the value that gives a maximum volume.

Step 6

Now, let's finish the problem by getting the remaining dimensions.

$$l = w = \sqrt{15} = 3.8730 \qquad h = \frac{45 - 15}{4\sqrt{15}} = 1.9365$$

The final dimensions are then,

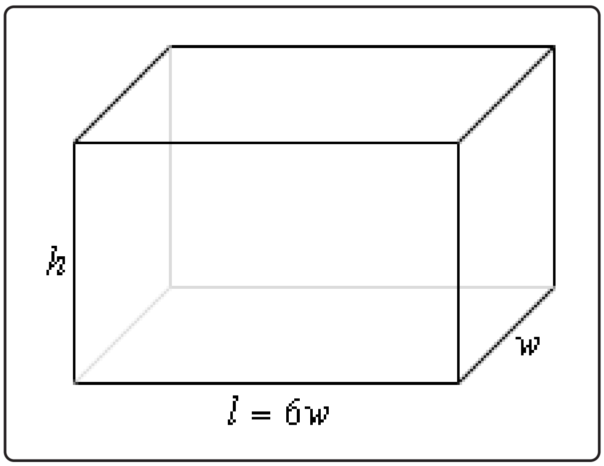
$$\boxed{l = w = 3.8730 \qquad h = 1.9365}$$

6. We want to build a box whose base length is 6 times the base width and the box will enclose 20 in^3. The cost of the material of the sides is \$3/in^2 and the cost of the top and bottom is \$15/in^2. Determine the dimensions of the box that will minimize the cost.

Step 1

The first step is to do a quick sketch of the problem. We could probably skip the sketch in this case, but that is a really bad habit to get into. For many of these problems a sketch is really convenient and it can be used to help us keep track of some of the important information in the problem and to "define" variables for the problem.

Here is the sketch for this problem.



Step 2

Next, we need to set up the constraint and equation that we are being asked to optimize.

We are told that the volume of the box must be 20 in^3 and so this is the constraint.

$$20 = lwh = 6w^2 h$$

We are being asked to minimize the cost and the cost function is,

$$C = 3\left[2\left(lh\right) + 2\left(wh\right)\right] + 15\left[2\left(lw\right)\right] = 3\left[12wh + 2wh\right] + 15\left[12w^2\right] = 42wh + 180w^2$$

Note as well that we went ahead and used fact that $l = 6w$ in both of these equations to reduce the three variables in the equation down to two variables.

Step 3

Now, let's solve the constraint for h (that will allow us to avoid dealing with roots).

$$h = \frac{10}{3w^2}$$

Plugging this into the cost function gives,

$$C\left(w\right) = 42w\left(\frac{10}{3w^2}\right) + 180w^2 = \frac{140}{w} + 180w^2$$

Step 4

Finding the critical point(s) for this shouldn't be too difficult at this point. Here is the derivative.

$$C'\left(w\right) = -\frac{140}{w^2} + 360w = \frac{360w^3 - 140}{w^2}$$

From this it looks like the only critical point is : $w = \sqrt[3]{\frac{7}{18}} = 0.7299$.

Note that $w = 0$ can't be a critical point because the function does not exist there.

Step 5

The second derivative of the volume function is,

$$C''(w) = \frac{280}{w^3} + 360$$

From this we can see that the second derivative is always positive for positive w (which we will always have for this case since w is the width of a box). Therefore, provided w is positive, $C(w)$ will always be concave up and so the single critical point we got in Step 4 must be a relative minimum and hence must be the value that gives a minimum cost.

Step 6

Now, let's finish the problem by getting the remaining dimensions.

$$l = 6w = 4.3794 \qquad h = \frac{10}{3(0.7299)^2} = 6.2568$$

The final dimensions are then,

$$\boxed{w = 0.7299 \qquad l = 4.3794 \qquad h = 6.2568}$$

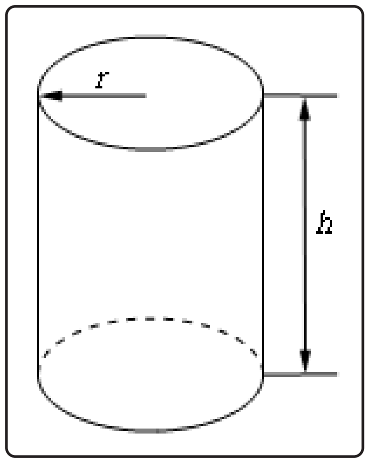
---

7. We want to construct a cylindrical can with a bottom but no top that will have a volume of 30 cm^3. Determine the dimensions of the can that will minimize the amount of material needed to construct the can.

Step 1

The first step is to do a quick sketch of the problem. We could probably skip the sketch in this case, but that is a really bad habit to get into. For many of these problems a sketch is really convenient and it can be used to help us keep track of some of the important information in the problem and to "define" variables for the problem.

Here is the sketch for this problem.



Step 2

Next, we need to set up the constraint and equation that we are being asked to optimize.

We are told that the volume of the can must be 30 cm^3 and so this is the constraint.

$$30 = \pi r^2 h$$

We are being asked to minimize the amount of material needed to construct the can,

$$A = 2\pi rh + \pi r^2$$

Recall that the can will have no top and so the second term will only be for the area of the bottom of the can.

Step 3

Now, le solve the constraint for h (that will allow us to avoid dealing with roots).

$$h = \frac{30}{\pi r^2}$$

Plugging this into the amount of material function gives,

$$A\left(r\right) = 2\pi r\left(\frac{30}{\pi r^2}\right) + \pi r^2 = \frac{60}{r} + \pi r^2$$

Step 4

Finding the critical point(s) for this shouldn't be too difficult at this point. Here is the derivative.

$$A'\left(r\right) = -\frac{60}{r^2} + 2\pi r = \frac{2\pi r^3 - 60}{r^2}$$

From this it looks like the only critical point is : $r = \sqrt[3]{\frac{60}{2\pi}} = 2.1216$.

Note that $r = 0$ can't be a critical point because the function does not exist there.

Be careful here and do not get into the habit of just eliminating the zero as a critical point. The only reason for eliminating it in this case is for physical reasons. If we had just given the equations without any physical reasoning it would have to be included in the rest of the work!

Step 5

The second derivative of the volume function is,

$$A''\left(r\right) = \frac{120}{r^3} + 2\pi$$

From this we can see that the second derivative is always positive for positive r (which we will always have for this case since r is the radius of a can). Therefore, provided r is positive, $A\left(r\right)$ will always be concave up and so the single critical point we got in Step 4 must be a relative minimum and hence must be the value that gives a minimum amount of material.

Step 6

Now, let's finish the problem by getting the height of the can.

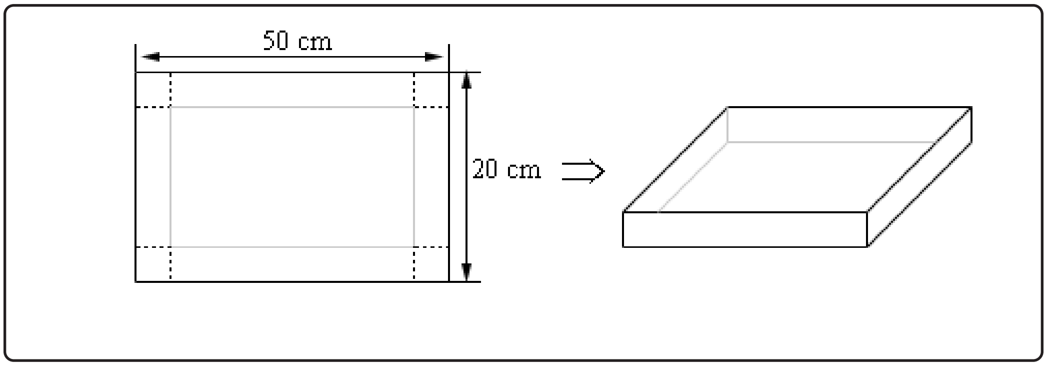
$$h = \frac{30}{\pi(2.1216)^2} = 2.1215$$

The final dimensions are then,

$$\boxed{r = 2.1216 \qquad h = 2.1215}$$

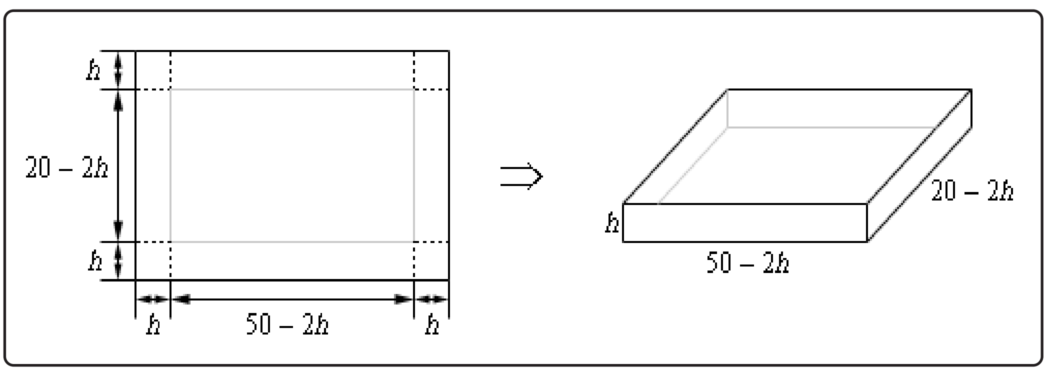
---

8. We have a piece of cardboard that is 50 cm by 20 cm and we are going to cut out the corners and fold up the sides to form a box. Determine the height of the box that will give a maximum volume.



Step 1

The first step is to do a quick sketch of the problem.



Step 2

As with the problem like this in the notes the constraint is really the size of the box and that has been taken into account in the figure so all we need to do is set up the volume equation that we want to maximize.

$$V(h) = h(50-2h)(20-2h) = 4h^3 - 140h^2 + 1000h$$

Step 3

Finding the critical point(s) for this shouldn't be too difficult at this point so here is that work,

$$V'(h) = 12h^2 - 280h + 1000 \qquad\qquad h = \frac{35 \pm 5\sqrt{19}}{3} = 4.4018,\ 18.9315$$

From the figure above, we can see that the limits on h must be $h = 0$ and $h = 10$ (the largest h could be is ½ the smaller side). Note that neither of these really make physical sense but they do provide limits on h.

So, we must have $0 \le h \le 10$ and this eliminates the second critical point and so the only critical point we need to worry about is $h = 4.4018$

Step 4

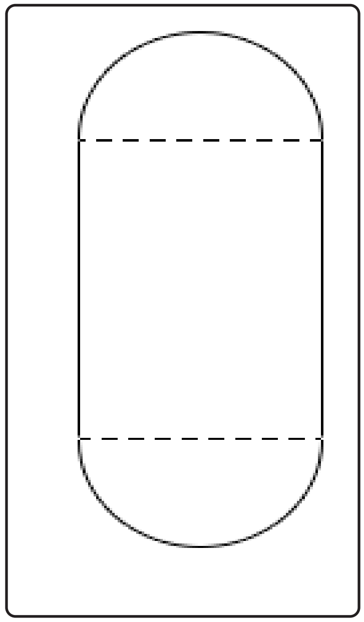
Because we have limits on h we can quickly check to see if we have maximum by plugging in the volume function.

$$V(0) = 0 \qquad\qquad V(4.4018) = 2030.34 \qquad\qquad V(10) = 0$$

So, we can see then that the height of the box will have to be $\boxed{h = 4.4018}$ in order to get a maximum volume.

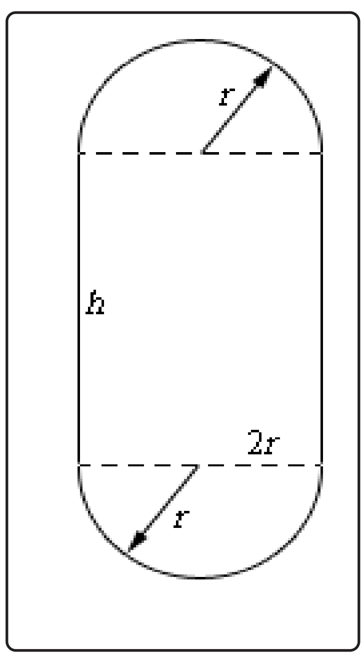
4.9 More Optimization

1. We want to construct a window whose middle is a rectangle and the top and bottom of the window are semi-circles. If we have 50 meters of framing material what are the dimensions of the window that will let in the most light?



Step 1

Let's start with a quick sketch of the window.



Step 2

Next, we need to set up the constraint and equation that we are being asked to optimize.

We are told that we have 50 meters of framing material (*i.e.* the perimeter of the window) and so that will be the constraint for this problem.

$$50 = 2h + 2(\pi r) = 2h + 2\pi r$$

We are being asked to maximize the amount of light being let in and that is simply the enclosed area or,

$$A = h(2r) + 2\left(\frac{1}{2}\pi r^2\right) = 2hr + \pi r^2$$

With both of these equations we were a little careful with the last term. In each case we needed either the perimeter or area of each semicircle and there were two of them. The end result of course is the equation of the perimeter/area of a whole circle, but we really should be careful setting these equations up and note just where everything is coming from.

Step 3

Now, let's solve the constraint for h.

$$h = 25 - \pi r$$

Plugging this into the area function gives,

$$A(r) = 2(25 - \pi r)r + \pi r^2 = 50r - \pi r^2$$

Step 4

Finding the critical point(s) for this shouldn't be too difficult at this point. Here is the derivative.

$$A'(r) = 50 - 2\pi r$$

From this it looks like we get a single critical points : $r = \frac{25}{\pi} = 7.9577$.

Step 5

The second derivative of the volume function is,

$$A''(r) = -2\pi$$

From this we can see that the second derivative is always negative. Therefore $A(r)$ will always be concave down and so the single critical point we got in Step 4 must be a relative maximum and hence must be the value that allows in the maximum amount of light.

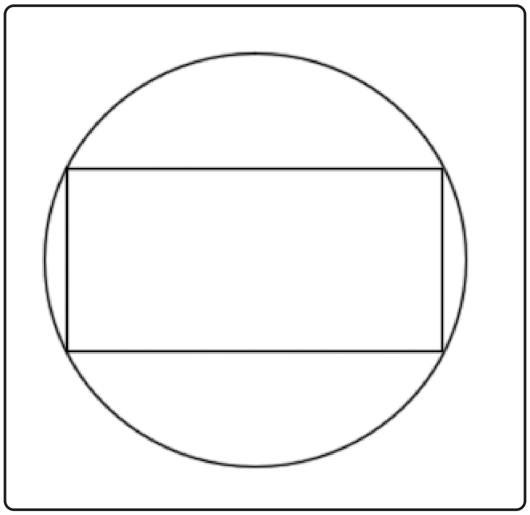
Step 6

Now, let's finish the problem by getting the radius of the semicircles.

$$h = 25 - \pi\left(\frac{25}{\pi}\right) = 0$$

Okay, what this means is that in fact the most light will come from not even having a rectangle between the semicircles and just having a circular window of radius $r = \frac{25}{\pi}$.

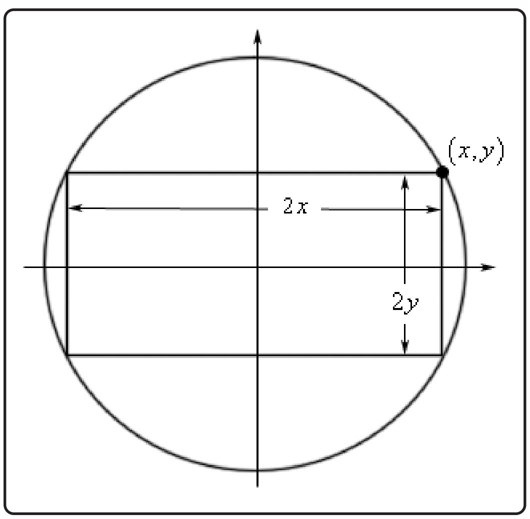
2. Determine the area of the largest rectangle that can be inscribed in a circle of radius 1.



Step 1

Let's start with a quick sketch of the circle and rectangle. Also, in order to make the work a little easier we went ahead and assumed that the circle was centered at the origin of the standard xy-coordinate system.

We've also defined a point (x, y) in the first quadrant. This is the point that we will be attempting to find when we get into the problems. If we know the coordinates of this point then the rectangle defined by the point, as shown in the figure, will be the one with the largest area.



Step 2

Next, we need to set up the constraint and equation that we are being asked to optimize.

Given our graph above we can easily determine the equation of the circle. This will also be the constraint of the problem because the corners of the rectangle must be on the circle.

$$x^2 + y^2 = 1$$

Also note that from the figure or equation we can clearly see that $-1 \le x \le 1$ and $-1 \le y \le 1$. One or both of these limits will be useful later on in the problem.

We are being asked to maximize the amount of the rectangle and using the definitions we see in the figure above the area is,

$$A = (2x)(2y) = 4xy$$

Step 3

We can solve the constraint for x or y. Either will lead to essentially the same work so we'll solve for x.

$$x = \pm\sqrt{1 - y^2}$$

Because we've defined the point on the circle to be in the 1^{st} quadrant we will use the "+" portion of this. Plugging this into the area function gives,

$$A(y) = 4y\sqrt{1 - y^2}$$

Step 4

Finding the critical point(s) for this shouldn't be too difficult at this point. Here is the derivative.

$$A'(y) = 4\sqrt{1 - y^2} - \frac{4y^2}{\sqrt{1 - y^2}} = \frac{4 - 8y^2}{\sqrt{1 - y^2}}$$

From this it looks like, from the numerator, we get the critical points,

$$y = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} = \pm 0.7071$$

From the denominator we get the critical points : $y = \pm 1$ and yes these are critical points because the function will exist at these points.

Before proceeding to the next step let's notice that because our point is in the first quadrant we know that y must be positive. This fact along with the limits on y we discussed in Step 2 tells us that we must have : $0 \le y \le 1$.

This in turn tells us that the only two critical points that we need to worry about are,

$$y = \frac{1}{\sqrt{2}} = 0.7071 \qquad\qquad y = 1$$

Step 5

Because we've got a range for possible critical points all we need to do to determine the maximum area is plug the end points and critical points into the area.

$$A(0) \qquad\qquad A\left(\frac{1}{\sqrt{2}}\right) = 2 \qquad\qquad A(1) = 0$$

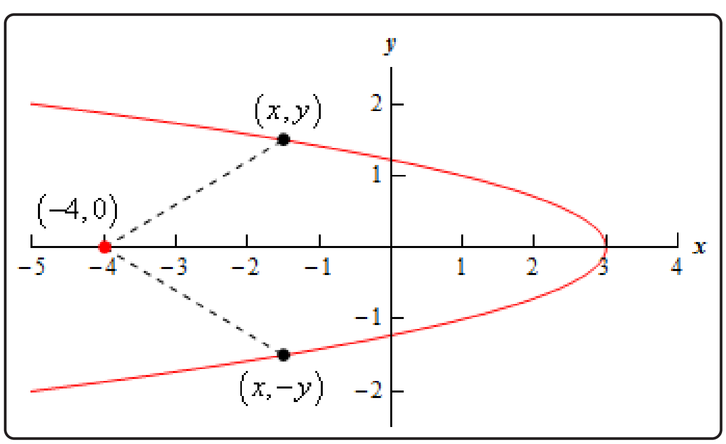
Step 6

So, the area of the largest rectangle that can be inscribed in the circle is : **2**.

3. Find the point(s) on $x = 3 - 2y^2$ that are closest to $(-4, 0)$.

Step 1

Let's start with a quick sketch of this situation. Below is a sketch of the graph of the function as well as the point $(-4, 0)$. As we can see we can expect to get two points as answers with the only difference being the sign on the y-coordinate.



Step 2

Next, we need to set up the constraint and equation that we are being asked to optimize.

In this case the constraint is simply the equation we are given. The point must lie on the graph and so must also satisfy the equation.

$$x = 3 - 2y^2$$

We are being asked to minimize the distance between a point (or points) on the graph and the point $(-4, 0)$. We can do this by looking at the distance between $(-4, 0)$ and (x, y). The distance between these two points is,

$$d = \sqrt{(x+4)^2 + y^2}$$

As we discussed in the notes for this section the point that minimizes the square of the distance will also minimize the distance itself and so to avoid dealing with the root we will minimize the square of the distance or,

$$d^2 = (x+4)^2 + y^2$$

Step 3

Now we have two choices on how to proceed from this point. The first option is to plug the equation we are given into the x in the distance squared and get a 4^{th} degree polynomial for y that we'll need to work with. The second is to solve the equation for y^2 and plug that into the distance squared and get a 2^{nd} degree polynomial for x that we'll need to work with. The second option gives a "nicer" polynomial to work with so we'll do that.

$$y^2 = \frac{1}{2}(3 - x) = \frac{3}{2} - \frac{1}{2}x$$

Plugging this into the distance squared gives,

$$f(x) = d^2 = (x+4)^2 + \frac{3}{2} - \frac{1}{2}x = x^2 + \frac{15}{2}x + \frac{35}{2}$$

Step 4

Finding the critical point(s) for this shouldn't be too difficult at this point. Here is the derivative.

$$f'(x) = 2x + \frac{15}{2}$$

From this it looks like we get a single critical point : $x = -\frac{15}{4} = -3.75$.

Step 5

The second derivative of the distance squared function is,

$$f''(x) = 2$$

From this we can see that the second derivative is always positive. Therefore, the distance squared will always be concave up and so the single critical point we got in Step 4 must be a relative minimum and hence must be the value of x that gives the points that are closest to $(-4, 0)$.

Step 6

Finally, we just need to determine the values y that give the actual points.

$$y^2 = \frac{3}{2} - \frac{1}{2}\left(-\frac{15}{4}\right) = \frac{27}{8} \qquad \Rightarrow \qquad y = \pm\sqrt{\frac{27}{8}} = \pm 1.8371$$

So, the two points on the graph that are closest to $(-4, 0)$ are,

$$\boxed{\left(-\frac{15}{4}, \sqrt{\frac{27}{8}}\right) \quad \& \quad \left(-\frac{15}{4}, -\sqrt{\frac{27}{8}}\right)}$$

4. An 80 cm piece of wire is cut into two pieces. One piece is bent into an equilateral triangle and the other will be bent into a rectangle with one side 4 times the length of the other side. Determine where, if anywhere, the wire should be cut to maximize the area enclosed by the two figures.

Step 1

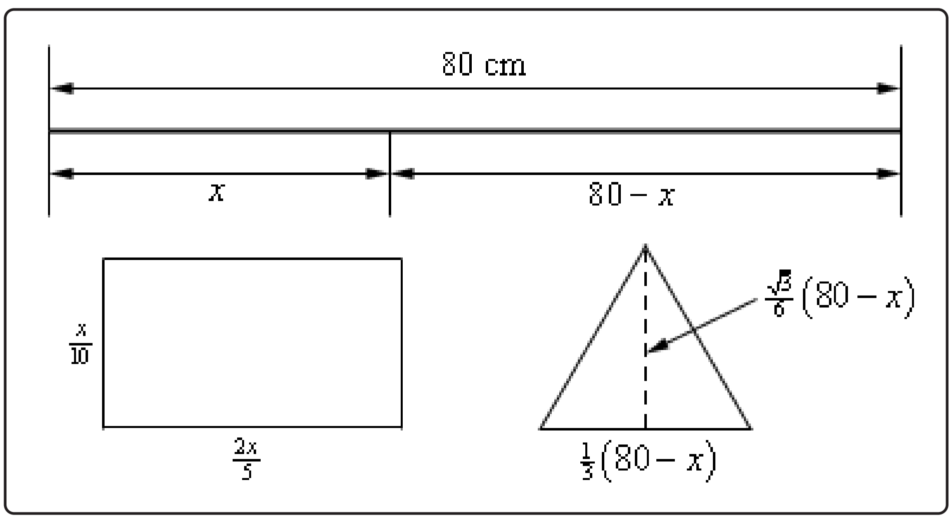
Before we do a sketch we'll need to do a little setup. Let's suppose that the length of the piece of wire that goes to the rectangle is x. This means that the length of the piece of wire going to the triangle is $80 - x$.

We know that the length of each side of the triangle are equal and so must have length $\frac{1}{3}(80 - x)$. We also know that the interior angles of the triangle are $\frac{\pi}{3}$ and so the height of the triangle is $\frac{1}{3}(80 - x)\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{6}(80 - x)$.

For the rectangle let's suppose that the length of the smaller side is L and so the length of the larger side is 4 L. Next, we know that the total perimeter of the rectangle is x and so we must have,

$$x = 2(L) + 2(4L) = 10L \qquad \to \qquad L = \frac{x}{10}$$

Now that we have all the various lengths of the figures in terms of x (which will make the work here a little easier) let's summarize everything up with the following figure.



Step 2

Next, we need to set up the constraint and equation that we are being asked to optimize.

This is one of those cases where we really don't have a constraint equation to work with.

The constraint is the length of the wire (80 cm), but we took that into account when we set up our figure above so there isn't anything to do with that in this case.

We are being asked to maximize the enclosed area of the two figures and so here is the total area of the enclosed figures.

$$A(x) = \left(\frac{x}{10}\right)\left(\frac{2x}{5}\right) + \frac{1}{2}\left[\frac{1}{3}(80-x)\right]\left[\frac{\sqrt{3}}{6}(80-x)\right] = \frac{x^2}{25} + \frac{\sqrt{3}}{36}(80-x)^2$$

Step 3

Finding the critical point(s) for this shouldn't be too difficult at this point (although the Algebra will be a little messy). Here is the derivative.

$$A'(x) = \frac{2x}{25} - \frac{\sqrt{3}}{18}(80-x)$$

From this it looks like we get a single critical point,

$$x = \frac{\frac{40\sqrt{3}}{9}}{\frac{2}{25} + \frac{\sqrt{3}}{18}} = 43.6828$$

Step 4

The second derivative of the area function is,

$$A''(x) = \frac{2}{25} + \frac{\sqrt{3}}{18}$$

From this we can see that the second derivative is always positive. Therefore $A(x)$ will always be concave up and so the single critical point we got in Step 4 must be a relative minimum and hence must be the value of x (*i.e.* the cut point) that will give the minimum enclosed area.

This is a problem however as we were asked for the maximum enclosed area. This is the reason for this step being in every problem that we've worked over the last couple of sections. Far too often students get to this point, get a single answer and then just assume that it must be the correct answer and don't bother doing any kind of checking to verify if it is the correct answer.

After all there was a single value so there is no choice for it to be correct. Right? Well, no. As we'll seen here it in fact is not the correct answer.

Step 5

So, what to do? We'll recall for the problem statement that we were asked to,

"Determine where, **if anywhere**, the wire should be cut to maximize the area enclosed by the two figures."

The "if anywhere" portion seems to suggest that we may not want to cut it at all. Maybe all of the wire should go to the rectangle (corresponding to $x = 80$ above) or maybe all of the wire should to the triangle (corresponding to $x = 0$ above).

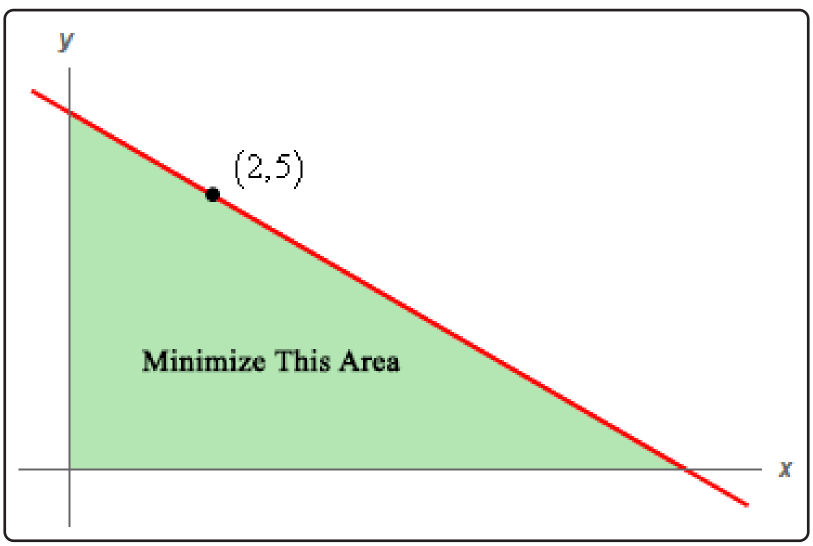
So, all we need to do is plug $x = 80$ and $x = 0$ into the area function and determine which will give the largest area.

$$A(0) = 307.92 \qquad \text{All wire goes to triangle.}$$
$$A(43.6828) = 139.785 \qquad \text{Wire goes to both triangle and rectangle.}$$
$$A(80) = 256 \qquad \text{All wire goes to rectangle}$$

Note that we included the critical point above just to make it really clear that it will not in fact give the maximum area. We didn't really need to include it here as we already knew it wouldn't work for us.

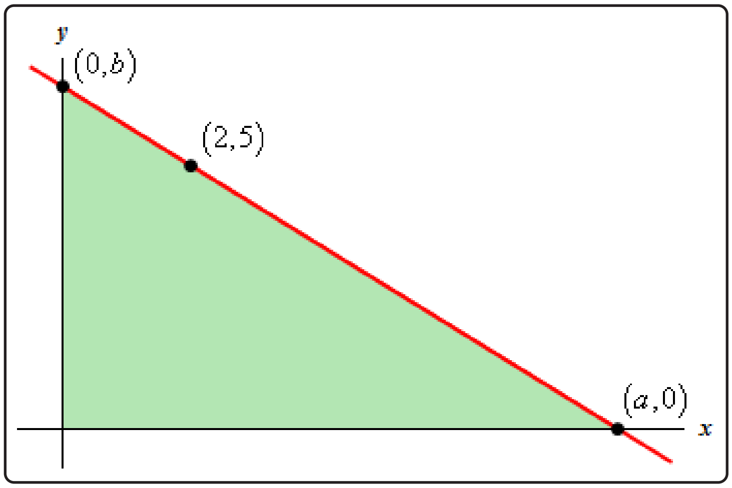
From the function evaluations above it looks like we'll need to take all of the wire and bend it into an equilateral triangle in order to get the maximum area.

5. A line through the point $(2, 5)$ forms a right triangle with the x-axis and y-axis in the 1^{st} quadrant. Determine the equation of the line that will minimize the area of this triangle.



Step 1

This problem may seem a little tricky at first. Here is a sketch of a line that goes through the point $(2,5)$, has an x-intercept of $(a,0)$ and a y-intercept of $(0,b)$.



Note that the only way we can get a triangle with the line, x-axis and y-axis as sides is to require that $a > 2$ and $b > 5$. If either of those are not true we will not have the triangle that we want.

Step 2

Next, we need to set up the constraint and equation that we are being asked to optimize.

We are being asked to minimize the area of the triangle shown above. In terms of the quantities given on the graph it is easy enough to get an equation for the area. The base length of the triangle is a and the height of the triangle is b. We don't have values for either of these but that isn't a problem. Here is the area of the triangle.

$$A = \frac{1}{2}ab$$

The constraint in this case is the equation of the line since that will define the hypotenuse of the triangle and hence also give both the base and height of the triangle. We need to write down the equation of the line, but we have three points on the line that we can use. Note however, that we really should use $(2,5)$ as one of the points because the line does need to go the point and using this point to write down the equation will give us that without any extra work.

The real question then is whether we should use the x or y-intercept for the second point when determining the slope of the line. It really doesn't matter which point that you use.

The work will be slightly different for each point but there will be no real difference in the difficulty of the problem.

We are going to use $(a, 0)$ for the second point. The slope of the line using this point is,

$$m = \frac{5}{2-a}$$

We already know that b is the y-intercept and so the equation of the line through the point is,

$$y = \frac{5}{2-a}x + b$$

Note that we definitely seem to have a problem here. Normally at this point we've got two equation and two unknowns. In this case we appear to have four unknowns : a, b, x and y. This isn't a problem as well see in the next step.

Step 3

We now need to solve the constraint for one of the unknowns in the area function, i.e either a or b. However, as we noted above we also have an x and y in the equation that will cause problems if they stay in the equation.

The point of this step is to get the area function down to a single variable. If we leave the x and y in the equation of the line we will end up with an area function with not one variable but three and that won't work for us.

What we really need is an equation involving only a and b that we can solve for one or the other and plug into the area function. Luckily this is easy to get. All we need to do is plug the x-intercept into the equation of the line to get,

$$0 = \frac{5}{2-a}a + b$$

Do you see why we couldn't have used the y-intercept here? If not, plug it in and you'll very quickly see why it won't work.

At this point we can easily solve the equation for b to get,

$$b = -\frac{5a}{2-a} = \frac{5a}{a-2}$$

To eliminate one of the minus signs we took the minus sign in front of the quotient and applied it to the denominator and simplified. This doesn't need to be done, but it does eliminate one of them.

Note that if we had used the y-intercept to determine the slope we would have found it to be easier at this step to solve for a instead. That is the only real difference in which point you use to find the slope.

Okay, let's put all this together. We know the value of b in terms of a so plug that into the area function to get,

$$A(a) = \frac{1}{2}(a)\left(\frac{5a}{a-2}\right) = \frac{5}{2}\frac{a^2}{a-2}$$

Step 4

Here is the derivative of the area function,

$$A'(a) = \frac{5}{2}\frac{a^2-4a}{(a-2)^2} = \frac{5}{2}\frac{a(a-4)}{(a-2)^2}$$

From this it looks like we get a three potential critical points : $a = 0$, $a = 2$ and $a = 4$.

We can't use $a = 0$ as the critical point because that will no longer form a triangle with both the x-axis and the y-axis as the problem asks for as noted in the first step.

We also can't use $a = 2$ for two reasons. First, it isn't actually a critical point because the area function doesn't exist at $a = 2$. This shouldn't be surprising given that if we used this point we wouldn't have a triangle anyway (again as we noted in the first step) and that is also the second reason for not using it.

This leaves only $a = 4$ as a potential critical point that we can use.

Step 5

The second derivative of the area function (after a little simplification) is,

$$A''(a) = \frac{20}{(a-2)^3}$$

From this we can see that the second derivative is always positive provided we have $a > 2$. However, as we noted in the first step this is required in order even work the problem. Therefore, the second derivative will always be positive for the range of a that we are working on. The area function will then will always be concave up for the range of a and $a = 4$ must give a minimum area.

Step 6

Now that we know the value of a we know that the slope and y-intercept are,

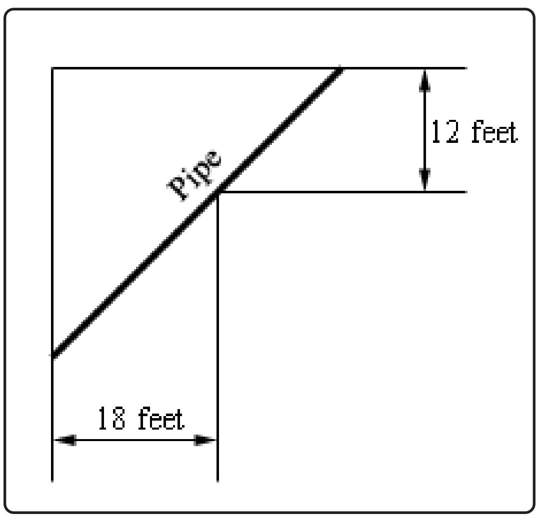
$$m = \frac{5}{2-4} = -\frac{5}{2} \qquad\qquad b = \frac{5\,(4)}{4-2} = 10$$

The equation of the line is then,

$$\boxed{y = -\frac{5}{2}x + 10}$$

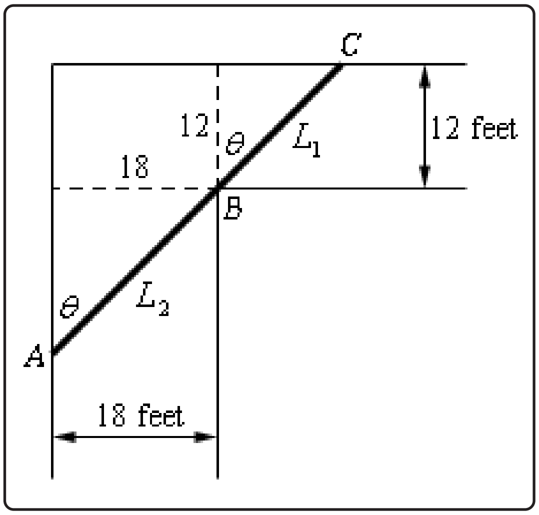
---

6. A piece of pipe is being carried down a hallway that is 18 feet wide. At the end of the hallway there is a right-angled turn and the hallway narrows down to 12 feet wide. What is the longest pipe (always keeping it horizontal) that can be carried around the turn in the hallway?



Step 1

Let's start with a quick sketch of the pipe and hallways with all the important quantities given.



Step 2

Next, we need to set up the constraint and equation that we are being asked to optimize.

As we discussed in the similar problem in the notes for this section we actually need to minimize the total length of the pipe. The equation we need to minimize is then,

$$L = L_1 + L_2$$

Also as we discussed in the notes problem with actually have two constraints : the widths of the two hallways. We can easily solve for these in terms of the angle θ.

$$L_1 = 12\sec\theta \qquad\qquad L_2 = 18\csc\theta$$

As discussed in the notes problem we also know that we must have $0 < \theta < \frac{\pi}{2}$.

Step 3

All we need to do here is plug our two constraints in the length function to get a function in terms of θ that we can minimize.

$$L(\theta) = 12\sec\theta + 18\csc\theta$$

Step 4

The derivative of the length function is,

$$L'(\theta) = 12\sec\theta\tan\theta - 18\csc\theta\cot\theta$$

Next, we need to set this equal to zero and solve this for θ to get the critical point that is in the range $0 < \theta < \frac{\pi}{2}$.

$$\begin{aligned} 12\sec\theta\tan\theta &= 18\csc\theta\cot\theta \\ \frac{\sec\theta\tan\theta}{\csc\theta\cot\theta} &= \frac{18}{12} \\ \tan^3\theta &= \frac{3}{2} \end{aligned}$$

The critical point that we need is then : $\theta = \tan^{-1}\left(\sqrt[3]{\frac{3}{2}}\right) = 0.8528$.

Step 5

Verifying that this is the value that gives the minimum is a little trickier than the other problems.

As noted in the notes for this section as we move $\theta \to 0$ we have $L \to \infty$ and as we move $\theta \to \frac{\pi}{2}$ we have $L \to \infty$. Therefore, on either side of $\theta = 0.8528$ radians the length of the pipe is increasing to infinity as we move towards the end of the range.

Therefore, this angle must give us the minimum length of the pipe and so is the largest pipe that we can fit around corner.

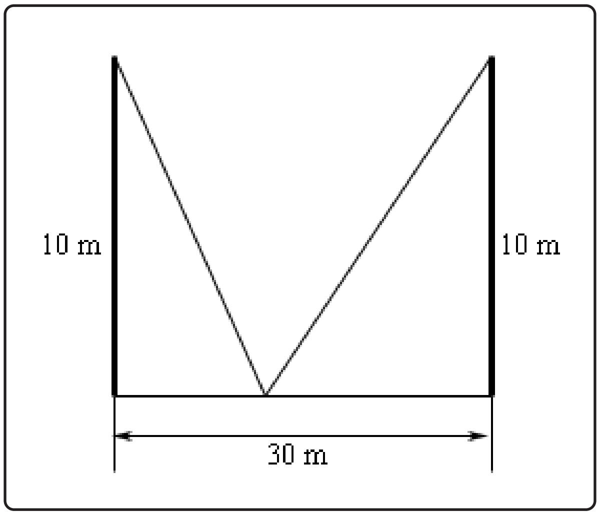
Step 6

The largest pipe that we can fit around the corner is then,

$$\boxed{L(0.8528) = 42.1409 \text{ feet}}$$

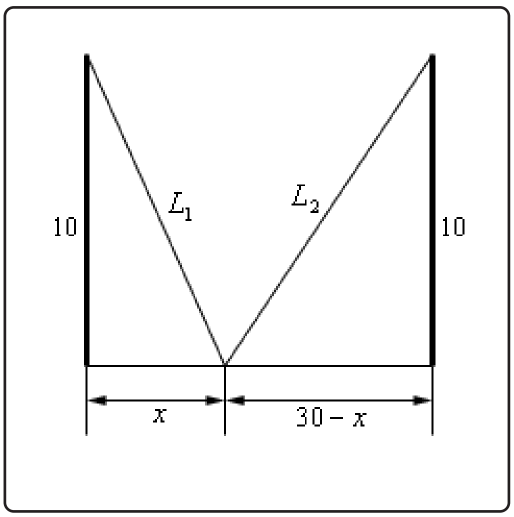
---

7. Two 10 meter tall poles are 30 meters apart. A length of wire is attached to the top of each pole and it is staked to the ground somewhere between the two poles. Where should the wire be staked so that the minimum amount of wire is used?



Step 1

Let's start with a quick of the situation with some more information added in.



Step 2

Next, we need to set up the constraint and equation that we are being asked to optimize.

We want to minimize the amount of wire and so the equation we need to minimize is,

$$L = L_1 + L_2$$

The constraint here is that the poles must be 30 meters apart. We can use this to determine the lengths of the individual wires in terms of x. Doing this gives,

$$L_1 = \sqrt{100 + x^2} \qquad\qquad L_2 = \sqrt{100 + (30 - x)^2}$$

Note that as well can also see that we need to require that $0 \leq x \leq 30$.

Step 3

All we need to do here is plug the lengths of the individual wires in the total length to get a function in terms of x that we can minimize.

$$L(x) = \sqrt{100 + x^2} + \sqrt{100 + (30 - x)^2}$$

Step 4

The derivative of the length function is,

$$L'(x) = \frac{x}{\sqrt{100 + x^2}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1000}}$$

Solving for the critical point(s) is going to be messy so here it goes.

$$\begin{aligned}
\frac{x}{\sqrt{100 + x^2}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1000}} &= 0 \\
\frac{x}{\sqrt{100 + x^2}} &= -\frac{x - 30}{\sqrt{x^2 - 60x + 1000}} \\
x\sqrt{x^2 - 60x + 1000} &= -(x - 30)\sqrt{100 + x^2} \\
x^2\left(x^2 - 60x + 1000\right) &= (x - 30)^2\left(100 + x^2\right) \\
x^4 - 60x^3 + 1000x^2 &= x^4 - 60x^3 + 1000x^2 - 6000x + 90000 \\
0 &= -6000x + 90000 \\
x &= 15
\end{aligned}$$

A quick check by plugging this back into the derivative shows that we do indeed get $L'(15) = 0$ and so this is a critical point and it is in the acceptable range of x.

Recall that because we squared both sides of the equation above it is possible to end up with answers that in fact are not solutions and so we have to go back and check in the original equation to make sure that they are solutions.

Step 5

Since we have a range of x's and the distance function is continuous in the range all we need to do is plug in the endpoints and the critical point to identify the minimum distance.

$$L(0) = 41.6228 \qquad L(15) = 36.0555 \qquad L(30) = 41.6228$$

Step 6

The wire should be staked midway between the poles to minimize the amount of wire.

4.10 L'Hospital's Rule and Indeterminate Forms

1. Use L'Hospital's Rule to evaluate $\displaystyle\lim_{x \to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$.

Step 1

The first step we should really do here is verify that L'Hospital's Rule can in fact be used on this limit.

This may seem like a silly step given that we are told to use L'Hospital's Rule. However, in later sections we won't be told to use it when/if it can be used. Therefore, we really need to get in the habit of checking that it can be used before applying it just to make sure that we can. If we apply L'Hospital's Rule to a problem that it can't be applied to then it's is almost assured that we will get the wrong answer (it's always possible you might get lucky and get the correct answer, but we will only be very lucky if it does).

So, a quick check shows us that,

$$\text{as } x \to 2 \qquad \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} \to \frac{0}{0}$$

and so this is a form that allows the use of L'Hospital's Rule.

Step 2

So, at this point let's just apply L'Hospital's Rule.

$$\lim_{x \to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \lim_{x \to 2} \frac{3x^2 - 14x + 10}{2x + 1}$$

Step 3

At this point all we need to do is try the limit and see if it can be done.

$$\lim_{x \to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \lim_{x \to 2} \frac{3x^2 - 14x + 10}{2x + 1} = \frac{-6}{5}$$

So, the limit can be done, and we done with the problem! The limit is then,

$$\lim_{x \to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \boxed{-\frac{6}{5}}$$

2. Use L'Hospital's Rule to evaluate $\displaystyle\lim_{w \to -4} \frac{\sin(\pi w)}{w^2 - 16}$.

Step 1

The first step we should really do here is verify that L'Hospital's Rule can in fact be used on this limit.

This may seem like a silly step given that we are told to use L'Hospital's Rule. However, in later sections we won't be told to use it when/if it can be used. Therefore, we really need to get in the habit of checking that it can be used before applying it just to make sure that we can. If we apply L'Hospital's Rule to a problem that it can't be applied to then it's is almost assured that we will get the wrong answer (it's always possible you might get lucky and get the correct answer, but we will only be very lucky if it does).

So, a quick check shows us that,

$$\text{as } w \to -4 \qquad \frac{\sin(\pi w)}{w^2 - 16} \to \frac{0}{0}$$

and so this is a form that allows the use of L'Hospital's Rule.

Step 2

So, at this point let's just apply L'Hospital's Rule.

$$\lim_{w \to -4} \frac{\sin(\pi w)}{w^2 - 16} = \lim_{w \to -4} \frac{\pi \cos(\pi w)}{2w}$$

Step 3

At this point all we need to do is try the limit and see if it can be done.

$$\lim_{w \to -4} \frac{\sin(\pi w)}{w^2 - 16} = \lim_{w \to -4} \frac{\pi \cos(\pi w)}{2w} = \frac{\pi \cos(-4\pi)}{-8} = \frac{\pi}{-8}$$

So, the limit can be done, and we done with the problem! The limit is then,

$$\lim_{w \to -4} \frac{\sin(\pi w)}{w^2 - 16} = \boxed{-\frac{\pi}{8}}$$

3. Use L'Hospital's Rule to evaluate $\displaystyle\lim_{t\to\infty}\frac{\ln(3t)}{t^2}$.

Step 1

The first step we should really do here is verify that L'Hospital's Rule can in fact be used on this limit.

This may seem like a silly step given that we are told to use L'Hospital's Rule. However, in later sections we won't be told to use it when/if it can be used. Therefore, we really need to get in the habit of checking that it can be used before applying it just to make sure that we can. If we apply L'Hospital's Rule to a problem that it can't be applied to then it's is almost assured that we will get the wrong answer (it's always possible you might get lucky and get the correct answer, but we will only be very lucky if it does).

So, a quick check shows us that,

$$\text{as } t\to\infty \qquad \frac{\ln(3t)}{t^2}\to\frac{\infty}{\infty}$$

and so this is a form that allows the use of L'Hospital's Rule.

Step 2

So, at this point let's just apply L'Hospital's Rule.

$$\lim_{t\to\infty}\frac{\ln(3t)}{t^2}=\lim_{t\to\infty}\frac{\frac{1}{t}}{2t}=\lim_{t\to\infty}\frac{1}{2t^2}$$

Don't forget to simplify after taking the derivatives. This can often be the difference between being able to do the problem or not.

Step 3

At this point all we need to do is try the limit and see if it can be done.

$$\lim_{t\to\infty}\frac{\ln(3t)}{t^2}=\lim_{t\to\infty}\frac{1}{2t^2}=0$$

So, the limit can be done, and we done with the problem! The limit is then,

$$\lim_{t\to\infty}\frac{\ln(3t)}{t^2}=\boxed{0}$$

4. Use L'Hospital's Rule to evaluate $\displaystyle\lim_{z\to 0}\frac{\sin(2z)+7z^2-2z}{z^2(z+1)^2}$.

Step 1

The first step we should really do here is verify that L'Hospital's Rule can in fact be used on this limit.

This may seem like a silly step given that we are told to use L'Hospital's Rule. However, in later sections we won't be told to use it when/if it can be used. Therefore, we really need to get in the habit of checking that it can be used before applying it just to make sure that we can. If we apply L'Hospital's Rule to a problem that it can't be applied to then it's is almost assured that we will get the wrong answer (it's always possible you might get lucky and get the correct answer, but we will only be very lucky if it does).

So, a quick check shows us that,

$$\text{as } z\to 0 \qquad \frac{\sin(2z)+7z^2-2z}{z^2(z+1)^2}\to\frac{0}{0}$$

and so this is a form that allows the use of L'Hospital's Rule.

Step 2

Before actually using L'Hospital's Rule it might be better if we multiply out the denominator to make the derivative (and later steps a little easier). Doing this gives,

$$\lim_{z\to 0}\frac{\sin(2z)+7z^2-2z}{z^2(z+1)^2}=\lim_{z\to 0}\frac{\sin(2z)+7z^2-2z}{z^4+2z^3+z^2}$$

Now let's apply L'Hospitals's Rule.

$$\lim_{z\to 0}\frac{\sin(2z)+7z^2-2z}{z^2(z+1)^2}=\lim_{z\to 0}\frac{2\cos(2z)+14z-2}{4z^3+6z^2+2z}$$

Step 3

At this point let's try the limit and see if it can be done. However, in this case, we can see that,

$$\text{as } z\to 0 \qquad \frac{2\cos(2z)+14z-2}{4z^3+6z^2+2z}\to\frac{0}{0}$$

Step 4

So, using L'Hospital's Rule doesn't give us a limit that we can do. However, the new limit is one that can use L'Hospital's Rule on so let's do that.

$$\lim_{z\to 0}\frac{\sin(2z)+7z^2-2z}{z^2(z+1)^2}=\lim_{z\to 0}\frac{-4\sin(2z)+14}{12z^2+12z+2}=\frac{14}{2}$$

Okay, the second L'Hospital's Rule gives us a limit we can do and so the answer is,

$$\lim_{z\to 0}\frac{\sin(2z)+7z^2-2z}{z^2(z+1)^2}=\boxed{7}$$

---

5. Use L'Hospital's Rule to evaluate $\displaystyle\lim_{x\to -\infty}\frac{x^2}{\mathbf{e}^{1-x}}$.

Step 1

The first step we should really do here is verify that L'Hospital's Rule can in fact be used on this limit.

This may seem like a silly step given that we are told to use L'Hospital's Rule. However, in later sections we won't be told to use it when/if it can be used. Therefore, we really need to get in the habit of checking that it can be used before applying it just to make sure that we can. If we apply L'Hospital's Rule to a problem that it can't be applied to then it's is almost assured that we will get the wrong answer (it's always possible you might get lucky and get the correct answer, but we will only be very lucky if it does).

So, a quick check shows us that,

$$\text{as } x\to -\infty \qquad \frac{x^2}{\mathbf{e}^{1-x}}\to\frac{\infty}{\infty}$$

and so this is a form that allows the use of L'Hospital's Rule.

Step 2

So, at this point let's just apply L'Hospital's Rule.

$$\lim_{x\to -\infty}\frac{x^2}{\mathbf{e}^{1-x}}=\lim_{x\to -\infty}\frac{2x}{-\mathbf{e}^{1-x}}$$

Step 3

At this point let's try the limit and see if it can be done. However, in this case, we can see that,

$$\text{as } x \to -\infty \qquad \frac{2x}{-\mathbf{e}^{1-x}} \to \frac{-\infty}{-\infty}$$

Step 4

So, using L'Hospital's Rule doesn't give us a limit that we can do. However, the new limit is one that can use L'Hospital's Rule on so let's do that.

$$\lim_{x\to-\infty}\frac{x^2}{\mathbf{e}^{1-x}} = \lim_{x\to-\infty}\frac{2x}{-\mathbf{e}^{1-x}} = \lim_{x\to-\infty}\frac{2}{\mathbf{e}^{1-x}} = 0$$

Okay, the second L'Hospital's Rule gives us a limit we can do and so the answer is,

$$\lim_{x\to-\infty}\frac{x^2}{\mathbf{e}^{1-x}} = \boxed{0}$$

6. Use L'Hospital's Rule to evaluate $\displaystyle\lim_{z\to\infty}\frac{z^2+\mathbf{e}^{4z}}{2z-\mathbf{e}^{z}}$.

Step 1

The first step we should really do here is verify that L'Hospital's Rule can in fact be used on this limit.

This may seem like a silly step given that we are told to use L'Hospital's Rule. However, in later sections we won't be told to use it when/if it can be used. Therefore, we really need to get in the habit of checking that it can be used before applying it just to make sure that we can. If we apply L'Hospital's Rule to a problem that it can't be applied to then it's is almost assured that we will get the wrong answer (it's always possible you might get lucky and get the correct answer, but we will only be very lucky if it does).

So, a quick check shows us that,

$$\text{as } z \to \infty \qquad \frac{z^2+\mathbf{e}^{4z}}{2z-\mathbf{e}^{z}} \to \frac{\infty}{-\infty}$$

and so this is a form that allows the use of L'Hospital's Rule.

Step 2

So, at this point let's just apply L'Hospital's Rule.

$$\lim_{z\to\infty} \frac{z^2 + \mathbf{e}^{4z}}{2z - \mathbf{e}^{z}} = \lim_{z\to\infty} \frac{2z + 4\mathbf{e}^{4z}}{2 - \mathbf{e}^{z}}$$

Step 3

At this point let's try the limit and see if it can be done. However, in this case, we can see that,

$$\text{as } z \to \infty \qquad \frac{2z + 4\mathbf{e}^{4z}}{2 - \mathbf{e}^{z}} \to \frac{\infty}{-\infty}$$

Step 4

So, using L'Hospital's Rule doesn't give us a limit that we can do. However, the new limit is one that can use L'Hospital's Rule on so let's do that.

$$\lim_{z\to\infty} \frac{z^2 + \mathbf{e}^{4z}}{2z - \mathbf{e}^{z}} = \lim_{z\to\infty} \frac{2z + 4\mathbf{e}^{4z}}{2 - \mathbf{e}^{z}} = \lim_{z\to\infty} \frac{2 + 16\mathbf{e}^{4z}}{-\mathbf{e}^{z}}$$

Step 5

Now, at this point we need to be careful. It looks like we are still in a case of an infinity divided by an infinity and that looks to continue forever if we keep applying L'Hospital's Rule. However, do not forget to do some basic simplifications where you can.

If we simplify we get the following.

$$\lim_{z\to\infty} \frac{z^2 + \mathbf{e}^{4z}}{2z - \mathbf{e}^{z}} = \lim_{z\to\infty} \left(2 + 16\mathbf{e}^{4z}\right)\left(-\mathbf{e}^{-z}\right) = \lim_{z\to\infty} \left(-2\mathbf{e}^{-z} - 16\mathbf{e}^{3z}\right)$$

and this is something that we can take the limit of.

So, the answer is,

$$\lim_{z\to\infty} \frac{z^2 + \mathbf{e}^{4z}}{2z - \mathbf{e}^{z}} = \lim_{z\to\infty} \left(-2\mathbf{e}^{-z} - 16\mathbf{e}^{3z}\right) = \boxed{-\infty}$$

Again, it cannot be stressed enough that you've got to do simplification where you can. For some of these problems that can mean the difference between being able to do the problem or not.

7. Use L'Hospital's Rule to evaluate $\lim\limits_{t\to\infty}\left[t\ln\left(1+\frac{3}{t}\right)\right]$.

Step 1

The first thing to notice here is that is not in a form that allows L'Hospital's Rule. L'Hospital's Rule only works on a certain class of rational functions and this is clearly not a rational function.

Note however that it is in the following indeterminate form,

$$\text{as } t\to\infty \qquad t\ln\left(1+\frac{3}{t}\right)\to(\infty)(0)$$

and as we discussed in the notes for this section we can always turn this kind of indeterminate form into a rational expression that will allow L'Hospital's Rule to be applied.

Step 2

The real question is do we move the first term or the second term to the denominator. From the looks of things, it appears that it would be best to move the first term to the denominator.

$$\lim_{t\to\infty}\left[t\ln\left(1+\frac{3}{t}\right)\right]=\lim_{t\to\infty}\frac{\ln\left(1+\frac{3}{t}\right)}{{}^{1}/_{t}}$$

Notice as well that,

$$\text{as } t\to\infty \qquad \frac{\ln\left(1+\frac{3}{t}\right)}{{}^{1}/_{t}}\to\frac{0}{0}$$

and we can use L'Hospital's Rule on this.

Step 3

Applying L'Hospital's Rule gives,

$$\lim_{t\to\infty}\left[t\ln\left(1+\frac{3}{t}\right)\right]=\lim_{t\to\infty}\frac{\ln\left(1+\frac{3}{t}\right)}{{}^{1}/_{t}}=\lim_{t\to\infty}\frac{\frac{-3/t^2}{1+3/t}}{-1/t^2}$$

Can you see why we chose to move the t to the denominator? Moving the logarithm would have left us with a very messy derivative to take! It might have ended up working okay for us, but the work would be greatly increased.

Step 4

Do not forget to simplify after we've taken the derivative. This problem becomes very simple if we do that.

$$\lim_{t\to\infty}\left[t\ln\left(1+\frac{3}{t}\right)\right] = \lim_{t\to\infty}\frac{\ln\left(1+\frac{3}{t}\right)}{{}^1/_t} = \lim_{t\to\infty}\frac{3}{1+{}^3/_t} = \boxed{3}$$

---

8. Use L'Hospital's Rule to evaluate $\lim\limits_{w\to0^+}\left[w^2\ln\left(4w^2\right)\right]$.

Step 1

The first thing to notice here is that is not in a form that allows L'Hospital's Rule. L'Hospital's Rule only works on a certain class of rational functions and this is clearly not a rational function.

Note however that it is in the following indeterminate form,

$$\text{as } w\to0^+ \qquad w^2\ln\left(4w^2\right)\to(0)(-\infty)$$

and as we discussed in the notes for this section we can always turn this kind of indeterminate form into a rational expression that will allow L'Hospital's Rule to be applied.

Step 2

The real question is do we move the first term or the second term to the denominator. From the looks of things, it appears that it would be best to move the first term to the denominator.

$$\lim_{w\to0^+}\left[w^2\ln\left(4w^2\right)\right] = \lim_{w\to0^+}\frac{\ln\left(4w^2\right)}{{}^1/_{w^2}}$$

Notice as well that,

$$\text{as } w\to0^+ \qquad \frac{\ln\left(4w^2\right)}{{}^1/_{w^2}}\to\frac{-\infty}{\infty}$$

and we can use L'Hospital's Rule on this.

Step 3

Applying L'Hospital's Rule gives,

$$\lim_{w\to 0^+}\left[w^2 \ln\left(4w^2\right)\right] = \lim_{w\to 0^+}\frac{\ln\left(4w^2\right)}{{}^{1}\!/_{w^2}} = \lim_{w\to 0^+}\frac{{}^{2}\!/_{w}}{-{}^{2}\!/_{w^3}}$$

Can you see why we chose to move the first term to the denominator? Moving the logarithm would have left us with a very messy derivative to take! It might have ended up working okay for us, but the work would be greatly increased.

Step 4

Do not forget to simplify after we've taken the derivative. This problem becomes very simple if we do that. In fact, it is the only way to actually get an answer for this problem. If we do not simplify will get stuck in a never-ending chain of infinity divided by infinity forms no matter how many times we apply L'Hospital's Rule.

$$\lim_{w\to 0^+}\left[w^2 \ln\left(4w^2\right)\right] = \lim_{w\to 0^+}\frac{\ln\left(4w^2\right)}{{}^{1}\!/_{w^2}} = \lim_{w\to 0^+}\left(-w^2\right) = \boxed{0}$$

---

9. Use L'Hospital's Rule to evaluate $\lim\limits_{x\to 1^+}\left[(x-1)\tan\left(\frac{\pi}{2}x\right)\right]$.

Step 1

The first thing to notice here is that is not in a form that allows L'Hospital's Rule. L'Hospital's Rule only works on a certain class of rational functions and this is clearly not a rational function.

Note however that it is in the following indeterminate form,

$$\text{as } x \to 1^+ \qquad (x-1)\tan\left(\frac{\pi}{2}x\right) \to (0)(-\infty)$$

and as we discussed in the notes for this section we can always turn this kind of indeterminate form into a rational expression that will allow L'Hospital's Rule to be applied.

Step 2

The real question is do we move the first term or the second term to the denominator. At first glance it might appear that neither term will be particularly useful in the denominator. In particular, if we move the tangent to the denominator we would end up needing to differentiate a term in the form $^1/_{\tan}$. That doesn't look to be all that fun to differentiate and we're liable to end up with a mess when we are done.

However, that is exactly the term we are going to move to the denominator for reasons that will quickly become apparent.

$$\lim_{x\to 1^+}\left[(x-1)\tan\left(\frac{\pi}{2}x\right)\right] = \lim_{x\to 1^+}\frac{x-1}{^1/_{\tan\left(\frac{\pi}{2}x\right)}} = \lim_{x\to 1^+}\frac{x-1}{\cot\left(\frac{\pi}{2}x\right)}$$

Step 3

With a little simplification after moving the tangent to the denominator we ended up with something that doesn't look all that bad. We'll also see that the remainder of this problem is going to be quite simple.

Before we proceed however we should notice as well that,

$$\text{as } x\to 1^+ \qquad \frac{x-1}{\cot\left(\frac{\pi}{2}x\right)} \to \frac{0}{0}$$

and we can use L'Hospital's Rule on this.

Step 4

Applying L'Hospital's Rule gives,

$$\lim_{x\to 1^+}\left[(x-1)\tan\left(\frac{\pi}{2}x\right)\right] = \lim_{x\to 1^+}\frac{x-1}{\cot\left(\frac{\pi}{2}x\right)} = \lim_{x\to 1^+}\frac{1}{-\frac{\pi}{2}\csc^2\left(\frac{\pi}{2}x\right)} = \boxed{-\frac{2}{\pi}}$$

10. Use L'Hospital's Rule to evaluate $\lim\limits_{y \to 0^+} [\cos(2y)]^{1/y^2}$.

Step 1

The first thing to notice here is that is not in a form that allows L'Hospital's Rule. L'Hospital's Rule only works on certain classes of rational functions and this is clearly not a rational function.

Note however that it is in the following indeterminate form,

$$\text{as } y \to 0^+ \qquad [\cos(2y)]^{1/y^2} \to 1^\infty$$

and as we discussed in the notes for this section we can do some manipulation on this to turn it into a problem that can be done with L'Hospital's Rule.

Step 2

First, let's define,

$$z = [\cos(2y)]^{1/y^2}$$

and take the log of both sides. We'll also do a little simplification.

$$\ln z = \ln\left([\cos(2y)]^{1/y^2}\right) = \frac{1}{y^2}\ln[\cos(2y)] = \frac{\ln[\cos(2y)]}{y^2}$$

Step 3

We can now take the limit as $y \to 0^+$ of this.

$$\lim_{y \to 0^+}[\ln z] = \lim_{y \to 0^+}\left[\frac{\ln[\cos(2y)]}{y^2}\right]$$

Before we proceed let's notice that we have the following,

$$\text{as } y \to 0^+ \qquad \frac{\ln[\cos(2y)]}{y^2} \to \frac{\ln(1)}{0} = \frac{0}{0}$$

and we have a limit that we can use L'Hospital's Rule on.

Step 4

Applying L'Hospital's Rule gives,

$$\lim_{y\to 0^+}[\ln z] = \lim_{y\to 0^+}\left[\frac{\ln[\cos(2y)]}{y^2}\right] = \lim_{y\to 0^+}\frac{{}^{-2\sin(2y)}/{}_{\cos(2y)}}{2y} = \lim_{y\to 0^+}\frac{-\tan(2y)}{y}$$

Step 5

We now have a limit that behaves like,

$$\text{as } y\to 0^+ \qquad \frac{-\tan(2y)}{y} \to \frac{0}{0}$$

and so we can use L'Hospital's Rule on this as well. Doing this gives,

$$\lim_{y\to 0^+}[\ln z] = \lim_{y\to 0^+}\frac{-\tan(2y)}{y} = \lim_{y\to 0^+}\frac{-2\sec^2(2y)}{1} = -2$$

Step 6

Now all we need to do is recall that,

$$z = \mathbf{e}^{\ln z}$$

This in turn means that we can do the original limit as follows,

$$\lim_{y\to 0^+}[\cos(2y)]^{1/y^2} = \lim_{y\to 0^+} z = \lim_{y\to 0^+}\mathbf{e}^{\ln z} = \mathbf{e}^{\lim\limits_{y\to 0^+}[\ln z]} = \boxed{\mathbf{e}^{-2}}$$

---

11. Use L'Hospital's Rule to evaluate $\lim\limits_{x\to\infty}[\mathbf{e}^x + x]^{1/x}$.

Step 1

The first thing to notice here is that is not in a form that allows L'Hospital's Rule. L'Hospital's Rule only works on certain classes of rational functions and this is clearly not a rational function.

Note however that it is in the following indeterminate form,

$$\text{as } x \to \infty \qquad [\mathbf{e}^x + x]^{1/x} \to \infty^0$$

and as we discussed in the notes for this section we can do some manipulation on this to turn it into a problem that can be done with L'Hospital's Rule.

Step 2

First, let's define,

$$z = [\mathbf{e}^x + x]^{1/x}$$

and take the log of both sides. We'll also do a little simplification.

$$\ln z = \ln\left([\mathbf{e}^x + x]^{1/x}\right) = \frac{1}{x}\ln[\mathbf{e}^x + x] = \frac{\ln[\mathbf{e}^x + x]}{x}$$

Step 3

We can now take the limit as $x \to \infty$ of this.

$$\lim_{x\to\infty}[\ln z] = \lim_{x\to\infty}\left[\frac{\ln[\mathbf{e}^x + x]}{x}\right]$$

Before we proceed let's notice that we have the following,

$$\text{as } x \to \infty \qquad \frac{\ln[\mathbf{e}^x + x]}{x} \to = \frac{\infty}{\infty}$$

and we have a limit that we can use L'Hospital's Rule on.

Step 4

Applying L'Hospital's Rule gives,

$$\lim_{x\to\infty}[\ln z] = \lim_{x\to\infty}\left[\frac{\ln[\mathbf{e}^x + x]}{x}\right] = \lim_{x\to\infty}\frac{{}^{\mathbf{e}^x+1}/_{\mathbf{e}^x+x}}{1} = \lim_{x\to\infty}\frac{\mathbf{e}^x + 1}{\mathbf{e}^x + x}$$

Step 5

We now have a limit that behaves like,

$$\text{as } x \to \infty \qquad \frac{\mathbf{e}^x + 1}{\mathbf{e}^x + x} \to \frac{\infty}{\infty}$$

and so we can use L'Hospital's Rule on this as well. Doing this gives,

$$\lim_{x\to\infty} [\ln z] = \lim_{x\to\infty} \frac{\mathbf{e}^x + 1}{\mathbf{e}^x + x} = \lim_{x\to\infty} \frac{\mathbf{e}^x}{\mathbf{e}^x + 1} = \lim_{x\to\infty} \frac{\mathbf{e}^x}{\mathbf{e}^x} = \lim_{x\to\infty} (1) = 1$$

Notice that we did have to use L'Hospital's Rule twice here and we also made sure to do some simplification so we could actually take the limit.

Step 6

Now all we need to do is recall that,

$$z = \mathbf{e}^{\ln z}$$

This in turn means that we can do the original limit as follows,

$$\lim_{x\to\infty} [\mathbf{e}^x + x]^{1/x} = \lim_{x\to\infty} z = \lim_{x\to\infty} \mathbf{e}^{\ln z} = \mathbf{e}^{\lim\limits_{x\to\infty} [\ln z]} = \boxed{\mathbf{e}}$$

4.11 Linear Approximations

1. Find a linear approximation to $f(x) = 3x\,\mathbf{e}^{2x-10}$ at $x = 5$.

Step 1

We'll need the derivative first as well as a couple of function evaluations.

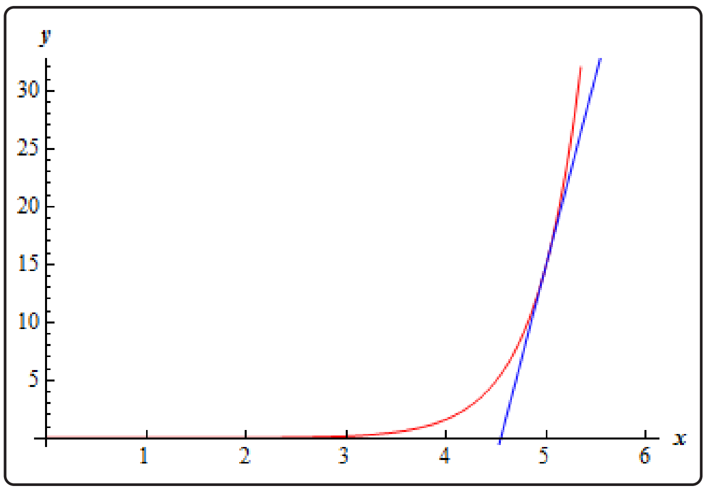
$$f'(x) = 3\mathbf{e}^{2x-10} + 6x\,\mathbf{e}^{2x-10} \qquad f(5) = 15 \qquad f'(5) = 33$$

Step 2

There really isn't much to do at this point other than write down the linear approximation.

$$\boxed{L(x) = 15 + 33(x-5) = 33x - 150}$$

While it wasn't asked for, here is a quick sketch of the function and the linear approximation.



2. Find a linear approximation to $h(t) = t^4 - 6t^3 + 3t - 7$ at $t = -3$.

Step 1

We'll need the derivative first as well as a couple of function evaluations.

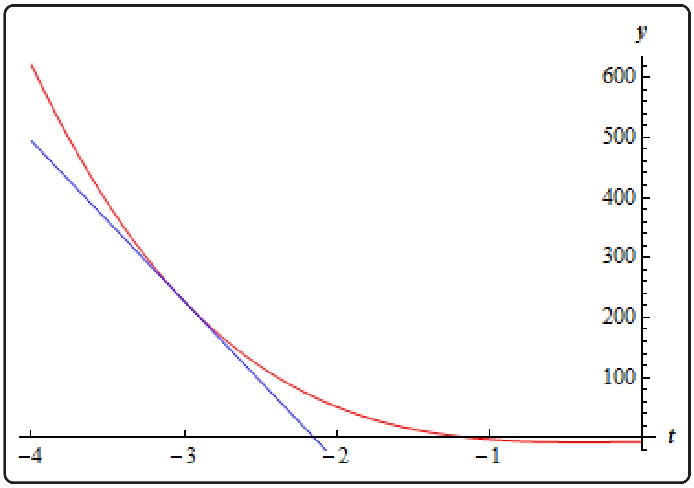
$$h'(t) = 4t^3 - 18t^2 + 3 \qquad h(-3) = 227 \qquad h'(-3) = -267$$

Step 2

There really isn't much to do at this point other than write down the linear approximation.

$$\boxed{L(t) = 227 - 267(t+3) = -267t - 574}$$

While it wasn't asked for, here is a quick sketch of the function and the linear approximation.



3. Find the linear approximation to $g(z) = \sqrt[4]{z}$ at $z = 2$. Use the linear approximation to approximate the value of $\sqrt[4]{3}$ and $\sqrt[4]{10}$. Compare the approximated values to the exact values.

Step 1

We'll need the derivative first as well as a couple of function evaluations.

$$g'(z) = \frac{1}{4}z^{-\frac{3}{4}} \qquad g(2) = 2^{\frac{1}{4}} \qquad g'(2) = \frac{1}{4}\left(2^{-\frac{3}{4}}\right)$$

Step 2

Here is the linear approximation.

$$\boxed{L(z) = 2^{\frac{1}{4}} + \frac{1}{4}\left(2^{-\frac{3}{4}}\right)(z-2)}$$

Step 3

Finally, here are the approximations of the values along with the exact values.

$L(3) = 1.33786$	$g(3) = 1.31607$	% Error : 1.65523
$L(10) = 2.37841$	$g(10) = 1.77828$	% Error : 33.7481

So, as we might have expected the farther from $z = 2$ we got the worse the approximation is. Recall that the approximation will generally be more accurate the closer to the point of the linear approximation.

4. Find the linear approximation to $f(t) = \cos(2t)$ at $t = \frac{1}{2}$. Use the linear approximation to approximate the value of $\cos(2)$ and $\cos(18)$. Compare the approximated values to the exact values.

Step 1

We'll need the derivative first as well as a couple of function evaluations.

$$f'(t) = -2\sin(2t) \qquad f\left(\frac{1}{2}\right) = \cos(1) \qquad f'\left(\frac{1}{2}\right) = -2\sin(1)$$

Step 2

Here is the linear approximation.

$$\boxed{L(t) = \cos(1) - 2\sin(1)\left(t - \frac{1}{2}\right) = 0.5403 - 1.6829\left(t - \frac{1}{2}\right)}$$

Make sure your calculator is set in radians! Remember that we use radians by default in this class.

Step 3

Now, if we want to approximate cos (2), that is equivalent to evaluating $f(1) = \cos(2)$, we need to evaluate the linear approximation at $t = 1$. Likewise, to approximate cos (18) we need to evaluate the linear approximation at $t = 9$.

So, here are the approximations of the values along with the exact values.

$L(1) = -0.301169$	$f(1) = -0.416147$	% Error : 27.6292
$L(9) = -13.7647$	$f(9) = 0.660317$	% Error : 2184.56

So, as we might have expected the farther from $t = \frac{1}{2}$ we got the worse the approximation is. Recall that the approximation will generally be more accurate the closer to the point of the linear approximation.

---

5. Without using any kind of computational aid use a linear approximation to estimate the value of $\mathbf{e}^{0.1}$.

Hint

This is really nothing more than Problem 3 and 4 from this section. The only difference is that you need to determine the function and the point for the linear approximation. The function should be pretty obvious given the value we are asked to estimate. There should also be a pretty obvious point to use given that we aren't supposed to use calculators/-computers.

Step 1

This is really the same problem as Problems 3 & 4 from this section. The difference is that we need to determine the function and point for the linear approximation.

Given the value we are being asked to estimate it should be fairly clear that the function should be,

$$\underline{f(x) = \mathbf{e}^{x}}$$

The point for the linear approximation should also be somewhat clear. With the function in hand it's now clear that we are being asked to use a linear approximation to estimate $f(0.1)$. So, we'll need a point that is close to $x = 0.1$ and one that we can evaluate in the function without a calculator. It therefore seems fairly clear that $x = 0$ would be a really nice point use for the linear approximation.

Step 2

At this point finding the linear approximation shouldn't be too bad so here is the work for that.

$$f'(x) = \mathbf{e}^x \qquad f(0) = 1 \qquad f'(0) = 1$$

The linear approximation is then,

$$\boxed{L(t) = 1 + (1)(x - 0) = x + 1}$$

Step 3

The estimation of $\mathbf{e}^{0.1}$ is then,

$$\boxed{\mathbf{e}^{0.1} \approx L(0.1) = 1.1}$$

For comparison purposes the exact value is $f(0.1) = 1.10517$ and so we have an error of 0.467884

4.12 Differentials

1. Compute the differential for $f(x) = x^2 - \sec(x)$.

Solution

There is not really a whole lot to this problem.

$$\boxed{df = (2x - \sec(x)\tan(x))\,dx}$$

Don't forget to tack on the dx at the end!

2. Compute the differential for $w = \mathbf{e}^{x^4 - x^2 + 4x}$.

Solution

There is not really a whole lot to this problem.

$$\boxed{dw = \left(4x^3 - 2x + 4\right)\mathbf{e}^{x^4 - x^2 + 4x}\,dx}$$

Don't forget to tack on the dx at the end!

3. Compute the differential for $h(z) = \ln(2z)\sin(2z)$.

Solution

There is not really a whole lot to this problem.

$$\boxed{dh = \left(\frac{1}{z}\sin(2z) + 2\ln(2z)\cos(2z)\right)dz}$$

Don't forget to tack on the dz at the end!

4. Compute dy and Δy for $y = \mathbf{e}^{x^2}$ as x changes from 3 to 3.01.

Step 1

First let's get the actual change, Δy.

$$\Delta y = \mathbf{e}^{3.01^2} - \mathbf{e}^{3^2} = 501.927$$

Step 2

Next, we'll need the differential.

$$dy = 2x\,\mathbf{e}^{x^2}\,dx$$

Step 3

As x changes from 3 to 3.01 we have $\Delta x = 3.01 - 3 = 0.01$ and we'll assume that $dx \approx \Delta x = 0.01$. The approximate change, dy, is then,

$$\boxed{dy = 2\,(3)\;\mathbf{e}^{3^2}\,(0.01) = 486.185}$$

Don't forget to use the "starting" value of x (*i.e.* $x = 3$) for all the x's in the differential.

---

5. Compute dy and Δy for $y = x^5 - 2x^3 + 7x$ as x changes from 6 to 5.9.

Step 1

First let's get the actual change, Δy.

$$\Delta y = \left(5.9^5 - 2\left(5.9^3\right) + 7\,(5.9)\right) - \left(6^5 - 2\left(6^3\right) + 7\,(6)\right) = -606.215$$

Step 2

Next, we'll need the differential.

$$dy = \left(5x^4 - 6x^2 + 7\right)dx$$

Step 3

As x changes from 6 to 5.9 we have $\Delta x = 5.9 - 6 = -0.1$ and we'll assume that $dx \approx \Delta x = -0.1$. The approximate change, dy, is then,

$$\boxed{dy = \left(5\left(6^4\right) - 6\left(6^2\right) + 7\right)(-0.1) = -627.1}$$

Don't forget to use the "starting" value of x (*i.e.* $x = 6$) for all the x's in the differential.

6. The sides of a cube are found to be 6 feet in length with a possible error of no more than 1.5 inches. What is the maximum possible error in the volume of the cube if we use this value of the length of the side to compute the volume?

Step 1

Let's get everything set up first.

If we let the side of the cube be denoted by x the volume is then,

$$V(x) = x^3$$

We are told that $x = 6$ and we can assume that $dx \approx \Delta x = \frac{1.5}{12} = 0.125$ (don't forget to convert the inches to feet!).

Step 2

We want to estimate the maximum error in the volume and so we can again assume that $\Delta V \approx dV$.

The differential is then,

$$dV = 3x^2 dx$$

The maximum error in the volume is then,

$$\boxed{\Delta V \approx dV = 3\left(6^2\right)(0.125) = 13.5\,\text{ft}^3}$$

4.13 Newtons Method

1. Use Newton's Method to determine x_2 for $f(x) = x^3 - 7x^2 + 8x - 3$ if $x_0 = 5$

Step 1

There really isn't that much to do with this problem. We know that the basic formula for Newton's Method is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

so all we need to do is run through this twice.

Here is the derivative of the function since we'll need that.

$$f'(x) = 3x^2 - 14x + 8$$

We just now need to run through the formula above twice.

Step 2

The first iteration through the formula for x_1 is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 5 - \frac{f(5)}{f'(5)} = 5 - \frac{-13}{13} = 6$$

Step 3

The second iteration through the formula for x_2 is,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 6 - \frac{f(6)}{f'(6)} = 6 - \frac{9}{32} = 5.71875$$

So, the answer for this problem is $\boxed{x_2 = 5.71875}$.

Although it was not asked for in the problem statement the actual root is 5.68577952609. Note as well that this did require some computational aid to get and it not something that you can, in general, get by hand.

2. Use Newton's Method to determine x_2 for $f(x) = x\cos(x) - x^2$ if $x_0 = 1$

Step 1

There really isn't that much to do with this problem. We know that the basic formula for Newton's Method is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

so all we need to do is run through this twice.

Here is the derivative of the function since we'll need that.

$$f'(x) = \cos(x) - x\sin(x) - 2x$$

We just now need to run through the formula above twice.

Step 2

The first iteration through the formula for x_1 is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-0.4596976941}{-2.301168679} = 0.8002329432$$

Don't forget that for us angles are always in radians so make sure your calculator is set to compute in radians.

Step 3

The second iteration through the formula for x_2 is,

$$\begin{aligned} x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} &= 0.8002329432 - \frac{f(0.8002329432)}{f'(0.8002329432)} \\ &= 0.8002329432 - \frac{-0.08297883948}{-1.478108132} = 0.7440943985 \end{aligned}$$

So, the answer for this problem is $\boxed{x_2 = 0.7440943985}$.

Although it was not asked for in the problem statement the actual root is 0.73908513322. Note as well that this did require some computational aid to get and it not something that you can, in general, get by hand.

3. Use Newton's Method to find the root of $x^4 - 5x^3 + 9x + 3 = 0$ accurate to six decimal places in the interval $[4, 6]$.

Step 1

First, recall that Newton's Method solves equation in the form $f(x) = 0$ and so it is (hopefully) fairly clear that we have,

$$f(x) = x^4 - 5x^3 + 9x + 3$$

Next, we are not given a starting value, x_0, but we were given an interval in which the root exists so we may as well use the midpoint of this interval as our starting point or, $x_0 = 5$. Note that this is not the only value we could use and if you use a different one (which is perfectly acceptable) then your values will be different from those here.

At this point all we need to do is run through Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until the answers agree to six decimal places.

Step 2

The first iteration through the formula for x_1 is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 5 - \frac{48}{134} = 4.641791045$$

Step 3

The second iteration through the formula for x_2 is,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 4.641791045 - \frac{8.950542057}{85.85891882} = 4.537543959$$

We'll need to keep going because even the first decimal is not correct yet.

Step 4

The third iteration through the formula for x_3 is,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 4.537543959 - \frac{0.6329967413}{73.85993168} = 4.528973727$$

At this point we are accurate to the first decimal place so we need to continue.

Step 5

The fourth iteration through the formula for x_4 is,

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 4.528973727 - \frac{0.004066133005}{72.91199944} = 4.52891796$$

At this point we are accurate to 4 decimal places so we need to continue.

Step 6

The fifth iteration through the formula for x_5 is,

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 4.52891796 - \frac{1.714694911 * 10^{-7}}{72.90585006} = 4.52891796$$

At this point we are accurate to 8 decimal places which is actually better than we asked and so we can officially stop and we can estimate that the root in the interval is,

$$\boxed{x \approx 4.52891796}$$

Using computational aids we found that the actual root in this interval is 4.52891795729. Note that this wasn't actually asked for in the problem and is only given for comparison purposes.

4. Use Newton's Method to find the root of $2x^2 + 5 = \mathbf{e}^x$ accurate to six decimal places in the interval $[3, 4]$.

Step 1

First, recall that Newton's Method solves equation in the form $f(x) = 0$ and so we'll need move everything to one side. Doing this gives,

$$f(x) = 2x^2 + 5 - \mathbf{e}^x$$

Note that we could have just as easily gone the other direction. All that would have done was change the signs on the function and derivative evaluations in the work below. The final answers however would not be changed.

Next, we are not given a starting value, x_0, but we were given an interval in which the root exists so we may as well use the midpoint of this interval as our starting point or, $x_0 = 3.5$. Note that this is not the only value we could use and if you use a different one (which is perfectly acceptable) then your values will be different that those here.

At this point all we need to do is run through Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until the answers agree to six decimal places.

Step 2

The first iteration through the formula for x_1 is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3.5 - \frac{-3.615451959}{-19.11545196} = 3.310862334$$

Step 3

The second iteration through the formula for x_2 is,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.310862334 - \frac{-0.4851319992}{-14.16530146} = 3.276614422$$

We'll need to keep going because even the first decimal is not correct yet.

Step 4

The second iteration through the formula for x_3 is,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.276614422 - \frac{-0.0135463486}{-13.37949281} = 3.275601951$$

At this point we are accurate to two decimal places so we need to continue.

Step 5

The second iteration through the formula for x_4 is,

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 3.275601951 - \frac{-0.00001152056596}{-13.356740003} = 3.275601089$$

At this point we are accurate to 6 decimal places which is what we were asked to do and so we can officially stop and we can estimate that the root in the interval is,

$$\boxed{x \approx 3.275601089}$$

Using computational aids we found that the actual root in this interval is 3.27560108885. Note that this wasn't actually asked for in the problem and is only given for comparison purposes and it does look like Newton's Method did a pretty good job as this is identical to the final iteration that we did.

5. Use Newton's Method to find all the roots of $x^3 - x^2 - 15x + 1 = 0$ accurate to six decimal places.

Hint

Can you use your knowledge of Algebra to determine how many roots this equation should have? Maybe a graph of the function could also be useful for this problem.

Step 1

First, recall that Newton's Method solves equation in the form $f(x) = 0$ and so it is (hopefully) fairly clear that we have,

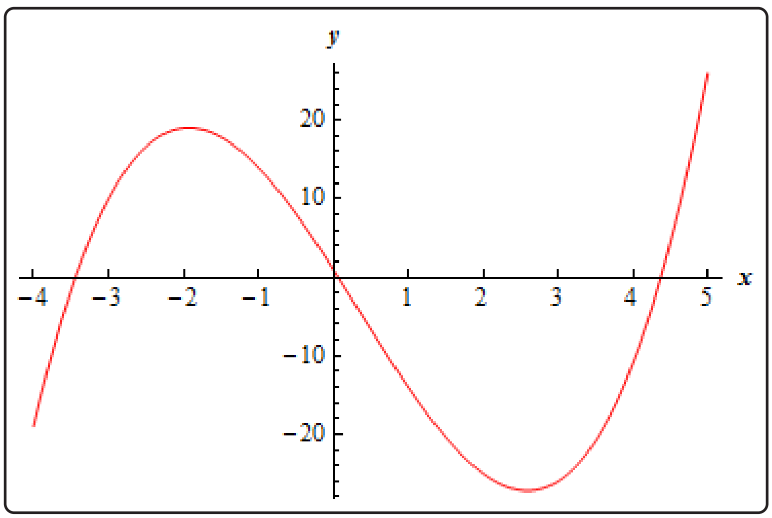
$$f(x) = x^3 - x^2 - 15x + 1$$

Next, we are not given a starting value, x_0 and unlike Problems 3 & 4 above we are not even given an interval to use as a way to determine a good possible value of x_0. We are also not even told how many roots we need to find.

Of course, if we recall our Algebra skills we can see that we have a cubic polynomial and so there should be at most three distinct roots of the equation (there may be some that repeat and so we may not have three distinct roots...). Knowing this all we really need to do to get potential starting values is to do a quick sketch of the function.

In determining a proper range of x values just keep in mind what we know about limits at infinity. Because the largest power of x is odd in this case we know that as $x \to \infty$ the graph should also be approaching positive infinity and as $x \to -\infty$ the graph should be approaching negative infinity. So, we can start with a large range of x's that gives the behavior we expect at the right/left ends of the graph and then narrow it down until we see the actual roots showing up on the graph.

Doing this gives,



So, it looks like we are going to have three roots here (*i.e.* the graph crosses the x-axis three times and so three roots...).

For each root we'll use the graph to pick a value of x_0 that is close to the root we are after (we'll go from left to right for the problem) and then run through Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until the answers agree to six decimal places.

Note as well that unlike Problems 3 & 4 we are not going to put in all the function evaluations for this problem. We'll leave that to you to check and verify our final answers for each iteration.

Step 2

For the left most root let's start with $x_0 = -3.5$. Here are the results of iterating through Newton's Method for this root.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -3.443478261 \qquad \text{No decimal places agree}$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -3.442146902 \qquad \text{Accurate to two decimal places}$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -3.44214617 \qquad \text{Accurate to six decimal places}$$

So, it looks like the estimate of the left most root is : $\boxed{x \approx -3.44214617}$.

Step 3

For the middle root let's start with $x_0 = 0$. Be careful with this root. From the graph we may be tempted to just say the root is zero. However, as we'll see the root is not zero. It is close to zero, but is not exactly zero!

Here are the results of iterating through Newton's Method for this root.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.06666666667 \qquad \text{No decimal places agree}$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.06639231824 \qquad \text{Accurate to three decimal places}$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.06639231426 \qquad \text{Accurate to eight decimal places}$$

So, it looks like the estimate of the middle root is : $\boxed{x \approx 0.06639231426}$.

Step 4

For the right most root let's start with $x_0 = 4.5$. Here are the results of iterating through Newton's Method for this root.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4.380952381 \qquad \text{No decimal places agree}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 4.375763556 \qquad \text{Accurate to one decimal place}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 4.375753856 \qquad \text{Accurate to four decimal places}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 4.375753856 \qquad \text{Accurate to nine decimal places}$$

So, it looks like the estimate of the right most root is : $\boxed{x \approx 4.375753856}$.

Step 5

Using computational aids we found that the actual roots of this equation to be,

$$x = -3.44214616993 \qquad x = 0.0663923142603 \qquad x = 4.37575385567$$

Note that these weren't actually asked for in the problem and are only given for comparison purposes.

As a final warning about Newton's Method, be careful to not assume that you'll get six (or better in some cases) decimal places of accuracy with just a few iterations.

These problems were chosen with the understanding that it would only take a few iterations of the method. There are problems and/or choices of x_0 for which it will take significantly more iterations to get any kind of real accuracy, provided the method even works for that equation and/or choice of x_0. Recall that we saw an example in the notes in which the method failed spectacularly.

6. Use Newton's Method to find all the roots of $2 - x^2 = \sin(x)$ accurate to six decimal places.

Hint

Can you use your knowledge what the graph of the left side and right side of this equation to determine how many roots this equation should have? Maybe a graph of the functions on the left and right side could also be useful for this problem.

Step 1

First, recall that Newton's Method solves equation in the form $f(x) = 0$ and so we'll need move everything to one side. Doing this gives,

$$f(x) = 2 - x^2 - \sin(x)$$

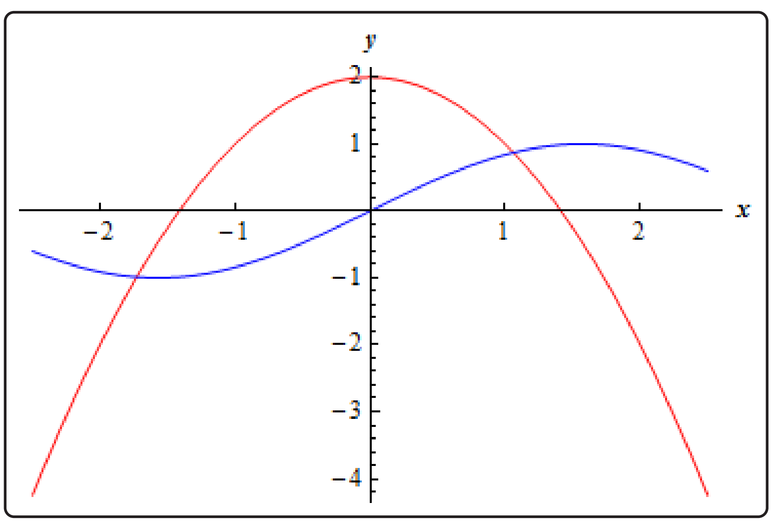
Note that we could have just as easily gone the other direction. All that would have done was change the signs on the function and derivative evaluations in the work below. The final answers however would not be changed.

Next, we are not given a starting value, x_0 and unlike Problems 3 & 4 above we are not even given an interval to use as a way to determine a good possible value of x_0. We are also not even told how many roots we need to find.

So, to estimate the number of roots of the equation let's take a look at each side of the equation and realize that each root will in fact be the point of intersection of the two curves on the left and right of the equal sign.

The left side of the original equation is a quadratic that will have its vertex at $x = 2$ and open downward while the right side is the sine function. Given what we know of these two functions we should expect there to be at most two roots where the quadratic, on its way down, intersects with the sine function. Because the quadratic will never turn around and start moving back upwards it should never intersect with the sine function again after those points.

So, let's graph both the quadratic and sine function to see if our intuition on this is correct. Doing this gives,



So, it looks like we guessed correctly and should have two roots here.

For each root we'll use the graph to pick a value of x_0 that is close to the root we are after (we'll go from left to right for the problem) and then run through Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until the answers agree to six decimal places.

Note as well that unlike Problems 3 & 4 we are not going to put in all the function evaluations for this problem. We'll leave that to you to check and verify our final answers for each iteration.

Also note that the analysis that we had to do to estimate the number of roots is something that does need to be done for these kinds of problems and it will differ for each equation. However, if you do have a basic knowledge of how most of the basic functions behave you can do this for most equations you'll be asked to deal with.

Step 2

For the left most root let's start with $x_0 = -1.5$. Here are the results of iterating through Newton's Method for this root.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1.755181948 \qquad \text{No decimal places agree}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1.728754674 \qquad \text{Accurate to one decimal place}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -1.728466353 \qquad \text{Accurate to three decimal places}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = -1.728466319 \qquad \text{Accurate to seven decimal places}$$

So, it looks like the estimate of the left most root is : $\boxed{x \approx -1.728466319}$.

Step 3

For the right most root let's start with $x_0 = 1$. Here are the results of iterating through

Newton's Method for this root.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.062405571 \qquad \text{No decimal places agree}$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.061549933 \qquad \text{Accurate to two decimal places}$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.061549775 \qquad \text{Accurate to six decimal places}$$

So, it looks like the estimate of the right most root is : $\boxed{x \approx 1.061549775}$.

Step 4

Using computational aids we found that the actual roots of this equation to be,

$$x = -1.72846631899718 \qquad x = 1.06154977463138$$

Note that these weren't actually asked for in the problem and are only given for comparison purposes.

As a final warning about Newton's Method, be careful to not assume that you'll get six (or better in some cases) decimal places of accuracy with just a few iterations.

These problems were chosen with the understanding that it would only take a few iterations of the method. There are problems and/or choices of x_0 for which it will take significantly more iterations to get any kind of real accuracy, provided the method even works for that equation and/or choice of x_0. Recall that we saw an example in the notes in which the method failed spectacularly.

4.14 Business Applications

1. A company can produce a maximum of 1500 widgets in a year. If they sell x widgets during the year then their profit, in dollars, is given by,

$$P\left(x\right) = 30,000,000 - 360,000x + 750x^2 - \frac{1}{3}x^3$$

How many widgets should they try to sell in order to maximize their profit?

Step 1

Because these are essentially the same type of problems that we did in the Absolute Extrema section we will not be doing a lot of explanation to the steps here. If you need some practice on absolute extrema problems you should check out some of the examples and/or practice problems there.

All we really need to do here is determine the absolute maximum of the profit function and the value of x that will give the absolute maximum.

Here is the derivative of the profit function and the critical point(s) since we'll need those for this problem.

$$P'\left(x\right) = -360,000 + 1500x - x^2 = -\left(x-1200\right)\left(x-300\right) = 0 \quad \Rightarrow \quad x = 300, \quad x = 1200$$

Step 2

From the problem statement we can see that we only want critical points that are in the interval $\left[0, 1500\right]$. As we can see both of the critical points from the above step are in this interval and so we'll need both of them.

Step 3

The next step is to evaluate the profit function at the critical points from the second step and at the end points of the given interval. Here are those function evaluations.

$$P\left(0\right) = 30,000,000 \qquad P\left(300\right) = -19,500,000$$
$$P\left(1200\right) = 102,000,000 \qquad P\left(1500\right) = 52,500,000$$

Step 4

From these evaluations we can see that they will need to sell 1200 widgets to maximize the profits.

---

2. A management company is going to build a new apartment complex. They know that if the complex contains x apartments the maintenance costs for the building, landscaping etc. will be,

$$C(x) = 4000 + 14x - 0.04x^2$$

The land they have purchased can hold a complex of at most 500 apartments. How many apartments should the complex have in order to minimize the maintenance costs?

Step 1

Because these are essentially the same type of problems that we did in the Absolute Extrema section we will not be doing a lot of explanation to the steps here. If you need some practice on absolute extrema problems you should check out some of the examples and/or practice problems there.

All we really need to do here is determine the absolute minimum of the maintenance function and the value of x that will give the absolute minimum.

Here is the derivative of the maintenance function and the critical point(s) since we'll need those for this problem.

$$C'(x) = 14 - 0.08x = \qquad \Rightarrow \qquad x = 175$$

Step 2

From the problem statement we can see that we only want critical points that are in the interval $[0, 500]$. As we can see both of the critical points from the above step are in this interval and so we'll need both of them.

Step 3

The next step is to evaluate the maintenance function at the critical points from the second step and at the end points of the given interval. Here are those function evaluations.

$$C(0) = 4000 \qquad C(175) = 5225 \qquad C(500) = 1000$$

Step 4

From these evaluations we can see that the complex should have 500 apartments to minimize the maintenance costs.

---

3. The production costs, in dollars, per day of producing x widgets is given by,

$$C(x) = 1750 + 6x - 0.04x^2 + 0.0003x^3$$

What is the marginal cost when $x = 175$ and $x = 300$? What do your answers tell you about the production costs?

Step 1

From the notes in this section we know that the marginal cost is simply the derivative of the cost function so let's start with that.

$$C'(x) = 6 - 0.08x + 0.0009x^2$$

Step 2

The marginal costs for each value of x is then,

$$\boxed{C'(175) = 19.5625 \qquad C'(300) = 63}$$

Step 3

From these computations we can see that is will cost approximately \$19.56 to produce the 176th widget and approximately \$63 to produce the 301st widget.

4. The production costs, in dollars, per month of producing x widgets is given by,

$$C(x) = 200 + 0.5x + \frac{10000}{x}$$

What is the marginal cost when $x = 200$ and $x = 500$? What do your answers tell you about the production costs?

Step 1

From the notes in this section we know that the marginal cost is simply the derivative of the cost function so let's start with that.

$$C'(x) = 0.5 - \frac{10000}{x^2}$$

Step 2

The marginal costs for each value of x is then,

$$\boxed{C'(200) = 0.25 \qquad C'(500) = 0.46}$$

Step 3

From these computations we can see that is will cost approximately 25 cents to produce the 201^{st} widget and approximately 46 cents to produce the 501^{st} widget.

5. The production costs, in dollars, per week of producing x widgets is given by,

$$C(x) = 4000 - 32x + 0.08x^2 + 0.00006x^3$$

and the demand function for the widgets is given by,

$$p(x) = 250 + 0.02x - 0.001x^2$$

What is the marginal cost, marginal revenue and marginal profit when $x = 200$ and $x = 400$? What do these numbers tell you about the cost, revenue and profit?

Step 1

First, we need to get the revenue and profit functions. From the notes for this section we know that these functions are,

$$\text{Revenue : } R(x) = x\,p(x) = 250x + 0.02x^2 - 0.001x^3$$
$$\text{Profit : } P(x) = R(x) - C(x) = -4000 + 282x - 0.06x^2 - 0.00106x^3$$

Step 2

From the notes in this section we know that the marginal cost, marginal revenue and marginal profit functions are simply the derivative of the cost, revenue and profit functions so let's start with those.

$$C'(x) = -32 + 0.16x + 0.00018x^2$$
$$R'(x) = 250 + 0.04x - 0.003x^2$$
$$P'(x) = 282 - 0.12x - 0.00318x^2$$

Step 3

The marginal cost, marginal revenue and marginal profit for each value of x is then,

$C'(200) = 7.2$	$R'(200) = 138$	$P'(200) = 130.8$
$C'(400) = 60.8$	$R'(400) = -214$	$P'(400) = -274.8$

Step 4

From these computations we can see that producing the 201^{st} widget will cost approximately $7.2 and will add approximately $138 in revenue and $130.8 in profit.

Likewise, producing the 401^{st} widget will cost approximately $60.8 and will see a decrease of approximately $214 in revenue and a decrease of $274.8 in profit

5 Integrals

In this chapter we will be looking at the third and final major topic that will be covered in this class, integrals. As with derivatives this chapter will be devoted almost exclusively to finding and computing integrals. Applications will be given in the following chapter. There are really two types of integrals that we'll be looking at in this chapter : Indefinite Integrals and Definite Integrals. The first half of this chapter is devoted to indefinite integrals and the last half is devoted to definite integrals.

As we investigate indefinite integrals we will see that as long as we understand basic differentiation we shouldn't have a lot of problems with basic indefinite integrals. The reason for this is that indefinite integration is basically "undoing" differentiation. In fact, indefinite integrals are sometimes called anti-derivatives to make this idea clear. Having said that however we will be using the phrase indefinite integral instead of anti-derivative as that is the more common phrase used.

We will also spend a fair amount of time learning the substitution rule for integrals. We will see that it is really just "undoing" the chain rule and so, again, if you understand the chain rule it will help when using the substitution rule. In addition, as we'll see as we go through the rest of the calculus course the substitution rule will come up time and again and so it is very important to make sure that we have that down so we don't have issues with it in later topics.

As we move over to investigating definite integrals we will quickly realize just how important it is to be able to do indefinite integrals. As we will see we will not be able to compute definite integrals unless we can fist compute indefinite integrals.

We will also take a look at an important interpretation of definite integrals. Namely, a definite integral can be interpreted as the net area between the graph of the function and the x-axis.

The following sections are the practice problems, with solutions, for this material.

5.1 Indefinite Integrals

1. Evaluate each of the following indefinite integrals.

(a) $\displaystyle\int 6x^5 - 18x^2 + 7\,dx$

(b) $\displaystyle\int 6x^5\,dx - 18x^2 + 7$

Hint

As long as you recall your derivative rules and the fact that all this problem is really asking is the for us to determine the function that we differentiated to get the integrand (*i.e.* the function inside the integral....) this problem shouldn't be too difficult.

Solutions

(a) $\displaystyle\int 6x^5 - 18x^2 + 7\,dx$

Solution

All we are being asked to do here is "undo" a differentiation and if you recall the basic differentiation rules for polynomials this shouldn't be too difficult. As we saw in the notes for this section all we really need to do is increase the exponent by one (so upon differentiation we get the correct exponent) and then fix up the coefficient to make sure that we will get the correct coefficient upon differentiation.

Here is the answer for this part.

$$\int 6x^5 - 18x^2 + 7\,dx = \boxed{x^6 - 6x^3 + 7x + c}$$

Don't forget the $+c$! Remember that the original function may have had a constant on it and the $+c$ is there to remind us of that.

Also, don't forget that you can easily check your answer by differentiating your answer and making sure that the result is the same as the integrand.

(b) $\displaystyle\int 6x^5\,dx - 18x^2 + 7$

Solution

This part is not really all that different from the first part. The only difference is the placement of the dx. Recall that one of the things that the dx tells us where to end the integration. So, in the part we are only going to integrate the first term.

Here is the answer for this part.

$$\int 6x^5\,dx - 18x^2 + 7 = \boxed{x^6 + c - 18x^2 + 7}$$

---

2. Evaluate each of the following indefinite integrals.

(a) $\displaystyle\int 40x^3 + 12x^2 - 9x + 14\,dx$

(b) $\displaystyle\int 40x^3 + 12x^2 - 9x\,dx + 14$

(c) $\displaystyle\int 40x^3 + 12x^2\,dx - 9x + 14$

Hint

As long as you recall your derivative rules and the fact that all this problem is really asking is the for us to determine the function that we differentiated to get the integrand (*i.e.* the function inside the integral....) this problem shouldn't be too difficult.

Solutions

(a) $\displaystyle\int 40x^3 + 12x^2 - 9x + 14\,dx$

Solution

All we are being asked to do here is "undo" a differentiation and if you recall the basic differentiation rules for polynomials this shouldn't be too difficult. As we saw in the notes for this section all we really need to do is increase the exponent by one (so upon differentiation we get the correct exponent) and then fix up the coefficient to make sure that we will get the correct coefficient upon differentiation.

Here is that answer for this part.

$$\int 40x^3 + 12x^2 - 9x + 14\,dx = \boxed{10x^4 + 4x^3 - \frac{9}{2}x^2 + 14\,x + c}$$

Don't forget the $+c$! Remember that the original function may have had a constant on it and the $+c$ is there to remind us of that.

Also, don't forget that you can easily check your answer by differentiating your answer and making sure that the result is the same as the integrand.

(b) $\int 40x^3 + 12x^2 - 9x\,dx + 14$

Solution

This part is not really all that different from the first part. The only difference is the placement of the dx. Recall that one of the things that the dx tells us where to end the integration. So, in the part we are only going to integrate the first term.

Here is the answer for this part.

$$\int 40x^3 + 12x^2 - 9x\,dx + 14 = \boxed{10x^4 + 4x^3 - \frac{9}{2}x^2 + c + 14}$$

(c) $\int 40x^3 + 12x^2\,dx - 9x + 14$

Solution

The only difference between this part and the previous part is that the location of the dx moved.

Here is the answer for this part.

$$\int 40x^3 + 12x^2\,dx - 9x + 14 = \boxed{10x^4 + 4x^3 + c - 9x + 14}$$

---

3. Evaluate $\displaystyle\int 12t^7 - t^2 - t + 3\,dt$.

Hint

As long as you recall your derivative rules and the fact that all this problem is really asking is the for us to determine the function that we differentiated to get the integrand (*i.e.* the function inside the integral....) this problem shouldn't be too difficult.

Solution

All we are being asked to do here is "undo" a differentiation and if you recall the basic differentiation rules for polynomials this shouldn't be too difficult. As we saw in the notes for this section all we really need to do is increase the exponent by one (so upon differentiation we get the correct exponent) and then fix up the coefficient to make sure that we will get the correct coefficient upon differentiation.

Here is the answer.

$$\int 12t^7 - t^2 - t + 3\,dt = \boxed{\frac{3}{2}t^8 - \frac{1}{3}t^3 - \frac{1}{2}t^2 + 3\,t + c}$$

Don't forget the $+c$! Remember that the original function may have had a constant on it and the $+c$ is there to remind us of that.

Also, don't forget that you can easily check your answer by differentiating your answer and making sure that the result is the same as the integrand.

---

4. Evaluate $\displaystyle\int 10w^4 + 9w^3 + 7w\;dw$.

Hint

As long as you recall your derivative rules and the fact that all this problem is really asking is the for us to determine the function that we differentiated to get the integrand (*i.e.* the function inside the integral....) this problem shouldn't be too difficult.

Solution

All we are being asked to do here is "undo" a differentiation and if you recall the basic differentiation rules for polynomials this shouldn't be too difficult. As we saw in the notes for this section all we really need to do is increase the exponent by one (so upon differentiation we get the correct exponent) and then fix up the coefficient to make sure that we will get the correct coefficient upon differentiation.

Here is the answer.

$$\int 10w^4 + 9w^3 + 7w\, dw = \boxed{2w^5 + \frac{9}{4}w^4 + \frac{7}{2}w^2 + c}$$

Don't forget the $+c$! Remember that the original function may have had a constant on it and the $+c$ is there to remind us of that.

Also, don't forget that you can easily check your answer by differentiating your answer and making sure that the result is the same as the integrand.

5. Evaluate $\int z^6 + 4z^4 - z^2\, dz$.

Hint

As long as you recall your derivative rules and the fact that all this problem is really asking is the for us to determine the function that we differentiated to get the integrand (*i.e.* the function inside the integral....) this problem shouldn't be too difficult.

Solution

All we are being asked to do here is "undo" a differentiation and if you recall the basic differentiation rules for polynomials this shouldn't be too difficult. As we saw in the notes for this section all we really need to do is increase the exponent by one (so upon differentiation we get the correct exponent) and then fix up the coefficient to make sure that we will get the correct coefficient upon differentiation.

Here is the answer.

$$\int z^6 + 4z^4 - z^2\, dz = \boxed{\frac{1}{7}z^7 + \frac{4}{5}z^5 - \frac{1}{3}z^3 + c}$$

Don't forget the $+c$! Remember that the original function may have had a constant on it and the $+c$ is there to remind us of that.

Also, don't forget that you can easily check your answer by differentiating your answer and making sure that the result is the same as the integrand.

6. Determine $f(x)$ given that $f'(x) = 6x^8 - 20x^4 + x^2 + 9$.

Hint

Remember that all indefinite integrals are asking us to do is "undo" a differentiation.

Solution

We know that indefinite integrals are asking us to undo a differentiation to so all we are really being asked to do here is evaluate the following indefinite integral.

$$f(x) = \int f'(x)\,dx = \int 6x^8 - 20x^4 + x^2 + 9\,dx = \boxed{\frac{2}{3}x^9 - 4x^5 + \frac{1}{3}x^3 + 9x + c}$$

Don't forget the $+c$! Remember that the original function may have had a constant on it and the $+c$ is there to remind us of that.

---

7. Determine $h(t)$ given that $h'(t) = t^4 - t^3 + t^2 + t - 1$.

Hint

Remember that all indefinite integrals are asking us to do is "undo" a differentiation.

Solution

We know that indefinite integrals are asking us to undo a differentiation to so all we are really being asked to do here is evaluate the following indefinite integral.

$$h(t) = \int h'(t)\,dt = \int t^4 - t^3 + t^2 + t - 1\,dt = \boxed{\frac{1}{5}t^5 - \frac{1}{4}t^4 + \frac{1}{3}t^3 + \frac{1}{2}t^2 - t + c}$$

Don't forget the $+c$! Remember that the original function may have had a constant on it and the $+c$ is there to remind us of that.

5.2 Computing Indefinite Integrals

1. Evaluate $\int 4x^6 - 2x^3 + 7x - 4\,dx$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int 4x^6 - 2x^3 + 7x - 4\,dx = \frac{4}{7}x^7 - \frac{2}{4}x^4 + \frac{7}{2}x^2 - 4x + c = \boxed{\frac{4}{7}x^7 - \frac{1}{2}x^4 + \frac{7}{2}x^2 - 4x + c}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

2. Evaluate $\int z^7 - 48z^{11} - 5z^{16}\,dz$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int z^7 - 48z^{11} - 5z^{16}\,dz = \frac{1}{8}z^8 - \frac{48}{12}z^{12} - \frac{5}{17}z^{17} + c = \boxed{\frac{1}{8}z^8 - 4z^{12} - \frac{5}{17}z^{17} + c}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

3. Evaluate $\int 10t^{-3} + 12t^{-9} + 4t^3\,dt$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int 10t^{-3} + 12t^{-9} + 4t^3\,dt = \frac{10}{-2}t^{-2} + \frac{12}{-8}t^{-8} + \frac{4}{4}t^4 + c = \boxed{-5t^{-2} - \frac{3}{2}t^{-8} + t^4 + c}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

---

4. Evaluate $\displaystyle\int w^{-2} + 10w^{-5} - 8\,dw$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int w^{-2} + 10w^{-5} - 8\,dw = \frac{1}{-1}w^{-1} + \frac{10}{-4}w^{-4} - 8w + c = \boxed{-w^{-1} - \frac{5}{2}w^{-4} - 8w + c}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

---

5. Evaluate $\displaystyle\int 12\,dy$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int 12\,dy = \boxed{12y + c}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

---

6. Evaluate $\displaystyle\int \sqrt[3]{w} + 10\,\sqrt[5]{w^3}\,dw$.

Hint

Don't forget to convert the roots to fractional exponents.

Step 1

We first need to convert the roots to fractional exponents.

$$\int \sqrt[3]{w} + 10\,\sqrt[5]{w^3}\,dw = \int w^{\frac{1}{3}} + 10\left(w^3\right)^{\frac{1}{5}}\,dw = \int w^{\frac{1}{3}} + 10w^{\frac{3}{5}}\,dw$$

Step 2

Once we've gotten the roots converted to fractional exponents there really isn't too much to do other than to evaluate the integral.

$$\int \sqrt[3]{w} + 10\,\sqrt[5]{w^3}\,dw = \int w^{\frac{1}{3}} + 10w^{\frac{3}{5}}\,dw = \frac{3}{4}w^{\frac{4}{3}} + 10\left(\frac{5}{8}\right)w^{\frac{8}{5}} + c$$
$$= \boxed{\frac{3}{4}w^{\frac{4}{3}} + \frac{25}{4}w^{\frac{8}{5}} + c}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

---

7. Evaluate $\int \sqrt{x^7} - 7\,\sqrt[6]{x^5} + 17\,\sqrt[3]{x^{10}}\,dx$.

Hint

Don't forget to convert the roots to fractional exponents.

Step 1

We first need to convert the roots to fractional exponents.

$$\int \sqrt{x^7} - 7\,\sqrt[6]{x^5} + 17\,\sqrt[3]{x^{10}}\,dx = \int x^{\frac{7}{2}} - 7\left(x^5\right)^{\frac{1}{6}} + 17\left(x^{10}\right)^{\frac{1}{3}}\,dx$$
$$= \int x^{\frac{7}{2}} - 7x^{\frac{5}{6}} + 17x^{\frac{10}{3}}\,dx$$

Step 2

Once we've gotten the roots converted to fractional exponents there really isn't too much to do other than to evaluate the integral.

$$\begin{aligned}\int \sqrt{x^7} - 7\sqrt[6]{x^5} + 17\sqrt[3]{x^{10}}\,dx &= \int x^{\frac{7}{2}} - 7x^{\frac{5}{6}} + 17x^{\frac{10}{3}}\,dx \\ &= \frac{2}{9}x^{\frac{9}{2}} - 7\left(\frac{6}{11}\right)x^{\frac{11}{6}} + 17\left(\frac{3}{13}\right)x^{\frac{13}{3}} + c \\ &= \boxed{\frac{2}{9}x^{\frac{9}{2}} - \frac{42}{11}x^{\frac{11}{6}} + \frac{51}{13}x^{\frac{13}{3}} + c}\end{aligned}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

---

8. Evaluate $\displaystyle\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}\,dx$.

Hint

Don't forget to move the x's in the denominator to the numerator with negative exponents.

Step 1

We first need to move the x's in the denominator to the numerator with negative exponents.

$$\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}\,dx = \int 4x^{-2} + 2 - \frac{1}{8}x^{-3}\,dx$$

Remember that the "8" in the denominator of the third term stays in the denominator and does not move up with the x.

Step 2

Once we've gotten the x's out of the denominator there really isn't too much to do other than to evaluate the integral.

$$\begin{aligned}\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}\,dx &= \int 4x^{-2} + 2 - \frac{1}{8}x^{-3}\,dx\\ &= 4\left(\frac{1}{-1}\right)x^{-1} + 2x - \frac{1}{8}\left(\frac{1}{-2}\right)x^{-2} + c\\ &= \boxed{-4x^{-1} + 2x + \frac{1}{16}x^{-2} + c}\end{aligned}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

9. Evaluate $\displaystyle\int \frac{7}{3y^6} + \frac{1}{y^{10}} - \frac{2}{\sqrt[3]{y^4}}\,dy$.

Hint

Don't forget to convert the root to a fractional exponents and move the y's in the denominator to the numerator with negative exponents.

Step 1

We first need to convert the root to a fractional exponent and move the y's in the denominator to the numerator with negative exponents.

$$\int \frac{7}{3y^6} + \frac{1}{y^{10}} - \frac{2}{\sqrt[3]{y^4}}\,dy = \int \frac{7}{3y^6} + \frac{1}{y^{10}} - \frac{2}{y^{\frac{4}{3}}}\,dy = \int \frac{7}{3}y^{-6} + y^{-10} - 2y^{-\frac{4}{3}}\,dy$$

Remember that the "3" in the denominator of the first term stays in the denominator and does not move up with the y.

Step 2

Once we've gotten the root converted to a fractional exponent and the y's out of the denominator there really isn't too much to do other than to evaluate the integral.

$$\begin{aligned}\int \frac{7}{3y^6} + \frac{1}{y^{10}} - \frac{2}{\sqrt[3]{y^4}}\,dy &= \int \frac{7}{3}y^{-6} + y^{-10} - 2y^{-\frac{4}{3}}\,dy\\ &= \frac{7}{3}\left(\frac{1}{-5}\right)y^{-5} + \left(\frac{1}{-9}\right)y^{-9} - 2\left(-\frac{3}{1}\right)y^{-\frac{1}{3}} + c\\ &= \boxed{-\frac{7}{15}y^{-5} - \frac{1}{9}y^{-9} + 6y^{-\frac{1}{3}} + c}\end{aligned}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

---

10. Evaluate $\displaystyle\int \left(t^2 - 1\right)\left(4 + 3t\right)\,dt$.

Hint

Remember that there is no "Product Rule" for integrals and so we'll need to eliminate the product before integrating.

Step 1

Since there is no "Product Rule" for integrals we'll need to multiply the terms out prior to integration.

$$\int \left(t^2 - 1\right)\left(4 + 3t\right)\,dt = \int 3t^3 + 4t^2 - 3t - 4\,dt$$

Step 2

At this point there really isn't too much to do other than to evaluate the integral.

$$\int \left(t^2 - 1\right)\left(4 + 3t\right)\,dt = \int 3t^3 + 4t^2 - 3t - 4\,dt = \boxed{\frac{3}{4}t^4 + \frac{4}{3}t^3 - \frac{3}{2}t^2 - 4t + c}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

11. Evaluate $\displaystyle\int \sqrt{z}\left(z^2 - \frac{1}{4z}\right)\,dz$.

Hint

Remember that there is no "Product Rule" for integrals and so we'll need to eliminate the product before integrating.

Step 1

Since there is no "Product Rule" for integrals we'll need to multiply the terms out prior to integration.

$$\int \sqrt{z}\left(z^2 - \frac{1}{4z}\right)dz = \int z^{\frac{5}{2}} - \frac{1}{4z^{\frac{1}{2}}}\,dz = \int z^{\frac{5}{2}} - \frac{1}{4}z^{-\frac{1}{2}}\,dz$$

Don't forget to convert the root to a fractional exponent and move the z's out of the denominator.

Step 2

At this point there really isn't too much to do other than to evaluate the integral.

$$\int \sqrt{z}\left(z^2 - \frac{1}{4z}\right)dz = \int z^{\frac{5}{2}} - \frac{1}{4}z^{-\frac{1}{2}}\,dz = \boxed{\frac{2}{7}z^{\frac{7}{2}} - \frac{1}{2}z^{\frac{1}{2}} + c}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

12. Evaluate $\displaystyle\int \frac{z^8 - 6z^5 + 4z^3 - 2}{z^4}\,dz$.

Hint

Remember that there is no "Quotient Rule" for integrals and so we'll need to eliminate the quotient before integrating.

Step 1

Since there is no "Quotient Rule" for integrals we'll need to break up the integrand and simplify a little prior to integration.

$$\int \frac{z^8 - 6z^5 + 4z^3 - 2}{z^4}\,dz = \int \frac{z^8}{z^4} - \frac{6z^5}{z^4} + \frac{4z^3}{z^4} - \frac{2}{z^4}\,dz = \int z^4 - 6z + \frac{4}{z} - 2z^{-4}\,dz$$

Step 2

At this point there really isn't too much to do other than to evaluate the integral.

$$\int \frac{z^8 - 6z^5 + 4z^3 - 2}{z^4}\,dz = \int z^4 - 6z + \frac{4}{z} - 2z^{-4}\,dz$$
$$= \boxed{\frac{1}{5}z^5 - 3z^2 + 4\ln|z| + \frac{2}{3}z^{-3} + c}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

13. Evaluate $\displaystyle\int \frac{x^4 - \sqrt[3]{x}}{6\sqrt{x}}\,dx$.

Hint

Remember that there is no "Quotient Rule" for integrals and so we'll need to eliminate the quotient before integrating.

Step 1

Since there is no "Quotient Rule" for integrals we'll need to break up the integrand and simplify a little prior to integration.

$$\int \frac{x^4 - \sqrt[3]{x}}{6\sqrt{x}}\,dx = \int \frac{x^4}{6x^{\frac{1}{2}}} - \frac{x^{\frac{1}{3}}}{6x^{\frac{1}{2}}}\,dx = \int \frac{1}{6}x^{\frac{7}{2}} - \frac{1}{6}x^{-\frac{1}{6}}\,dx$$

Don't forget to convert the roots to fractional exponents!

Step 2

At this point there really isn't too much to do other than to evaluate the integral.

$$\int \frac{x^4 - \sqrt[3]{x}}{6\sqrt{x}}\,dx = \int \frac{1}{6}x^{\frac{7}{2}} - \frac{1}{6}x^{-\frac{1}{6}}\,dx = \boxed{\frac{1}{27}x^{\frac{9}{2}} - \frac{1}{5}x^{\frac{5}{6}} + c}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

---

14. Evaluate $\int \sin(x) + 10\csc^2(x)\,dx$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int \sin(x) + 10\csc^2(x)\,dx = \boxed{-\cos(x) - 10\cot(x) + c}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

---

15. Evaluate $\int 2\cos(w) - \sec(w)\tan(w)\,dw$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int 2\cos(w) - \sec(w)\tan(w)\,dw = \boxed{2\sin(w) - \sec(w) + c}$$

Don't forget to add on the $+c$ since we know that we are asking what function did we differentiate to get the integrand and the derivative of a constant is zero and so we do need to add that onto the answer.

---

16. Evaluate $\displaystyle\int 12 + \csc(\theta)\left[\sin(\theta) + \csc(\theta)\right]\,d\theta$.

Hint

From previous problems in this set we should know how to deal with the product in the integrand.

Step 1

Before doing the integral we need to multiply out the product and don't forget the definition of cosecant in terms of sine.

$$\begin{aligned}\int 12 + \csc(\theta)\left[\sin(\theta) + \csc(\theta)\right]\,d\theta &= \int 12 + \csc(\theta)\sin(\theta) + \csc^2(\theta)\,d\theta \\ &= \int 13 + \csc^2(\theta)\,d\theta\end{aligned}$$

Recall that,

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

and so,

$$\csc(\theta)\sin(\theta) = 1$$

Doing this allows us to greatly simplify the integrand and, in fact, allows us to actually do the integral. Without this simplification we would not have been able to integrate the second term with the knowledge that we currently have.

Step 2

At this point there really isn't too much to do other than to evaluate the integral.

$$\int 12 + \csc(\theta)\left[\sin(\theta) + \csc(\theta)\right]\,d\theta = \int 13 + \csc^2(\theta)\,d\theta = \boxed{13\theta - \cot(\theta) + c}$$

Don't forget that with trig functions some terms can be greatly simplified just by recalling the definition of the trig functions and/or their relationship with the other trig functions.

17. Evaluate $\displaystyle\int 4\mathbf{e}^z + 15 - \frac{1}{6z}\,dz$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int 4\mathbf{e}^z + 15 - \frac{1}{6z}\,dz = \int 4\mathbf{e}^z + 15 - \frac{1}{6}\frac{1}{z}\,dz = \boxed{4\mathbf{e}^z + 15z - \frac{1}{6}\ln|z| + c}$$

Be careful with the "6" in the denominator of the third term. The "best" way of dealing with it in this case is to split up the third term as we've done above and then integrate.

Note that the "best" way to do a problem is always relative for many Calculus problems. There are other ways of dealing with this term (later section material) and so what one person finds the best another may not. For us, this seems to be an easy way to deal with the 6 and not overly complicate the integration process.

---

18. Evaluate $\displaystyle\int t^3 - \frac{\mathbf{e}^{-t} - 4}{\mathbf{e}^{-t}}\,dt$.

Hint

From previous problems in this set we should know how to deal with the quotient in the integrand.

Step 1

Before doing the integral we need to break up the quotient and do some simplification.

$$\int t^3 - \frac{\mathbf{e}^{-t} - 4}{\mathbf{e}^{-t}}\,dt = \int t^3 - \frac{\mathbf{e}^{-t}}{\mathbf{e}^{-t}} + \frac{4}{\mathbf{e}^{-t}}\,dt = \int t^3 - 1 + 4\mathbf{e}^t\,dt$$

Make sure that you correctly distribute the minus sign when breaking up the second term and don't forget to move the exponential in the denominator of the third term (after splitting up the integrand) to the numerator and changing the sign on the t to a "$+$" in the process.

Step 2

At this point there really isn't too much to do other than to evaluate the integral.

$$\int t^3 - \frac{\mathbf{e}^{-t} - 4}{\mathbf{e}^{-t}}\,dt = \int t^3 - 1 + 4\mathbf{e}^t\,dt = \boxed{\frac{1}{4}t^4 - t + 4\mathbf{e}^t + c}$$

19. Evaluate $\displaystyle\int \frac{6}{w^3} - \frac{2}{w}\,dw$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int \frac{6}{w^3} - \frac{2}{w}\,dw = \int 6w^{-3} - \frac{2}{w}\,dw = \boxed{-3w^{-2} - 2\ln|w| + c}$$

20. Evaluate $\displaystyle\int \frac{1}{1+x^2} + \frac{12}{\sqrt{1-x^2}}\,dx$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int \frac{1}{1+x^2} + \frac{12}{\sqrt{1-x^2}}\,dx = \boxed{\tan^{-1}(x) + 12\sin^{-1}(x) + c}$$

Note that because of the similarity of the derivative of inverse sine and inverse cosine an alternate answer is,

$$\int \frac{1}{1+x^2} + \frac{12}{\sqrt{1-x^2}}\,dx = \boxed{\tan^{-1}(x) - 12\cos^{-1}(x) + c}$$

21. Evaluate $\displaystyle\int 6\cos(z) + \frac{4}{\sqrt{1-z^2}}\,dz$.

Solution

There really isn't too much to do other than to evaluate the integral.

$$\int 6\cos(z) + \frac{4}{\sqrt{1-z^2}}\,dz = \boxed{6\sin(z) + 4\sin^{-1}(z) + c}$$

Note that because of the similarity of the derivative of inverse sine and inverse cosine an alternate answer is,

$$\int 6\cos(z) + \frac{4}{\sqrt{1-z^2}}\,dz = \boxed{6\sin(z) - 4\cos^{-1}(z) + c}$$

22. Determine $f(x)$ given that $f'(x) = 12x^2 - 4x$ and $f(-3) = 17$.

Hint

We know that integration is simply asking what function we differentiated to get the integrand and so we should be able to use this idea to arrive at a general formula for the function.

Step 1

Recall from the notes in this section that we saw,

$$f(x) = \int f'(x)\, dx$$

and so to arrive at a general formula for $f(x)$ all we need to do is integrate the derivative that we've been given in the problem statement.

$$f(x) = \int 12x^2 - 4x\, dx = 4x^3 - 2x^2 + c$$

Don't forget the $+c$!

Hint

To determine the value of the constant of integration, c, we have the value of the function at $x = -3$.

Step 2

Because we have the condition that $f(-3) = 17$ we can just plug $x = -3$ into our answer from the previous step, set the result equal to 17 and solve the resulting equation for c.

Doing this gives,

$$17 = f(-3) = -126 + c \qquad \Rightarrow \qquad c = 143$$

The function is then,

$$\boxed{f(x) = 4x^3 - 2x^2 + 143}$$

23. Determine $g(z)$ given that $g'(z) = 3z^3 + \frac{7}{2\sqrt{z}} - \mathbf{e}^z$ and $g(1) = 15 - \mathbf{e}$.

Hint

We know that integration is simply asking what function we differentiated to get the integrand and so we should be able to use this idea to arrive at a general formula for the function.

Step 1

Recall from the notes in this section that we saw,

$$g(z) = \int g'(z)\ dz$$

and so to arrive at a general formula for $g(z)$ all we need to do is integrate the derivative that we've been given in the problem statement.

$$g(z) = \int 3z^3 + \frac{7}{2}z^{-\frac{1}{2}} - \mathbf{e}^z\, dz = \frac{3}{4}z^4 + 7z^{\frac{1}{2}} - \mathbf{e}^z + c$$

Don't forget the $+c$!

Hint

To determine the value of the constant of integration, c, we have the value of the function at $z = 1$.

Step 2

Because we have the condition that $g(1) = 15 - \mathbf{e}$ we can just plug $z = 1$ into our answer from the previous step, set the result equal to $15 - \mathbf{e}$ and solve the resulting equation for c.

Doing this gives,

$$15 - \mathbf{e} = g(1) = \frac{31}{4} - \mathbf{e} + c \qquad \Rightarrow \qquad c = \frac{29}{4}$$

The function is then,

$$\boxed{g(z) = \frac{3}{4}z^4 + 7z^{\frac{1}{2}} - \mathbf{e}^z + \frac{29}{4}}$$

24. Determine $h(t)$ given that $h''(t) = 24t^2 - 48t + 2$, $h(1) = -9$ and $h(-2) = -4$.

Hint

We know how to find $h(t)$ from $h'(t)$ but we don't have that. We should however be able to determine the general formula for $h'(t)$ from $h''(t)$ which we are given.

Step 1

Because we know that the 2^{nd} derivative is just the derivative of the 1^{st} derivative we know that,

$$h'(t) = \int h''(t)\ dt$$

and so to arrive at a general formula for $h'(t)$ all we need to do is integrate the 2^{nd} derivative that we've been given in the problem statement.

$$h'(t) = \int 24t^2 - 48t + 2\ dt = 8t^3 - 24t^2 + 2t + c$$

Don't forget the $+c$!

Hint

From the previous two problems you should be able to determine a general formula for $h(t)$. Just don't forget that c is just a constant!

Step 2

Now, just as we did in the previous two problems, all that we need to do is integrate the 1^{st} derivative (which we found in the first step) to determine a general formula for $h(t)$.

$$h(t) = \int 8t^3 - 24t^2 + 2t + c\ dt = 2t^4 - 8t^3 + t^2 + ct + d$$

Don't forget that c is just a constant and so it will integrate just like we were integrating 2 or 4 or any other number. Also, the constant of integration from this step is liable to be different that the constant of integration from the first step and so we'll need to make sure to call it something different, d in this case.

Hint

To determine the value of the constants of integration, c and d, we have the value of the function at two values that should help with that.

Step 3

Now, we know the value of the function at two values of z. So let's plug both of these into the general formula for $h(t)$ that we found in the previous step to get,

$$-9 = h(1) = -5 + c + d$$
$$-4 = h(-2) = 100 - 2c + d$$

Solving this system of equations (you do remember your Algebra class right?) for c and d gives,

$$c = \frac{100}{3} \qquad d = -\frac{112}{3}$$

The function is then,

$$\boxed{h(t) = 2t^4 - 8t^3 + t^2 + \frac{100}{3}t - \frac{112}{3}}$$

5.3 Substitution Rule for Indefinite Integrals

1. Evaluate $\displaystyle\int (8x-12)\left(4x^2-12x\right)^4\,dx$.

Hint

Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1

In this case it looks like we should use the following as our substitution.

$$u = 4x^2 - 12x$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the x's are replaced with u's we need to compute the differential so we can eliminate the dx as well as the remaining x's in the integrand.

$$du = (8x-12)\ dx$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that.

Step 3

Doing the substitution and evaluating the integral gives,

$$\int (8x-12)\left(4x^2-12x\right)^4\,dx = \int u^4\,du = \frac{1}{5}u^5 + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int (8x-12)\left(4x^2-12x\right)^4 dx = \boxed{\frac{1}{5}\left(4x^2-12x\right)^5+c}$$

---

2. Evaluate $\displaystyle\int 3t^{-4}\left(2+4t^{-3}\right)^{-7} dt$.

Hint

Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1

In this case it looks like we should use the following as our substitution.

$$u = 2+4t^{-3}$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the t's are replaced with u's we need to compute the differential so we can eliminate the dt as well as the remaining t's in the integrand.

$$du = -12t^{-4}\,dt$$

To help with the substitution let's do a little rewriting of this to get,

$$3t^{-4}\,dt = -\frac{1}{4}du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int 3t^{-4}\left(2+4t^{-3}\right)^{-7}dt = -\frac{1}{4}\int u^{-7}\,du = \frac{1}{24}u^{-6} + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int 3t^{-4}\left(2+4t^{-3}\right)^{-7}dt = \boxed{\frac{1}{24}\left(2+4t^{-3}\right)^{-6} + c}$$

---

3. Evaluate $\displaystyle\int (3-4w)\left(4w^2-6w+7\right)^{10}dw$.

Hint

Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1

In this case it looks like we should use the following as our substitution.

$$u = 4w^2 - 6w + 7$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the w's are replaced with u's we need to compute the differential so we can eliminate the dw as well as the remaining w's in the integrand.

$$du = (8w - 6)\,dw$$

To help with the substitution let's do a little rewriting of this to get,

$$du = -2\,(3 - 4w)\,dw \qquad \Rightarrow \qquad (3 - 4w)\,dw = -\frac{1}{2}du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int (3 - 4w)\left(4w^2 - 6w + 7\right)^{10} dw = -\frac{1}{2}\int u^{10}\,du = -\frac{1}{22}u^{11} + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int (3 - 4w)\left(4w^2 - 6w + 7\right)^{10} dw = \boxed{-\frac{1}{22}\left(4w^2 - 6w + 7\right)^{11} + c}$$

4. Evaluate $\int 5(z-4)\ \sqrt[3]{z^2-8z}\,dz$.

Hint

Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1

In this case it looks like we should use the following as our substitution.

$$u = z^2 - 8z$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the z's are replaced with u's we need to compute the differential so we can eliminate the dz as well as the remaining z's in the integrand.

$$du = (2z-8)\,dz$$

To help with the substitution let's do a little rewriting of this to get,

$$du = (2z-8)\,dz = 2(z-4)\,dz \qquad \Rightarrow \qquad (z-4)\,dz = \frac{1}{2}du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int 5(z-4)\ \sqrt[3]{z^2-8z}\,dz = \frac{5}{2}\int u^{\frac{1}{3}}\,du = \frac{15}{8}u^{\frac{4}{3}} + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int 5(z-4)\sqrt[3]{z^2-8z}\,dz = \boxed{\frac{15}{8}\left(z^2-8z\right)^{\frac{4}{3}}+c}$$

---

5. Evaluate $\displaystyle\int 90x^2 \sin\left(2+6x^3\right)\,dx$.

Hint

Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1

In this case it looks like we should use the following as our substitution.

$$u = 2+6x^3$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the x's are replaced with u's we need to compute the differential so we can eliminate the dx as well as the remaining x's in the integrand.

$$du = 18x^2dx$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that. When doing the substitution just notice that $90 = (18)(5)$.

Step 3

Doing the substitution and evaluating the integral gives,

$$\int 90x^2 \sin\left(2 + 6x^3\right)\, dx = \int 5 \sin\left(u\right)\, du = -5\cos\left(u\right) + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int 90x^2 \sin\left(2 + 6x^3\right)\, dx = \boxed{-5\cos\left(2 + 6x^3\right) + c}$$

---

6. Evaluate $\int \sec\left(1 - z\right) \tan\left(1 - z\right)\, dz$.

Hint

Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1

In this case it looks like we should use the following as our substitution.

$$u = 1 - z$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the z's are replaced with u's we need to compute the differential so we can eliminate the dz as well as the remaining z's in the integrand.

$$du = -dz$$

To help with the substitution let's do a little rewriting of this to get,

$$dz = -du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int \sec(1-z)\tan(1-z)\,dz = -\int \sec(u)\tan(u)\,du = -\sec(u) + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int \sec(1-z)\tan(1-z)\,dz = \boxed{-\sec(1-z) + c}$$

---

7. Evaluate $\int \left(15t^{-2} - 5t\right) \cos\left(6t^{-1} + t^2\right)\, dt$.

Hint

Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1

In this case it looks like we should use the following as our substitution.

$$u = 6t^{-1} + t^2$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the t's are replaced with u's we need to compute the differential so we can eliminate the dt as well as the remaining t's in the integrand.

$$du = \left(-6t^{-2} + 2t\right) dt$$

To help with the substitution let's do a little rewriting of this to get,

$$du = \left(-6t^{-2} + 2t\right) dt = -2\left(\frac{5}{5}\right)\left(3t^{-2} - t\right) dt \qquad \Rightarrow \qquad \left(15t^{-2} - 5t\right) dt = -\frac{5}{2} du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int \left(15t^{-2} - 5t\right) \cos\left(6t^{-1} + t^2\right)\, dt = -\frac{5}{2}\int \cos\left(u\right)\, du = -\frac{5}{2}\sin\left(u\right) + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int \left(15t^{-2} - 5t\right) \cos\left(6t^{-1} + t^2\right)\, dt = \boxed{-\frac{5}{2}\sin\left(6t^{-1} + t^2\right) + c}$$

---

8. Evaluate $\displaystyle\int \left(7y - 2y^3\right) \mathbf{e}^{y^4 - 7y^2}\, dy$.

Hint

Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1

In this case it looks like we should use the following as our substitution.

$$u = y^4 - 7y^2$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the y's are replaced with u's we need to compute the differential so we can eliminate the dy as well as the remaining y's in the integrand.

$$du = \left(4y^3 - 14y\right) dy$$

To help with the substitution let's do a little rewriting of this to get,

$$du = \left(4y^3 - 14y\right) dy = -2\left(7y - 2y^3\right) dy \qquad \Rightarrow \qquad \left(7y - 2y^3\right) dy = -\frac{1}{2} du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int \left(7y - 2y^3\right) \mathbf{e}^{y^4 - 7y^2}\, dy = -\frac{1}{2}\int \mathbf{e}^u\, du = -\frac{1}{2}\mathbf{e}^u + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int \left(7y - 2y^3\right) \mathbf{e}^{y^4 - 7y^2}\, dy = \boxed{-\frac{1}{2}\mathbf{e}^{y^4 - 7y^2} + c}$$

---

9. Evaluate $\displaystyle\int \frac{4w + 3}{4w^2 + 6w - 1}\, dw$.

Hint

What is the derivative of the denominator?

Step 1

In this case it looks like we should use the following as our substitution.

$$u = 4w^2 + 6w - 1$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the w's are replaced with u's we need to compute the differential so we can eliminate the dw as well as the remaining w's in the integrand.

$$du = (8w + 6)\,dw$$

To help with the substitution let's do a little rewriting of this to get,

$$du = 2\,(4w + 3)\,dw \qquad \Rightarrow \qquad (4w + 3)\,dw = \frac{1}{2}du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int \frac{4w + 3}{4w^2 + 6w - 1}\,dw = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int \frac{4w + 3}{4w^2 + 6w - 1}\,dw = \boxed{\frac{1}{2}\ln\left|4w^2 + 6w - 1\right| + c}$$

---

10. Evaluate $\int \left(\cos(3t) - t^2\right)\left(\sin(3t) - t^3\right)^5 dt$.

Hint

Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1

In this case it looks like we should use the following as our substitution.

$$u = \sin(3t) - t^3$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the t's are replaced with u's we need to compute the differential so we can eliminate the dt as well as the remaining t's in the integrand.

$$du = \left(3\cos(3t) - 3t^2\right) dt$$

To help with the substitution let's do a little rewriting of this to get,

$$du = 3\left(\cos(3t) - t^2\right) dt \qquad \Rightarrow \qquad \left(\cos(3t) - t^2\right) dt = \frac{1}{3} du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int \left(\cos(3t) - t^2\right)\left(\sin(3t) - t^3\right)^5 dt = \frac{1}{3}\int u^5\, du = \frac{1}{18}u^6 + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int \left(\cos(3t) - t^2\right)\left(\sin(3t) - t^3\right)^5 dt = \boxed{\frac{1}{18}\left(\sin(3t) - t^3\right)^6 + c}$$

---

11. Evaluate $\int 4\left(\frac{1}{z} - \mathbf{e}^{-z}\right)\cos\left(\mathbf{e}^{-z} + \ln z\right)\, dz$.

Hint

Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1

In this case it looks like we should use the following as our substitution.

$$u = \mathbf{e}^{-z} + \ln z$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the z's are replaced with u's we need to compute the differential so we can eliminate the dz as well as the remaining z's in the integrand.

$$du = \left(-\mathbf{e}^{-z} + \frac{1}{z}\right) dt$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that.

Step 3

Doing the substitution and evaluating the integral gives,

$$\int 4\left(\frac{1}{z} - \mathbf{e}^{-z}\right)\cos\left(\mathbf{e}^{-z} + \ln z\right)\,dz = \int 4\cos(u)\,du = 4\sin(u) + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int 4\left(\frac{1}{z} - \mathbf{e}^{-z}\right)\cos\left(\mathbf{e}^{-z} + \ln z\right)\,dz = \boxed{4\sin\left(\mathbf{e}^{-z} + \ln z\right) + c}$$

12. Evaluate $\int \sec^2(v)\,\mathbf{e}^{1+\tan(v)}\,dv$.

Hint

Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1

In this case it looks like we should use the following as our substitution.

$$u = 1 + \tan(v)$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the v's are replaced with u's we need to compute the differential so we can eliminate the dv as well as the remaining v's in the integrand.

$$du = \sec^2(v)\,dv$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that.

Step 3

Doing the substitution and evaluating the integral gives,

$$\int \sec^2(v)\,\mathbf{e}^{1+\tan(v)}\,dv = \int \mathbf{e}^u\,du = \mathbf{e}^u + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int \sec^2(v)\,\mathbf{e}^{1+\tan(v)}\,dv = \boxed{\mathbf{e}^{1+\tan(v)} + c}$$

---

13. Evaluate $\displaystyle\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x) - 5}\,dx$.

Hint

Recall that if there is a term in the integrand (or a portion of a term) with an "obvious" inside function then there is at least a chance that the "inside" function is the substitution that we need.

Step 1

In this case it looks like we should use the following as our substitution.

$$u = \cos^2(2x) - 5$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the x's are replaced with u's we need to compute the differential so we can eliminate the dx as well as the remaining x's in the integrand.

$$du = -4\cos(2x)\sin(2x)\,dx$$

To help with the substitution let's do a little rewriting of this to get,

$$du = -4\cos(2x)\sin(2x)\,dx = -2(2)\left(\frac{5}{5}\right)\cos(2x)\sin(2x)\,dx$$
$$\Rightarrow \quad 10\cos(2x)\sin(2x)\,dx = -\frac{5}{2}du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x) - 5}\,dx = -\frac{5}{2}\int u^{\frac{1}{2}}\,du = -\frac{5}{3}u^{\frac{3}{2}} + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x)-5}\,dx = \boxed{-\frac{5}{3}\left(\cos^2(2x)-5\right)^{\frac{3}{2}}+c}$$

---

14. Evaluate $\displaystyle\int \frac{\csc(x)\cot(x)}{2-\csc(x)}\,dx$.

Hint

What is the derivative of the denominator?

Step 1

In this case it looks like we should use the following as our substitution.

$$u = 2-\csc(x)$$

Hint

Recall that after the substitution all the original variables in the integral should be replaced with u's.

Step 2

Because we need to make sure that all the x's are replaced with u's we need to compute the differential so we can eliminate the dx as well as the remaining x's in the integrand.

$$du = \csc(x)\cot(x)\,dx$$

Recall that, in most cases, we will also need to do a little manipulation of the differential prior to doing the substitution. In this case we don't need to do that.

Step 3

Doing the substitution and evaluating the integral gives,

$$\int \frac{\csc(x)\cot(x)}{2-\csc(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int \frac{\csc(x)\cot(x)}{2-\csc(x)}\,dx = \boxed{\ln|2-\csc(x)| + c}$$

---

15. Evaluate $\displaystyle\int \frac{6}{7+y^2}\,dy$.

Hint

Be careful with this substitution. The integrand should look somewhat familiar, so maybe we should try to put it into a more familiar form.

Step 1

The integrand looks an awful lot like the derivative of the inverse tangent.

$$\frac{d}{du}\left(\tan^{-1}(u)\right) = \frac{1}{1+u^2}$$

So, let's do a little rewrite to make the integrand look more like this.

$$\int \frac{6}{7+y^2}\,dy = \int \frac{6}{7\left(1+\frac{1}{7}y^2\right)}\,dy = \frac{6}{7}\int \frac{1}{1+\frac{1}{7}y^2}\,dy$$

Hint

One more little rewrite of the integrand should make this look almost exactly like the derivative the inverse tangent and the substitution should then be fairly obvious.

Step 2

Let's do one more rewrite of the integrand.

$$\int \frac{6}{7+y^2}\,dy = \frac{6}{7}\int \frac{1}{1+\left(\frac{y}{\sqrt{7}}\right)^2}\,dy$$

At this point we can see that the following substitution will work for us.

$$u = \frac{y}{\sqrt{7}} \qquad \to \qquad du = \frac{1}{\sqrt{7}}dy \qquad \to \qquad dy = \sqrt{7}du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int \frac{6}{7+y^2}\,dy = \frac{6}{7}\left(\sqrt{7}\right)\int \frac{1}{1+u^2}\,du = \frac{6}{\sqrt{7}}\tan^{-1}(u) + c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int \frac{6}{7+y^2}\,dy = \frac{6}{7}\left(\sqrt{7}\right)\int \frac{1}{1+u^2}\,du = \boxed{\frac{6}{\sqrt{7}}\tan^{-1}\left(\frac{y}{\sqrt{7}}\right) + c}$$

Substitutions for inverse trig functions can be a little tricky to spot when you are first start doing them. Once you do enough of them however they start to become a little easier to spot.

16. Evaluate $\displaystyle\int \frac{1}{\sqrt{4-9w^2}}\,dw$.

Hint

Be careful with this substitution. The integrand should look somewhat familiar, so maybe we should try to put it into a more familiar form.

Step 1

The integrand looks an awful lot like the derivative of the inverse sine.

$$\frac{d}{du}\left(\sin^{-1}(u)\right) = \frac{1}{\sqrt{1-u^2}}$$

So, let's do a little rewrite to make the integrand look more like this.

$$\int \frac{1}{\sqrt{4-9w^2}}\,dw = \int \frac{1}{\sqrt{4\left(1-\frac{9}{4}w^2\right)}}\,dw = \frac{1}{2}\int \frac{1}{\sqrt{1-\frac{9}{4}w^2}}\,dw$$

Hint

One more little rewrite of the integrand should make this look almost exactly like the derivative the inverse sine and the substitution should then be fairly obvious.

Step 2

Let's do one more rewrite of the integrand.

$$\int \frac{1}{\sqrt{4-9w^2}}\,dw = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{3w}{2}\right)^2}}\,dw$$

At this point we can see that the following substitution will work for us.

$$u = \frac{3w}{2} \qquad \rightarrow \qquad du = \frac{3}{2}dw \qquad \rightarrow \qquad dw = \frac{2}{3}du$$

Step 3

Doing the substitution and evaluating the integral gives,

$$\int \frac{1}{\sqrt{4-9w^2}}\,dw = \frac{1}{2}\left(\frac{2}{3}\right)\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{3}\sin^{-1}(u)+c$$

Hint

Don't forget that the original variable in the integrand was not u!

Step 4

Finally, don't forget to go back to the original variable!

$$\int \frac{1}{\sqrt{4-9w^2}}\,dw = \boxed{\frac{1}{3}\sin^{-1}\left(\frac{3w}{2}\right)+c}$$

Substitutions for inverse trig functions can be a little tricky to spot when you are first start doing them. Once you do enough of them however they start to become a little easier to spot.

---

17. Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{3x}{1+9x^2}\,dx$

(b) $\displaystyle\int \frac{3x}{(1+9x^2)^4}\,dx$

(c) $\displaystyle\int \frac{3}{1+9x^2}\,dx$

Hint

Make sure you pay attention to each of these and note the differences between each integrand and how that will affect the substitution and/or answer.

Solutions

(a) $\displaystyle\int \frac{3x}{1+9x^2}\,dx$

Solution

In this case it looks like the substitution should be

$$u = 1+9x^2$$

Here is the differential for this substitution.

$$du = 18x\,dx \qquad \Rightarrow \qquad 3x\,dx = \frac{1}{6}du$$

The integral is then,

$$\int \frac{3x}{1+9x^2}\,dx = \frac{1}{6}\int \frac{1}{u}\,du = \frac{1}{6}\ln|u| + c = \boxed{\frac{1}{6}\ln\left|1+9x^2\right| + c}$$

(b) $\displaystyle\int \frac{3x}{\left(1+9x^2\right)^4}\,dx$

Solution

The substitution and differential work for this part are identical to the previous part.

$$u = 1+9x^2 \qquad du = 18x\,dx \qquad \Rightarrow \qquad 3x\,dx = \frac{1}{6}du$$

Here is the integral for this part,

$$\begin{aligned}\int \frac{3x}{\left(1+9x^2\right)^4}\,dx &= \frac{1}{6}\int \frac{1}{u^4}\,du = \frac{1}{6}\int u^{-4}\,du \\ &= -\frac{1}{18}u^{-3} + c = \boxed{-\frac{1}{18}\frac{1}{\left(1+9x^2\right)^3} + c}\end{aligned}$$

Be careful to not just turn every integral of functions of the form of 1/(something) into logarithms! This is one of the more common mistakes that students often make.

(c) $\displaystyle\int \frac{3}{1+9x^2}\,dx$

Solution

Because we no longer have an x in the numerator this integral is very different from the previous two. Let's do a quick rewrite of the integrand to make the substitution clearer.

$$\int \frac{3}{1+9x^2}\,dx = \int \frac{3}{1+(3x)^2}\,dx$$

So, this looks like an inverse tangent problem that will need the substitution.

$$u = 3x \qquad \rightarrow \qquad du = 3dx$$

The integral is then,

$$\int \frac{3}{1+9x^2}\,dx = \int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + c = \boxed{\tan^{-1}(3x) + c}$$

5.4 More Substitution Rule

1. Evaluate $\displaystyle\int 4\sqrt{5+9t} + 12(5+9t)^7\,dt$.

Hint

Each term seems to require the same substitution and recall that the same substitution can be used in multiple terms of an integral if we need to.

Step 1

Don't get too excited about the fact that there are two terms in this integrand. Each term requires the same substitution,

$$u = 5+9t$$

so we'll simply use that in both terms.

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 2

Here is the differential work for the substitution.

$$du = 9\,dt \qquad \to \qquad dt = \frac{1}{9}du$$

Doing the substitution and evaluating the integral gives,

$$\int \left[4u^{\frac{1}{2}} + 12u^7\right]\left(\frac{1}{9}\right)du = \frac{1}{9}\left[\frac{8}{3}u^{\frac{3}{2}} + \frac{3}{2}u^8\right] + c = \boxed{\frac{1}{9}\left[\frac{8}{3}(5+9t)^{\frac{3}{2}} + \frac{3}{2}(5+9t)^8\right] + c}$$

Be careful when dealing with the dt substitution here. Make sure that the $\frac{1}{9}$ gets multiplied times the whole integrand and not just one of the terms. You can do this either by using parenthesis (as we've done here) or pulling the $\frac{1}{9}$ completely out of the integral.

Do not forget to go back to the original variable after evaluating the integral!

2. Evaluate $\displaystyle\int 7x^3 \cos\left(2+x^4\right) - 8x^3 \mathbf{e}^{2+x^4}\,dx$.

Hint

Each term seems to require the same substitution and recall that the same substitution can be used in multiple terms of an integral if we need to.

Step 1

Don't get too excited about the fact that there are two terms in this integrand. Each term requires the same substitution,

$$u = 2 + x^4$$

so we'll simply use that in both terms.

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 2

Here is the differential work for the substitution.

$$du = 4x^3\,dx \qquad \to \qquad x^3 dx = \frac{1}{4}du$$

Before doing the actual substitution it might be convenient to factor an x^3 out of the integrand as follows.

$$\int 7x^3 \cos\left(2+x^4\right) - 8x^3 \mathbf{e}^{2+x^4}\,dx = \int \left[7\cos\left(2+x^4\right) - 8\mathbf{e}^{2+x^4}\right] x^3 dx$$

Doing this should make the differential part (*i.e.* the du part) of the substitution clearer.

Now, doing the substitution and evaluating the integral gives,

$$\begin{aligned}\int 7x^3 \cos\left(2+x^4\right) - 8x^3 \mathbf{e}^{2+x^4}\,dx &= \frac{1}{4}\int 7\cos\left(u\right) - 8\mathbf{e}^{u} du \\ &= \frac{1}{4}\left[7\sin\left(u\right) - 8\mathbf{e}^{u}\right] + c \\ &= \boxed{\frac{1}{4}\left[7\sin\left(2+x^4\right) - 8\mathbf{e}^{2+x^4}\right] + c}\end{aligned}$$

Be careful when dealing with the dx substitution here. Make sure that the $\frac{1}{4}$ gets multiplied times the whole integrand and not just one of the terms. You can do this either by using parenthesis around the whole integrand or pulling the $\frac{1}{4}$ completely out of the integral (as we've done here).

Do not forget to go back to the original variable after evaluating the integral!

---

3. Evaluate $\displaystyle\int \frac{6\mathbf{e}^{7w}}{\left(1-8\mathbf{e}^{7w}\right)^3} + \frac{14\mathbf{e}^{7w}}{1-8\mathbf{e}^{7w}}\,dw$.

Hint

Each term seems to require the same substitution and recall that the same substitution can be used in multiple terms of an integral if we need to.

Step 1

Don't get too excited about the fact that there are two terms in this integrand. Each term requires the same substitution,

$$u = 1 - 8\mathbf{e}^{7w}$$

so we'll simply use that in both terms.

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 2

Here is the differential work for the substitution.

$$du = -56\mathbf{e}^{7w}dw \qquad \to \qquad \mathbf{e}^{7w}dw = -\frac{1}{56}du$$

Before doing the actual substitution it might be convenient to factor an $\mathbf{e}^{7w}$ out of the integrand as follows.

$$\int \frac{6\mathbf{e}^{7w}}{\left(1-8\mathbf{e}^{7w}\right)^3} + \frac{14\mathbf{e}^{7w}}{1-8\mathbf{e}^{7w}}\,dw = \int \left[\frac{6}{\left(1-8\mathbf{e}^{7w}\right)^3} + \frac{14}{1-8\mathbf{e}^{7w}}\right]\mathbf{e}^{7w}\,dw$$

Doing this should make the differential part (*i.e.* the du part) of the substitution clearer.

Now, doing the substitution and evaluating the integral gives,

$$\int \frac{6\mathbf{e}^{7w}}{\left(1-8\mathbf{e}^{7w}\right)^3} + \frac{14\mathbf{e}^{7w}}{1-8\mathbf{e}^{7w}}\,dw = -\frac{1}{56}\int 6u^{-3} + \frac{14}{u}\,du = -\frac{1}{56}\left(-3u^{-2} + 14\ln|u|\right) + c$$
$$= \boxed{-\frac{1}{56}\left(-3\left(1-8\mathbf{e}^{7w}\right)^{-2} + 14\ln\left|1-8\mathbf{e}^{7w}\right|\right) + c}$$

Be careful when dealing with the dw substitution here. Make sure that the $-\frac{1}{56}$ gets multiplied times the whole integrand and not just one of the terms. You can do this either by using parenthesis around the whole integrand or pulling the $-\frac{1}{56}$ completely out of the integral (as we've done here).

Do not forget to go back to the original variable after evaluating the integral!

---

4. Evaluate $\displaystyle\int x^4 - 7x^5 \cos\left(2x^6+3\right)\,dx$.

Hint

Recall that terms that do not need substitutions should not be in the integral when the substitution is being done. At this point we should know how to "break" integrals up so that we can get the terms that require a substitution into a one integral and those that don't into another integral.

Step 1

Clearly the first term does not need a substitution while the second term does need a substitution. So, we'll first need to split up the integral as follows.

$$\int x^4 - 7x^5\cos\left(2x^6+3\right)\,dx = \int x^4\,dx - \int 7x^5\cos\left(2x^6+3\right)\,dx$$

Step 2

The substitution needed for the second integral is then,

$$u = 2x^6 + 3$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 3

Here is the differential work for the substitution.

$$du = 12x^5 dx \qquad \to \qquad x^5 dx = \frac{1}{12} du$$

Now, doing the substitution and evaluating the integrals gives,

$$\begin{aligned}\int x^4 - 7x^5 \cos\left(2x^6 + 3\right)\, dx &= \int x^4\, dx - \frac{7}{12}\int \cos(u)\, du = \frac{1}{5}x^5 - \frac{7}{12}\sin(u) + c \\ &= \boxed{\frac{1}{5}x^5 - \frac{7}{12}\sin\left(2x^6 + 3\right) + c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

---

5. Evaluate $\displaystyle\int \mathbf{e}^z + \frac{4\sin(8z)}{1 + 9\cos(8z)}\, dz$.

Hint

Recall that terms that do not need substitutions should not be in the integral when the substitution is being done. At this point we should know how to "break" integrals up so that we can get the terms that require a substitution into a one integral and those that don't into another integral.

Step 1

Clearly the first term does not need a substitution while the second term does need a substitution. So, we'll first need to split up the integral as follows.

$$\int \mathbf{e}^z + \frac{4\sin(8z)}{1 + 9\cos(8z)}\, dz = \int \mathbf{e}^z\, dz + \int \frac{4\sin(8z)}{1 + 9\cos(8z)}\, dz$$

Step 2

The substitution needed for the second integral is then,

$$u = 1 + 9\cos(8z)$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 3

Here is the differential work for the substitution.

$$du = -72\sin(8z)\,dz \qquad \to \qquad \sin(8z)\,dz = -\frac{1}{72}du$$

Now, doing the substitution and evaluating the integrals gives,

$$\int \mathbf{e}^z + \frac{4\sin(8z)}{1+9\cos(8z)}\,dz = \int \mathbf{e}^z\,dz - \frac{4}{72}\int \frac{1}{u}\,du = \boxed{\mathbf{e}^z - \frac{1}{18}\ln|1+9\cos(8z)| + c}$$

Do not forget to go back to the original variable after evaluating the integral!

---

6. Evaluate $\displaystyle\int 20\mathbf{e}^{2-8w}\sqrt{1+\mathbf{e}^{2-8w}} + 7w^3 - 6\sqrt[3]{w}\,dw$.

Hint

Recall that terms that do not need substitutions should not be in the integral when the substitution is being done. At this point we should know how to "break" integrals up so that we can get the terms that require a substitution into a one integral and those that don't into another integral.

Step 1

Clearly the first term needs a substitution while the second and third terms don't. So, we'll first need to split up the integral as follows.

$$\int 20\mathbf{e}^{2-8w}\sqrt{1+\mathbf{e}^{2-8w}} + 7w^3 - 6\sqrt[3]{w}\,dw = \int 20\mathbf{e}^{2-8w}\sqrt{1+\mathbf{e}^{2-8w}}\,dw + \int 7w^3 - 6\sqrt[3]{w}\,dw$$

Step 2

The substitution needed for the first integral is then,

$$u = 1 + \mathbf{e}^{2-8w}$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 3

Here is the differential work for the substitution.

$$du = -8\mathbf{e}^{2-8w}dw \qquad \to \qquad \mathbf{e}^{2-8w}dw = -\frac{1}{8}du$$

Now, doing the substitutions and evaluating the integrals gives,

$$\begin{aligned}\int 20\mathbf{e}^{2-8w}\sqrt{1+\mathbf{e}^{2-8w}} + 7w^3 - 6\sqrt[3]{w}\,dw &= -\frac{20}{8}\int u^{\frac{1}{2}}\,du + \int 7w^3 - 6w^{\frac{1}{3}}\,dw\\ &= -\frac{5}{3}u^{\frac{3}{2}} + \frac{7}{4}w^4 - \frac{9}{2}w^{\frac{4}{3}} + c\\ &= \boxed{-\frac{5}{3}\left(1+\mathbf{e}^{2-8w}\right)^{\frac{3}{2}} + \frac{7}{4}w^4 - \frac{9}{2}w^{\frac{4}{3}} + c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

7. Evaluate $\displaystyle\int (4+7t)^3 - 9t\,\sqrt[4]{5t^2+3}\,dt$.

Hint

You can only do one substitution per integral. At this point we should know how to "break" integrals up so that we can get the terms that require different substitutions into different integrals.

Step 1

Clearly each term needs a separate substitution. So, we'll first need to split up the integral as follows.

$$\int (4+7t)^3 - 9t\,\sqrt[4]{5t^2+3}\,dt = \int (4+7t)^3\,dt - \int 9t\,\sqrt[4]{5t^2+3}\,dt$$

Step 2

The substitutions needed for each integral are then,

$$u = 4+7t \qquad\qquad v = 5t^2+3$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 3

Here is the differential work for each substitution.

$$du = 7dt \quad \to \quad dt = \frac{1}{7}du \qquad\qquad dv = 10t\,dt \quad \to \quad t\,dt = \frac{1}{10}dv$$

Now, doing the substitutions and evaluating the integrals gives,

$$\begin{aligned}\int (4+7t)^3\,dt - \int 9t\,\sqrt[4]{5t^2+3}\,dt &= \frac{1}{7}\int u^3\,du - \frac{9}{10}\int v^{\frac{1}{4}}\,dv = \frac{1}{28}u^4 - \frac{18}{25}v^{\frac{5}{4}} + c \\ &= \boxed{\frac{1}{28}(4+7t)^4 - \frac{18}{25}\left(5t^2+3\right)^{\frac{5}{4}} + c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

8. Evaluate $\displaystyle\int \frac{6x - x^2}{x^3 - 9x^2 + 8} - \csc^2\left(\frac{3x}{2}\right)\,dx$.

Hint

You can only do one substitution per integral. At this point we should know how to "break" integrals up so that we can get the terms that require different substitutions into different integrals.

Step 1

Clearly each term needs a separate substitution. So, we'll first need to split up the integral as follows.

$$\int \frac{6x - x^2}{x^3 - 9x^2 + 8} - \csc^2\left(\frac{3x}{2}\right)\,dx = \int \frac{6x - x^2}{x^3 - 9x^2 + 8}\,dx - \int \csc^2\left(\frac{3x}{2}\right)\,dx$$

Step 2

The substitutions needed for each integral are then,

$$u = x^3 - 9x^2 + 8 \qquad\qquad v = \frac{3x}{2}$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 3

Here is the differential work for each substitution.

$$du = \left(3x^2 - 18x\right)dx = -3\left(6x - x^2\right)dx \qquad \to \qquad \left(6x - x^2\right)dx = -\frac{1}{3}du$$

$$dv = \frac{3}{2}\,dx \qquad \to \qquad dx = \frac{2}{3}dv$$

Now, doing the substitutions and evaluating the integrals gives,

$$\begin{aligned}\int \frac{6x - x^2}{x^3 - 9x^2 + 8} - \csc^2\left(\frac{3x}{2}\right)\,dx &= -\frac{1}{3}\int \frac{1}{u}\,du - \frac{2}{3}\int \csc^2(v)\,dv \\ &= -\frac{1}{3}\ln|u| + \frac{2}{3}\cot(v) + c \\ &= \boxed{-\frac{1}{3}\ln\left|x^3 - 9x^2 + 8\right| + \frac{2}{3}\cot\left(\frac{3x}{2}\right) + c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

9. Evaluate $\displaystyle\int 7(3y+2)\left(4y+3y^2\right)^3 + \sin(3+8y)\,dy$.

Hint

You can only do one substitution per integral. At this point we should know how to "break" integrals up so that we can get the terms that require different substitutions into different integrals.

Step 1

Clearly each term needs a separate substitution. So, we'll first need to split up the integral as follows.

$$\begin{aligned}\int 7(3y+2)\left(4y+3y^2\right)^3 + \sin(3+8y)\,dy = &\int 7(3y+2)\left(4y+3y^2\right)^3\,dy \\ &+ \int \sin(3+8y)\,dy\end{aligned}$$

Step 2

The substitutions needed for each integral are then,

$$u = 4y + 3y^2 \qquad\qquad v = 3 + 8y$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 3

Here is the differential work for each substitution.

$$du = (4+6y)\,dy = 2(3y+2)\,dy \qquad \to \qquad (3y+2)\,dy = \frac{1}{2}du$$

$$dv = 8\,dy \qquad \to \qquad dy = \frac{1}{8}dv$$

Now, doing the substitutions and evaluating the integrals gives,

$$\begin{aligned}\int 7(3y+2)\left(4y+3y^2\right)^3 + \sin(3+8y)\,dy &= \frac{7}{2}\int u^3 du + \frac{1}{8}\int \sin(v)\,dv \\ &= \frac{7}{8}u^4 - \frac{1}{8}\cos(v) + c \\ &= \boxed{\frac{7}{8}\left(4y+3y^2\right)^4 - \frac{1}{8}\cos(3+8y) + c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

---

10. Evaluate $\displaystyle\int \sec^2(2t)\left[9 + 7\tan(2t) - \tan^2(2t)\right]\,dt$.

Hint

Don't let this one fool you. This is simply an integral that requires you to use the same substitution more than once.

Step 1

This integral can be a little daunting at first glance. To do it all we need to notice is that the derivative of $\tan(x)$ is $\sec^2(x)$ and we can notice that there is a $\sec^2(2t)$ times the remaining portion of the integrand and that portion only contains constants and tangents.

So, it looks like the substitution is then,

$$u = \tan(2t)$$

If you aren't comfortable with the basic substitution mechanics you should work some problems in the previous section as we'll not be putting in as much detail with regards to the basics in this section. The problems in this section are intended for those that are fairly comfortable with the basic mechanics of substitutions and will involve some more "advanced" substitutions.

Step 2

Here is the differential work for the substitution.

$$du = 2\sec^2(2t)\,dt \qquad \to \qquad \sec^2(2t)\,dt = \frac{1}{2}du$$

Now, doing the substitution and evaluating the integrals gives,

$$\begin{aligned}\int \sec^2(2t)\left[9 + 7\tan(2t) - \tan^2(2t)\right]\,dt &= \frac{1}{2}\int 9 + 7u - u^2 du = \frac{1}{2}\left(9u + \frac{7}{2}u^2 - \frac{1}{3}u^3\right) + c \\ &= \boxed{\frac{1}{2}\left(9\tan(2t) + \frac{7}{2}\tan^2(2t) - \frac{1}{3}\tan^3(2t)\right) + c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

11. Evaluate $\displaystyle\int \frac{8-w}{4w^2+9}\,dw$.

Hint

With the integrand written as it is here this problem can't be done.

Step 1

As written we can't do this problem. In order to do this integral we'll need to rewrite the integral as follows.

$$\int \frac{8-w}{4w^2+9}\,dw = \int \frac{8}{4w^2+9}\,dw - \int \frac{w}{4w^2+9}\,dw$$

Step 2

Now, the first integral looks like it might be an inverse tangent (although we'll need to do a rewrite of that integral) and the second looks like it's a logarithm (with a quick substitution).

So, here is the rewrite on the first integral.

$$\int \frac{8-w}{4w^2+9}\,dw = \frac{8}{9}\int \frac{1}{\frac{4}{9}w^2+1}\,dw - \int \frac{w}{4w^2+9}\,dw$$

Step 3

Now we'll need a substitution for each integral. Here are the substitutions we'll need for each integral.

$$u = \frac{2}{3}w \quad \left(\text{so } u^2 = \frac{4}{9}w^2\right) \qquad\qquad v = 4w^2+9$$

Step 4

Here is the differential work for the substitution.

$$du = \frac{2}{3}dw \quad \to \quad dw = \frac{3}{2}du \qquad\qquad dv = 8w\,dw \quad \to \quad w\,dw = \frac{1}{8}dv$$

Now, doing the substitutions and evaluating the integrals gives,

$$\begin{aligned}\int \frac{8-w}{4w^2+9}\,dw &= \frac{8}{9}\left(\frac{3}{2}\right)\int \frac{1}{u^2+1}\,du - \frac{1}{8}\int \frac{1}{v}\,dv = \frac{4}{3}\tan^{-1}(u) - \frac{1}{8}\ln|v| + c \\ &= \boxed{\frac{4}{3}\tan^{-1}\left(\frac{2}{3}w\right) - \frac{1}{8}\ln\left|4w^2+9\right| + c}\end{aligned}$$

Do not forget to go back to the original variable after evaluating the integral!

---

12. Evaluate $\displaystyle\int \frac{7x+2}{\sqrt{1-25x^2}}\,dx$.

Hint

With the integrand written as it is here this problem can't be done.

Step 1

As written we can't do this problem. In order to do this integral we'll need to rewrite the integral as follows.

$$\int \frac{7x+2}{\sqrt{1-25x^2}}\,dx = \int \frac{7x}{\sqrt{1-25x^2}}\,dx + \int \frac{2}{\sqrt{1-25x^2}}\,dx$$

Step 2

Now, the second integral looks like it might be an inverse sine (although we'll need to do a rewrite of that integral) and the first looks like a simple substitution will work for us.

So, here is the rewrite on the second integral.

$$\int \frac{7x+2}{\sqrt{1-25x^2}}\,dx = \int \frac{7x}{\sqrt{1-25x^2}}\,dx + 2\int \frac{1}{\sqrt{1-(5x)^2}}\,dx$$

Step 3

Now we'll need a substitution for each integral. Here are the substitutions we'll need for each integral.

$$u = 1-25x^2 \qquad v = 5x \qquad \left(\text{so } v^2 = 25x^2\right)$$

Step 4

Here is the differential work for the substitution.

$$du = -50x\,dx \quad \to \quad x\,dx = -\frac{1}{50}du \qquad dv = 5dx \quad \to \quad dx = \frac{1}{5}dv$$

Now, doing the substitutions and evaluating the integrals gives,

$$\int \frac{7x+2}{\sqrt{1-25x^2}}\,dx = -\frac{7}{50}\int u^{-\frac{1}{2}}\,du + \frac{2}{5}\int \frac{1}{\sqrt{1-v^2}}\,dv = -\frac{7}{25}u^{\frac{1}{2}} + \frac{2}{5}\sin^{-1}(v) + c$$

$$= \boxed{-\frac{7}{25}\left(1-25x^2\right)^{\frac{1}{2}} + \frac{2}{5}\sin^{-1}(5x) + c}$$

Do not forget to go back to the original variable after evaluating the integral!

13. Evaluate $\displaystyle\int z^7 \left(8+3z^4\right)^8 dz$.

Hint

Use the "obvious" substitution and don't forget that the substitution can be used more than once and in different ways.

Step 1

Okay, the "obvious" substitution here is probably,

$$u = 8+3z^4 \qquad \to \qquad du = 12z^3 dz \qquad \to \qquad z^3 dz = \frac{1}{12}du$$

however, that doesn't look like it might work because of the z^7.

Step 2

Let's do a quick rewrite of the integrand.

$$\int z^7 \left(8+3z^4\right)^8 dz = \int z^4 z^3 \left(8+3z^4\right)^8 dz = \int z^4 \left(8+3z^4\right)^8 z^3 dz$$

Step 3

Now, notice that we can convert all of the z's in the integrand except apparently for the z^4 that is in the front. However, notice from the substitution that we can solve for z^4 to get,

$$z^4 = \frac{1}{3}(u-8)$$

Step 4

With this we can now do the substitution and evaluate the integral.

$$\int z^7 \left(8+3z^4\right)^8 dz = \frac{1}{12}\int \frac{1}{3}\left(u-8\right)u^8 du = \frac{1}{36}\int u^9 - 8u^8 du = \frac{1}{36}\left(\frac{1}{10}u^{10} - \frac{8}{9}u^9\right) + c$$

$$= \boxed{\frac{1}{36}\left(\frac{1}{10}\left(8+3z^4\right)^{10} - \frac{8}{9}\left(8+3z^4\right)^9\right) + c}$$

5.5 Area Problem

1. Estimate the area of the region between $f(x) = x^3 - 2x^2 + 4$ the x-axis on $[1, 4]$ using $n = 6$ and using,

   (a) the right end points of the subintervals for the height of the rectangles,

   (b) the left end points of the subintervals for the height of the rectangles and,

   (c) the midpoints of the subintervals for the height of the rectangles.

Solutions

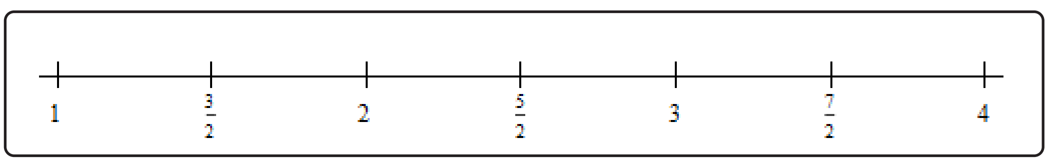
(a) The right end points of the subintervals for the height of the rectangles.

Solution

The widths of each of the subintervals for this problem are,

$$\Delta x = \frac{4-1}{6} = \frac{1}{2}$$

We don't need to actually graph the function to do this problem. It would probably help to have a number line showing subintervals however. Here is that number line.



In this case we're going to be using right end points of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the x-axis is then approximately,

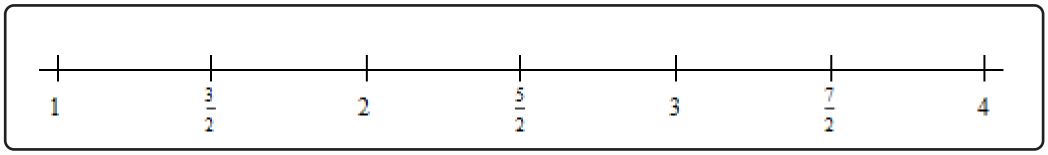
$$\begin{aligned} \text{Area} &\approx \frac{1}{2}f\left(\frac{3}{2}\right) + \frac{1}{2}f(2) + \frac{1}{2}f\left(\frac{5}{2}\right) + \frac{1}{2}f(3) + \frac{1}{2}f\left(\frac{7}{2}\right) + \frac{1}{2}f(4) \\ &= \frac{1}{2}\left(\frac{23}{8}\right) + \frac{1}{2}(4) + \frac{1}{2}\left(\frac{57}{8}\right) + \frac{1}{2}(13) + \frac{1}{2}\left(\frac{179}{8}\right) + \frac{1}{2}(36) \\ &= \boxed{\frac{683}{16} = 42.6875} \end{aligned}$$

(b) The left end points of the subintervals for the height of the rectangles.

Solution

As we found in the previous part the widths of each of the subintervals are $\Delta x = \frac{1}{2}$.

Here is a copy of the number line showing the subintervals to help with the problem.



In this case we're going to be using left end points of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the x-axis is then approximately,

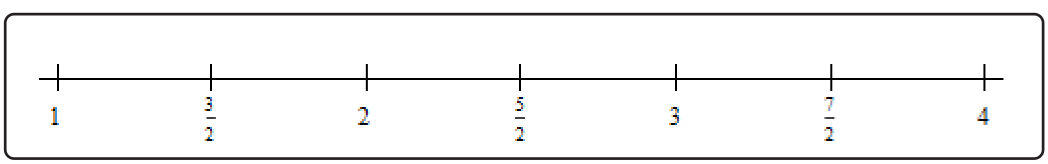
$$\begin{aligned}\text{Area} &\approx \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right) + \frac{1}{2}f(2) + \frac{1}{2}f\left(\frac{5}{2}\right) + \frac{1}{2}f(3) + \frac{1}{2}f\left(\frac{7}{2}\right)\\ &= \frac{1}{2}(3) + \frac{1}{2}\left(\frac{23}{8}\right) + \frac{1}{2}(4) + \frac{1}{2}\left(\frac{57}{8}\right) + \frac{1}{2}(13) + \frac{1}{2}\left(\frac{179}{8}\right)\\ &= \boxed{\frac{419}{16} = 26.1875}\end{aligned}$$

(c) The midpoints of the subintervals for the height of the rectangles.

Solution

As we found in the first part the widths of each of the subintervals are $\Delta x = \frac{1}{2}$.

Here is a copy of the number line showing the subintervals to help with the problem.



In this case we're going to be using midpoints of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the x-axis is then approximately,

$$\begin{aligned}\text{Area} &\approx \frac{1}{2}f\left(\frac{5}{4}\right)+\frac{1}{2}f\left(\frac{7}{4}\right)+\frac{1}{2}f\left(\frac{9}{4}\right)+\frac{1}{2}f\left(\frac{11}{4}\right)+\frac{1}{2}f\left(\frac{13}{4}\right)+\frac{1}{2}f\left(\frac{15}{4}\right)\\ &=\frac{1}{2}\left(\frac{181}{64}\right)+\frac{1}{2}\left(\frac{207}{64}\right)+\frac{1}{2}\left(\frac{337}{64}\right)+\frac{1}{2}\left(\frac{619}{64}\right)+\frac{1}{2}\left(\frac{1101}{64}\right)+\frac{1}{2}\left(\frac{1831}{64}\right)\\ &=\boxed{\frac{1069}{32}=33.40625}\end{aligned}$$

---

2. Estimate the area of the region between $g\left(x\right)=4-\sqrt{x^2+2}$ the x-axis on $[-1,3]$ using $n=6$ and using,

   (a) the right end points of the subintervals for the height of the rectangles,

   (b) the left end points of the subintervals for the height of the rectangles and,

   (c) the midpoints of the subintervals for the height of the rectangles.

Solutions

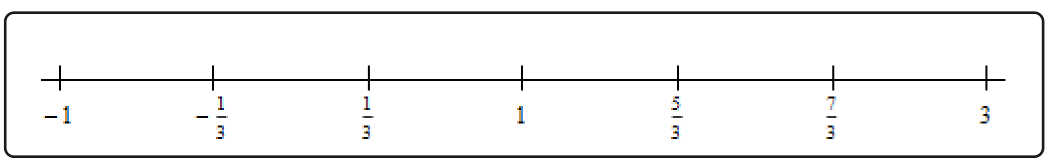
(a) The right end points of the subintervals for the height of the rectangles.

Solution

The widths of each of the subintervals for this problem are,

$$\Delta x=\frac{3-(-1)}{6}=\frac{2}{3}$$

We don't need to actually graph the function to do this problem. It would probably help to have a number line showing subintervals however. Here is that number line.



In this case we're going to be using right end points of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the x-axis is then approximately,

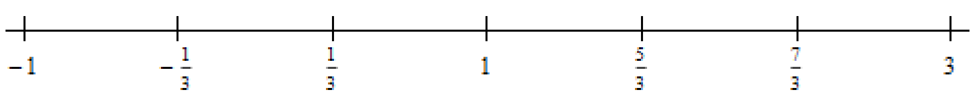
$$\begin{aligned}\text{Area} &\approx \frac{2}{3}f\left(-\frac{1}{3}\right)+\frac{2}{3}f\left(\frac{1}{3}\right)+\frac{2}{3}f\left(1\right)+\frac{2}{3}f\left(\frac{5}{3}\right)+\frac{2}{3}f\left(\frac{7}{3}\right)+\frac{2}{3}f\left(3\right)\\ &=\frac{2}{3}\left(4-\frac{\sqrt{19}}{3}\right)+\frac{2}{3}\left(4-\frac{\sqrt{19}}{3}\right)+\frac{2}{3}\left(4-\sqrt{3}\right)\frac{2}{3}\left(4-\frac{\sqrt{43}}{3}\right)+\\ &\qquad +\frac{2}{3}\left(4-\frac{\sqrt{67}}{3}\right)+\frac{2}{3}\left(4-\sqrt{11}\right)\\ &=\boxed{7.420752}\end{aligned}$$

(b) The left end points of the subintervals for the height of the rectangles.

Solution

As we found in the previous part the widths of each of the subintervals are $\Delta x=\frac{2}{3}$.

Here is a copy of the number line showing the subintervals to help with the problem.



In this case we're going to be using left end points of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the x-axis is then approximately,

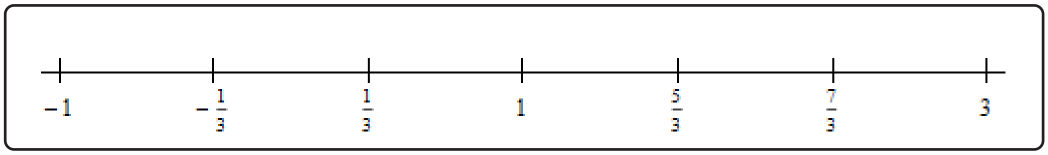
$$\begin{aligned}\text{Area} &\approx \frac{2}{3}f\left(-1\right)+\frac{2}{3}f\left(-\frac{1}{3}\right)+\frac{2}{3}f\left(\frac{1}{3}\right)+\frac{2}{3}f\left(1\right)+\frac{2}{3}f\left(\frac{5}{3}\right)+\frac{2}{3}f\left(\frac{7}{3}\right)\\ &=\frac{2}{3}\left(4-\sqrt{3}\right)+\frac{2}{3}\left(4-\frac{\sqrt{19}}{3}\right)+\frac{2}{3}\left(4-\frac{\sqrt{19}}{3}\right)+\frac{2}{3}\left(4-\sqrt{3}\right)\\ &\qquad +\frac{2}{3}\left(4-\frac{\sqrt{43}}{3}\right)+\frac{2}{3}\left(4-\frac{\sqrt{67}}{3}\right)\\ &=\boxed{8.477135}\end{aligned}$$

(c) The midpoints of the subintervals for the height of the rectangles.

Solution

As we found in the first part the widths of each of the subintervals are $\Delta x = \frac{2}{3}$.

Here is a copy of the number line showing the subintervals to help with the problem.



In this case we're going to be using midpoints of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the x-axis is then approximately,

$$\begin{aligned}\text{Area} &\approx \frac{2}{3}f\left(-\frac{2}{3}\right) + \frac{2}{3}f(0) + \frac{2}{3}f\left(\frac{2}{3}\right) + \frac{2}{3}f\left(\frac{4}{3}\right) + \frac{2}{3}f(2) + \frac{2}{3}f\left(\frac{8}{3}\right)\\ &= \frac{2}{3}\left(4 - \frac{\sqrt{22}}{3}\right) + \frac{2}{3}\left(4 - \sqrt{2}\right) + \frac{2}{3}\left(4 - \frac{\sqrt{22}}{3}\right) + \frac{2}{3}\left(4 - \frac{\sqrt{34}}{3}\right)\\ &\qquad + \frac{2}{3}\left(4 - \sqrt{6}\right) + \frac{2}{3}\left(4 - \frac{\sqrt{82}}{3}\right)\\ &= \boxed{8.031494}\end{aligned}$$

---

3. Estimate the area of the region between $h(x) = -x\cos\left(\frac{x}{3}\right)$ the x-axis on $[0, 3]$ using $n = 6$ and using,

(a) the right end points of the subintervals for the height of the rectangles,

(b) the left end points of the subintervals for the height of the rectangles and,

(c) the midpoints of the subintervals for the height of the rectangles.

Solutions

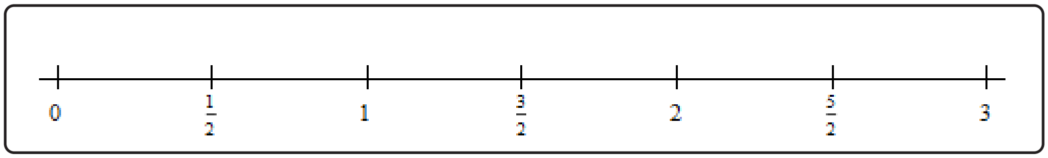
(a) The right end points of the subintervals for the height of the rectangles.

Solution

The widths of each of the subintervals for this problem are,

$$\Delta x = \frac{3-0}{6} = \frac{1}{2}$$

We don't need to actually graph the function to do this problem. It would probably help to have a number line showing subintervals however. Here is that number line.



In this case we're going to be using right end points of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the x-axis is then approximately,

$$\begin{aligned}\text{Area} &\approx \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f\left(1\right) + \frac{1}{2}f\left(\frac{3}{2}\right) + \frac{1}{2}f\left(2\right) + \frac{1}{2}f\left(\frac{5}{2}\right) + \frac{1}{2}f\left(3\right)\\ &= \frac{1}{2}\left(-\frac{1}{2}\cos\left(\frac{1}{6}\right)\right) + \frac{1}{2}\left(-\cos\left(\frac{1}{3}\right)\right) + \frac{1}{2}\left(-\frac{3}{2}\cos\left(\frac{1}{2}\right)\right)\\ &\qquad + \frac{1}{2}\left(-2\cos\left(\frac{2}{3}\right)\right) + \frac{1}{2}\left(-\frac{5}{2}\cos\left(\frac{5}{6}\right)\right) + \frac{1}{2}\left(-3\cos\left(1\right)\right)\\ &= \boxed{-3.814057}\end{aligned}$$

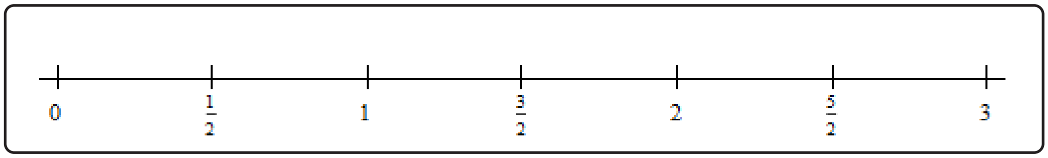
Do not get excited about the negative area here. As we discussed in this section this just means that the graph, in this case, is below the x-axis as you could verify if you'd like to.

(b) The left end points of the subintervals for the height of the rectangles.

Solution

As we found in the previous part the widths of each of the subintervals are $\Delta x = \frac{1}{2}$.

Here is a copy of the number line showing the subintervals to help with the problem.



In this case we're going to be using left end points of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the x-axis is then approximately,

$$\begin{aligned}
\text{Area} &\approx \frac{1}{2}f(0) + \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right) + \frac{1}{2}f(2) + \frac{1}{2}f\left(\frac{5}{2}\right) \\
&= +\frac{1}{2}(0) + \frac{1}{2}\left(-\frac{1}{2}\cos\left(\frac{1}{6}\right)\right) + \frac{1}{2}\left(-\cos\left(\frac{1}{3}\right)\right) + \frac{1}{2}\left(-\frac{3}{2}\cos\left(\frac{1}{2}\right)\right) \\
&\qquad + \frac{1}{2}\left(-2\cos\left(\frac{2}{3}\right)\right) + \frac{1}{2}\left(-\frac{5}{2}\cos\left(\frac{5}{6}\right)\right) \\
&= \boxed{-3.003604}
\end{aligned}$$

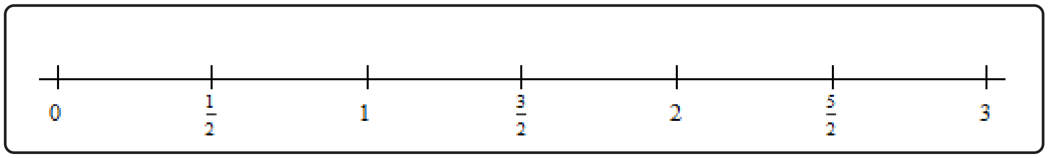
Do not get excited about the negative area here. As we discussed in this section this just means that the graph, in this case, is below the x-axis as you could verify if you'd like to.

(c) The midpoints of the subintervals for the height of the rectangles.

Solution

As we found in the first part the widths of each of the subintervals are $\Delta x = \frac{1}{2}$.

Here is a copy of the number line showing the subintervals to help with the problem.



In this case we're going to be using midpoints of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the x-axis is then approximately,

$$\begin{aligned}\text{Area} &\approx \frac{1}{2}f\left(\frac{1}{4}\right)+\frac{1}{2}f\left(\frac{3}{4}\right)+\frac{1}{2}f\left(\frac{5}{4}\right)+\frac{1}{2}f\left(\frac{7}{4}\right)+\frac{1}{2}f\left(\frac{9}{4}\right)+\frac{1}{2}f\left(\frac{11}{4}\right)\\ &=\frac{1}{2}\left(-\frac{1}{4}\cos\left(\frac{1}{12}\right)\right)+\frac{1}{2}\left(-\frac{3}{4}\cos\left(\frac{1}{4}\right)\right)+\frac{1}{2}\left(-\frac{5}{4}\cos\left(\frac{5}{12}\right)\right)\\ &\qquad+\frac{1}{2}\left(-\frac{7}{4}\cos\left(\frac{7}{12}\right)\right)+\frac{1}{2}\left(-\frac{9}{4}\cos\left(\frac{3}{4}\right)\right)+\frac{1}{2}\left(-\frac{11}{4}\cos\left(\frac{11}{12}\right)\right)\\ &=\boxed{-3.449532}\end{aligned}$$

Do not get excited about the negative area here. As we discussed in this section this just means that the graph, in this case, is below the x-axis as you could verify if you'd like to.

---

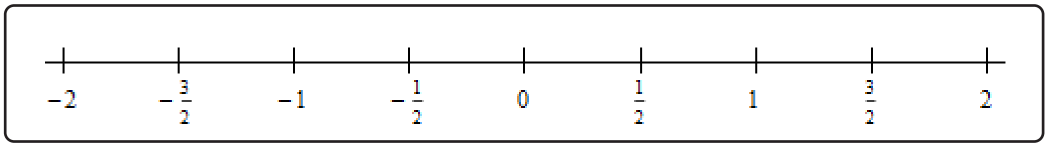
4. Estimate the net area between $f(x)=8x^2-x^5-12$ and the x-axis on $[-2,2]$ using $n=8$ and the midpoints of the subintervals for the height of the rectangles. Without looking at a graph of the function on the interval does it appear that more of the area is above or below the x-axis?

Step 1

First let's estimate the area between the function and the x-axis on the interval. The widths of each of the subintervals for this problem are,

$$\Delta x=\frac{2-(-2)}{8}=\frac{1}{2}$$

We don't need to actually graph the function to do this problem. It would probably help to have a number line showing subintervals however. Here is that number line.



Now, we'll be using midpoints of each of these subintervals to determine the height of each of the rectangles.

The area between the function and the x-axis is then approximately,

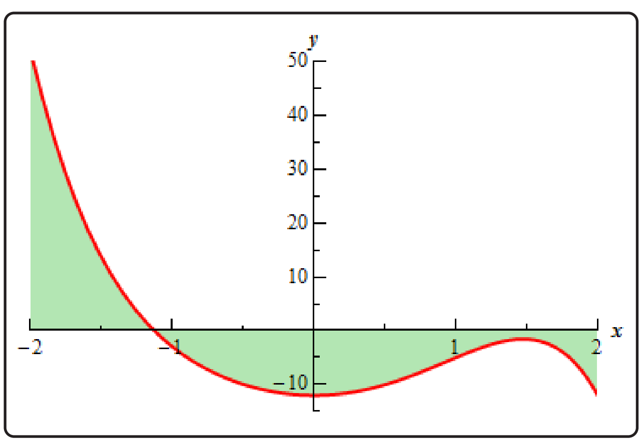
$$\begin{aligned}\text{Area} &\approx \frac{1}{2}f\left(-\frac{7}{4}\right)+\frac{1}{2}f\left(-\frac{5}{4}\right)+\frac{1}{2}f\left(-\frac{3}{4}\right)+\frac{1}{2}f\left(-\frac{1}{4}\right)+\frac{1}{2}f\left(\frac{1}{4}\right)+\frac{1}{2}f\left(\frac{3}{4}\right)\\&\qquad+\frac{1}{2}f\left(\frac{5}{4}\right)+\frac{1}{2}f\left(\frac{7}{4}\right)\\&=\boxed{-6}\end{aligned}$$

We'll leave it to you to check all the function evaluations. They get a little messy, but after all the arithmetic is done we get a net area of -6.

Step 2

Now, as we (hopefully) recall from the discussion in this section area above the x-axis is positive and area below the x-axis is negative. In this case we have estimated that the net area is -6 and so, assuming that our estimate is accurate, it looks like we should have more area is below the x-axis as above it.

For reference purposes here is the graph of the function with the area shaded in and as we can see it does appear that there is slightly more area below as above the x-axis.



5.6 Definition of the Definite Integral

1. Use the definition of the definite integral to evaluate the integral. Use the right end point of each interval for x_i^*.

$$\int_1^4 2x + 3\,dx$$

Step 1

The width of each subinterval will be,

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

The subintervals for the interval $[1, 4]$ are then,

$$\left[1, 1+\frac{3}{n}\right], \left[1+\frac{3}{n}, 1+\frac{6}{n}\right], \left[1+\frac{6}{n}, 1+\frac{9}{n}\right], \ldots, \left[1+\frac{3(i-1)}{n}, 1+\frac{3i}{n}\right], \ldots, \left[1+\frac{3(n-1)}{n}, 4\right]$$

From this it looks like the right end point, and hence x_i^*, of the general subinterval is,

$$x_i^* = 1 + \frac{3i}{n}$$

Step 2

The summation in the definition of the definite integral is then,

$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \sum_{i=1}^{n}\left[2\left(1+\frac{3i}{n}\right)+3\right]\left[\frac{3}{n}\right] = \sum_{i=1}^{n}\left[\frac{15}{n}+\frac{18i}{n^2}\right]$$

Step 3

Now we need to use the formulas from the Summation Notation section in the Extras chapter to "evaluate" the summation.

$$\begin{aligned}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x &= \sum_{i=1}^{n}\frac{15}{n} + \sum_{i=1}^{n}\frac{18i}{n^2} = \frac{1}{n}\sum_{i=1}^{n}15 + \frac{18}{n^2}\sum_{i=1}^{n} i \\ &= \frac{1}{n}(15n) + \frac{18}{n^2}\left(\frac{n(n+1)}{2}\right) = 15 + \frac{9n+9}{n}\end{aligned}$$

Step 4

Finally, we can use the definition of the definite integral to determine the value of the integral.

$$\int_1^4 2x + 3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \lim_{n\to\infty}\left[15 + \frac{9n+9}{n}\right] = \lim_{n\to\infty}\left[24 + \frac{9}{n}\right] = \boxed{24}$$

2. Use the definition of the definite integral to evaluate the integral. Use the right end point of each interval for x_i^*.

$$\int_0^1 6x\left(x-1\right)\,dx$$

Step 1

The width of each subinterval will be,

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

The subintervals for the interval $[0,1]$ are then,

$$\left[0,\frac{1}{n}\right],\left[\frac{1}{n},\frac{2}{n}\right],\left[\frac{2}{n},\frac{3}{n}\right],\ldots,\left[\frac{i-1}{n},\frac{i}{n}\right],\ldots,\left[\frac{n-1}{n},1\right]$$

From this it looks like the right end point, and hence x_i^*, of the general subinterval is,

$$x_i^* = \frac{i}{n}$$

Step 2

The summation in the definition of the definite integral is then,

$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \sum_{i=1}^{n}\left[\left(\frac{6i}{n}\right)\left(\frac{i}{n}-1\right)\right]\left[\frac{1}{n}\right] = \sum_{i=1}^{n}\left[\frac{6i^2}{n^3} - \frac{6i}{n^2}\right]$$

Step 3

Now we need to use the formulas from the Summation Notation section in the Extras chapter to "evaluate" the summation.

$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \sum_{i=1}^{n}\left[\frac{6i^2}{n^3}\right] - \sum_{i=1}^{n}\left[\frac{6i}{n^2}\right] = \frac{6}{n^3}\sum_{i=1}^{n} i^2 - \frac{6}{n^2}\sum_{i=1}^{n} i$$
$$= \frac{6}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{6}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{2n^2+3n+1}{n^2} - \frac{3n+3}{n}$$

Step 4

Finally, we can use the definition of the definite integral to determine the value of the integral.

$$\int_0^1 6x(x-1)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \lim_{n\to\infty}\left[\frac{2n^2+3n+1}{n^2} - \frac{3n+3}{n}\right]$$
$$= 2 - 3 = \boxed{-1}$$

3. Evaluate : $\displaystyle\int_4^4 \frac{\cos\left(\mathbf{e}^{3x}+x^2\right)}{x^4+1}\,dx$

Solution

There really isn't much to this problem other than use Property 2 from the notes on this section.

$$\int_4^4 \frac{\cos\left(\mathbf{e}^{3x}+x^2\right)}{x^4+1}\,dx = \boxed{0}$$

4. Determine the value of $\displaystyle\int_{11}^{6} 9f(x)\,dx$ given that $\displaystyle\int_{6}^{11} f(x)\,dx = -7$.

Solution

There really isn't much to this problem other than use the properties from the notes of this section until we get the given interval at which point we use the given value.

$$\begin{aligned}\int_{11}^{6} 9f(x)\,dx &= 9\int_{11}^{6} f(x)\,dx && \text{Property 3}\\ &= -9\int_{6}^{11} f(x)\,dx && \text{Property 1}\\ &= -9(-7) = \boxed{63}\end{aligned}$$

5. Determine the value of $\int_{6}^{11} 6g(x) - 10f(x)\,dx$ given that $\int_{6}^{11} f(x)\,dx = -7$ and $\int_{6}^{11} g(x)\,dx = 24$.

Solution

There really isn't much to this problem other than use the properties from the notes of this section until we get the given intervals at which point we use the given values.

$$\begin{aligned}\int_{6}^{11} 6g(x) - 10f(x)\,dx &= \int_{6}^{11} 6g(x)\,dx - \int_{6}^{11} 10f(x)\,dx && \text{Property 4}\\ &= 6\int_{6}^{11} g(x)\,dx - 10\int_{6}^{11} f(x)\,dx && \text{Property 3}\\ &= 6(24) - 10(-7) = \boxed{214}\end{aligned}$$

6. Determine the value of $\int_{2}^{9} f(x)\,dx$ given that $\int_{5}^{2} f(x)\,dx = 3$ and $\int_{5}^{9} f(x)\,dx = 8$.

Step 1

First we need to use Property 5 from the notes of this section to break up the integral into two integrals that use the same limits as the integrals given in the problem statement.

Note that we won't worry about whether the limits are in correct place at this point.

$$\int_{2}^{9} f(x)\,dx = \int_{2}^{5} f(x)\,dx + \int_{5}^{9} f(x)\,dx$$

Step 2

Finally, all we need to do is use Property 1 from the notes of this section to interchange the limits on the first integral so they match up with the limits on the given integral. We can then use the given values to determine the value of the integral.

$$\int_2^9 f(x)\,dx = -\int_5^2 f(x)\,dx + \int_5^9 f(x)\,dx = -(3) + 8 = \boxed{5}$$

---

7. Determine the value of $\displaystyle\int_{-4}^{20} f(x)\,dx$ given that $\displaystyle\int_{-4}^{0} f(x)\,dx = -2$, $\displaystyle\int_{31}^{0} f(x)\,dx = 19$ and $\displaystyle\int_{20}^{31} f(x)\,dx = -21$.

Step 1

First we need to use Property 5 from the notes of this section to break up the integral into three integrals that use the same limits as the integrals given in the problem statement.

Note that we won't worry about whether the limits are in correct place at this point.

$$\int_{-4}^{20} f(x)\,dx = \int_{-4}^{0} f(x)\,dx + \int_{0}^{31} f(x)\,dx + \int_{31}^{20} f(x)\,dx$$

Step 2

Finally, all we need to do is use Property 1 from the notes of this section to interchange the limits on the second and third integrals so they match up with the limits on the given integral. We can then use the given values to determine the value of the integral.

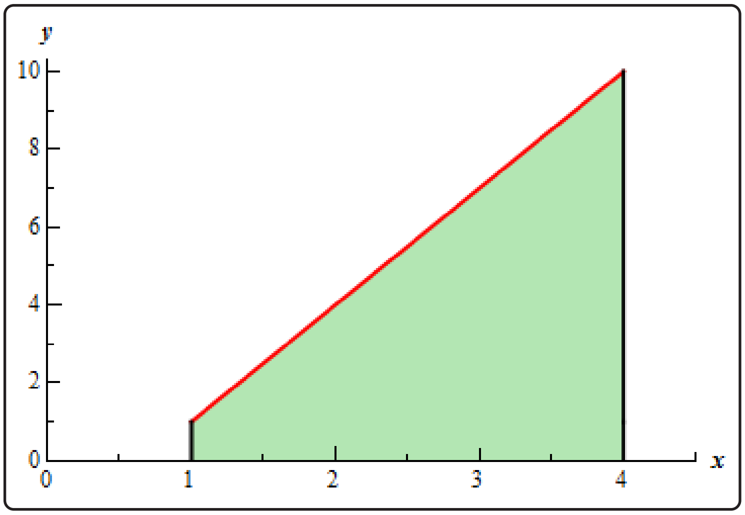
$$\int_{-4}^{20} f(x)\,dx = \int_{-4}^{0} f(x)\,dx - \int_{31}^{0} f(x)\,dx - \int_{20}^{31} f(x)\,dx = -2 - (19) - (-21) = \boxed{0}$$

---

8. For $\displaystyle\int_1^4 3x - 2\,dx$ sketch the graph of the integrand and use the area interpretation of the definite integral to determine the value of the integral.

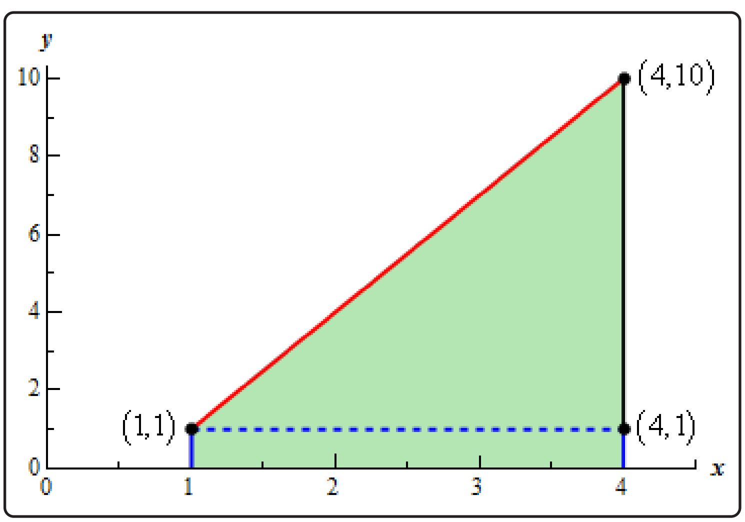
Step 1

Here is the graph of the integrand, $f(x) = 3x - 2$, on the interval $[1, 4]$.



Step 2

Now, we know that the integral is simply the area between the line and the x-axis and so we should be able to use basic area formulas to help us determine the value of the integral. Here is a "modified" graph that will help with this.



From this sketch we can see that we can think of this area as a rectangle with width 3 and height 1 and a triangle with base 3 and height 9. The value of the integral will then be the sum of the areas of the rectangle and the triangle.

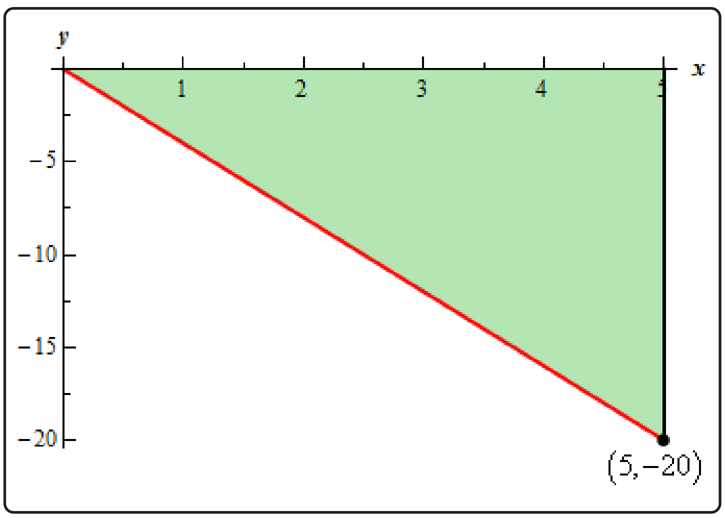
Here is the value of the integral,

$$\int_1^4 3x - 2\,dx = (3)(1) + \frac{1}{2}(3)(9) = \boxed{\frac{33}{2}}$$

9. For $\displaystyle\int_0^5 -4x\,dx$ sketch the graph of the integrand and use the area interpretation of the definite integral to determine the value of the integral.

Step 1

Here is the graph of the integrand, $f(x) = -4x$ on the interval $[0, 5]$.



Step 2

Now, we know that the integral is simply the area between the line and the x-axis and so we should be able to use basic area formulas to help us determine the value of the integral.

In this case we can see the area is clearly a triangle with base 5 and height 20. However, we need to be a little careful here and recall that area that is below the x-axis is considered to be negative area and so we'll need to keep that in mind when we do the area computation.

Here is the value of the integral,

$$\int_0^5 -4x\,dx = -\frac{1}{2}(5)(20) = \boxed{-50}$$

10. Differentiate the following integral with respect to x.

$$\int_4^x 9\cos^2\left(t^2 - 6t + 1\right)\,dt$$

Solution

This is nothing more than a quick application of the Fundamental Theorem of Calculus, Part I.

The derivative is,

$$\frac{d}{dx}\left[\int_4^x 9\cos^2\left(t^2 - 6t + 1\right)\,dt\right] = \boxed{9\cos^2\left(x^2 - 6x + 1\right)}$$

11. Differentiate the following integral with respect to x.

$$\int_7^{\sin(6x)} \sqrt{t^2 + 4}dt$$

Solution

This is nothing more than a quick application of the Fundamental Theorem of Calculus, Part I.

Note however, that because the upper limit is not just x we'll need to use the Chain Rule, with the "inner function" as $\sin(6x)$.

The derivative is,

$$\frac{d}{dx}\left[\int_7^{\sin(6x)} \sqrt{t^2 + 4}dt\right] = \boxed{6\cos(6x)\sqrt{\sin^2(6x) + 4}}$$

12. Differentiate the following integral with respect to x.

$$\int_{3x^2}^{-1} \frac{\mathbf{e}^t - 1}{t} dt$$

Solution

This is nothing more than a quick application of the Fundamental Theorem of Calculus, Part I.

Note however, that we'll need to interchange the limits to get the lower limit to a number and the x's in the upper limit as required by the theorem. Also, note that because the upper limit is not just x we'll need to use the Chain Rule, with the "inner function" as $3x^2$.

The derivative is,

$$\frac{d}{dx}\left[\int_{3x^2}^{-1} \frac{\mathbf{e}^t - 1}{t} dt\right] = \frac{d}{dx}\left[-\int_{-1}^{3x^2} \frac{\mathbf{e}^t - 1}{t} dt\right] = -(6x)\frac{\mathbf{e}^{3x^2} - 1}{3x^2} = \boxed{\frac{2 - 2\mathbf{e}^{3x^2}}{x}}$$

5.7 Computing Definite Integrals

1. Evaluate each of the following integrals.

(a) $\displaystyle\int \cos(x) - \frac{3}{x^5}\,dx$

(b) $\displaystyle\int_{-3}^{4} \cos(x) - \frac{3}{x^5}\,dx$

(c) $\displaystyle\int_{1}^{4} \cos(x) - \frac{3}{x^5}\,dx$

Solutions

(a) $\displaystyle\int \cos(x) - \frac{3}{x^5}\,dx$

Solution

This is just an indefinite integral and by this point we should be comfortable doing them so here is the answer to this part.

$$\int \cos(x) - \frac{3}{x^5}\,dx = \int \cos(x) - 3x^{-5}\,dx = \sin(x) + \frac{3}{4}x^{-4} + c$$
$$= \boxed{\sin(x) + \frac{3}{4x^4} + c}$$

Don't forget to add on the $+c$ since we are doing an indefinite integral!

(b) $\displaystyle\int_{-3}^{4} \cos(x) - \frac{3}{x^5}\,dx$

Solution

Recall that in order to do a definite integral the integrand (*i.e.* the function we are integrating) must be continuous on the interval over which we are integrating, $[-3,4]$ in this case.

We can clearly see that the second term will have division by zero at $x = 0$ and $x = 0$ is in the interval over which we are integrating and so this function is not continuous on the interval over which we are integrating.

Therefore, this integral cannot be done.

(c) $\displaystyle\int_1^4 \cos(x) - \frac{3}{x^5}\,dx$

Solution

Now, the function still has a division by zero problem in the second term at $x = 0$. However, unlike the previous part $x = 0$ does not fall in the interval over which we are integrating, $[1, 4]$ in this case.

This integral can therefore be done. Here is the work for this integral.

$$\begin{aligned}\int_1^4 \cos(x) - \frac{3}{x^5}\,dx = \int_1^4 \cos(x) - 3x^{-5}\,dx &= \left(\sin(x) + \frac{3}{4x^4}\right)\Bigg|_1^4 \\ &= \sin(4) + \frac{3}{4(4^4)} - \left(\sin(1) - \frac{3}{4(1^4)}\right) \\ &= \sin(4) + \frac{3}{1024} - \left(\sin(1) - \frac{3}{4}\right) \\ &= \boxed{\sin(4) - \sin(1) - \frac{765}{1024}}\end{aligned}$$

2. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_1^6 12x^3 - 9x^2 + 2\,dx$$

Step 1

First we need to integrate the function.

$$\int_1^6 12x^3 - 9x^2 + 2\,dx = biggl.\left(3x^4 - 3x^3 + 2x\right)\Big|_1^6$$

Recall that we don't need to add the $+c$ in the definite integral case as it will just cancel in the next step.

Step 2

The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_1^6 12x^3 - 9x^2 + 2\,dx = 3252 - 2 = \boxed{3250}$$

---

3. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-2}^{1} 5z^2 - 7z + 3\,dz$$

Step 1

First we need to integrate the function.

$$\int_{-2}^{1} 5z^2 - 7z + 3\,dz = \left(\frac{5}{3}z^3 - \frac{7}{2}z^2 + 3z\right)\bigg|_{-2}^{1}$$

Recall that we don't need to add the $+c$ in the definite integral case as it will just cancel in the next step.

Step 2

The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{-2}^{1} 5z^2 - 7z + 3\,dz = \frac{7}{6} - \left(-\frac{100}{3}\right) = \boxed{\frac{69}{2}}$$

---

4. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{3}^{0} 15w^4 - 13w^2 + w\,dw$$

Step 1

First, do not get excited about the fact that the lower limit of integration is a larger number than the upper limit of integration. The problem works in exactly the same way.

So, we need to integrate the function.

$$\int_{3}^{0} 15w^4 - 13w^2 + w\,dw = \left(3w^5 - \frac{13}{3}w^3 + \frac{1}{2}w^2\right)\bigg|_{3}^{0}$$

Recall that we don't need to add the $+c$ in the definite integral case as it will just cancel in the next step.

Step 2

The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{3}^{0} 15w^4 - 13w^2 + w\,dw = 0 - \frac{1233}{2} = \boxed{-\frac{1233}{2}}$$

---

5. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{1}^{4} \frac{8}{\sqrt{t}} - 12\sqrt{t^3}\,dt$$

Step 1

First we need to integrate the function.

$$\int_{1}^{4} \frac{8}{\sqrt{t}} - 12\sqrt{t^3}\,dt = \int_{1}^{4} 8t^{-\frac{1}{2}} - 12t^{\frac{3}{2}}\,dt = \left(16t^{\frac{1}{2}} - \frac{24}{5}t^{\frac{5}{2}}\right)\bigg|_{1}^{4}$$

Recall that we don't need to add the $+c$ in the definite integral case as it will just cancel in the next step.

Step 2

The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_1^4 \frac{8}{\sqrt{t}} - 12\sqrt{t^3}\,dt = -\frac{608}{5} - \frac{56}{5} = \boxed{-\frac{664}{5}}$$

---

6. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_1^2 \frac{1}{7z} + \frac{\sqrt[3]{z^2}}{4} - \frac{1}{2z^3}\,dz$$

Step 1

First we need to integrate the function.

$$\int_1^2 \frac{1}{7z} + \frac{\sqrt[3]{z^2}}{4} - \frac{1}{2z^3}\,dz = \int_1^2 \frac{1}{7}\frac{1}{z} + \frac{1}{4}z^{\frac{2}{3}} - \frac{1}{2}z^{-3}\,dz = \left(\frac{1}{7}\ln|z| + \frac{3}{20}z^{\frac{5}{3}} + \frac{1}{4}z^{-2}\right)\Big|_1^2$$

Recall that we don't need to add the $+c$ in the definite integral case as it will just cancel in the next step.

Step 2

The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_1^2 \frac{1}{7z} + \frac{\sqrt[3]{z^2}}{4} - \frac{1}{2z^3}\,dz = \left(\frac{1}{7}\ln(2) + \frac{3}{20}\left(2^{\frac{5}{3}}\right) + \frac{1}{16}\right) - \left(\frac{1}{7}\ln(1) + \frac{2}{5}\right)$$
$$= \boxed{\frac{1}{7}\ln(2) + \frac{3}{20}\left(2^{\frac{5}{3}}\right) - \frac{27}{80}}$$

Don't forget that $\ln(1) = 0$! Also, don't get excited about "messy" answers like this. They happen on occasion.

7. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-2}^{4} x^6 - x^4 + \frac{1}{x^2}\,dx$$

Solution

In this case note that the third term will have division by zero at $x = 0$ and this is in the interval we are integrating over, $[-2, 4]$ and hence is not continuous on this interval.

Therefore, this integral cannot be done.

8. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-4}^{-1} x^2\left(3 - 4x\right)\,dx$$

Step 1

In this case we'll first need to multiply out the integrand before we actually do the integration. Doing that integrating the function gives,

$$\int_{-4}^{-1} x^2\left(3 - 4x\right)\,dx = \int_{-4}^{-1} 3x^2 - 4x^3\,dx = \left(x^3 - x^4\right)\Big|_{-4}^{-1}$$

Recall that we don't need to add the $+c$in the definite integral case as it will just cancel in the next step.

Step 2

The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{-4}^{-1} x^2 (3 - 4x)\, dx = -2 - (-320) = \boxed{318}$$

---

9. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{2}^{1} \frac{2y^3 - 6y^2}{y^2}\, dy$$

Step 1

In this case we'll first need to simplify the integrand to remove the quotient before we actually do the integration. Doing that integrating the function gives,

$$\int_{2}^{1} \frac{2y^3 - 6y^2}{y^2}\, dy = \int_{2}^{1} 2y - 6\, dy = \left(y^2 - 6y\right)\Big|_{2}^{1}$$

Do not get excited about the fact that the lower limit of integration is larger than the upper limit of integration. This will happen on occasion and the integral works in exactly the same manner as we've been doing them.

Also, recall that we don't need to add the $+c$ in the definite integral case as it will just cancel in the next step.

Step 2

The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_{2}^{1} \frac{2y^3 - 6y^2}{y^2}\, dy = -5 - (-8) = \boxed{3}$$

10. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_0^{\frac{\pi}{2}} 7\sin(t) - 2\cos(t)\, dt$$

Step 1

First we need to integrate the function.

$$\int_0^{\frac{\pi}{2}} 7\sin(t) - 2\cos(t)\, dt = \left(-7\cos(t) - 2\sin(t)\right)\Big|_0^{\frac{\pi}{2}}$$

Recall that we don't need to add the $+c$ in the definite integral case as it will just cancel in the next step.

Step 2

The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_0^{\frac{\pi}{2}} 7\sin(t) - 2\cos(t)\, dt = -2 - (-7) = \boxed{5}$$

11. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_0^{\pi} \sec(z)\tan(z) - 1\, dz$$

Solution

Be careful with this integral. Recall that,

$$\sec(z) = \frac{1}{\cos(z)} \qquad \tan(z) = \frac{\sin(z)}{\cos(z)}$$

Also recall that $\cos\left(\frac{\pi}{2}\right) = 0$ and that $x = \frac{\pi}{2}$ is in the interval we are integrating over, $[0, \pi]$ and hence is not continuous on this interval.

Therefore, this integral cannot be done.

It is often easy to overlook these kinds of division by zero problems in integrands when the integrand is not explicitly written as a rational expression. So, be careful and don't forget that division by zero can sometimes be "hidden" in the integrand!

---

12. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sec^2(w) - 8\csc(w)\cot(w)\;dw$$

Step 1

First notice that even though we do have some "hidden" rational expression here (in the definitions of the trig functions) neither cosine nor sine is zero in the interval we are integrating over and so both terms are continuous over the interval.

Therefore all we need to do integrate the function.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sec^2(w) - 8\csc(w)\cot(w)\;dw = \left(2\tan(w) + 8\csc(w)\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

Recall that we don't need to add the $+c$ in the definite integral case as it will just cancel in the next step.

Step 2

The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\begin{aligned}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sec^2(w) - 8\csc(w)\cot(w)\;dw &= \left(\frac{16}{\sqrt{3}} + 2\sqrt{3}\right) - \left(16 + \frac{2}{\sqrt{3}}\right) \\ &= \boxed{\frac{14}{\sqrt{3}} + 2\sqrt{3} - 16}\end{aligned}$$

13. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_0^2 \mathbf{e}^x + \frac{1}{x^2+1}\,dx$$

Step 1

First we need to integrate the function.

$$\int_0^2 \mathbf{e}^x + \frac{1}{x^2+1}\,dx = \left(\mathbf{e}^x + \tan^{-1}(x)\right)\Big|_0^2$$

Recall that we don't need to add the $+c$in the definite integral case as it will just cancel in the next step.

Step 2

The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\int_0^2 \mathbf{e}^x + \frac{1}{x^2+1}\,dx = \left(\mathbf{e}^2 + \tan^{-1}(2)\right) - \left(\mathbf{e}^0 + \tan^{-1}(0)\right) = \boxed{\mathbf{e}^2 + \tan^{-1}(2) - 1}$$

Note that $\tan^{-1}(0) = 0$ but $\tan^{-1}(2)$ doesn't have a "nice" answer and so was left as is.

14. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-5}^{-2} 7\mathbf{e}^y + \frac{2}{y}\,dy$$

Step 1

First we need to integrate the function.

$$\int_{-5}^{-2} 7\mathbf{e}^y + \frac{2}{y}\,dy = \left(7\mathbf{e}^y + 2\ln|y|\right)\Big|_{-5}^{-2}$$

Recall that we don't need to add the $+c$ in the definite integral case as it will just cancel in the next step.

Step 2

The final step is then just to do the evaluation.

We'll leave the basic arithmetic to you to verify and only show the results of the evaluation. Make sure that you evaluate the upper limit first and then subtract off the evaluation at the lower limit.

Here is the answer for this problem.

$$\begin{aligned}\int_{-5}^{-2} 7\mathbf{e}^{y} + \frac{2}{y}\,dy &= \left(7\mathbf{e}^{-2} + 2\ln\left|-2\right|\right) - \left(7\mathbf{e}^{-5} + 2\ln\left|-5\right|\right)\\ &= \boxed{7\left(\mathbf{e}^{-2} - \mathbf{e}^{-5}\right) + 2\left(\ln\left(2\right) - \ln\left(5\right)\right)}\end{aligned}$$

---

15. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{0}^{4} f\left(t\right)\,dt \text{ where } f\left(t\right) = \begin{cases} 2t & t > 1 \\ 1 - 3t^{2} & t \le 1 \end{cases}$$

Hint

Recall that integrals we can always "break up" an integral as follows,

$$\int_{a}^{b} f\left(x\right)\,dx = \int_{a}^{c} f\left(x\right)\,dx + \int_{c}^{b} f\left(x\right)\,dx$$

See if you can find a good choice for c that will make this integral doable.

Step 1

This integral can't be done as a single integral give the obvious change of the function at $t = 1$ which is in the interval over which we are integrating. However, recall that we can always break up an integral at any point and $t = 1$ seems to be a good point to do this.

Breaking up the integral at $t = 1$ gives,

$$\int_0^4 f(t)\, dt = \int_0^1 f(t)\, dt + \int_1^4 f(t)\, dt$$

So, in the first integral we have $0 \le t \le 1$ and so we can use $f(t) = 1 - 3t^2$ in the first integral. Likewise, in the second integral we have $1 \le t \le 4$ and so we can use $f(t) = 2t$ in the second integral.

Making these function substitutions gives,

$$\int_0^4 f(t)\, dt = \int_0^1 1 - 3t^2\, dt + \int_1^4 2t\, dt$$

Step 2

All we need to do at this point is evaluate each integral. Here is that work.

$$\int_0^4 f(t)\, dt = \int_0^1 1 - 3t^2\, dt + \int_1^4 2t\, dt = \left(t - t^3\right)\Big|_0^1 + t^2\Big|_1^4 = [0 - 0] + [16 - 1] = \boxed{15}$$

16. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$\int_{-6}^{1} g(z)\, dz$ where $g(z) = \begin{cases} 2 - z & z > -2 \\ 4\mathbf{e}^z & z \le -2 \end{cases}$

Hint

Recall that integrals we can always "break up" an integral as follows,

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

See if you can find a good choice for c that will make this integral doable.

Step 1

This integral can't be done as a single integral give the obvious change of the function at $z = -2$ which is in the interval over which we are integrating. However, recall that we can always break up an integral at any point and $z = -2$ seems to be a good point to do this.

Breaking up the integral at $z = -2$ gives,

$$\int_{-6}^{1} g(z)\,dz = \int_{-6}^{-2} g(z)\,dz + \int_{-2}^{1} g(z)\,dz$$

So, in the first integral we have $-6 \le z \le -2$ and so we can use $g(z) = 4\mathbf{e}^{z}$ in the first integral. Likewise, in the second integral we have $-2 \le z \le 1$ and so we can use $g(z) = 2 - z$ in the second integral.

Making these function substitutions gives,

$$\int_{-6}^{1} g(z)\,dz = \int_{-6}^{-2} 4\mathbf{e}^{z}\,dz + \int_{-2}^{1} 2 - z\,dz$$

Step 2

All we need to do at this point is evaluate each integral. Here is that work.

$$\begin{aligned}\int_{-6}^{1} g(z)\,dz &= \int_{-6}^{-2} 4\mathbf{e}^{z}\,dz + \int_{-2}^{1} 2 - z\,dz = \left(4\mathbf{e}^{z}\right)\Big|_{-6}^{-2} + \left(2z - \frac{1}{2}z^{2}\right)\Big|_{-2}^{1} \\ &= \left[4\mathbf{e}^{-2} - 4\mathbf{e}^{-6}\right] + \left[\frac{3}{2} - (-6)\right] = \boxed{4\mathbf{e}^{-2} - 4\mathbf{e}^{-6} + \frac{15}{2}}\end{aligned}$$

17. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{3}^{6} |2x - 10|\,dx$$

Hint

In order to do this integral we need to “remove” the absolute value bars from the integrand and we should know how to do that by this point.

Step 1

We’ll need to “remove” the absolute value bars in order to do this integral. However, in order to do that we’ll need to know where $2x - 10$ is positive and negative.

Since $2x - 10$ is the equation of a line is should be fairly clear that we have the following positive/negative nature of the function.

$$\begin{aligned} x < 5 \qquad &\Rightarrow \qquad 2x - 10 < 0 \\ x > 5 \qquad &\Rightarrow \qquad 2x - 10 > 0 \end{aligned}$$

Step 2

So, to remove the absolute value bars all we need to do then is break the integral up at $x = 5$.

$$\int_3^6 |2x - 10| \, dx = \int_3^5 |2x - 10| \, dx + \int_5^6 |2x - 10| \, dx$$

So, in the first integral we have $3 \le x \le 5$ and so we have $|2x - 10| = -(2x - 10)$ in the first integral. Likewise, in the second integral we have $5 \le x \le 6$ and so we have $|2x - 10| = 2x - 10$ in the second integral. Or,

$$\int_3^6 |2x - 10| \, dx = \int_3^5 -(2x - 10) \, dx + \int_5^6 2x - 10 \, dx$$

Step 3

All we need to do at this point is evaluate each integral. Here is that work.

$$\int_3^6 |2x - 10| \, dx = \int_3^5 -2x + 10 \, dx + \int_5^6 2x - 10 \, dx = \left(-x^2 + 10x\right)\Big|_3^5 + \left(x^2 - 10x\right)\Big|_5^6$$
$$= [25 - 21] + [-24 - (-25)] = \boxed{5}$$

---

18. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-1}^0 |4w + 3| \, dw$$

Hint

In order to do this integral we need to "remove" the absolute value bars from the integrand and we should know how to do that by this point.

Step 1

We'll need to "remove" the absolute value bars in order to do this integral. However, in order to do that we'll need to know where $4w + 3$ is positive and negative.

Since $4w + 3$ is the equation a line is should be fairly clear that we have the following positive/negative nature of the function.

$$\begin{aligned} w &< -\frac{3}{4} & \Rightarrow & & 4w + 3 < 0 \\ w &> -\frac{3}{4} & \Rightarrow & & 4w + 3 > 0 \end{aligned}$$

Step 2

So, to remove the absolute value bars all we need to do then is break the integral up at $w = -\frac{3}{4}$.

$$\int_{-1}^{0} |4w + 3| \, dw = \int_{-1}^{-\frac{3}{4}} |4w + 3| \, dw + \int_{-\frac{3}{4}}^{0} |4w + 3| \, dw$$

So, in the first integral we have $-1 \le w \le -\frac{3}{4}$ and so we have $|4w + 3| = -(4w + 3)$ in the first integral. Likewise, in the second integral we have $-\frac{3}{4} \le w \le 0$ and so we have $|4w + 3| = 4w + 3$ in the second integral. Or,

$$\int_{-1}^{0} |4w + 3| \, dw = \int_{-1}^{-\frac{3}{4}} -(4w + 3) \, dw + \int_{-\frac{3}{4}}^{0} 4w + 3 \, dw$$

Step 3

All we need to do at this point is evaluate each integral. Here is that work.

$$\begin{aligned} \int_{-1}^{0} |4w + 3| \, dw &= \int_{-1}^{-\frac{3}{4}} -4w - 3 \, dw + \int_{-\frac{3}{4}}^{0} 4w + 3 \, dw \\ &= \left(-2w^2 - 3w\right)\Big|_{-1}^{-\frac{3}{4}} + \left(2w^2 + 3w\right)\Big|_{-\frac{3}{4}}^{0} \\ &= \left[\frac{9}{8} - 1\right] + \left[0 - \left(-\frac{9}{8}\right)\right] = \boxed{\frac{5}{4}} \end{aligned}$$

5.8 Substitution Rule for Definite Integrals

1. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_0^1 3\left(4x+x^4\right)\left(10x^2+x^5-2\right)^6\,dx$$

Step 1

The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,

$$u = 10x^2 + x^5 - 2$$

Step 2

Here is the actual substitution work for this problem.

$$du = \left(20x+5x^4\right)dx = 5\left(4x+x^4\right)dx \qquad \to \qquad \left(4x+x^4\right)dx = \frac{1}{5}du$$

$$x=0: u=-2 \qquad\qquad x=1: u=9$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_0^1 3\left(4x+x^4\right)\left(10x^2+x^5-2\right)^6\,dx = \frac{3}{5}\int_{-2}^{9} u^6\,du$$

Step 3

The integral is then,

$$\int_0^1 3\left(4x + x^4\right)\left(10x^2 + x^5 - 2\right)^6 dx = \frac{3}{35}u^7\Big|_{-2}^{9} = \frac{3}{35}\left(4,782,969 - (-128)\right)$$

$$= \boxed{\frac{14,349,291}{35}}$$

Do not get excited about “messy” or “large” answers. They will happen on occasion so don’t worry about them when the happen.

---

2. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_0^{\frac{\pi}{4}} \frac{8\cos(2t)}{\sqrt{9 - 5\sin(2t)}}\,dt$$

Step 1

The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,

$$u = 9 - 5\sin(2t)$$

Step 2

Here is the actual substitution work for this problem.

$$du = -10\cos(2t)\,dt \qquad \to \qquad \cos(2t)\,dt = -\frac{1}{10}du$$

$$t = 0 : u = 9 \qquad\qquad t = \frac{\pi}{4} : u = 4$$

As we did in the notes for this section we are also going to convert the limits to u’s to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_0^{\frac{\pi}{4}} \frac{8\cos(2t)}{\sqrt{9-5\sin(2t)}}\,dt = -\frac{8}{10}\int_9^4 u^{-\frac{1}{2}}\,du$$

Step 3

The integral is then,

$$\int_0^{\frac{\pi}{4}} \frac{8\cos(2t)}{\sqrt{9-5\sin(2t)}}\,dt = -\frac{8}{5}u^{\frac{1}{2}}\Big|_9^4 = -\frac{16}{5} - \left(-\frac{24}{5}\right) = \boxed{\frac{8}{5}}$$

3. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_\pi^0 \sin(z)\cos^3(z)\,dz$$

Step 1

The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,

$$u = \cos(z)$$

Step 2

Here is the actual substitution work for this problem.

$$\begin{aligned} du &= -\sin(z)\,dz & \to & \quad \sin(z)\,dz = -du \\ z &= \pi : u = -1 & & \quad z = 0 : u = 1 \end{aligned}$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{\pi}^{0} \sin(z)\cos^3(z)\,dz = -\int_{-1}^{1} u^3\,du$$

Step 3

The integral is then,

$$\int_{\pi}^{0} \sin(z)\cos^3(z)\,dz = -\frac{1}{4}u^4\bigg|_{-1}^{1} = -\frac{1}{4} - \left(-\frac{1}{4}\right) = \boxed{0}$$

---

4. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{1}^{4} \sqrt{w}\,\mathbf{e}^{1-\sqrt{w^3}}\,dw$$

Step 1

The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,

$$u = 1 - w^{\frac{3}{2}}$$

Step 2

Here is the actual substitution work for this problem.

$$du = -\frac{3}{2}w^{\frac{1}{2}}dw \qquad \to \qquad \sqrt{w}\,dw = -\frac{2}{3}du$$

$$w = 1 : u = 0 \qquad\qquad w = 4 : u = -7$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_1^4 \sqrt{w}\,\mathbf{e}^{1-\sqrt{w^3}}\,dw = -\frac{2}{3}\int_0^{-7} \mathbf{e}^u\,du$$

Step 3

The integral is then,

$$\int_1^4 \sqrt{w}\,\mathbf{e}^{1-\sqrt{w^3}}\,dw = -\frac{2}{3}\mathbf{e}^u\bigg|_0^{-7} = -\frac{2}{3}\mathbf{e}^{-7} - \left(-\frac{2}{3}\mathbf{e}^0\right) = \boxed{\frac{2}{3}\left(1-\mathbf{e}^{-7}\right)}$$

---

5. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-4}^{-1} \sqrt[3]{5-2y} + \frac{7}{5-2y}\,dy$$

Step 1

The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,

$$u = 5-2y$$

Step 2

Here is the actual substitution work for this problem.

$$du = -2\,dy \qquad \to \qquad dy = -\frac{1}{2}du$$

$$y = -4 : u = 13 \qquad\qquad y = -1 : u = 7$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{-4}^{-1} \sqrt[3]{5-2y} + \frac{7}{5-2y}\,dy = -\frac{1}{2}\int_{13}^{7} u^{\frac{1}{3}} + \frac{7}{u}\,du$$

Step 3

The integral is then,

$$\begin{aligned}\int_{-4}^{-1} \sqrt[3]{5-2y} + \frac{7}{5-2y}\,dy &= \left(-\frac{1}{2}\left[\frac{3}{4}u^{\frac{4}{3}} + 7\ln|u|\right]\right)\Bigg|_{13}^{7} \\ &= -\frac{3}{8}7^{\frac{4}{3}} - \frac{7}{2}\ln|7| - \left(-\frac{3}{8}13^{\frac{4}{3}} - \frac{7}{2}\ln|13|\right) \\ &= \boxed{\frac{3}{8}\left(13^{\frac{4}{3}} - 7^{\frac{4}{3}}\right) + \frac{7}{2}\left(\ln(13) - \ln(7)\right)}\end{aligned}$$

6. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-1}^{2} x^3 + \mathbf{e}^{\frac{1}{4}x}\,dx$$

Step 1

The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

Before setting up the substitution we'll need to break up the integral because the first term doesn't need a substitution. Doing this gives,

$$\int_{-1}^{2} x^3 + \mathbf{e}^{\frac{1}{4}x}\,dx = \int_{-1}^{2} x^3\,dx + \int_{-1}^{2} \mathbf{e}^{\frac{1}{4}x}\,dx$$

The substitution for the second integral is then,

$$u = \frac{1}{4}x$$

Step 2

Here is the actual substitution work for this second integral.

$$du = \frac{1}{4}dx \qquad \to \qquad dx = 4du$$

$$x = -1 : u = -\frac{1}{4} \qquad\qquad x = 2 : u = \frac{1}{2}$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{-1}^{2} x^3 + \mathbf{e}^{\frac{1}{4}x}\,dx = \int_{-1}^{2} x^3\,dx + 4\int_{-\frac{1}{4}}^{\frac{1}{2}} \mathbf{e}^u\,du$$

Step 3

The integral is then,

$$\int_{-1}^{2} x^3 + \mathbf{e}^{\frac{1}{4}x}\,dx = \frac{1}{4}x^4\bigg|_{-1}^{2} + 4\mathbf{e}^u\big|_{-\frac{1}{4}}^{\frac{1}{2}} = \left(4 - \frac{1}{4}\right) + \left(4\mathbf{e}^{\frac{1}{2}} - 4\mathbf{e}^{-\frac{1}{4}}\right)$$

$$= \boxed{\frac{15}{4} + 4\mathbf{e}^{\frac{1}{2}} - 4\mathbf{e}^{-\frac{1}{4}}}$$

---

7. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{\pi}^{\frac{3\pi}{2}} 6\sin(2w) - 7\cos(w)\,dw$$

Step 1

The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

Before setting up the substitution we'll need to break up the integral because the second term doesn't need a substitution. Doing this gives,

$$\int_{\pi}^{\frac{3\pi}{2}} 6\sin(2w) - 7\cos(w)\,dw = \int_{\pi}^{\frac{3\pi}{2}} 6\sin(2w)\,dw - \int_{\pi}^{\frac{3\pi}{2}} 7\cos(w)\,dw$$

The substitution for the first integral is then,

$$u = 2w$$

Step 2

Here is the actual substitution work for this first integral.

$$du = 2dw \qquad \to \qquad dw = \frac{1}{2}du$$

$$w = \pi : u = 2\pi \qquad\qquad w = \frac{3\pi}{2} : u = 3\pi$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{\pi}^{\frac{3\pi}{2}} 6\sin(2w) - 7\cos(w)\,dw = 3\int_{2\pi}^{3\pi} \sin(u)\,du - \int_{\pi}^{\frac{3\pi}{2}} 7\cos(w)\,dw$$

Step 3

The integral is then,

$$\begin{aligned}\int_{\pi}^{\frac{3\pi}{2}} 6\sin(2w) - 7\cos(w)\,dw = -3\cos(u)\Big|_{2\pi}^{3\pi} - 7\sin(w)\Big|_{\pi}^{\frac{3\pi}{2}} &= (3-(-3)) + (7-0)\\ &= \boxed{13}\end{aligned}$$

8. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{1}^{5} \frac{2x^3 + x}{x^4 + x^2 + 1} - \frac{x}{x^2 - 4}\,dx$$

Solution

Be very careful with this problem. Recall that we can only do definite integrals if the integrand (*i.e.* the function we are integrating) is continuous on the interval over which we are integrating.

In this case the second term has division by zero at $x = 2$ and so is not continuous on $[1, 5]$ and therefore this integral can't be done.

---

9. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-2}^{0} t\sqrt{3 + t^2} + \frac{3}{(6t - 1)^2}\,dt$$

Step 1

The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

Before setting up the substitution we'll need to break up the integral because each term requires a different substitution. Doing this gives,

$$\int_{-2}^{0} t\sqrt{3 + t^2} + \frac{3}{(6t - 1)^2}\,dt = \int_{-2}^{0} t\sqrt{3 + t^2}\,dt + \int_{-2}^{0} \frac{3}{(6t - 1)^2}\,dt$$

The substitution for each integral is then,

$$u = 3 + t^2 \qquad\qquad v = 6t - 1$$

Step 2

Here is the actual substitution work for this first integral.

$$du = 2t\,dt \qquad \to \qquad t\,dt = \frac{1}{2}du$$

$$t = -2 : u = 7 \qquad\qquad t = 0 : u = 3$$

Here is the actual substitution work for the second integral.

$$dv = 6\,dt \qquad \to \qquad dt = \frac{1}{6}dv$$
$$t = -2 : v = -13 \qquad\qquad t = 0 : v = -1$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{-2}^{0} t\sqrt{3+t^2} + \frac{3}{(6t-1)^2}\,dt = \frac{1}{2}\int_{7}^{3} u^{\frac{1}{2}}\,du + \frac{3}{6}\int_{-13}^{-1} v^{-2}\,dv$$

Step 3

The integral is then,

$$\int_{-2}^{0} t\sqrt{3+t^2} + \frac{3}{(6t-1)^2}\,dt = \frac{1}{3}u^{\frac{3}{2}}\bigg|_{7}^{3} - \frac{1}{2}v^{-1}\bigg|_{-13}^{-1} = \frac{1}{3}\left(3^{\frac{3}{2}} - 7^{\frac{3}{2}}\right) - \frac{1}{2}\left(-1 - \left(-\frac{1}{13}\right)\right)$$
$$= \boxed{\frac{1}{3}\left(3^{\frac{3}{2}} - 7^{\frac{3}{2}}\right) + \frac{6}{13}}$$

10. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-2}^{1} (2-z)^3 + \sin(\pi z)\,[3 + 2\cos(\pi z)]^3\,dz$$

Step 1

The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

Before setting up the substitution we'll need to break up the integral because each term requires a different substitution. Doing this gives,

$$\int_{-2}^{1} (2-z)^3 + \sin(\pi z)\,[3+2\cos(\pi z)]^3\,dz = \int_{-2}^{1}(2-z)^3 dz + \int_{-2}^{1}\sin(\pi z)\,[3+2\cos(\pi z)]^3\,dz$$

The substitution for each integral is then,

$$u = 2 - z \qquad\qquad v = 3 + 2\cos(\pi z)$$

Step 2

Here is the actual substitution work for this first integral.

$$du = -dz \quad \rightarrow \quad dz = -du$$
$$z=-2: u=4 \qquad\qquad z=1: u=1$$

Here is the actual substitution work for the second integral.

$$dv = -2\pi\sin(\pi z)\ dz \quad \rightarrow \quad \sin(\pi z)\,dz = -\frac{1}{2\pi}dv$$
$$z=-2: v=5 \qquad\qquad z=1: v=1$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{-2}^{1} (2-z)^3 + \sin(\pi z)\,[3+2\cos(\pi z)]^3\,dz = -\int_{4}^{1} u^3 du - \frac{1}{2\pi}\int_{5}^{1} v^3\,dv$$

Step 3

The integral is then,

$$\begin{aligned}\int_{-2}^{1} (2-z)^3 + \sin(\pi z)\,[3+2\cos(\pi z)]^3\,dz &= -\frac{1}{4}u^4\Big|_{4}^{1} - \frac{1}{8\pi}v^4\Big|_{5}^{1}\\ &= -\frac{1}{4}(1-256) - \frac{1}{8\pi}(1-625)\\ &= \boxed{\frac{255}{4}+\frac{78}{\pi}}\end{aligned}$$

6 Applications of Integrals

The previous chapter dealt exclusively with the computation of definite and indefinite integrals as well as some discussion of their properties and interpretations. It is now time to start looking at some applications of integrals. Note as well that we should probably say applications of **definite** integrals as that is really what we'll be looking at in this section.

In addition, we should note that there are a lot of different applications of (definite) integrals out there. We will look at the ones that can easily be done with the knowledge we have at our disposal at this point. Once we have covered the next chapter, Integration Techniques, we will be able to take a look at a few more applications of integrals. At this point we would not be able to compute many of the integrals that arise in those later applications.

In this chapter we'll take a look at using integrals to compute the average value of a function and the work required to move an object over a given distance. In addition we will take a look at a couple of geometric applications of integrals. In particular we will use integrals to compute the area that is between two curves and note that this application should not be too surprising given one of the major interpretations of the definite integral. We will also see how to compute the volume of some solids. We will compute the volume of solids of revolution, *i.e.* a solid obtained by rotating a curve about a given axis. In addition, we will compute the volume of some slightly more general solids in which the cross sections can be easily described with nice 2D geometric formulas (*i.e.* rectangles, triangles, circles, *etc.*).

The following sections are the practice problems, with solutions, for this material.

6.1 Average Function Value

1. Determine f_{avg} for $f(x) = 8x - 3 + 5\mathbf{e}^{2-x}$ on $[0, 2]$.

Solution

There really isn't all that much to this problem other than use the formula given in the notes for this section.

$$f_{\text{avg}} = \frac{1}{2-0}\int_0^2 8x - 3 + 5\mathbf{e}^{2-x}\,dx = \frac{1}{2}\left(4x^2 - 3x - 5\mathbf{e}^{2-x}\right)\Big|_0^2 = \boxed{\frac{1}{2}\left(5 + 5\mathbf{e}^2\right)}$$

Note that we are assuming your integration skills are pretty good at this point and won't be showing many details of the actual integration process. This includes not showing substitutions such as the substitution needed for the third term (you did catch that correct?).

2. Determine f_{avg} for $f(x) = \cos(2x) - \sin\left(\frac{x}{2}\right)$ on $\left[-\frac{\pi}{2}, \pi\right]$.

Solution

There really isn't all that much to this problem other than use the formula given in the notes for this section.

$$f_{\text{avg}} = \frac{1}{\pi - \left(-\frac{\pi}{2}\right)}\int_{-\frac{\pi}{2}}^{\pi} \cos(2x) - \sin\left(\frac{x}{2}\right)dx = \frac{2}{3\pi}\left(\frac{1}{2}\sin(2x) + 2\cos\left(\frac{x}{2}\right)\right)\Big|_{-\frac{\pi}{2}}^{\pi}$$

$$= \boxed{-\frac{2\sqrt{2}}{3\pi}}$$

Note that we are assuming your integration skills are pretty good at this point and won't be showing many details of the actual integration process. This includes not showing either of the substitutions needed for the integral (you did catch both of them correct?).

3. Find f_{avg} for $f(x) = 4x^2 - x + 5$ on $[-2, 3]$ and determine the value(s) of c in $[-2, 3]$ for which $f(c) = f_{\text{avg}}$.

Step 1

First, we need to use the formula for the notes in this section to find f_{avg}.

$$f_{\text{avg}} = \frac{1}{3-(-2)} \int_{-2}^{3} 4x^2 - x + 5\, dx = \frac{1}{5}\left(\frac{4}{3}x^3 - \frac{1}{2}x^2 + 5x\right)\Bigg|_{-2}^{3} = \boxed{\frac{83}{6}}$$

Step 2

Note that for the second part of this problem we are really just asking to find the value of c that satisfies the Mean Value Theorem for Integrals.

There really isn't much to do here other than solve $f(c) = f_{\text{avg}}$.

$$4c^2 - c + 5 = \frac{83}{6}$$

$$4c^2 - c - \frac{53}{6} = 0 \qquad \Rightarrow \qquad c = \frac{1 \pm \sqrt{1 - 4(4)\left(-\frac{53}{6}\right)}}{2(4)} = \frac{1 \pm \sqrt{\frac{427}{3}}}{2(4)} = \boxed{-1.3663,\ \ 1.6163}$$

So, unlike the example from the notes both of the numbers that we found here are in the interval and so are both included in the answer.

6.2 Area Between Curves

1. Determine the area below $f(x) = 3 + 2x - x^2$ and above the x-axis.

Hint

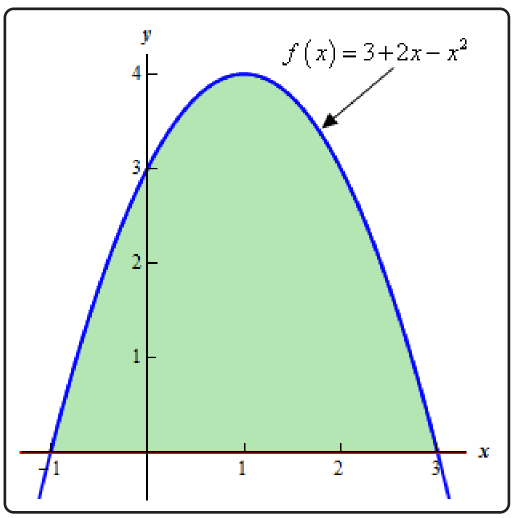
It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the upper/lower functions and the limits for the integral.

Step 1

Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.



Step 2

It should be clear from the graph that the upper function is the parabola (*i.e.* $y = 3 + 2x - x^2$) and the lower function is the x-axis (*i.e.* $y = 0$).

Since we weren't given any limits on x in the problem statement we'll need to get those. From the graph it looks like the limits are (probably) $-1 \le x \le 3$. However, we should never just assume that our graph is accurate or that we were able to read it accurately.

For all we know the limits are close to those we guessed from the graph but are in fact slightly different.

So, to determine if we guessed the limits correctly from the graph let's find them directly. The limits are where the parabola crosses the x-axis and so all we need to do is set the parabola equal to zero (*i.e.* where it crosses the line $y = 0$) and solve. Doing this gives,

$$3 + 2x - x^2 = 0 \qquad \to \qquad -(x+1)(x-3) = 0 \qquad \to \qquad x = -1, \quad x = 3$$

So, we did guess correctly, but it never hurts to be sure. That is especially true here where finding them directly takes almost no time.

Step 3

At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-1}^{3} 3 + 2x - x^2 \, dx = \left(3x + x^2 - \frac{1}{3}x^3\right)\Big|_{-1}^{3} = \boxed{\frac{32}{3}}$$

---

2. Determine the area to the left of $g(y) = 3 - y^2$ and to the right of $x = -1$.

Hint

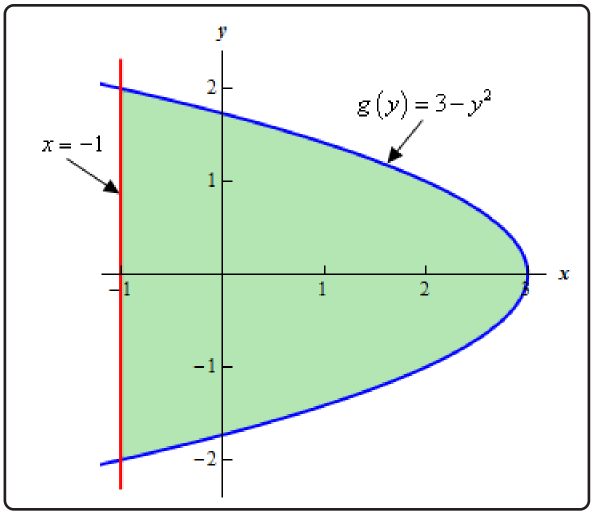
It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the right/left functions and the limits for the integral.

Step 1

Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.



Step 2

It should be clear from the graph that the right function is the parabola (*i.e.* $x = 3 - y^2$) and the left function is the line $x = -1$.

Since we weren't given any limits on y in the problem statement we'll need to get those. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the limits from the graph. This is especially true when the intersection points of the two curves (*i.e.* the limits on y that we need) do not occur on an axis (as they don't in this case).

So, to determine the intersection points correctly we'll need to find them directly. The intersection points are where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$3 - y^2 = -1 \qquad \to \qquad y^2 = 4 \qquad \to \qquad y = -2, \quad y = 2$$

So, the limits on y are : $-2 \le y \le 2$.

Step 3

At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-2}^{2} 3 - y^2 - (-1)\,dy = \int_{-2}^{2} 4 - y^2 dy = \left(4y - \frac{1}{3}y^3\right)\Bigg|_{-2}^{2} = \boxed{\frac{32}{3}}$$

3. Determine the area of the region bounded by $y = x^2 + 2$, $y = \sin(x)$, $x = -1$ and $x = 2$.

Hint

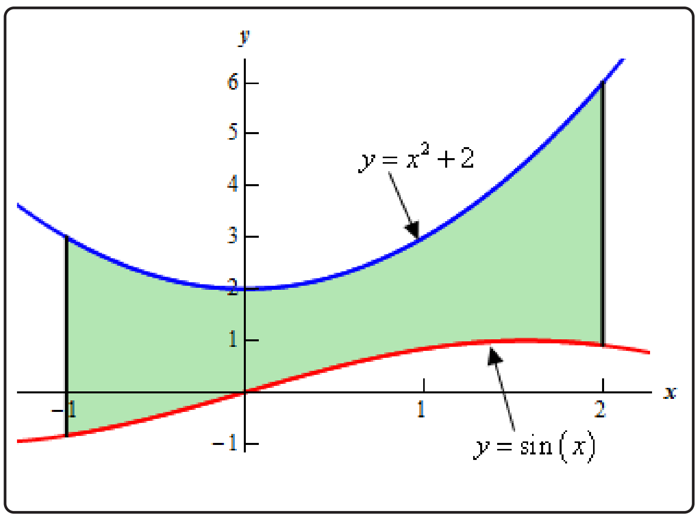
It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the upper/lower functions and the limits for the integral.

Step 1

Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.



Step 2

It should be clear from the graph that the upper function is $y = x^2 + 2$ and the lower function is $y = \sin(x)$.

Next, we were given limits on x in the problem statement and we can see that the two curves do not intersect in that range. Note that this is something that we can't always guarantee and so we need the graph to verify if the curves intersect or not. We should never just assume that because limits on x were given in the problem statement that the curves will not intersect anywhere between the given limits.

So, because the curves do not intersect we will be able to find the area with a single integral using the limits : $-1 \le x \le 2$.

Step 3

At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-1}^{2} x^2 + 2 - \sin(x)\,dx = \left(\frac{1}{3}x^3 + 2x + \cos(x)\right)\Big|_{-1}^{2}$$
$$= \boxed{9 + \cos(2) - \cos(1) = 8.04355}$$

Don't forget to set your calculator to radians if you take the answer to a decimal.

---

4. Determine the area of the region bounded by $y = \dfrac{8}{x}$, $y = 2x$ and $x = 4$.

Hint

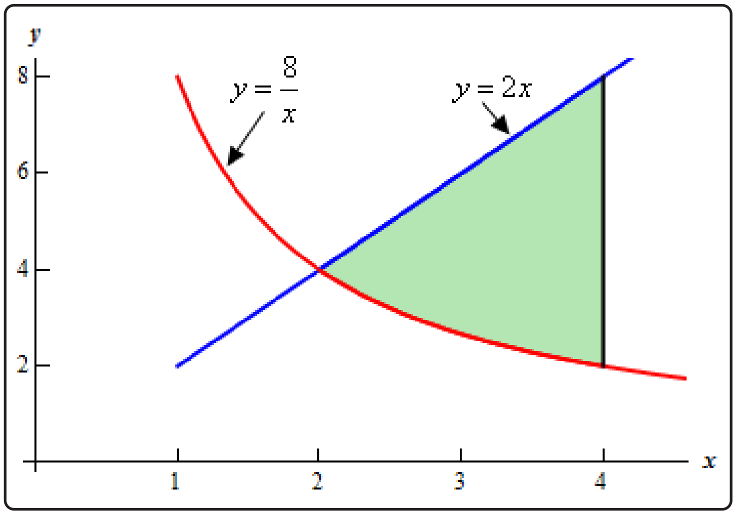
It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the upper/lower functions and the limits for the integral.

Step 1

Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.



Step 2

For this problem we were only given one limit on x (*i.e.* $x = 4$). To determine just what the region we are after recall that we are after a *bounded* region. This means that one of the given curves must be on each boundary of the region.

Therefore, we can't use any portion of the region to the right of the line $x = 4$ because there will never be a boundary on the right of that region.

We also can't take any portion of the region to the left of the intersection point. Because the first function is not continuous at $x = 0$ we can't use any region that includes $x = 0$. Therefore, any portion of the region to the left of the intersection point would have to stop prior to the y-axis and any region like that would not have any of the given curves on the left boundary.

The region is then the one shown in graph above. We will take the region to the left of the line $x = 4$ and to the right of the intersection point.

Step 3

We now need to determine the intersection point. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the coordinates from the graph. This is especially true when the intersection point of the two curves does not occur on an axis (as they don't in this case).

So, to determine the intersection point correctly we'll need to find it directly. The intersection point is where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$\frac{8}{x} = 2x \qquad \rightarrow \qquad x^2 = 4 \qquad \rightarrow \qquad x = -2, \quad x = 2$$

Note that while we got two answers here the negative value does not make any sense because to get to that value we would have to go through $x = 0$ and as we discussed above the bounded region cannot contain $x = 0$.

Therefore the limits on x are $2 \le x \le 4$.

It should also be clear from the graph and the limits above that the upper function is $y = 2x$ and the lower function is $y = \frac{8}{x}$.

Step 3

At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_2^4 2x - \frac{8}{x}\,dx = \left(x^2 - 8\ln|x|\right)\Big|_2^4 = \boxed{12 - 8\ln(4) + 8\ln(2) = 6.4548}$$

5. Determine the area of the region bounded by $x = 3 + y^2$, $x = 2 - y^2$, $y = 1$ and $y = -2$.

Hint

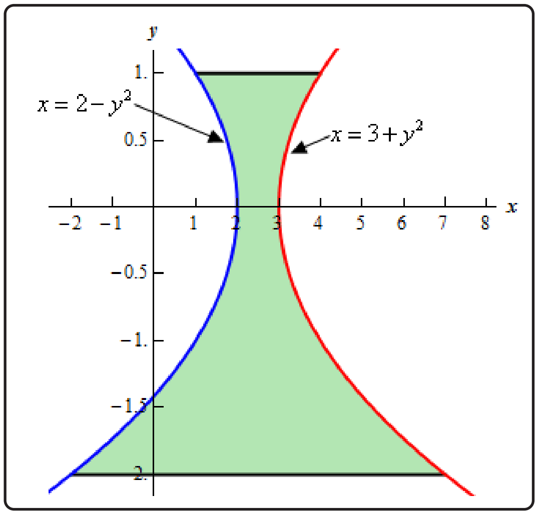
It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the right/left functions and the limits for the integral.

Step 1

Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.



Step 2

It should be clear from the graph that the right function is $x = 3 + y^2$ and the left function is $x = 2 - y^2$.

Next, we were given limits on y in the problem statement and we can see that the two curves do not intersect in that range. Note that this is something that we can't always guarantee and so we need the graph to verify if the curves intersect or not. We should never just assume that because limits on y were given in the problem statement that the curves will not intersect anywhere between the given limits.

So, because the curves do not intersect we will be able to find the area with a single integral using the limits : $-2 \le y \le 1$.

Step 3

At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-2}^{1} 3 + y^2 - (2 - y^2)\,dy = \int_{-2}^{1} 1 + 2y^2\,dy = \left(y + \frac{2}{3}y^3\right)\Big|_{-2}^{1} = \boxed{9}$$

---

6. Determine the area of the region bounded by $x = y^2 - y - 6$ and $x = 2y + 4$.

Hint

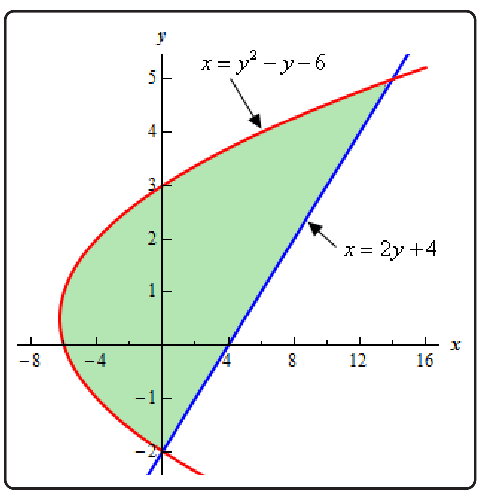
It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the right/left functions and the limits for the integral.

Step 1

Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.



Note that we won't include any portion of the region above the top intersection point or below the bottom intersection point. The region needs to be bounded by one of the given curves on each boundary. If we went past the top intersection point we would not have an upper bound on the region. Likewise, if we went past the bottom intersection point we would not have a lower bound on the region.

Step 2

It should be clear from the graph that the right function is $x = 2y + 4$ and the left function is $x = y^2 - y - 6$.

Since we weren't given any limits on y in the problem statement we'll need to get those. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the coordinates from the graph. This is especially true when the intersection points of the two curves (*i.e.* the limits on y that we need) do not occur on an axis (as they don't in this case).

So, to determine the intersection points correctly we'll need to find them directly. The intersection points are where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$y^2 - y - 6 = 2y + 4 \quad \to \quad y^2 - 3y - 10 = (y - 5)(y + 2) = 0 \quad \to \quad y = -2, \quad y = 5$$

Therefore the limits on y are $-2 \le y \le 5$.

Note that you may well have found the intersection points in the first step to help with the graph if you were graphing by hand which is not a bad idea with faced with graphing this kind of region.

Step 3

At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-2}^{5} 2y + 4 - \left(y^2 - y - 6\right)\,dy = \int_{-2}^{5} 10 + 3y - y^2\,dy = \left(10y + \frac{3}{2}y^2 - \frac{1}{3}y^3\right)\Bigg|_{-2}^{5}$$

$$= \boxed{\frac{343}{6}}$$

7. Determine the area of the region bounded by $y = x\sqrt{x^2+1}$, $y = \mathbf{e}^{-\frac{1}{2}x}$, $x = -3$ and the y-axis.

Hint

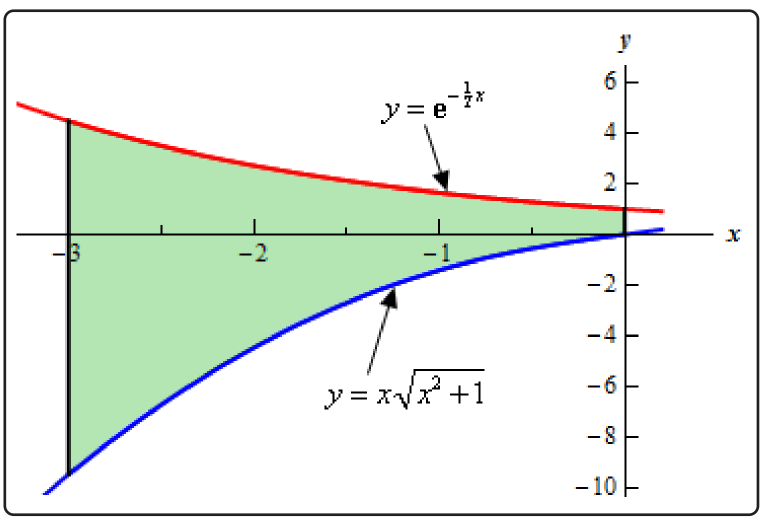
It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the upper/lower functions and the limits for the integral.

Step 1

Let's start off with getting a sketch of the region we want to find the area of.

Note that using a graphing calculator or computer may be needed to deal with the first equation, however you should be able to sketch the graph of the second equation by hand.

Here is a sketch of the bounded region we want to find the area of.



Step 2

It should be clear from the graph that the upper function is $y = \mathbf{e}^{-\frac{1}{2}x}$ and the lower function is $y = x\sqrt{x^2+1}$.

Next, we were given limits on x in the problem statement (recall that the y-axis is just the line $x = 0$!) and we can see that the two curves do not intersect in that range. Note that this is something that we can't always guarantee and so we need the graph to verify if the curves intersect or not. We should never just assume that because limits on x were given in the problem statement that the curves will not intersect anywhere between the given limits.

So, because the curves do not intersect we will be able to find the area with a single integral using the limits : $-3 \le x \le 0$.

Step 3

At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques, including substitution since that will be needed here, so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-3}^{0} \mathbf{e}^{-\frac{1}{2}x} - x\sqrt{x^2+1}\,dx = \left(-2\mathbf{e}^{-\frac{1}{2}x} - \frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}}\right)\bigg|_{-3}^{0}$$

$$= \boxed{-\frac{7}{3} + 2\mathbf{e}^{\frac{3}{2}} + \frac{1}{3}10^{\frac{3}{2}} = 17.17097}$$

---

8. Determine the area of the region bounded by $y = 4x + 3$, $y = 6 - x - 2x^2$, $x = -4$ and $x = 2$.

Hint

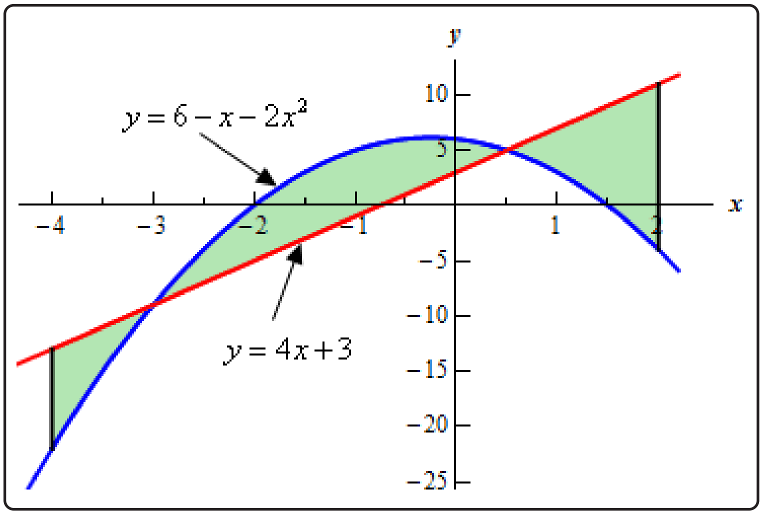
It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the upper/lower functions and the limits for the integral.

Step 1

Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.



Step 2

In the problem statement we were given two limits on x. However, as seen in the sketch of the graph above the curves intersect in this region and the upper/lower functions differ depending on what range of x's we are looking for.

Therefore we'll need to find the intersection points. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the coordinates from the graph. This is especially true when the intersection points of the two curves do not occur on an axis (as they don't in this case).

So, to determine the intersection points correctly we'll need to find them directly. The intersection points are where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$6 - x - 2x^2 = 4x + 3 \quad \to \quad 2\,x^2 + 5x - 3 = (2x - 1)\,(x + 3) = 0 \quad \to \quad x = -3, \; x = \frac{1}{2}$$

Note that you may well have found the intersection points in the first step to help with the graph if you were graphing by hand which is not a bad idea with faced with graphing this kind of region.

So, from the graph then it looks like we'll need three integrals since there are three ranges of x ($-4 \le x \le -3$, $-3 \le x \le \frac{1}{2}$ and $\frac{1}{2} \le x \le 2$) for which the upper/lower functions are different.

Step 3

At this point there isn't much to do other than step up the integrals and evaluate them.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$\begin{aligned} A &= \int_{-4}^{-3} 4x+3-\left(6-x-2x^2\right)\,dx + \int_{-3}^{\frac{1}{2}} 6-x-2x^2-(4x+3)\,dx \\ &\qquad\qquad + \int_{\frac{1}{2}}^{2} 4x+3-\left(6-x-2x^2\right)\,dx \\ &= \int_{-4}^{-3} 2x^2+5x-3\,dx + \int_{-3}^{\frac{1}{2}} 3-5x-2x^2\,dx + \int_{\frac{1}{2}}^{2} 2x^2+5x-3\,dx \\ &= \left(\frac{2}{3}x^3+\frac{5}{2}x^2-3x\right)\Bigg|_{-4}^{-3} + \left(3x-\frac{5}{2}x^2-\frac{2}{3}x^3\right)\Bigg|_{-3}^{\frac{1}{2}} + \left(\frac{2}{3}x^3+\frac{5}{2}x^2-3x\right)\Bigg|_{\frac{1}{2}}^{2} \\ &= \frac{25}{6}+\frac{343}{24}+\frac{81}{8} = \boxed{\frac{343}{12}} \end{aligned}$$

---

9. Determine the area of the region bounded by $y = \dfrac{1}{x+2}$, $y = (x+2)^2$, $x = -\frac{3}{2}$, $x = 1$.

Hint

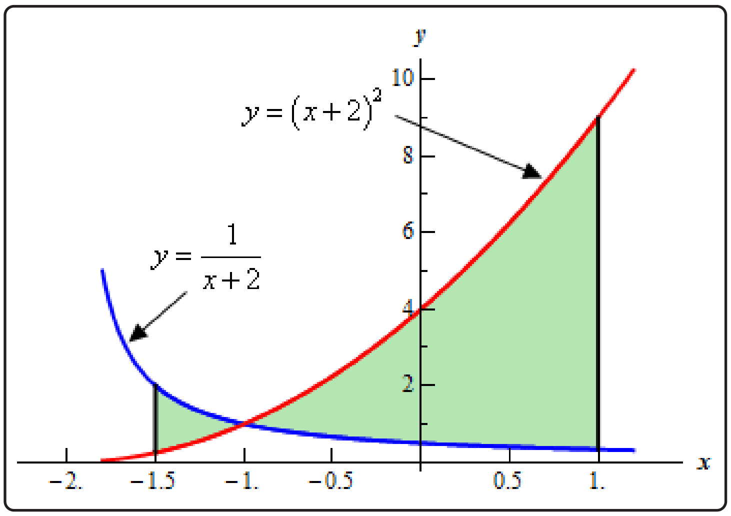
It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the upper/lower functions and the limits for the integral.

Step 1

Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.



Step 2

In the problem statement we were given two limits on x. However, as seen in the sketch of the graph above the curves intersect in this region and the upper/lower functions differ depending on what range of x's we are looking for.

Therefore, we'll need to find the intersection point. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the coordinates from the graph. This is especially true when the intersection point of the two curves does not occur on an axis (as they don't in this case).

So, to determine the intersection points correctly we'll need to find it directly. The intersection point is where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$\frac{1}{x+2} = (x+2)^2 \qquad \to \qquad (x+2)^3 = 1 \qquad \to \qquad x+2 = \sqrt[3]{1} = 1 \qquad \to \qquad x = -1$$

So, from the graph then it looks like we'll need two integrals since there are two ranges of x ($-\frac{3}{2} \le x \le -1$ and $-1 \le x \le 1$) for which the upper/lower functions are different.

Step 3

At this point there isn't much to do other than step up the integrals and evaluate them.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-\frac{3}{2}}^{-1} \frac{1}{x+2} - (x+2)^2\,dx + \int_{-1}^{1} (x+2)^2 - \frac{1}{x+2}\,dx$$

$$= \left(\ln|x+2| - \frac{1}{3}(x+2)^3\right)\Big|_{-\frac{3}{2}}^{-1} + \left(\frac{1}{3}(x+2)^3 - \ln|x+2|\right)\Big|_{-1}^{1}$$

$$= \left[-\frac{7}{24} - \ln\left(\frac{1}{2}\right)\right] + \left[\frac{26}{3} - \ln(3)\right] = \boxed{\frac{67}{8} - \ln\left(\frac{1}{2}\right) - \ln(3) = 7.9695}$$

10. Determine the area of the region bounded by $x = y^2 + 1$, $x = 5$, $y = -3$ and $y = 3$.

Hint

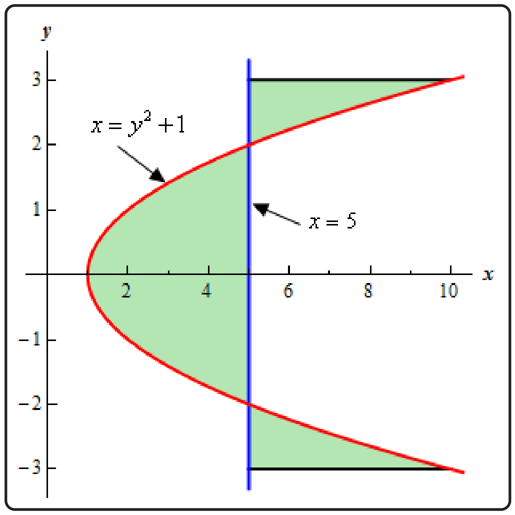
It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the right/left functions and the limits for the integral.

Step 1

Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.



Step 2

In the problem statement we were given two limits on y. However, as seen in the sketch of the graph above the curves intersect in this region and the right/left functions differ depending on what range of y's we are looking for.

Therefore, we'll need to find the intersection points. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the coordinates from the graph. This is especially true when the intersection points of the two curves do not occur on an axis (as they don't in this case).

So, to determine the intersection points correctly we'll need to find them directly. The intersection points are where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$y^2 + 1 = 5 \qquad \rightarrow \qquad y^2 = 4 \qquad \rightarrow \qquad y = -2, \quad y = 2$$

Note that you may well have found the intersection points in the first step to help with the graph if you were graphing by hand which is not a bad idea with faced with graphing this kind of region.

So, from the graph then it looks like we'll need three integrals since there are three ranges of x ($-3 \le x \le -2$, $-2 \le x \le 2$ and $2 \le x \le 3$) for which the right/left functions are different.

Step 3

At this point there isn't much to do other than step up the integrals and evaluate them.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$\begin{aligned} A &= \int_{-3}^{-2} y^2 + 1 - 5\,dy + \int_{-2}^{2} 5 - \left(y^2 + 1\right)\,dy + \int_{2}^{3} y^2 + 1 - 5\,dy \\ &= \int_{-3}^{-2} y^2 - 4\,dy + \int_{-2}^{2} 4 - y^2\,dy + \int_{2}^{3} y^2 - 4\,dy \\ &= \left(\frac{1}{3}y^3 - 4y\right)\Big|_{-3}^{-2} + \left(4y - \frac{1}{3}y^3\right)\Big|_{-2}^{2} + \left(\frac{1}{3}y^3 - 4y\right)\Big|_{2}^{3} \\ &= \frac{7}{3} + \frac{32}{3} + \frac{7}{3} = \boxed{\frac{46}{3}} \end{aligned}$$

11. Determine the area of the region bounded by $x = \mathbf{e}^{1+2y}$, $x = \mathbf{e}^{1-y}$, $y = -2$ and $y = 1$.

Hint

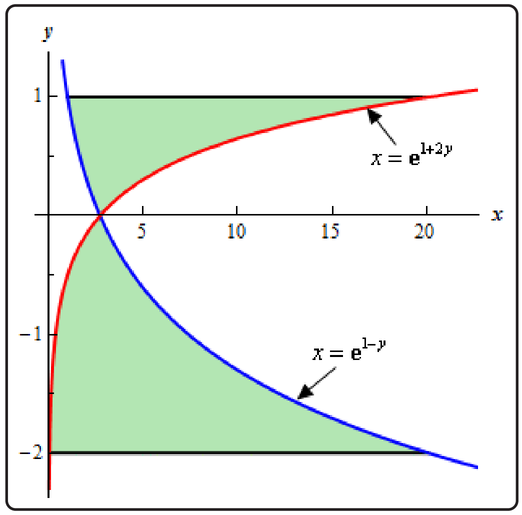
It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the right/left functions and the limits for the integral.

Step 1

Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.



Step 2

In the problem statement we were given two limits on y. However, as seen in the sketch of the graph above the curves intersect in this region and the right/left functions differ depending on what range of y's we are looking for.

Therefore, we'll need to find the intersection point. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the coordinates from the graph. In this case it seems pretty clear from the graph that the intersection point lies on the x-axis (and so we can guess the point we need is $y = 0$). However, for all we know the actual intersection point is slightly above or slightly below the x-axis and the scale of the graph just makes this hard to see.

So, to determine the intersection points correctly we'll need to find it directly. The intersection point is where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$\mathbf{e}^{1+2y} = \mathbf{e}^{1-y} \qquad \to \qquad \frac{\mathbf{e}^{1+2y}}{\mathbf{e}^{1-y}} = 1 \qquad \to \qquad \mathbf{e}^{3y} = 1 \qquad \to \qquad y = 0$$

So, from the graph then it looks like we'll need two integrals since there are two ranges of x ($-2 \le x \le 0$ and $0 \le x \le 1$) for which the right/left functions are different.

Step 3

At this point there isn't much to do other than step up the integrals and evaluate them.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$\begin{aligned} A &= \int_{-2}^{0} \mathbf{e}^{1-y} - \mathbf{e}^{1+2y}\,dy + \int_{0}^{1} \mathbf{e}^{1+2y} - \mathbf{e}^{1-y}\,dy \\ &= \left(-\mathbf{e}^{1-y} - \frac{1}{2}\mathbf{e}^{1+2y}\right)\Bigg|_{-2}^{0} + \left(\frac{1}{2}\mathbf{e}^{1+2y} + \mathbf{e}^{1-y}\right)\Bigg|_{0}^{1} \\ &= \left[\mathbf{e}^{3} + \frac{1}{2}\mathbf{e}^{-3} - \frac{3}{2}\mathbf{e}\right] + \left[1 + \frac{1}{2}\mathbf{e}^{3} - \frac{3}{2}\mathbf{e}\right] = \boxed{22.9983} \end{aligned}$$

6.3 Solids of Revolution / Method of Rings

1. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 3$ and the y-axis about the y-axis.

Hint

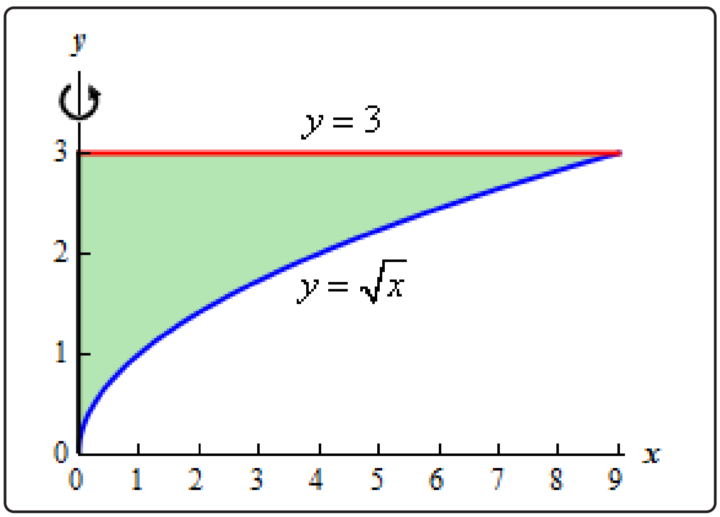
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



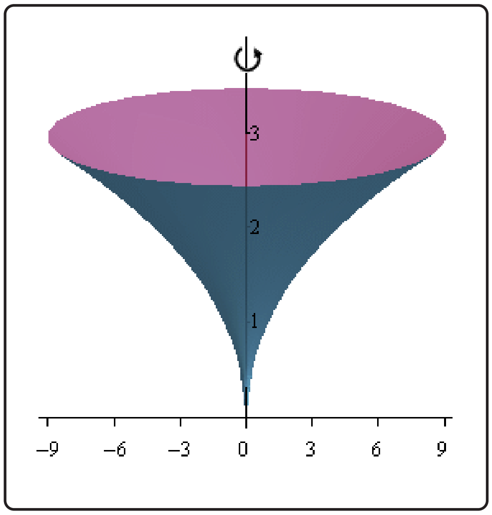
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.

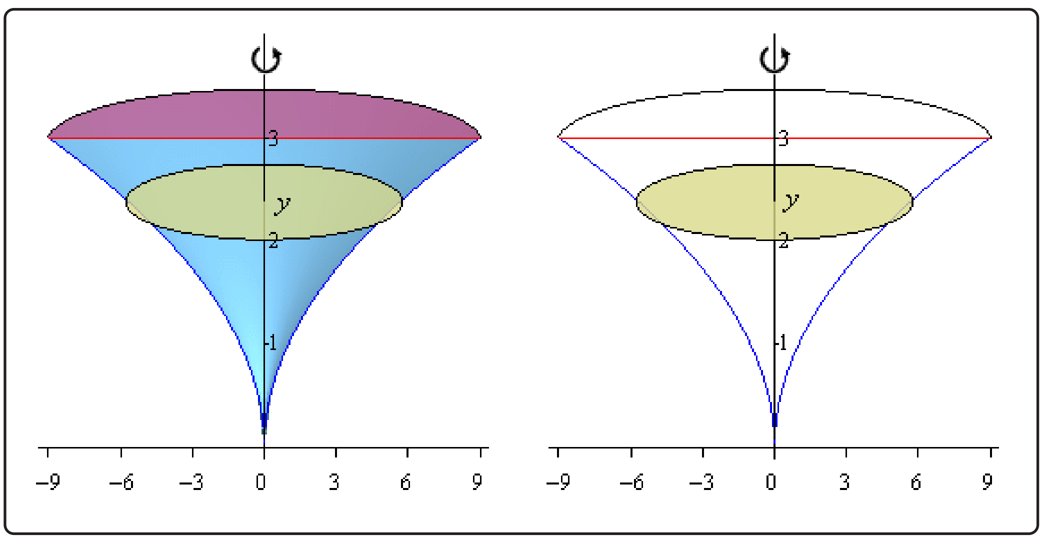
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative disk can be of great help when we go to write down the area formula. Also, getting the representative disk can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative disk. The image on the left shows a representative disk with the front half of the solid cut away and the image on the right shows a representative disk with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



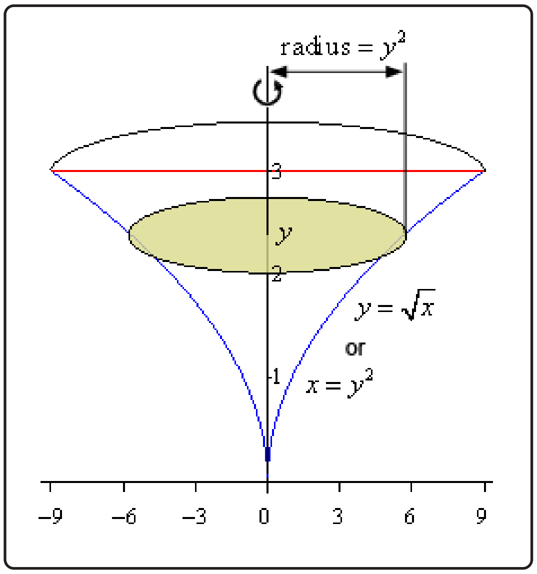
Hint

Determine a formula for the area of the ring.

Step 3

We now need to find a formula for the area of the disk. Because we are using disks that are centered on the y-axis we know that the area formula will need to be in terms of y. This in turn means that we'll need to rewrite the equation of the boundary curve to get into terms of y.

Here is another sketch of a representative disk with all of the various quantities we need put into it.



As we can see from the sketch the disk is centered on the y-axis and placed at some y. The radius of the disk is the distance from the y-axis to the curve defining the edge of the solid. In other words,

$$\text{Radius } = y^2$$

The area of the disk is then,

$$A(y) = \pi(\text{Radius})^2 = \pi\left(y^2\right)^2 = \pi y^4$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

For the limits on the integral we can see that the "first" disk in the solid would occur at $y = 0$ and the "last" disk would occur at $y = 3$. Our limits are then : $0 \le y \le 3$.

The volume is then,

$$V = \int_0^3 \pi y^4 \, dy = \frac{1}{5}\pi y^5 \bigg|_0^3 = \boxed{\frac{243}{5}\pi}$$

---

2. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $y = 7 - x^2$, $x = -2$, $x = 2$ and the x-axis about the x-axis.

Hint

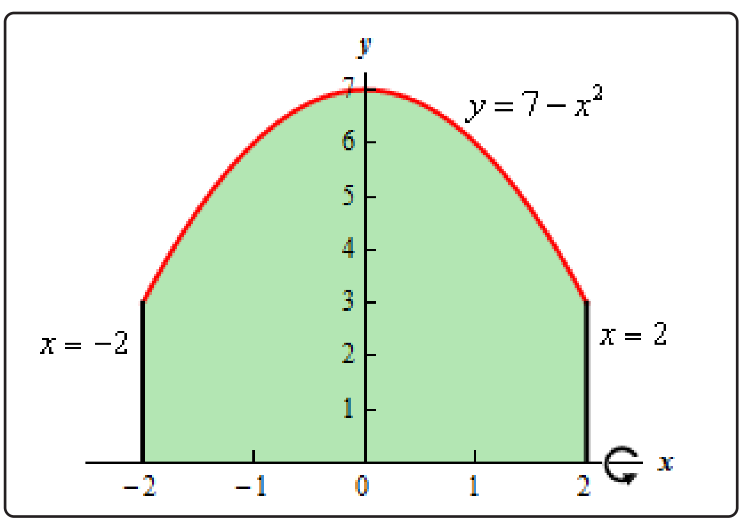
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



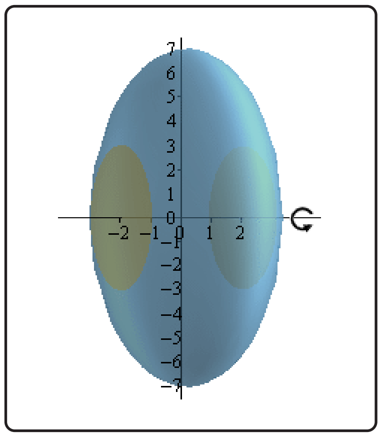
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative disk.

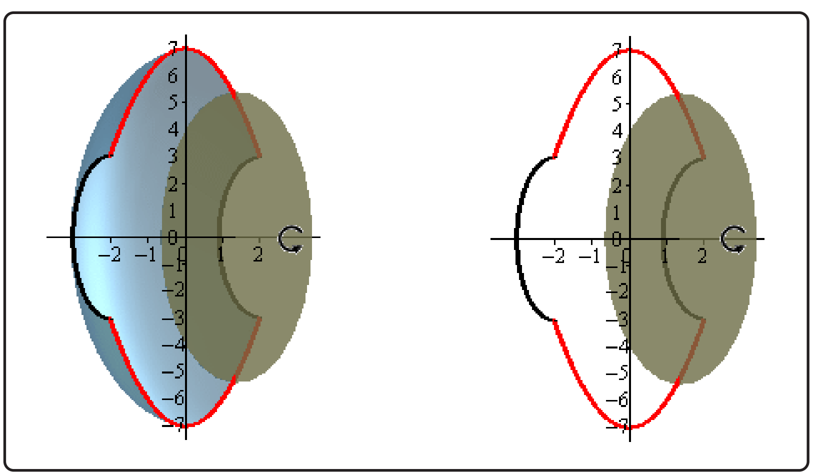
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative disk can be of great help when we go to write down the area formula. Also, getting the representative disk can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative disk. The image on the left shows a representative disk with the front half of the solid cut away and the image on the right shows a representative disk with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



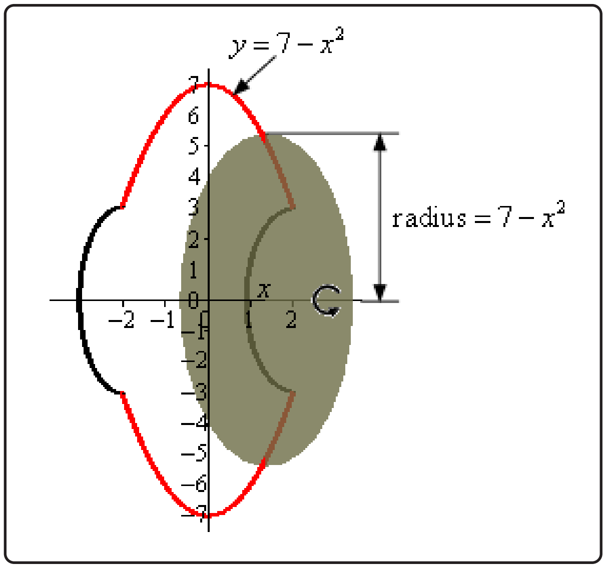
Hint

Determine a formula for the area of the disk.

Step 3

We now need to find a formula for the area of the disk. Because we are using disks that are centered on the x-axis we know that the area formula will need to be in terms of x. Therefore, the equation of the curve will need to be in terms of x (which in this case it already is).

Here is another sketch of a representative disk with all of the various quantities we need put into it.



As we can see from the sketch the disk is centered on the x-axis and placed at some x. The radius of the disk is the distance from the x-axis to the curve defining the edge of the solid. In other words,

$$\text{Radius } = 7 - x^2$$

The area of the disk is then,

$$A(x) = \pi(\text{Radius})^2 = \pi\left(7 - x^2\right)^2 = \pi\left(49 - 14x^2 + x^4\right)$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

For the limits on the integral we can see that the "first" disk in the solid would occur at $x = -2$ and the "last" disk would occur at $x = 2$. Our limits are then : $-2 \le x \le 2$.

The volume is then,

$$V = \int_{-2}^{2} \pi \left(49 - 14x^2 + x^4\right)\,dx = \pi\left(49x - \frac{14}{3}x^3 + \frac{1}{5}x^5\right)\Big|_{-2}^{2} = \boxed{\frac{2012}{15}\pi}$$

---

3. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $x = y^2 - 6y + 10$ and $x = 5$ about the y-axis.

Hint

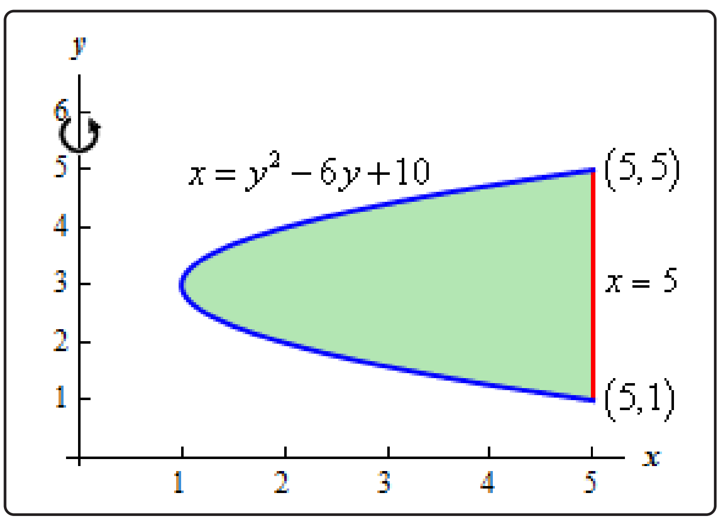
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



Here is the work used to determine the intersection points (we'll need these later).

$$\begin{aligned} y^2 - 6y + 10 &= 5 \\ y^2 - 6y + 5 &= 0 \\ (y-5)(y-1) &= 0 \qquad \Rightarrow \qquad y = 1, \quad y = 5 \qquad \Rightarrow \qquad (5,1) \;\; \& \;\; (5,5) \end{aligned}$$

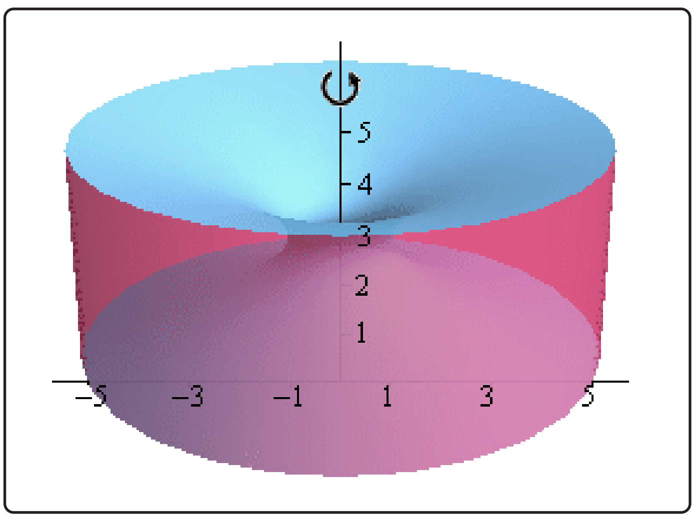
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.

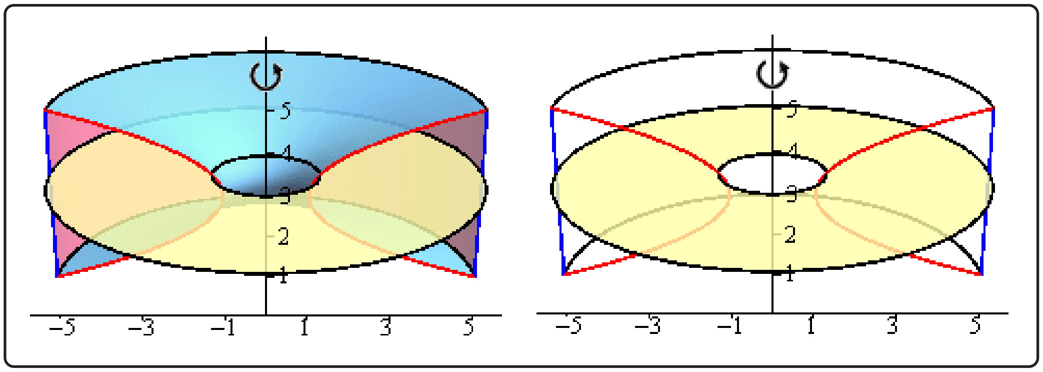
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative ring can be of great help when we go to write down the area formula. Also, getting the representative ring can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative ring. The image on the left shows a representative ring with the front half of the solid cut away and the image on the right shows a representative ring with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



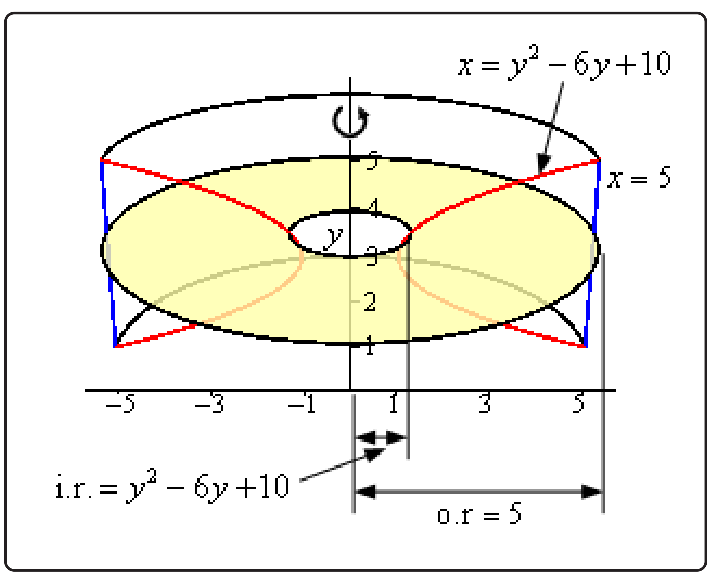
Hint

Determine a formula for the area of the ring.

Step 3

We now need to find a formula for the area of the ring. Because we are using rings that are centered on the y-axis we know that the area formula will need to be in terms of y. Therefore, the equation of the curves will need to be in terms of y (which in this case they already are).

Here is another sketch of a representative ring with all of the various quantities we need put into it.



As we can see from the sketch the ring is centered on the y-axis and placed at some y. The inner radius of the ring is the distance from the y-axis to the curve defining the inner edge of the solid. The outer radius of the ring is the distance from the y-axis to the curve defining the outer edge of the solid. In other words,

$$\text{Inner Radius} = y^2 - 6y + 10 \qquad \text{Outer Radius} = 5$$

The area of the ring is then,

$$\begin{aligned} A(x) &= \pi\left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2\right] \\ &= \pi\left[(5)^2 - \left(y^2 - 6y + 10\right)^2\right] = \pi\left(-75 + 120y - 56y^2 + 12y^3 - y^4\right) \end{aligned}$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the intersection points shown in the graph from Step 1 we can see that the "first" ring in the solid would occur at $y = 1$ and the "last" ring would occur at $y = 5$. Our limits are then : $1 \le y \le 5$.

The volume is then,

$$\begin{aligned} V &= \int_1^5 \pi\left(-75 + 120y - 56y^2 + 12y^3 - y^4\right)\,dy \\ &= \pi\left(-75y + 60y^2 - \frac{56}{3}y^3 + 3y^4 - \frac{1}{5}y^5\right)\Bigg|_1^5 = \boxed{\frac{1088}{15}\pi} \end{aligned}$$

---

4. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $y = 2x^2$ and $y = x^3$ about the x-axis.

Hint

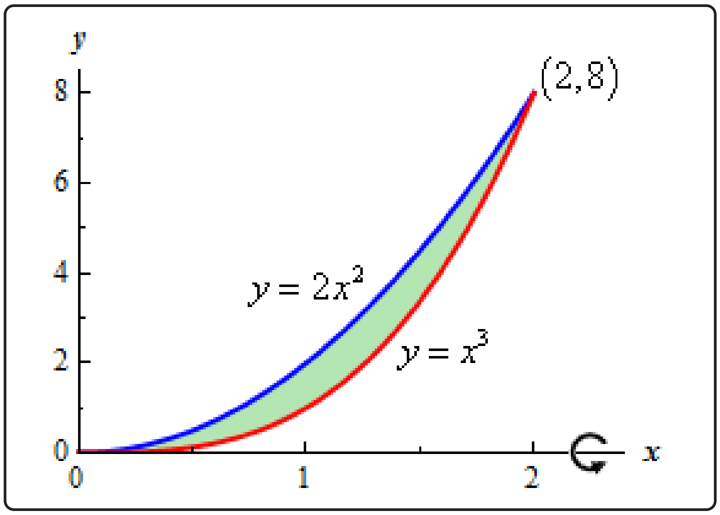
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



Here is the work used to determine the intersection points (we'll need these later).

$$\begin{aligned} x^3 &= 2x^2 \\ x^3 - 2x^2 &= 0 \\ x^2(x-2) &= 0 \qquad \Rightarrow \qquad x = 0, \quad x = 2 \qquad \Rightarrow \qquad (0,0) \ \ \& \ \ (2,8) \end{aligned}$$

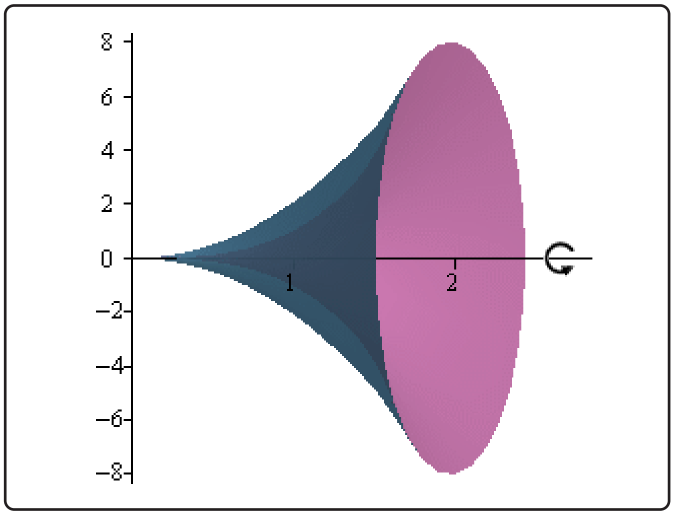
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.

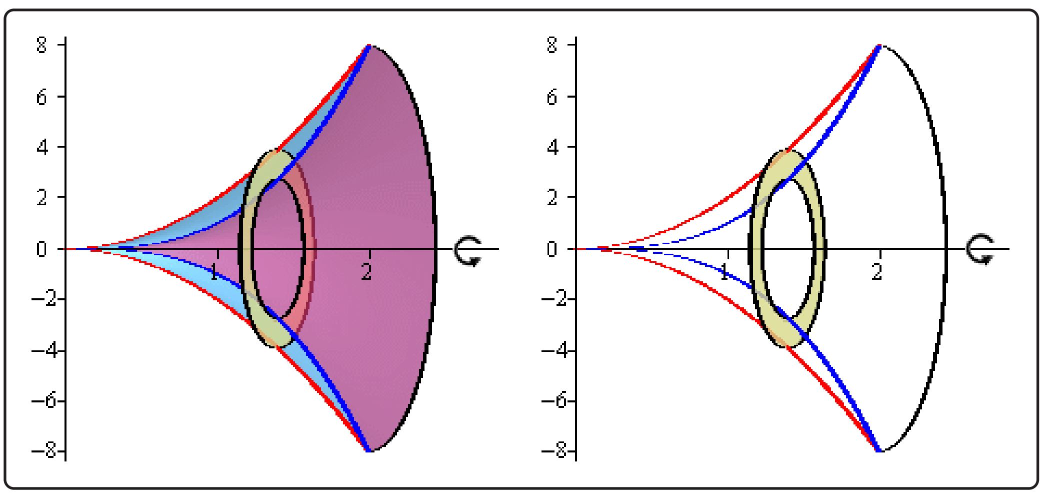
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative ring can be of great help when we go to write down the area formula. Also, getting the representative ring can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative ring. The image on the left shows a representative ring with the front half of the solid cut away and the image on the right shows a representative ring with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



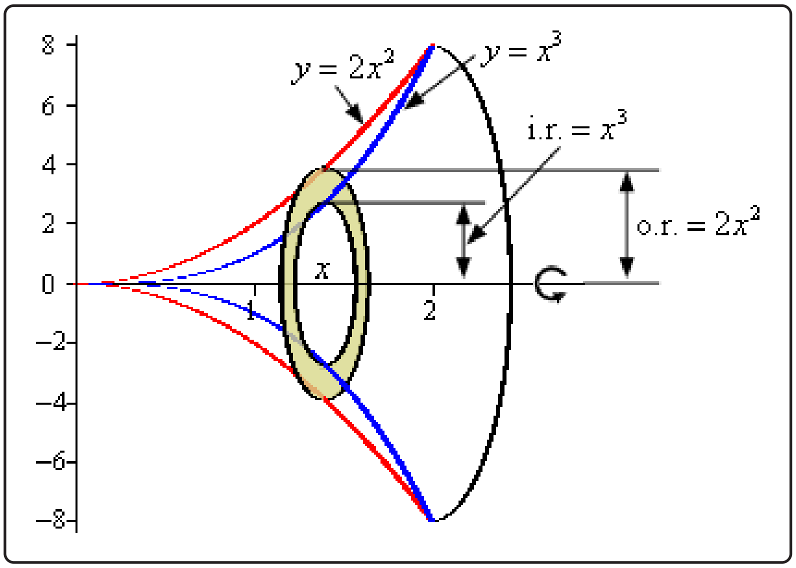
Hint

Determine a formula for the area of the ring.

Step 3

We now need to find a formula for the area of the ring. Because we are using rings that are centered on the x-axis we know that the area formula will need to be in terms of x. Therefore, the equation of the curves will need to be in terms of x (which in this case they already are).

Here is another sketch of a representative ring with all of the various quantities we need put into it.



As we can see from the sketch the ring is centered on the x-axis and placed at some x. The inner radius of the ring is the distance from the x-axis to the curve defining the inner edge of the solid. The outer radius of the ring is the distance from the x-axis to the curve defining the outer edge of the solid. In other words,

$$\text{Inner Radius} = x^3 \qquad\qquad \text{Outer Radius} = 2x^2$$

The area of the ring is then,

$$\begin{aligned} A(x) &= \pi\left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2\right] \\ &= \pi\left[\left(2x^2\right)^2 - \left(x^3\right)^2\right] = \pi\left(4x^4 - x^6\right) \end{aligned}$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the intersection points shown in the graph from Step 1 we can see that the "first" ring in the solid would occur at $x = 0$ and the "last" ring would occur at $x = 2$. Our limits are then : $0 \le x \le 2$.

The volume is then,

$$V = \int_0^2 \pi \left(4x^4 - x^6\right)\, dx = \pi \left(\frac{4}{5}x^5 - \frac{1}{7}x^7\right)\Big|_0^2 = \boxed{\frac{256}{35}\pi}$$

---

5. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $y = 6\mathbf{e}^{-2x}$ and $y = 6 + 4x - 2x^2$ between $x = 0$ and $x = 1$ about the line $y = -2$.

Hint

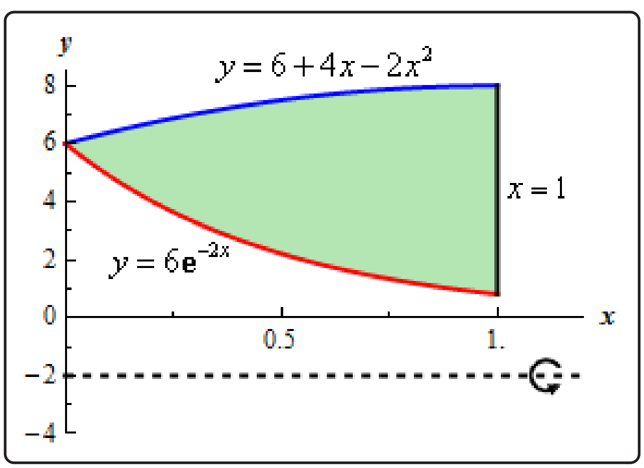
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



For the intersection point on the left a quick check by plugging $x = 0$ into both equations shows that the intersection point is in fact $(0, 6)$ as we might have guessed from the graph. We'll be needing this point in a bit. From the sketch of the region it is also clear that there is no intersection point on the right.

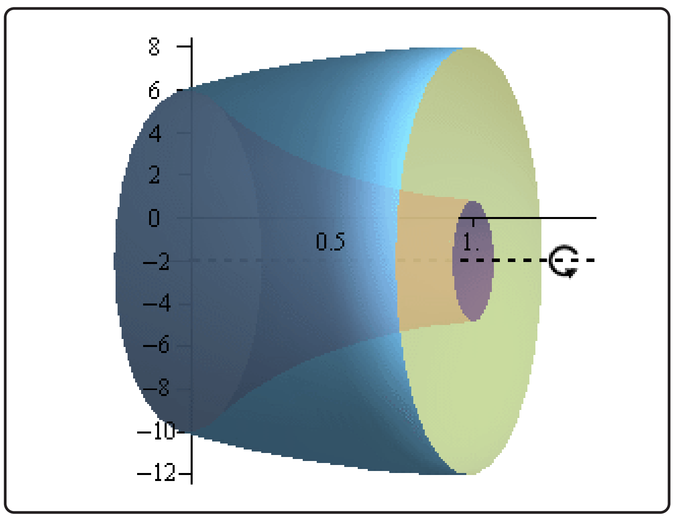
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.

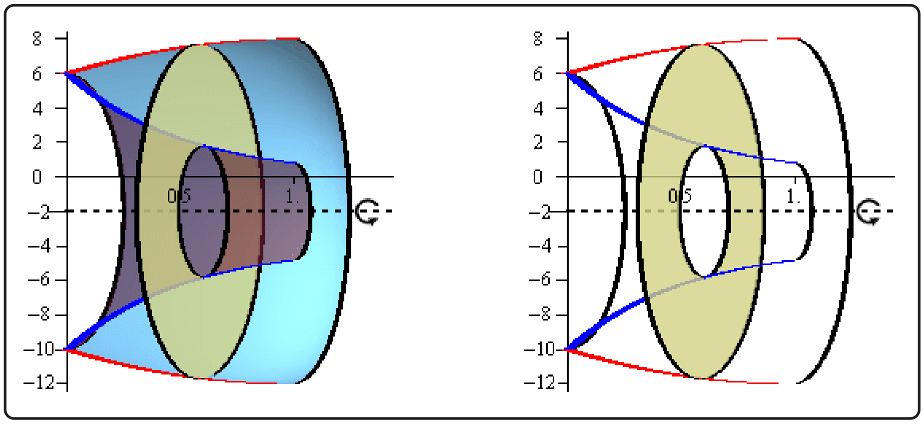
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative ring can be of great help when we go to write down the area formula. Also, getting the representative ring can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative ring. The image on the left shows a representative ring with the front half of the solid cut away and the image on the right shows a representative ring with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



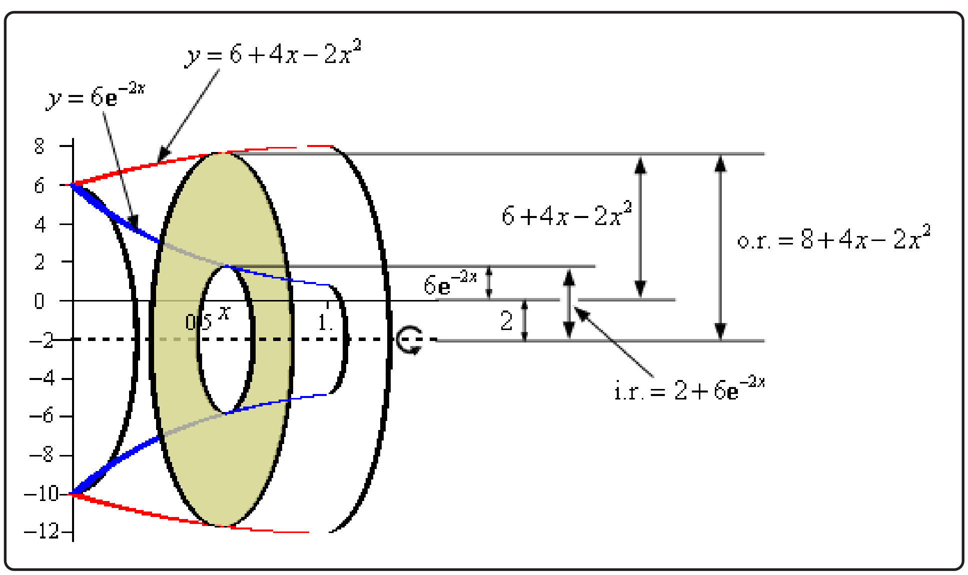
Hint

Determine a formula for the area of the ring.

Step 3

We now need to find a formula for the area of the ring. Because we are using rings that are centered on a horizontal axis (*i.e.* parallel to the x-axis) we know that the area formula will need to be in terms of x. Therefore, the equations of the curves will need to be in terms of x (which in this case they already are).

Here is another sketch of a representative ring with all of the various quantities we need put into it.



From the sketch we can see the ring is centered on the line $y = -2$ and placed at some x.

The inner radius of the ring is the distance from the axis of rotation to the x-axis (a distance of 2) followed by the distance from the x-axis to the curve defining the inner edge of the solid (a distance of $6\mathbf{e}^{-2x}$).

Likewise, the outer radius of the ring is the distance from the axis of rotation to the x-axis (again, a distance of 2) followed by the distance from the x-axis to the curve defining the outer edge of the solid (a distance of $6 + 4x - 2x^2$).

So, the inner and outer radii are,

$$\text{Inner Radius} = 2 + 6\mathbf{e}^{-2x} \qquad \text{Outer Radius} = 2 + 6 + 4x - 2x^2 = 8 + 4x - 2x^2$$

The area of the ring is then,

$$\begin{aligned} A(x) &= \pi\left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2\right] \\ &= \pi\left[\left(8 + 4x - 2x^2\right)^2 - \left(2 + 6\mathbf{e}^{-2x}\right)^2\right] \\ &= \pi\left(60 + 64x - 16x^2 - 16x^3 + 4x^4 - 24\mathbf{e}^{-2x} - 36\mathbf{e}^{-4x}\right) \end{aligned}$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" ring in the solid would occur at $x = 0$ and the "last" ring would occur at $x = 1$. Our limits are then : $0 \le x \le 1$.

The volume is then,

$$\begin{aligned} V &= \int_0^1 \pi\left(60 + 64x - 16x^2 - 16x^3 + 4x^4 - 24\mathbf{e}^{-2x} - 36\mathbf{e}^{-4x}\right) dx \\ &= \pi\left(60x + 32x^2 - \frac{16}{3}x^3 - 4x^4 + \frac{4}{5}x^5 + 12\mathbf{e}^{-2x} + 9\mathbf{e}^{-4x}\right)\Bigg|_0^1 \\ &= \boxed{\left(\frac{937}{15} + 12\mathbf{e}^{-2} + 9\mathbf{e}^{-4}\right)\pi} \end{aligned}$$

6. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $y = 10 - 6x + x^2$, $y = -10 + 6x - x^2$, $x = 1$ and $x = 5$ about the line $y = 8$.

Hint

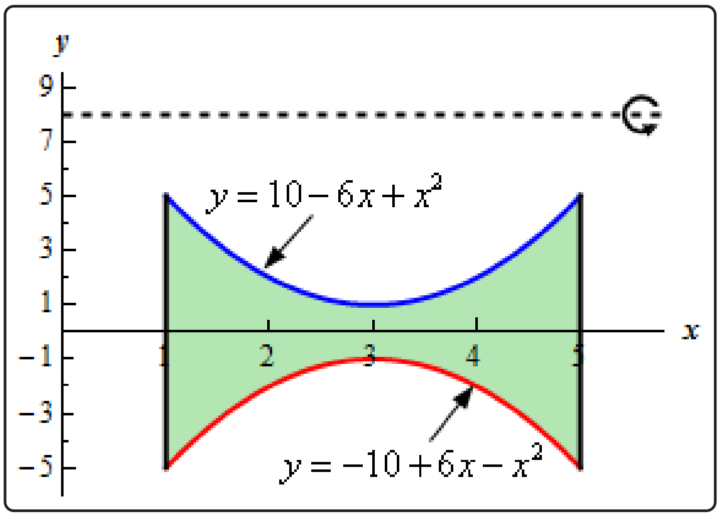
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



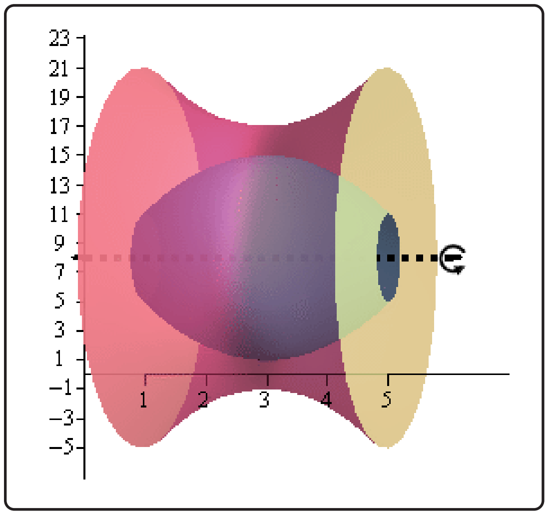
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.

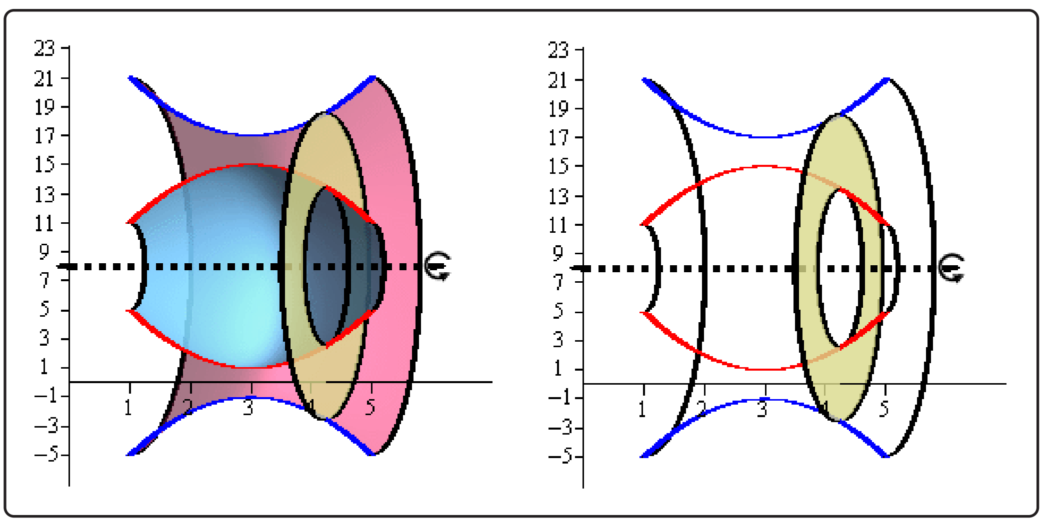
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative ring can be of great help when we go to write down the area formula. Also, getting the representative ring can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative ring. The image on the left shows a representative ring with the front half of the solid cut away and the image on the right shows a representative ring with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



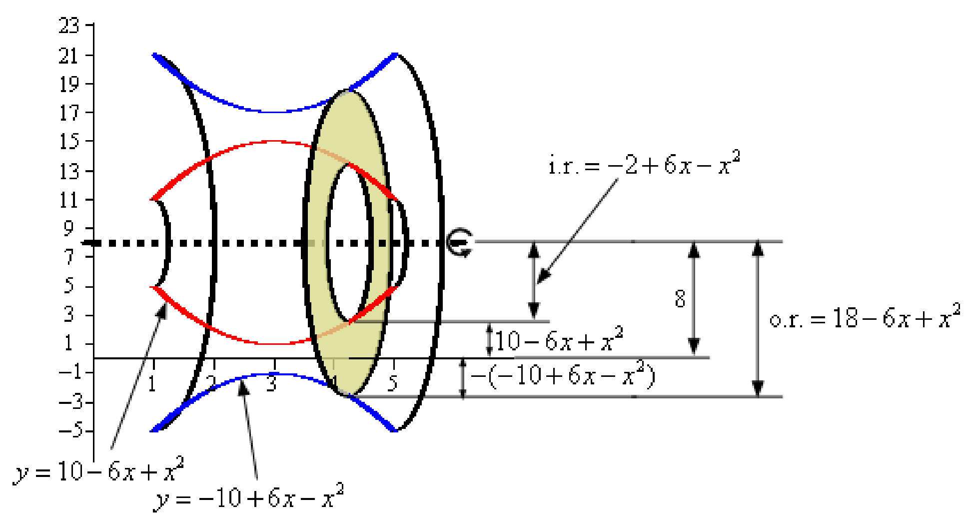
Hint

Determine a formula for the area of the ring.

Step 3

We now need to find a formula for the area of the ring. Because we are using rings that are centered on a horizontal axis (*i.e.* parallel to the x-axis) we know that the area formula will need to be in terms of x. Therefore, the equations of the curves will need to be in terms of x (which in this case they already are).

Here is another sketch of a representative ring with all of the various quantities we need put into it.



From the sketch we can see the ring is centered on the line $y = 8$ and placed at some x.

The inner radius of the ring is then the distance from the axis of rotation to the curve defining the inner edge of the solid. To determine a formula for this first notice that the axis of rotation is a distance of 8 from the x-axis. Next, the curve defining the inner edge of the solid is a distance of $y = 10 - 6x + x^2$ from the x-axis. The inner radius is then the difference between these two distances or,

$$\text{Inner Radius} = 8 - \left(10 - 6x + x^2\right) = -2 + 6x - x^2$$

The outer radius is computed in a similar manner. It is the distance from the axis of rotation to the x-axis (a distance of 8) and then it continues below the x-axis until it reaches the curve defining the outer edge of the solid. So, we need to add these two distances but we need to be careful because the "lower" function is in fact negative value and so the distance of the point on the lower function from the x-axis is in fact : $-\left(-10+6x-x^2\right)$ as is shown on the sketch. The negative in front of the equation makes sure that the negative value of the function is turned into a positive quantity (which we need for our distance). The outer radius is then the sum of these two distances or,

$$\text{Outer Radius} = 8-\left(-10+6x-x^2\right) = 18-6x+x^2$$

The area of the ring is then,

$$\begin{aligned} A(x) &= \pi\left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2\right] \\ &= \pi\left[\left(18-6x+x^2\right)^2 - \left(-2+6x-x^2\right)^2\right] = \pi\left(320-192x+32x^2\right) \end{aligned}$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" ring in the solid would occur at $x=1$ and the "last" ring would occur at $x=5$. Our limits are then : $1 \le x \le 5$.

The volume is then,

$$V = \int_1^5 \pi\left(320-192x+32x^2\right)\,dx = \pi\left(320x-96x^2+\frac{32}{3}x^3\right)\Bigg|_1^5 = \boxed{\frac{896}{3}\pi}$$

---

7. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $x=y^2-4$ and $x=6-3y$ about the line $x=24$.

Hint

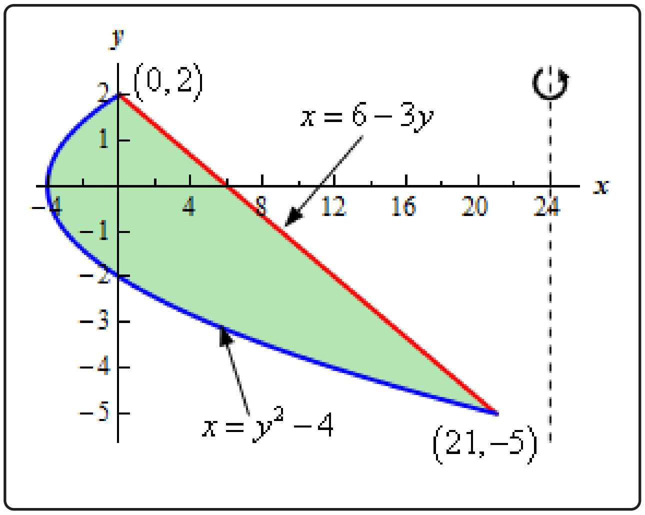
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



To get the intersection points shown on the sketch all we need to do is set the two equations equal and solve (we'll need these in a bit).

$$\begin{aligned} y^2 - 4 &= 6 - 3y \\ y^2 + 3y - 10 &= 0 \\ (y+5)(y-2) &= 0 \qquad \Rightarrow \qquad y = -5, \quad y = 2 \qquad \Rightarrow \qquad (21,-5) \;\; \& \;\; (0,2) \end{aligned}$$

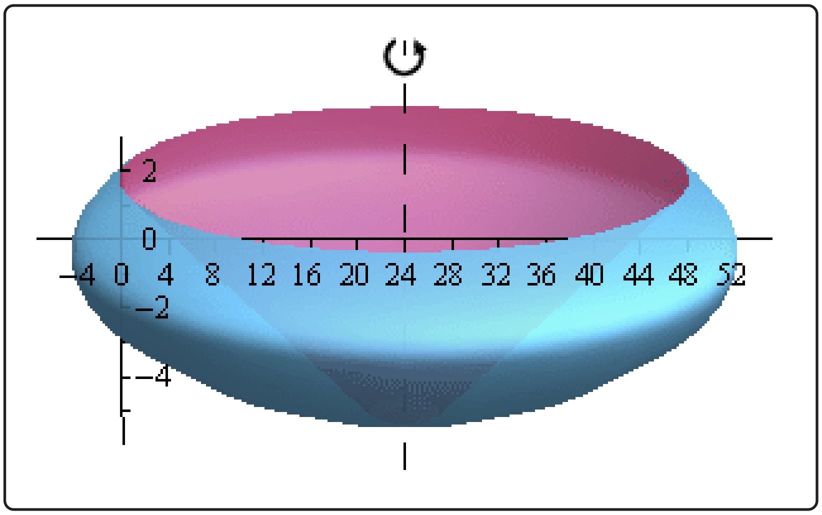
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.

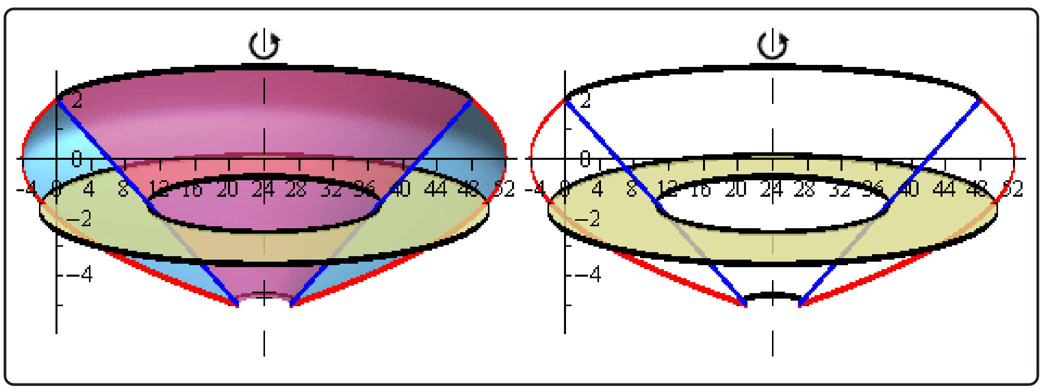
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative ring can be of great help when we go to write down the area formula. Also, getting the representative ring can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative ring. The image on the left shows a representative ring with the front half of the solid cut away and the image on the right shows a representative ring with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



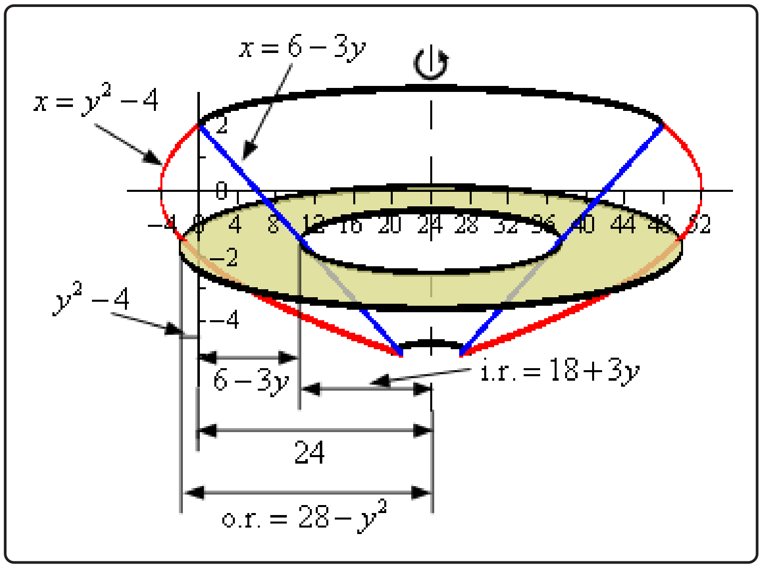
Hint

Determine a formula for the area of the ring.

Step 3

We now need to find a formula for the area of the ring. Because we are using rings that are centered on a vertical axis (*i.e.* parallel to the y-axis) we know that the area formula will need to be in terms of y. Therefore, the equation of the curves will need to be in terms of y (which in this case they already are).

Here is another sketch of a representative ring with all of the various quantities we need put into it.



From the sketch we can see the ring is centered on the line $x = 24$ and placed at some y.

The inner radius of the ring is then the distance from the axis of rotation to the curve defining the inner edge of the solid. To determine a formula for this first notice that the axis of rotation is a distance of 24 from the y-axis. Next, the curve defining the inner edge of the solid is a distance of $x = 6 - 3y$ from the y-axis. The inner radius is then the difference between these two distances or,

$$\text{Inner Radius} = 24 - (6 - 3y) = 18 + 3y$$

The outer radius is computed in a similar manner but is a little trickier. In this case the curve defining the outer edge of the solid occurs on both the left and right of the y-axis.

Let's first look at the case as shown in the sketch above. In this case the value of the function defining the outer edge of the solid is to the left of the y-axis and so has a negative value. The distance of this point from the y-axis is then $-\left(y^2 - 4\right)$ where the minus sign turns the negative function value into a positive value that we need for distance. The outer radius for this case is then the sum of the distance of the axis of rotation to the y-axis (a distance of 24) and the distance of the curve defining the outer edge to the y-axis (which we found above).

If the curve defining the outer edge of the solid is to the right of the y-axis then it will have a positive value and so the distance of points on the curve and the y-axis is just $y^2 - 4$. We don't need the minus sign in this case because the function value is already positive, which we need for distance. The outer radius in this case is then the distance from the axis of rotation to the y-axis (a distance of 24) minus this new distance.

Nicely enough in either case the outer radius is then,

$$\text{Outer Radius} = 24 - \left(y^2 - 4\right) = 28 - y^2$$

Note that in cases like this where the curve defining an edge has both positive and negative values the final equation of the radius (inner or outer depending on the problem) will be the same. You just need to be careful in setting up the case you choose to look at. If you get the first case set up correctly you won't need to do the second as the formula will be the same.

The area of the ring is then,

$$\begin{aligned} A\left(x\right) &= \pi\left[\left(\text{Outer Radius}\right)^2 - \left(\text{Inner Radius}\right)^2\right] \\ &= \pi\left[\left(28 - y^2\right)^2 - \left(18 + 3y\right)^2\right] = \pi\left(460 - 108y - 65y^2 + y^4\right) \end{aligned}$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the intersection points of the two curves we found in Step 1 we can see that the "first" ring in the solid would occur at $y = -5$ and the "last" ring would occur at $y = 2$. Our limits are then : $-5 \le y \le 2$.

The volume is then,

$$\begin{aligned} V &= \int_{-5}^{2} \pi\left(460 - 108y - 65y^2 + y^4\right)\,dy = \pi\left(460y - 54y^2 - \frac{65}{3}y^3 + \frac{1}{5}y^5\right)\Bigg|_{-5}^{2} \\ &= \boxed{\frac{31556}{15}\pi} \end{aligned}$$

8. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $y = 2x + 1$, $x = 4$ and $y = 3$ about the line $x = -4$.

Hint

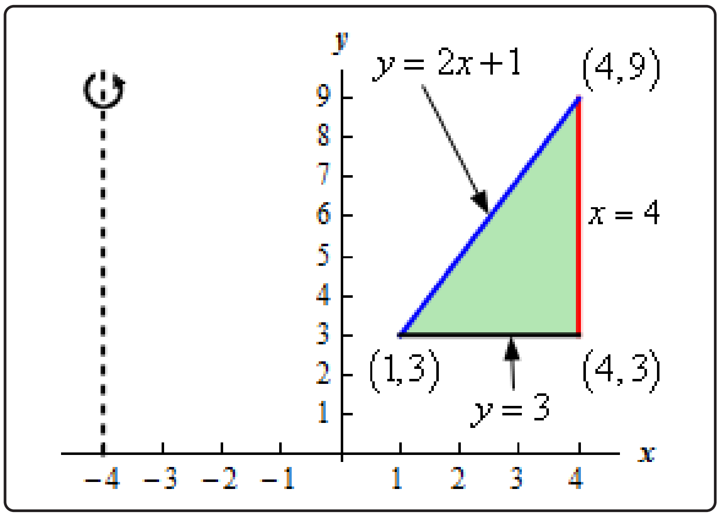
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



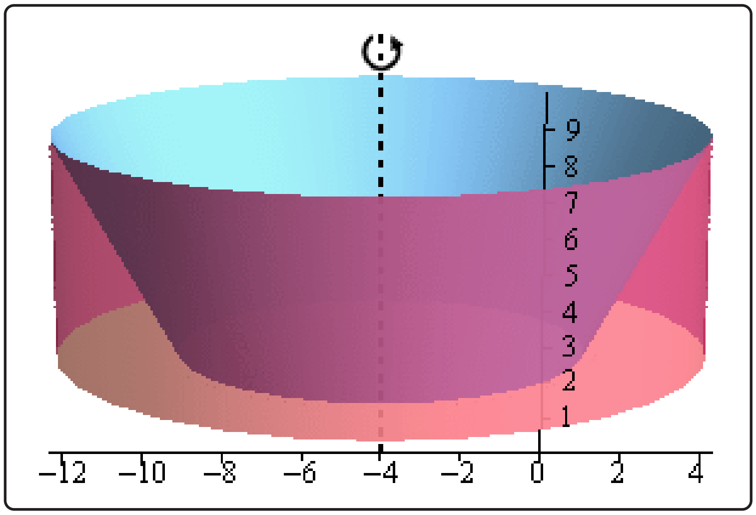
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.

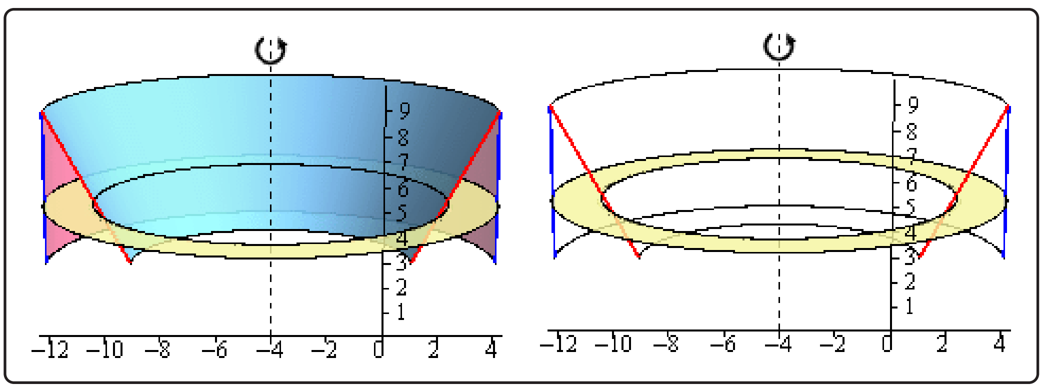
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative ring can be of great help when we go to write down the area formula. Also, getting the representative ring can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative ring. The image on the left shows a representative ring with the front half of the solid cut away and the image on the right shows a representative ring with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



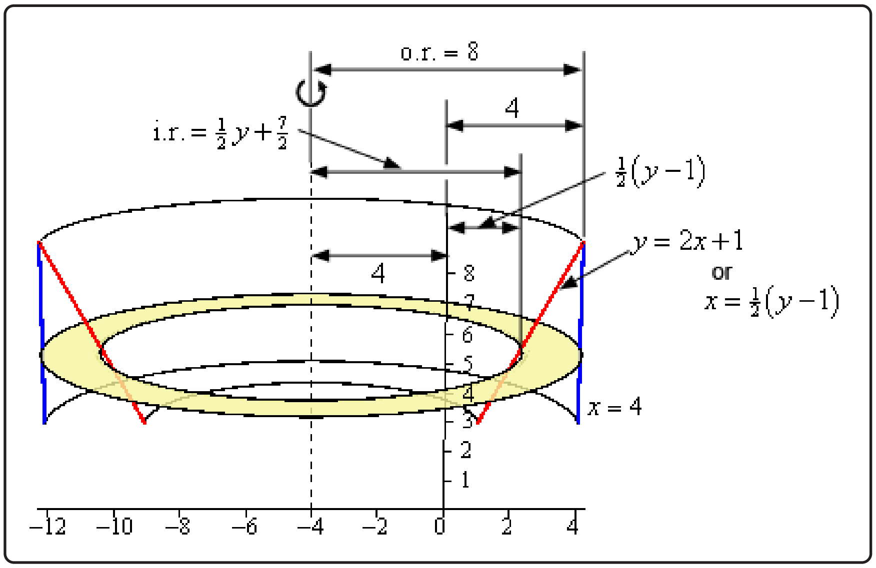
Hint

Determine a formula for the area of the ring.

Step 3

We now need to find a formula for the area of the ring. Because we are using rings that are centered on a vertical axis (*i.e.* parallel to the y-axis) we know that the area formula will need to be in terms of y. Therefore, the equations of the curves will need to be in terms of y and so we'll need to rewrite the equation of the line to be in terms of y.

Here is another sketch of a representative ring with all of the various quantities we need put into it.



From the sketch we can see the ring is centered on the line $x = -4$ and placed at some y.

The inner radius of the ring is the distance from the axis of rotation to the y-axis (a distance of 4) followed by the distance from the y-axis to the curve defining the inner edge of the solid (a distance of $\frac{1}{2}(y-1)$).

Likewise, the outer radius of the ring is the distance from the axis of rotation to the y-axis (again, a distance of 4) followed by the distance from the y-axis to the curve defining the outer edge of the solid (a distance of 4).

So, the inner and outer radii are,

$$\text{Inner Radius} = 4 + \frac{1}{2}(y-1) = \frac{1}{2}y + \frac{7}{2} \qquad \text{Outer Radius} = 4 + 4 = 8$$

The area of the ring is then,

$$\begin{aligned} A(x) &= \pi\left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2\right] \\ &= \pi\left[(8)^2 - \left(\frac{1}{2}y + \frac{7}{2}\right)^2\right] = \pi\left(\frac{207}{4} - \frac{7}{2}y - \frac{1}{4}y^2\right) \end{aligned}$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the intersection points of the two curves we found in Step 1 we can see that the "first" ring in the solid would occur at $y = 3$ and the "last" ring would occur at $y = 9$. Our limits are then : $3 \le y \le 9$.

The volume is then,

$$V = \int_3^9 \pi \left(\frac{207}{4} - \frac{7}{2}y - \frac{1}{4}y^2 \right) dy = \pi \left(\frac{207}{4}y - \frac{7}{4}y^2 - \frac{1}{12}y^3 \right) \Bigg|_3^9 = \boxed{126\pi}$$

6.4 Solids of Revolution / Method of Cylinders

1. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $x = (y-2)^2$, the x-axis and the y-axis about the x-axis.

Hint

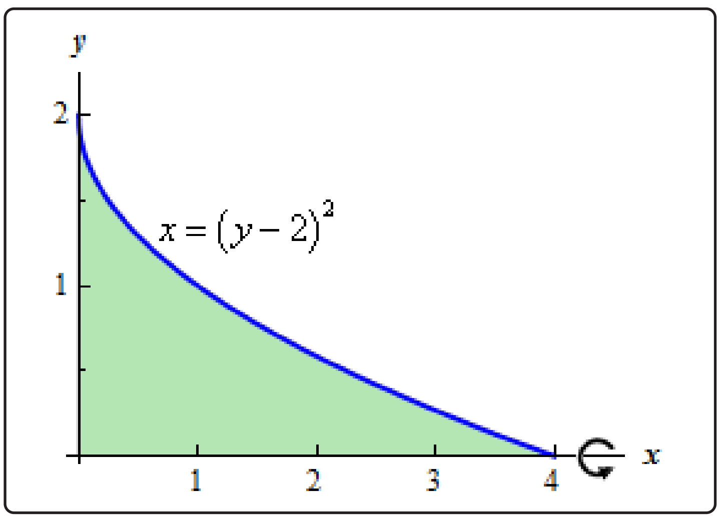
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



Note that we only used the lower half of the parabola here because if we also included the upper half there would be nothing to bound the region above it. Therefore, in order for the x-axis and y-axis to be bounding curves we have to use the portion below the lower half of the parabola.

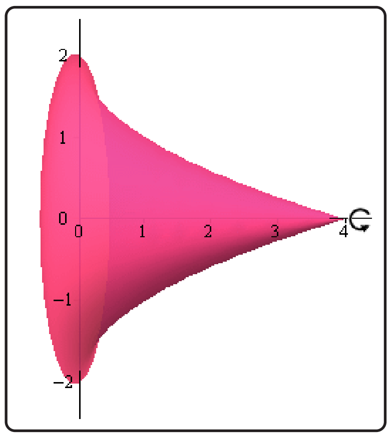
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.

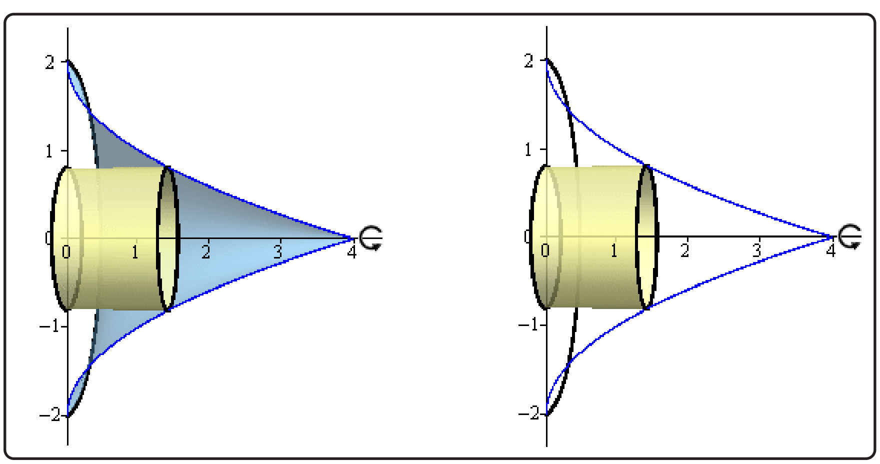
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



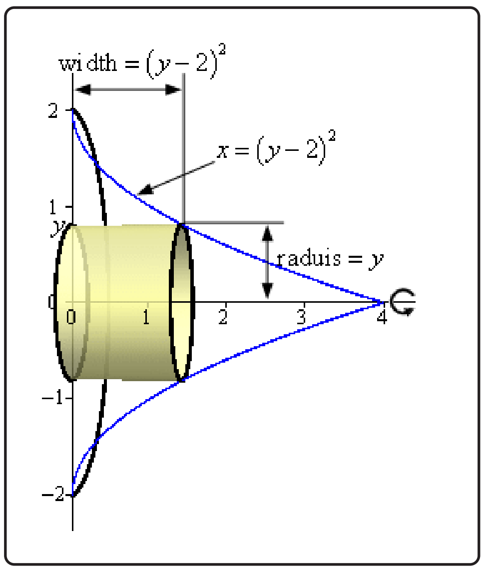
Hint

Determine a formula for the surface area of the cylinder.

Step 3

We now need to find a formula for the surface area of the cylinder. Because we are using cylinders that are centered on the x-axis we know that the area formula will need to be in terms of y. Therefore, the equation of the curves will need to be in terms of y (which in this case they already are).

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.



From the sketch we can see the cylinder is centered on the x-axis and the upper edge of the cylinder is at some y.

The radius of the cylinder is just the distance from the x-axis to the upper edge of the cylinder (*i.e.* y). The width of the cylinder is the distance from the y-axis to the curve defining the edge of the solid (a distance of $(y-2)^2$).

So, the radius and width of the cylinder are,

$$\text{Radius} = y \qquad\qquad \text{Width} = (y-2)^2$$

The area of the cylinder is then,

$$A(y) = 2\pi\,(\text{Radius})\,(\text{Width}) = 2\pi\,(y)\,(y-2)^2 = 2\pi\left(4y - 4y^2 + y^3\right)$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $y = 0$ and the "last" cylinder would occur at $y = 2$. Our limits are then : $0 \le y \le 2$.

The volume is then,

$$V = \int_0^2 2\pi \left(4y - 4y^2 + y^3\right)\, dy = 2\,\pi \left(2y^2 - \frac{4}{3}y^3 + \frac{1}{4}y^4\right)\Big|_0^2 = \boxed{\frac{8}{3}\pi}$$

2. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $y = \frac{1}{x}$, $x = \frac{1}{2}$, $x = 4$ and the x-axis about the y-axis.

Hint

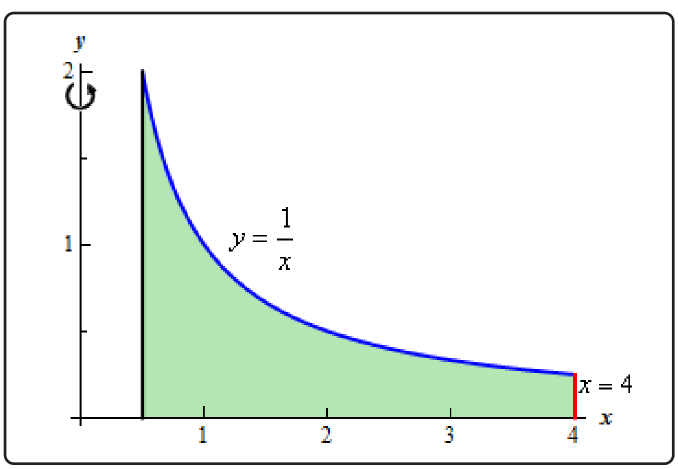
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



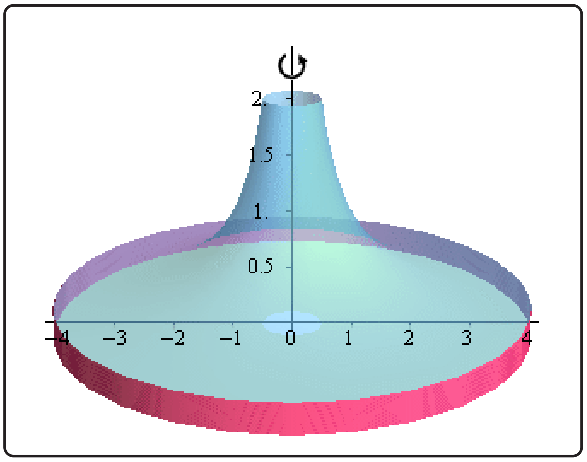
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.

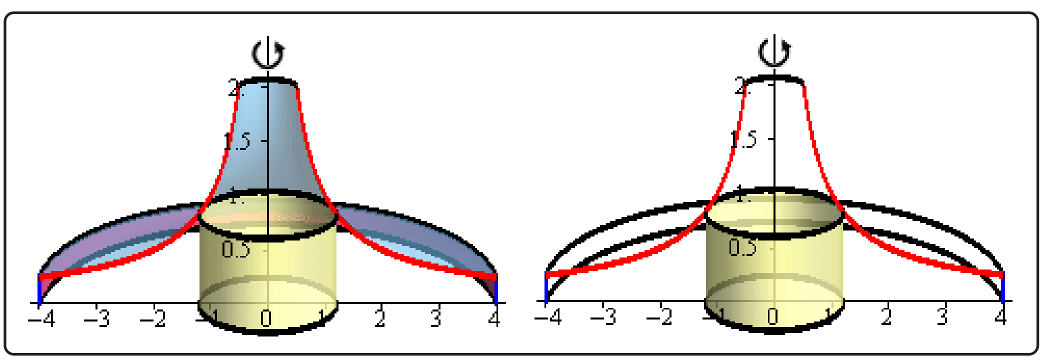
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



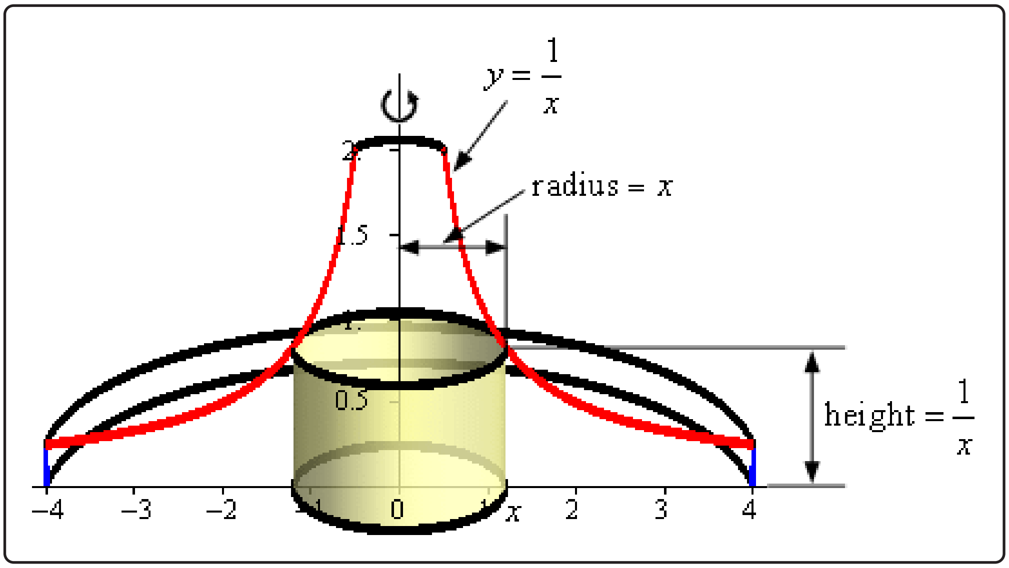
Hint

Determine a formula for the surface area of the cylinder.

Step 3

We now need to find a formula for the surface area of the cylinder. Because we are using cylinders that are centered on the y-axis we know that the area formula will need to be in terms of x. Therefore, the equation of the curves will need to be in terms of x (which in this case they already are).

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.



From the sketch we can see the cylinder is centered on the y-axis and the right edge of the cylinder is at some x.

The radius of the cylinder is just the distance from the y-axis to the right edge of the cylinder (*i.e.* x). The height of the cylinder is the distance from the x-axis to the curve defining the edge of the solid (a distance of $\frac{1}{x}$).

So, the radius and width of the cylinder are,

$$\text{Radius} = x \qquad\qquad \text{Height} = \frac{1}{x}$$

The area of the cylinder is then,

$$A(x) = 2\pi\,(\text{Radius})\,(\text{Height}) = 2\pi\,(x)\left(\frac{1}{x}\right) = 2\pi$$

Do not expect all the variables to cancel out in the area formula. It may happen on occasion, as it did here, but it is rare with it does.

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $x = \frac{1}{2}$ and the "last" cylinder would occur at $x = 4$. Our limits are then : $\frac{1}{2} \le x \le 4$.

The volume is then,

$$V = \int_{\frac{1}{2}}^{4} 2\pi\,dx = 2\pi\,(x)\Big|_{\frac{1}{2}}^{4} = \boxed{7\pi}$$

---

3. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $y = 4x$ and $y = x^3$ about the y-axis. For this problem assume that $x \ge 0$.

Hint

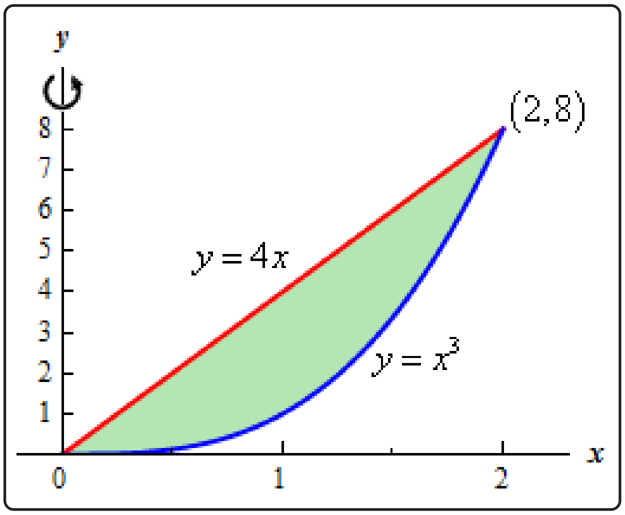
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



To get the intersection points shown on the graph, which we'll need in a bit, all we need to do is set the equations equal to each other and solve.

$$\begin{aligned} x^3 &= 4x \\ x^3 - 4x &= 0 \\ x\left(x^2-4\right) &= 0 \qquad \Rightarrow \qquad x = 0, \quad x = \pm 2 \qquad \Rightarrow \qquad (0,0) \;\;\&\;\; (2,8) \end{aligned}$$

Note that the problem statement said to assume that $x \geq 0$ and so we won't use the $x = -2$ intersection point.

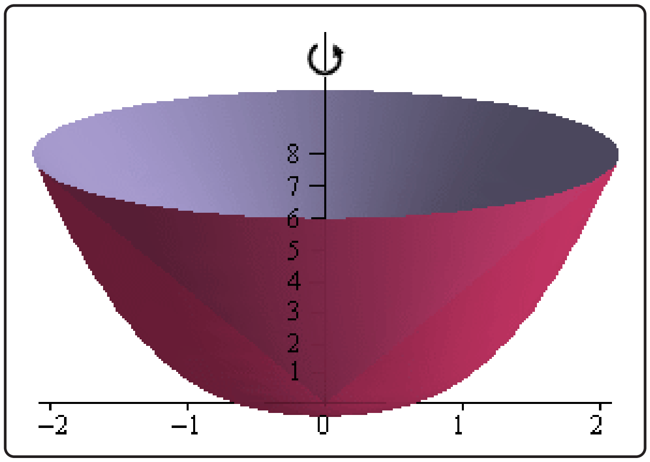
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.

Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



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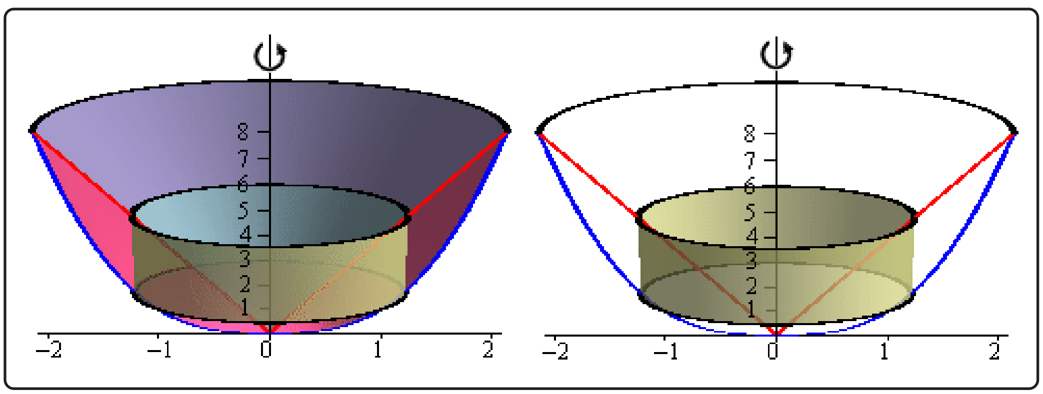


Hint

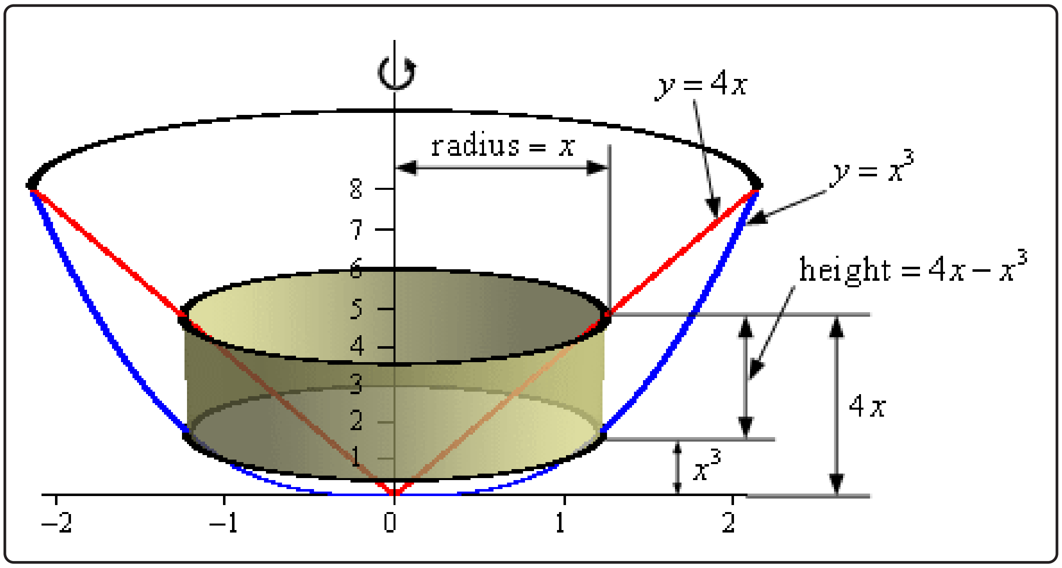
Determine a formula for the surface area of the cylinder.

Step 3

We now need to find a formula for the surface area of the cylinder. Because we are using cylinders that are centered on the y-axis we know that the area formula will need to be in terms of x. Therefore, the equation of the curves will need to be in terms of x

(which in this case they already are).

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.



From the sketch we can see the cylinder is centered on the y-axis and the right edge of the cylinder is at some x.

The radius of the cylinder is just the distance from the y-axis to the right edge of the cylinder (*i.e.* x).

The top of the cylinder is on the curve defining the upper portion of the solid and is a distance of $4x$ from the x-axis. The bottom of the cylinder is on the curve defining the lower portion of the solid and is a distance of x^3 from the x-axis. The height then is the difference of these two.

So, the radius and height of the cylinder are,

$$\text{Radius} = x \qquad\qquad \text{Height} = 4x - x^3$$

The area of the cylinder is then,

$$A(x) = 2\pi\,(\text{Radius})\,(\text{Height}) = 2\pi\,(x)\left(4x - x^3\right) = 2\pi\left(4x^2 - x^4\right)$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $x = 0$ and the "last" cylinder would occur at $x = 2$. Our limits are then : $0 \le x \le 2$.

The volume is then,

$$V = \int_0^2 2\pi\left(4x^2 - x^4\right)\,dx = 2\,\pi\left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Bigg|_0^2 = \boxed{\frac{128}{15}\pi}$$

---

4. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $y = 4x$ and $y = x^3$ about the x-axis. For this problem assume that $x \ge 0$.

Hint

Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

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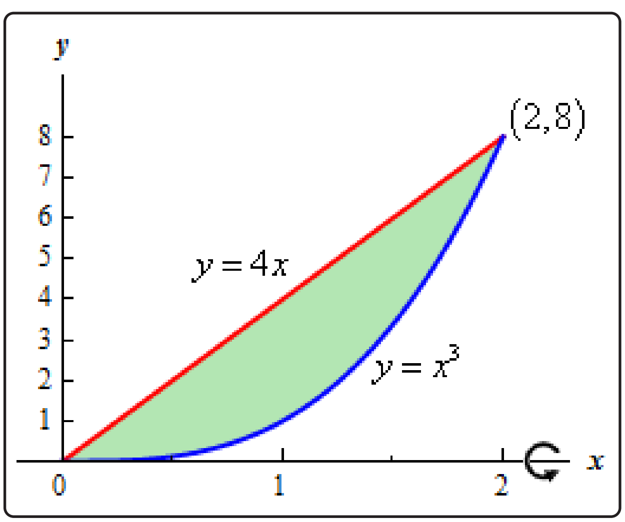


To get the intersection points shown on the graph, which we'll need in a bit, all we need to do is set the equations equal to each other and solve.

$$\begin{aligned} x^3 &= 4x \\ x^3 - 4x &= 0 \\ x\left(x^2-4\right) &= 0 \qquad \Rightarrow \qquad x=0, \quad x=\pm 2 \qquad \Rightarrow \qquad (0,0) \;\;\&\;\; (2,8) \end{aligned}$$

Note that the problem statement said to assume that $x \geq 0$ and so we won't use the $x = -2$ intersection point.

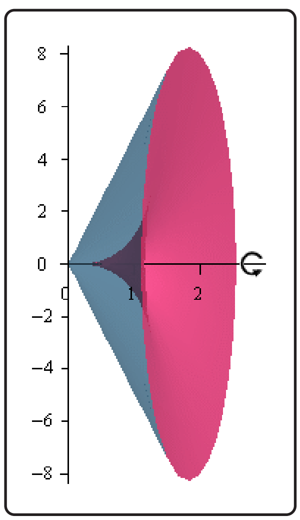
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.

Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



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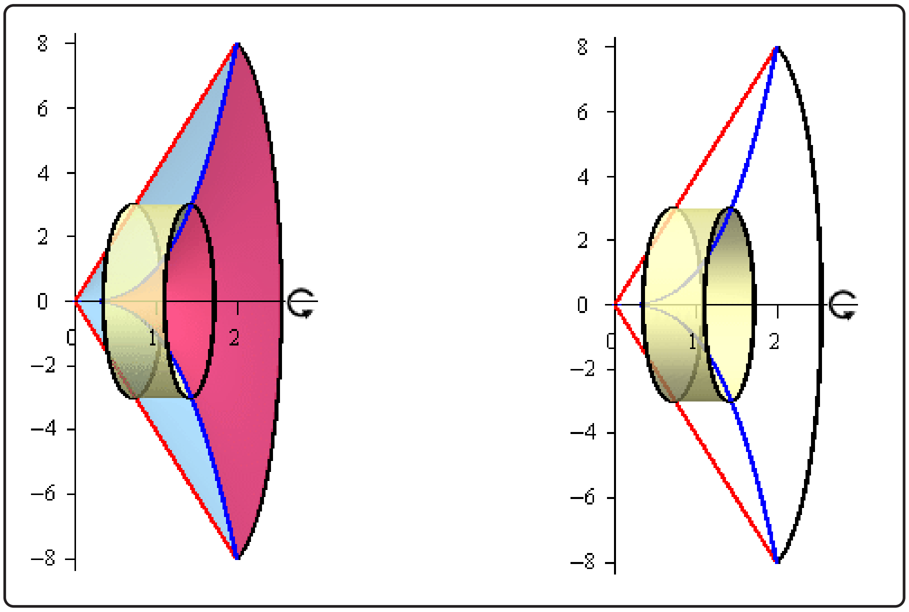


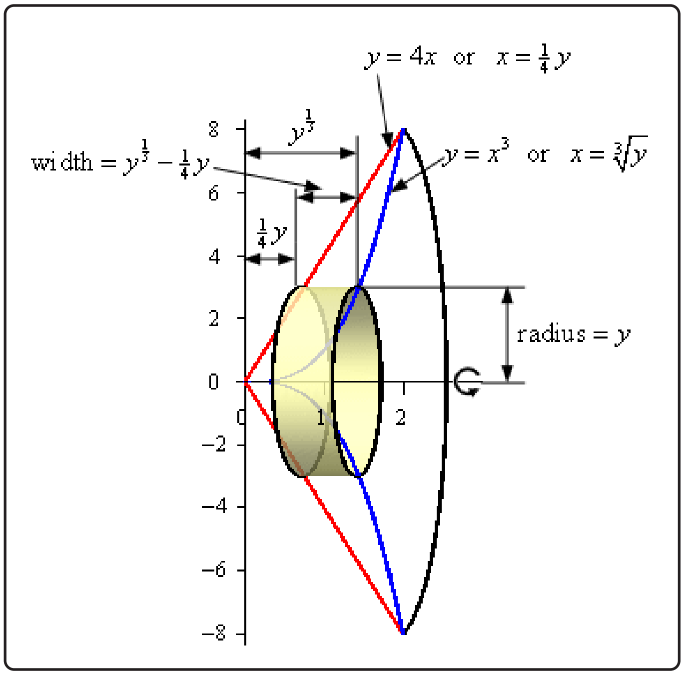
Hint

Determine a formula for the surface area of the cylinder.

Step 3

We now need to find a formula for the surface area of the cylinder. Because we are using cylinders that are centered on the x-axis we know that the area formula will need to be in terms of y. Therefore, we'll need to rewrite the equations of the curves in terms of y.

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.



From the sketch we can see the cylinder is centered on the x-axis and the upper edge of the cylinder is at some y.

The radius of the cylinder is just the distance from the x-axis to the upper edge of the cylinder (*i.e.* y).

The right edge of the cylinder is on the curve defining the right portion of the solid and is a distance of $y^{\frac{1}{3}}$ from the y-axis. The left edge of the cylinder is on the curve defining the left portion of the solid and is a distance of $\frac{1}{4}y$ from the y-axis. The height then is the difference of these two.

So, the radius and width of the cylinder are,

$$\text{Radius} = y \qquad\qquad \text{Width} = y^{\frac{1}{3}} - \frac{1}{4}y$$

The area of the cylinder is then,

$$A(y) = 2\pi\,(\text{Radius})\,(\text{Height}) = 2\pi\,(y)\left(y^{\frac{1}{3}} - \frac{1}{4}y\right) = 2\pi\left(y^{\frac{4}{3}} - \frac{1}{4}y^2\right)$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $y = 0$ and the "last" cylinder would occur at $y = 8$. Our limits are then : $0 \le y \le 8$.

The volume is then,

$$V = \int_0^8 2\pi\left(y^{\frac{4}{3}} - \frac{1}{4}y^2\right)\,dy = 2\,\pi\left(\frac{3}{7}y^{\frac{7}{3}} - \frac{1}{12}y^3\right)\Bigg|_0^8 = \boxed{\frac{512}{21}\pi}$$

---

5. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $y = 2x + 1$, $y = 3$ and $x = 4$ about the line $y = 10$.

Hint

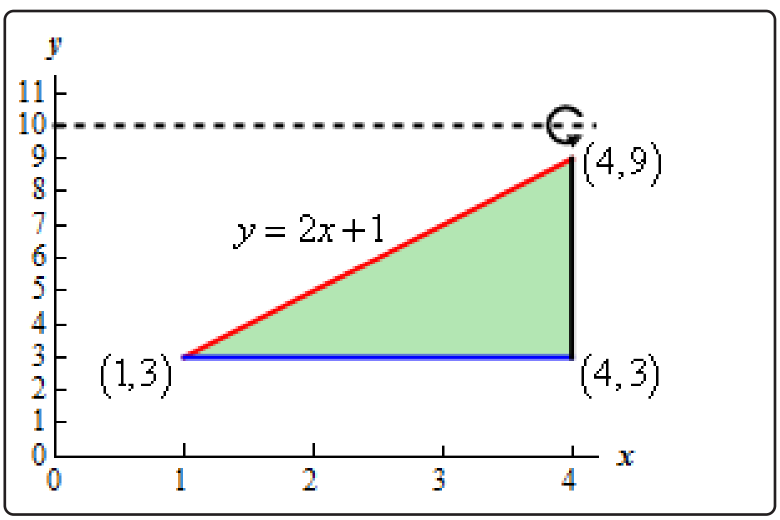
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



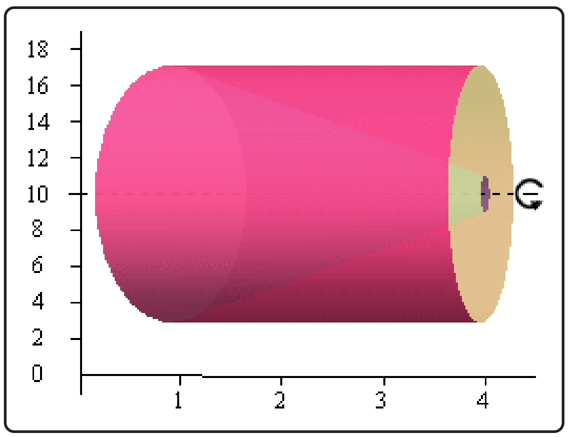
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.

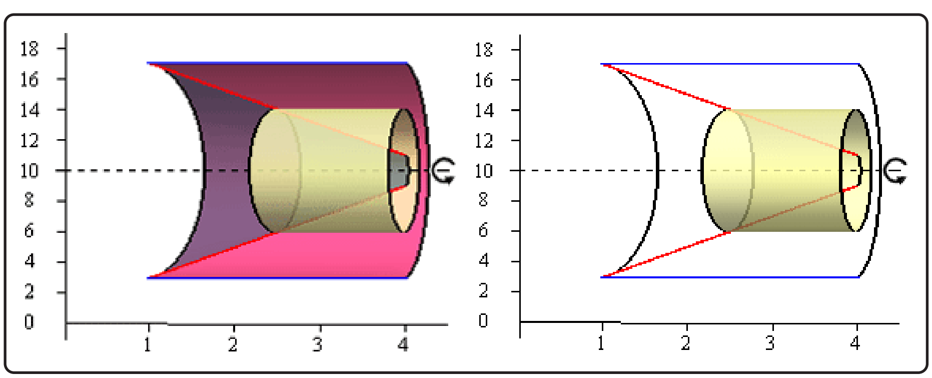
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



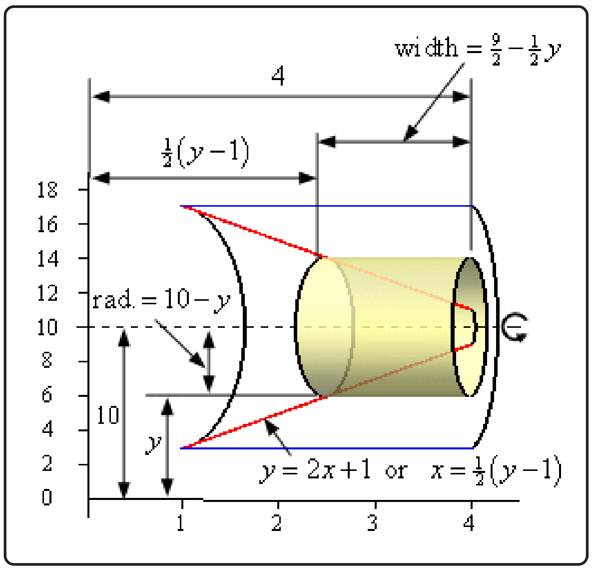
Hint

Determine a formula for the surface area of the cylinder.

Step 3

We now need to find a formula for the surface area of the cylinder. Because we are using cylinders that are centered on a horizontal axis (*i.e.* parallel to the x-axis) we know that the area formula will need to be in terms of y. Therefore, we'll need to rewrite the equations of the curves in terms of y.

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.



From the sketch we can see the cylinder is centered on the line $y = 10$ and the lower edge of the cylinder is at some y.

The radius of the cylinder is just the distance from the axis of rotation to the lower edge of the cylinder (*i.e.* $10 - y$).

The right edge of the cylinder is on the curve defining the right portion of the solid and is a distance of 4 from the y-axis. The left edge of the cylinder is on the curve defining the left portion of the solid and is a distance of $\frac{1}{2}(y-1)$ from the y-axis. The width then is the difference of these two.

So, the radius and width of the cylinder are,

$$\text{Radius} = 10 - y \qquad \text{Width} = 4 - \frac{1}{2}(y-1) = \frac{9}{2} - \frac{1}{2}y$$

The area of the cylinder is then,

$$A(y) = 2\pi\,(\text{Radius})\,(\text{Height}) = 2\pi\,(10-y)\left(\frac{9}{2} - \frac{1}{2}y\right) = 2\pi\left(45 - \frac{19}{2}y + \frac{1}{2}y^2\right)$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $y = 3$ and the "last" cylinder would occur at $y = 9$. Our limits are then : $3 \le y \le 9$.

The volume is then,

$$V = \int_3^9 2\pi\left(45 - \frac{19}{2}y + \frac{1}{2}y^2\right)dy = 2\,\pi\left(45y - \frac{19}{4}y^2 + \frac{1}{6}y^3\right)\Big|_3^9 = \boxed{90\pi}$$

6. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $x = y^2 - 4$ and $x = 6 - 3y$ about the line $y = -8$.

Hint

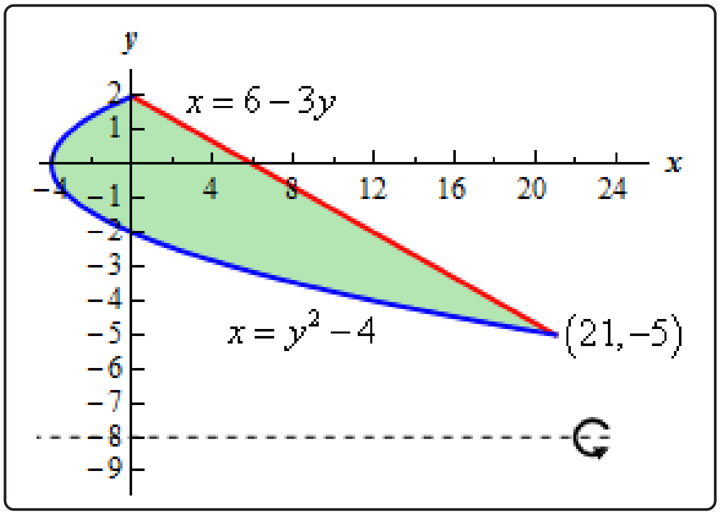
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



To get the intersection points shown above, which we'll need in a bit, all we need to do is set the two equations equal and solve.

$$\begin{aligned} y^2 - 4 &= 6 - 3y \\ y^2 + 3y - 10 &= 0 \\ (y+5)(y-2) &= 0 \qquad \Rightarrow \qquad y = -5, \;\; y = 2 \qquad \Rightarrow \qquad (21,-5) \;\; \& \;\; (0,2) \end{aligned}$$

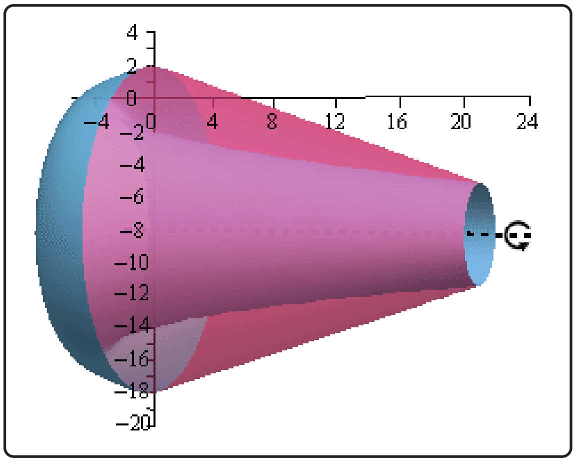
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.

Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



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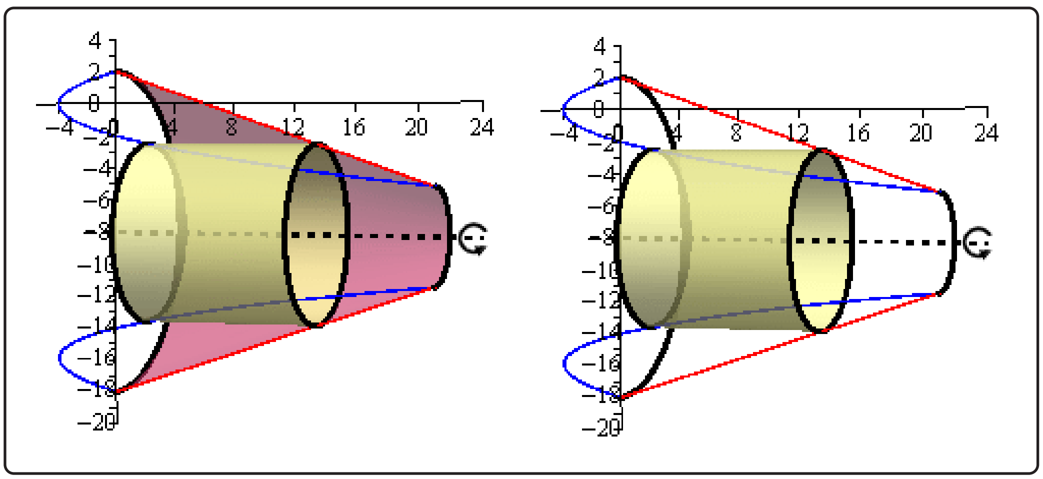


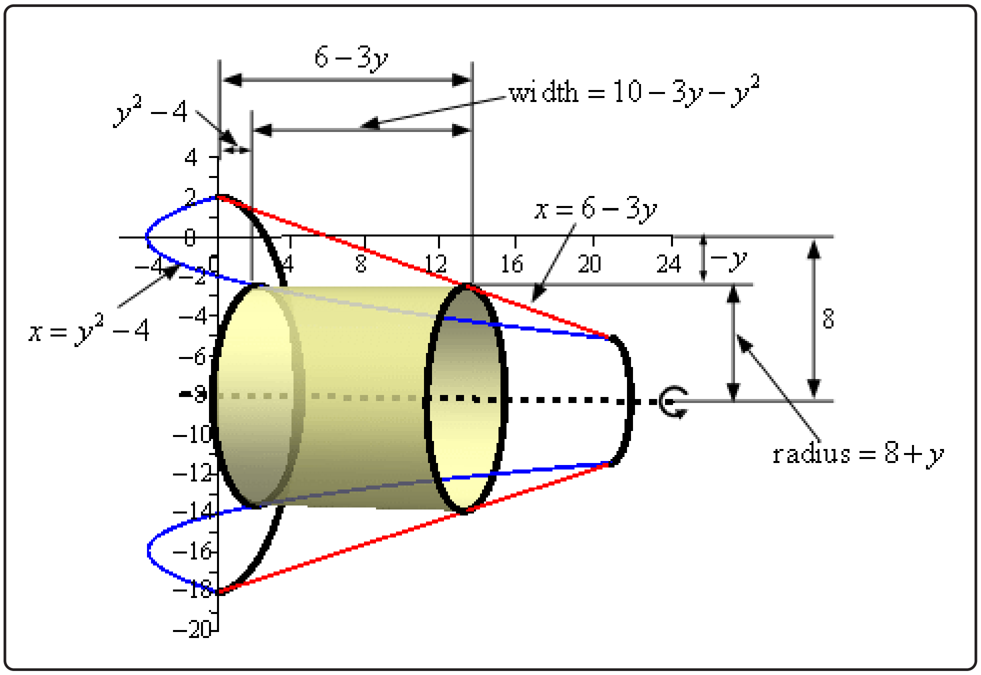
Hint

Determine a formula for the surface area of the cylinder.

Step 3

We now need to find a formula for the surface area of the cylinder. Because we are using cylinders that are centered on a horizontal axis (*i.e.* parallel to the x-axis) we know that the area formula will need to be in terms of y. Therefore, the equations of the curves will need to be in terms of y (which in this case they already are).

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.



From the sketch we can see the cylinder is centered on the line $y=-8$ and the upper edge of the cylinder is at some y.

The radius of the cylinder is a little tricky for this problem.

First, notice that the axis of rotation is a distance of 8 below the x-axis. Next, the upper edge of the cylinder is at some y however because y is negative at the point where we drew the cylinder that means that the distance of the upper edge below the x-axis is in fact $-y$. The minus is needed to turn this into a positive quantity that we need for distance. The radius for this cylinder is then the difference of these two distances or,

$$8-(-y)=8+y$$

Now, note that when the upper edge of the cylinder rises above the x-axis the distance of the upper edge above the x-axis will be just y. This time because y is positive we don't need the minus sign (and in fact don't want it because that would turn the distance into a negative quantity). The radius is then the distance of the axis of rotation from the x-axis (still a distance of 8) plus by the distance of the upper edge above the x-axis (which is y) or,

$$8 + y$$

In either case we get the same radius.

The right edge of the cylinder is on the curve defining the right portion of the solid and is a distance of $6 - 3y$ from the y-axis. The left edge of the cylinder is on the curve defining the left portion of the solid and is a distance of $y^2 - 4$ from the y-axis. The width then is the difference of these two.

So, the radius and width of the cylinder are,

$$\text{Radius} = 8 + y \qquad \text{Width} = 6 - 3y - \left(y^2 - 4\right) = 10 - 3y - y^2$$

The area of the cylinder is then,

$$A\left(y\right) = 2\pi \left(\text{Radius}\right)\left(\text{Height}\right) = 2\pi\left(8 + y\right)\left(10 - 3y - y^2\right) = 2\pi\left(80 - 14y - 11y^2 - y^3\right)$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the “first” cylinder in the solid would occur at $y = -5$ and the “last” cylinder would occur at $y = 2$. Our limits are then : $-5 \le y \le 2$.

The volume is then,

$$V = \int_{-5}^{2} 2\pi\left(80 - 14y - 11y^2 - y^3\right)\,dy = 2\,\pi\left(80y - 7y^2 - \frac{11}{3}y^3 - \frac{1}{4}y^4\right)\Bigg|_{-5}^{2}$$
$$= \boxed{\frac{4459}{6}\pi}$$

7. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 6x + 9$ and $y = -x^2 + 6x - 1$ about the line $x = 8$.

Hint

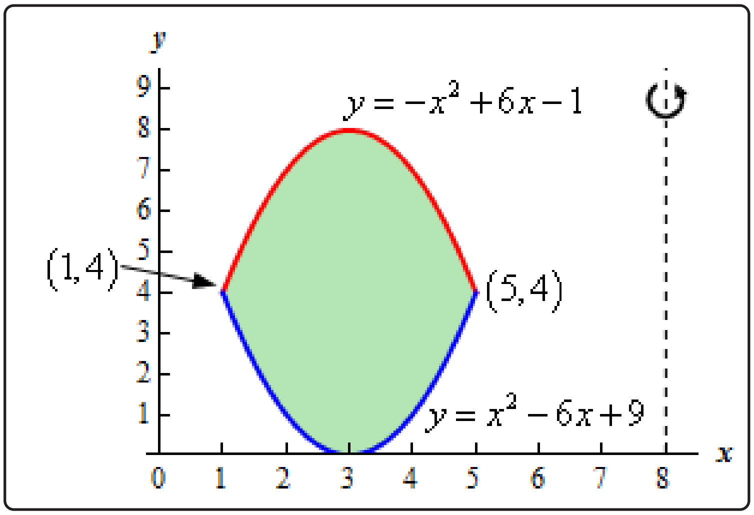
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



To get the intersection points shown above, which we'll need in a bit, all we need to do is set the two equations equal and solve.

$$\begin{aligned} x^2 - 6x + 9 &= -x^2 + 6x - 1 \\ 2x^2 - 12x + 10 &= 0 \\ 2(x-1)(x-5) &= 0 \qquad \Rightarrow \qquad x = 1, \;\; x = 5 \qquad \Rightarrow \qquad (1,4) \;\; \& \;\; (5,4) \end{aligned}$$

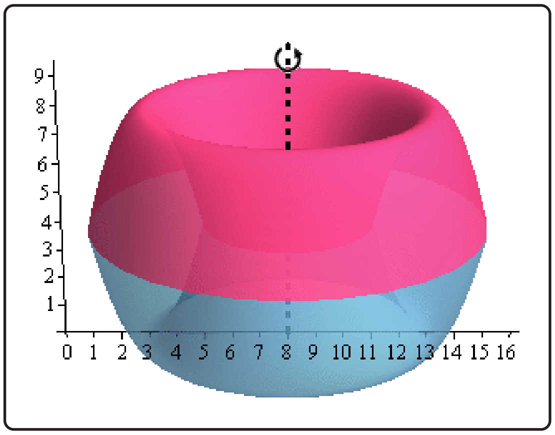
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.

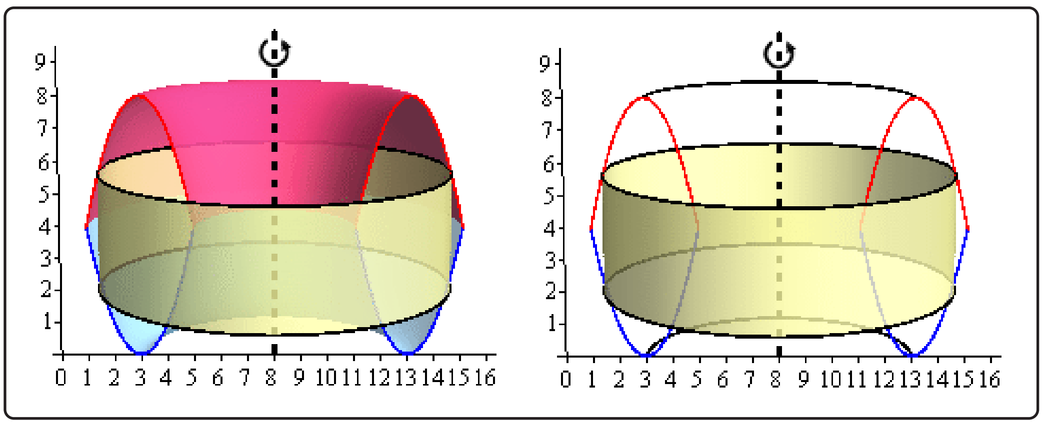
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



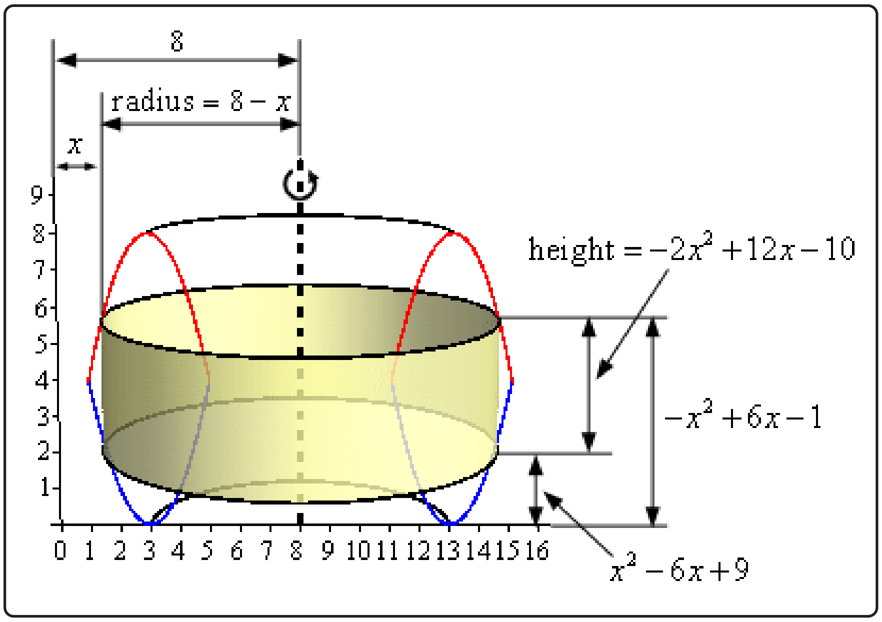
Hint

Determine a formula for the surface area of the cylinder.

Step 3

We now need to find a formula for the surface area of the cylinder. Because we are using cylinders that are centered on a vertical axis (*i.e.* parallel to the y-axis) we know that the area formula will need to be in terms of x. Therefore, the equations of the curves will need to be in terms of x (which in this case they already are).

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.



Note that we put all the height "lines" on the mirrored curves and not the actual curves. This was done so we could put them in a place that didn't interfere with the y-axis.

From the sketch we can see the cylinder is centered on the line $x = 8$ and the left edge of the cylinder is at some x.

The radius of the cylinder is just the distance from the axis of rotation to the left edge of the cylinder (*i.e.* $8 - x$).

The upper edge of the cylinder is on the curve defining the upper portion of the solid and is a distance of $-x^2 + 6x - 1$ from the x-axis. The lower edge of the cylinder is on the curve defining the lower portion of the solid and is a distance of $x^2 - 6x + 9$ from the *x*-axis. The height then is the difference of these two.

So the radius and width of the cylinder are,

$$\text{Radius} = 8 - x \qquad \text{Width} = -x^2 + 6x - 1 - \left(x^2 - 6x + 9\right) = -2x^2 + 12x - 10$$

The area of the cylinder is then,

$$A(x) = 2\pi\,(\text{Radius})\,(\text{Height})$$
$$= 2\pi\,(8-x)\left(-2x^2+12x-10\right) = 2\pi\left(-80+106x-28x^2+2x^3\right)$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $x = 1$ and the "last" cylinder would occur at $x = 5$. Our limits are then : $1 \le x \le 5$.

The volume is then,

$$V = \int_1^5 2\pi\left(-80+106x-28x^2+2x^3\right)\,dx = 2\,\pi\left(-80x+53x^2-\frac{28}{3}x^3+\frac{1}{2}x^4\right)\Big|_1^5$$
$$= \boxed{\frac{640}{3}\pi}$$

---

8. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $y = \dfrac{\mathbf{e}^{\frac{1}{2}x}}{x+2}$, $y = 5 - \dfrac{1}{4}x$, $x = -1$ and $x = 6$ about the line $x = -2$.

Hint

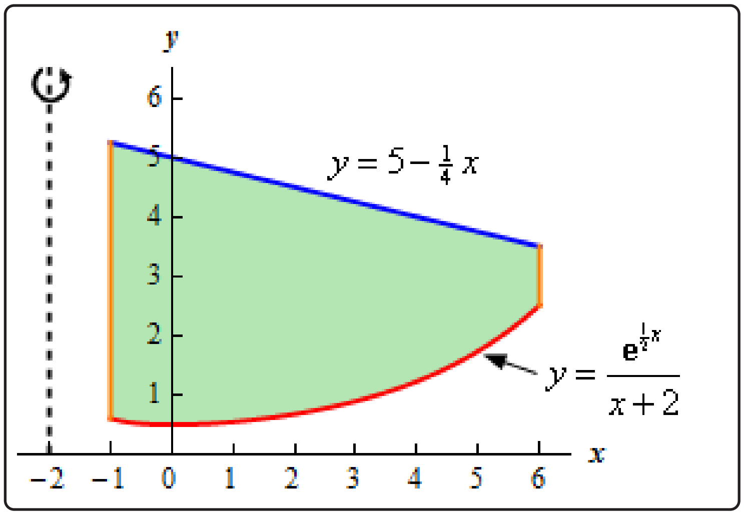
Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.



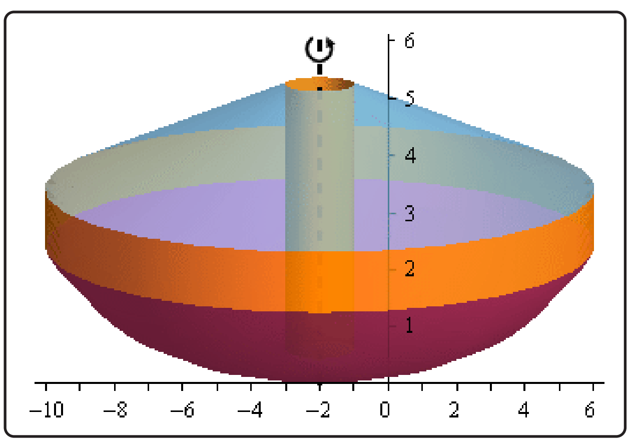
Hint

Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.

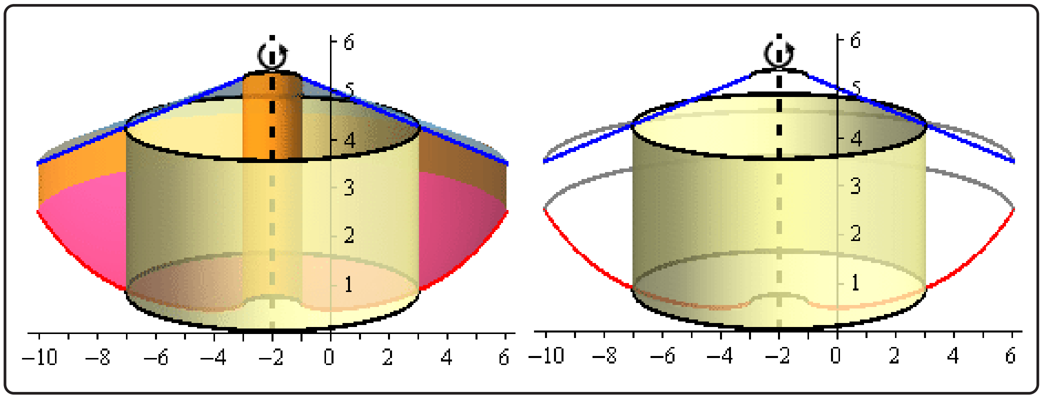
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2

Here is a sketch of the solid of revolution.



Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).



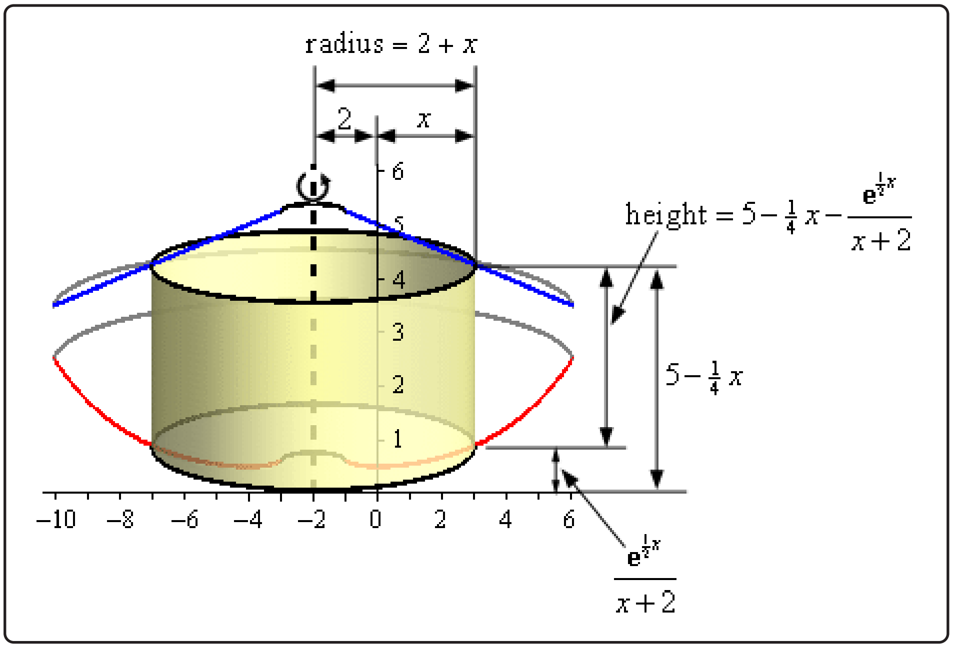
Hint

Determine a formula for the surface area of the cylinder.

Step 3

We now need to find a formula for the surface area of the cylinder. Because we are using cylinders that are centered on a vertical axis (*i.e.* parallel to the y-axis) we know that the area formula will need to be in terms of x. Therefore, the equations of the curves will need to be in terms of x (which in this case they already are).

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.



From the sketch we can see the cylinder is centered on the line $x = -2$ and the right edge of the cylinder is at some x.

We need to be a little careful with the radius here since the right edge of the cylinder can be on both the left and right side of the y-axis depending on where it cuts into the solid.

If the right edge of the cylinder cuts into the object to the right of the y-axis, as shown in the sketch above, then the radius is the distance of the axis of rotation to the y-axis (a distance of 2) plus the distance from the y-axis to the right edge of the cylinder (a distance of x). Therefore, in this case, the radius is $2 + x$.

On the other hand, if the right edge of the cylinder cuts into the solid to the left of the y-axis then the radius will be the distance from the axis of rotation to the y-axis (a distance of 2) minus the distance of the right edge of the cylinder to the y-axis. However, in this case, the value of the x that defines the right edge is a negative value and so the distance of the right edge of the cylinder to the y-axis must be $-x$. The minus sign in needed to turn this into a positive quantity. Therefore the radius in this case is $2 - (-x) = 2 + x$, the same as in the first case.

The upper edge of the cylinder is on the curve defining the upper portion of the solid and is a distance of $5 - \frac{1}{4}x$ from the x-axis. The lower edge of the cylinder is on the curve defining the lower portion of the solid and is a distance of $\frac{\mathbf{e}^{\frac{1}{2}x}}{x+2}$ from the x-axis. The height then is the difference of these two.

So, the radius and width of the cylinder are,

$$\text{Radius} = 2 + x \qquad \text{Width} = 5 - \frac{1}{4}x - \frac{\mathbf{e}^{\frac{1}{2}x}}{x+2}$$

The area of the cylinder is then,

$$\begin{aligned} A\left(x\right) = 2\pi\left(\text{Radius}\right)\left(\text{Height}\right) &= 2\pi\left(2+x\right)\left(5 - \frac{1}{4}x - \frac{\mathbf{e}^{\frac{1}{2}x}}{x+2}\right) \\ &= 2\pi\left(10 + \frac{9}{2}x - \frac{1}{4}x^2 - \mathbf{e}^{\frac{1}{2}x}\right) \end{aligned}$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $x = -1$ and the "last" cylinder would occur at $x = 6$. Our limits are then : $-1 \le x \le 6$.

The volume is then,

$$\begin{aligned} V &= \int_{-1}^{6} 2\pi\left(10 + \frac{9}{2}x - \frac{1}{4}x^2 - \mathbf{e}^{\frac{1}{2}x}\right)\,dx \\ &= 2\,\pi\left(10x + \frac{9}{4}x^2 - \frac{1}{12}x^3 - 2\mathbf{e}^{\frac{1}{2}x}\right)\bigg|_{-1}^{6} = \boxed{2\pi\left(\frac{392}{3} + 2\mathbf{e}^{-\frac{1}{2}} - 2\mathbf{e}^{3}\right)} \end{aligned}$$

6.5 More Volume Problems

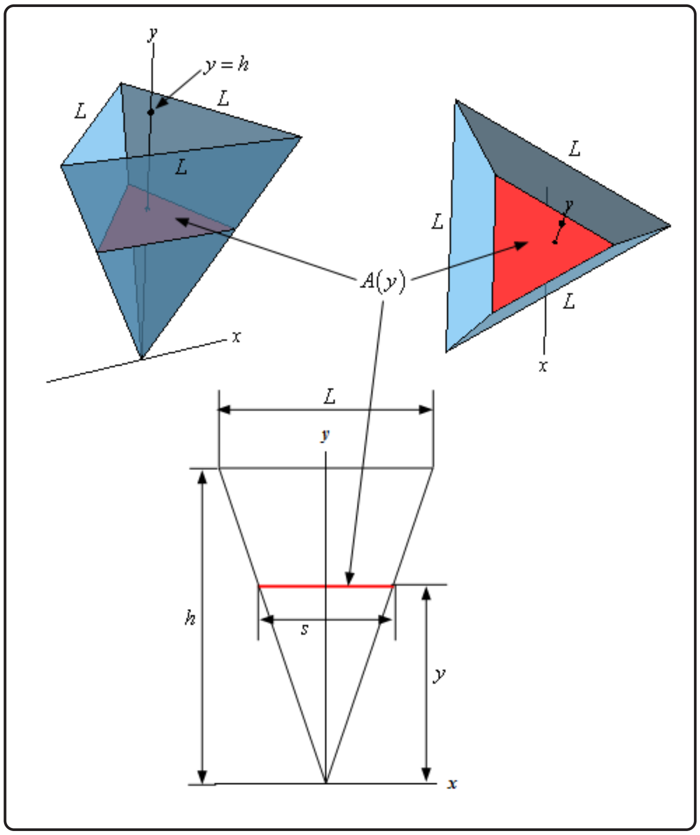
1. Find the volume of a pyramid of height h whose base is an equilateral triangle of length L.

Hint

If possible, try to get a sketch of what the pyramid looks like. These can be difficult to sketch on occasion but if we can get a sketch it will help to set up the problem.

Step 1

Okay, let's start with a sketch of the pyramid. These can be difficult to sketch but having the sketch will help greatly with the set up portion of the problem.



We've got several sketches here. In each sketch we've shown a representative cross-sectional area (shown in red). Because the cross-section can be placed at any point on the y-axis the area of the cross-section will be a function of y as indicated in the image.

The sketch in the upper right we see the pyramid from the "front" and the sketch in the upper left we see pyramid from the "top". Note that we set the point of the pyramid at the origin and drew the pyramid upwards. This was done to make the set up for the problem a little easier. Also we sketched the pyramid so that one of the sides of the pyramid was parallel to the x-axis. This was done only so we could draw in the bottom sketch (which we'll get to in a second) and have the images match up, so to speak.

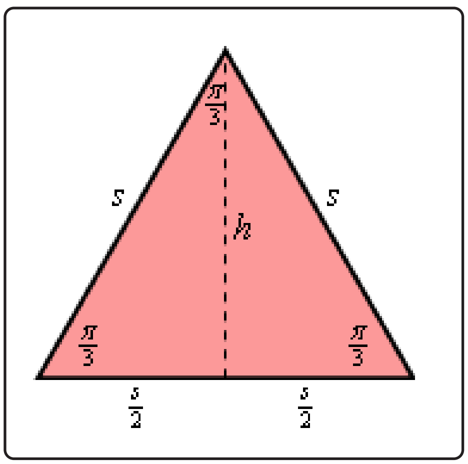
The bottom sketch is a sketch of the side of the pyramid that is parallel to the x-axis. It also has all of the various quantities that we'll need shown. The representative cross-section here is indicated by the red line on the sketch.

Hint

Determine a formula for the cross-sectional area in terms of y.

Step 2

Let's start off with a sketch of what a typical cross-section looks like.



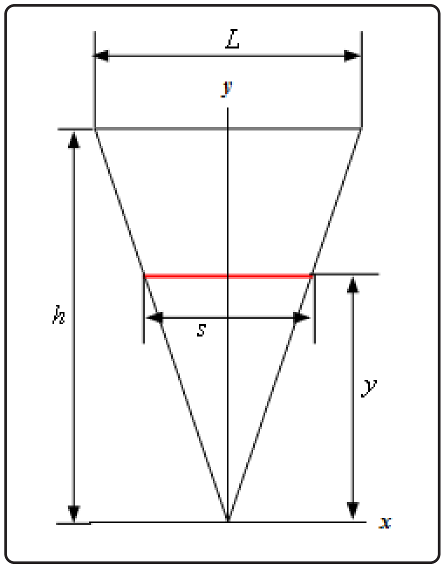
In this case we know that the cross-sections are equilateral triangles and so all of the interior angles are $\frac{\pi}{3}$ and we know that all the sides are the same length, let's say s. In the sketch above notice that since we have an equilateral triangle we know that the dashed line (representing the height of the triangle) will divide the base of the triangle into equal length portions, *i.e.* $\frac{s}{2}$. Also, from basic right triangle trig (each "half" of the cross-section is a right triangle right?) we can see that we can write the height in terms of s as follows,

$$\tan\left(\frac{\pi}{3}\right) = \frac{h}{{}^{s}/_{2}} \qquad \Rightarrow \qquad h = \frac{s}{2}\tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}s$$

Therefore, in terms of s the area of each cross-section is,

$$\text{Area} = \frac{1}{2}(s)\left(\frac{\sqrt{3}}{2}s\right) = \frac{\sqrt{3}}{4}s^2$$

Now, we know from the sketches in Step 1 that the cross-sectional area should be a function of y. So, if we could determine a relationship between s and y we'd have what we need. Let's revisit one of the sketches from Step 1.



From this we can see that we have two similar triangles. The overall side (base L and height h) as well as the "lower" portion formed by the red line representing the cross-sectional area (base s and height y).

Because these two triangles are similar triangles we know the following ratios must be equal.

$$\frac{s}{y} = \frac{L}{h} \qquad \Rightarrow \qquad s = \frac{L}{h}y$$

We now have a relationship between s and y so plug this into the area formula from above to get the area of the cross-section in terms of y.

$$A(y) = \frac{\sqrt{3}}{4}\left(\frac{L}{h}y\right)^2 = \frac{\sqrt{3}L^2}{4h^2}y^2$$

Hint

All we need to do now is determine the volume itself.

Step 3

Finally, we need the volume itself. We know that the volume is found by evaluating the following integral.

$$V = \int_c^d A\left(y\right)\,dy$$

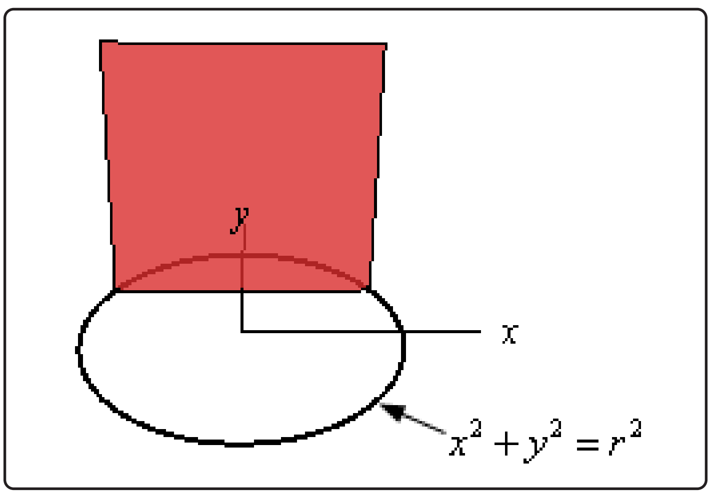
We already have a formula for $A\left(y\right)$ from Step 2 and from the sketches in Step 1 we can see that the "first" cross-section will occur at $y = 0$ and that the "last" cross-section will occur at $y = h$ and so these are the limits for the integral.

The volume is then,

$$V = \int_0^h \frac{\sqrt{3}L^2}{4h^2}y^2\,dy = \frac{\sqrt{3}L^2}{4h^2}\int_0^h y^2\,dy = \frac{\sqrt{3}L^2}{4h^2}\left(\frac{1}{3}y^3\right)\Bigg|_0^h = \boxed{\frac{\sqrt{3}\,L^2 h}{12}}$$

Do not get excited about the h and L in the integral and area formula. These are just constants. The only letter that is actually changing is y. Because the h and L are constants we can factor them out of the integral as we did with the actual numbers.

2. Find the volume of the solid whose base is a disk of radius r and whose cross-sections are squares. See figure below to see a sketch of the cross-sections.

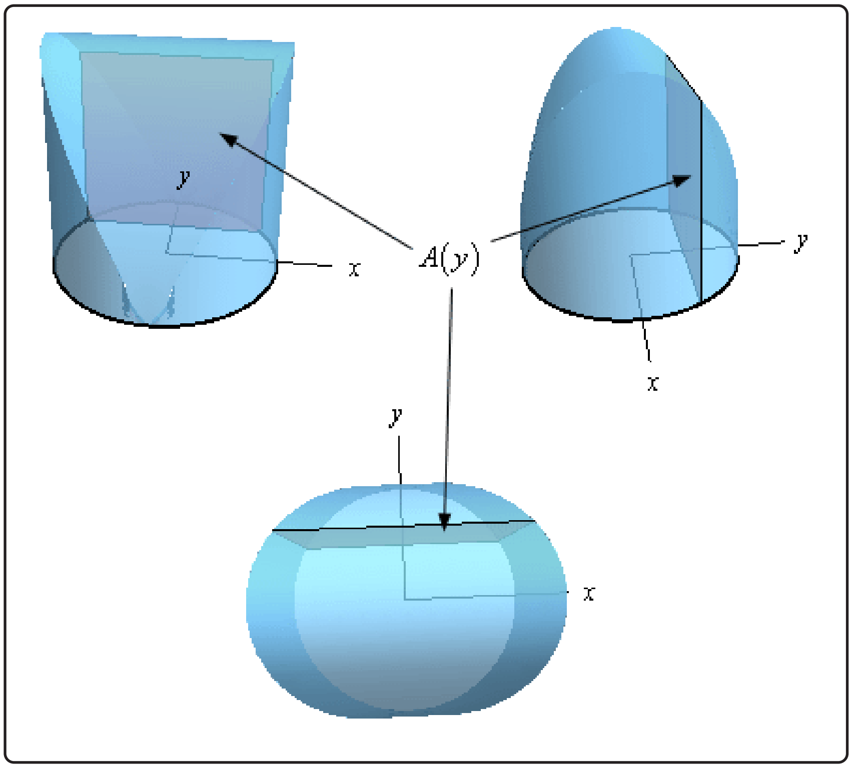


Hint

While it's not strictly needed for this problem a sketch of the solid might be interesting to see just what the solid looks like.

Step 1

Here are a couple of sketches of the solid from three different angles. For reference the positive x-axis and positive y-axis are shown.



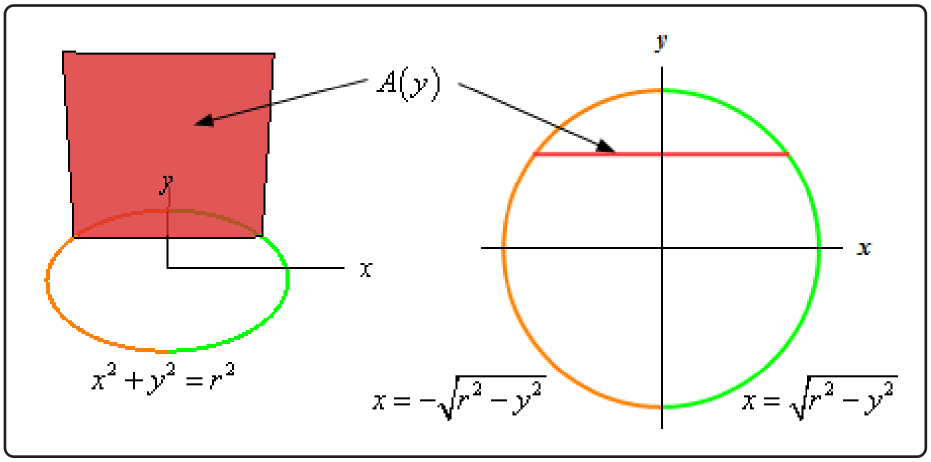
Because the cross-section is perpendicular to the y-axis as we move the cross-section along the y-axis we'll change its area and so the cross-sectional area will be a function of y, *i.e.* $A(y)$.

Hint

Determine a formula for the cross-sectional area in terms of y.

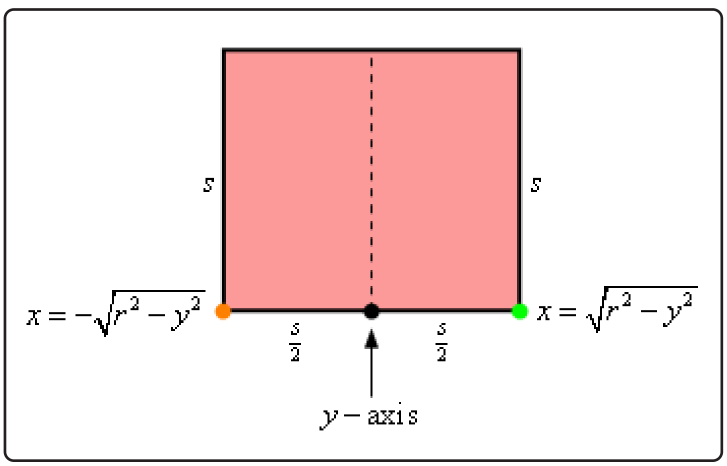
Step 2

While the sketches above are nice to get a feel for what the solid looks like, what we really need is just a sketch of the cross-section. So, here's a couple of sketches of the cross-sectional area.



The sketch on the left is really just the graph given in the problem statement with the only difference that we colored the right/left sides so it will match with the sketch on the right. The sketch on the right looks at the cross-section from directly above and is shown by the red line.

Let's get a quick sketch of just the cross-section and let's call the length of the side of each square s.



Now, along the bottom we've denoted the y-axis location in the cross-section with a black dot and the orange and green dots represent where the left and right portions of the circle are at. We can also see that, assuming the cross-section is placed at some y, the green dot must be a distance of $\sqrt{r^2 - y^2}$ from the y-axis. Likewise, the orange dot must also be a distance of $\sqrt{r^2 - y^2}$ from the y-axis (recall we want the distance to be positive here and so we drop the minus sign from the function to get a positive distance).

Now, we know that the area of the square is simply s^2 and from the discussion above we see that,

$$\frac{s}{2} = \sqrt{r^2 - y^2} \qquad \Rightarrow \qquad s = 2\sqrt{r^2 - y^2}$$

So, a formula for the area of the cross-section in terms of y is,

$$A(y) = s^2 = \left(2\sqrt{r^2 - y^2}\right)^2 = 4\left(r^2 - y^2\right)$$

Hint

All we need to do now is determine the volume itself.

Step 3

Finally, we need the volume itself. We know that the volume is found by evaluating the following integral.

$$V = \int_c^d A(y)\,dy$$

We already have a formula for $A(y)$ from Step 2 and from the sketches in Step 1 we can see that the "first" cross-section will occur at $y = -r$ and that the "last" cross-section will occur at $y = r$ and so these are the limits for the integral.

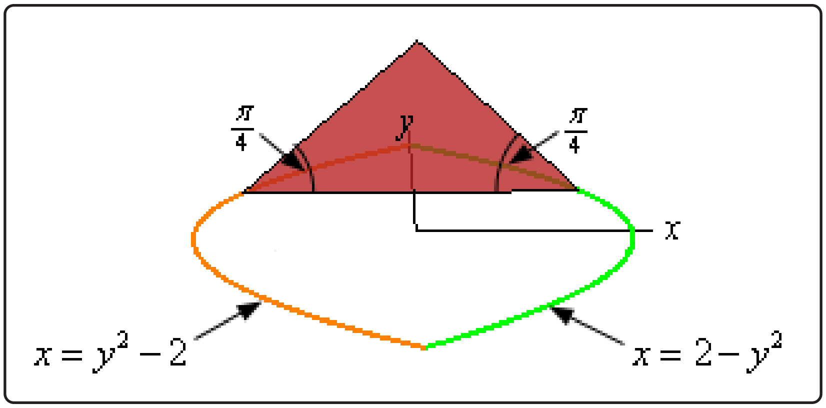
The volume is then,

$$V = \int_{-r}^{r} 4\left(r^2 - y^2\right)\,dy = 4\left(yr^2 - \frac{1}{3}y^3\right)\bigg|_{-r}^{r} = \boxed{\frac{16}{3}r^3}$$

Do not get excited about the r integral and area formula. It is just a constant. The only letter that is actually changing is y.

---

3. Find the volume of the solid whose base is the region bounded by $x = 2 - y^2$ and $x = y^2 - 2$ and whose cross-sections are isosceles triangles with the base perpendicular to the y-axis and the angle between the base and the two sides of equal length is $\frac{\pi}{4}$. See figure below to see a sketch of the cross-sections.

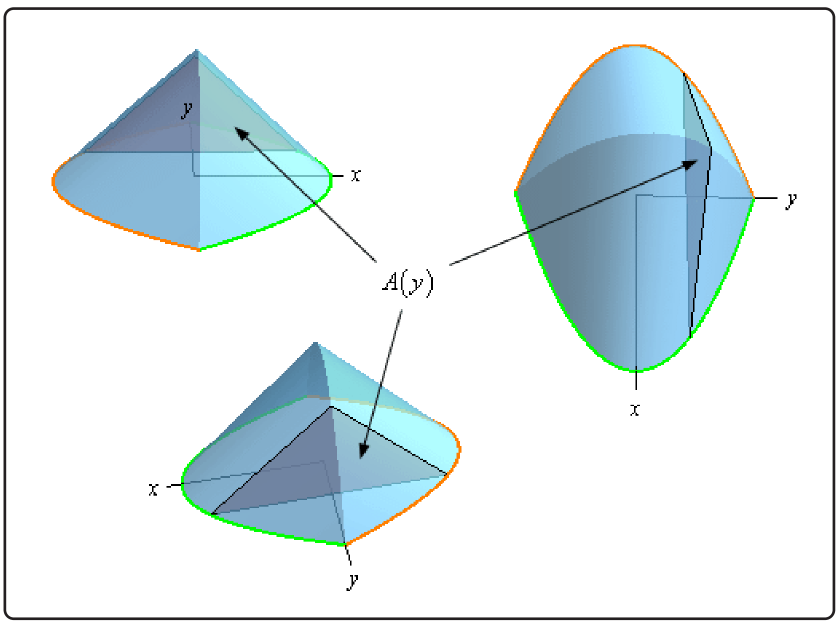


Hint

While it's not strictly needed for this problem a sketch of the solid might be interesting to see just what the solid looks like.

Step 1

Here are a couple of sketches of the solid from three different angles. For reference the positive x-axis and positive y-axis are shown.



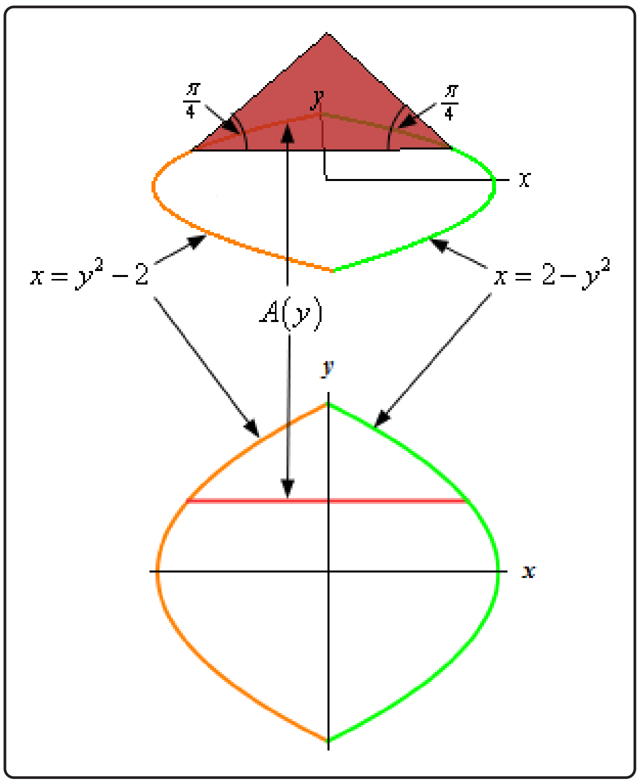
Because the cross-section is perpendicular to the y-axis as we move the cross-section along the y-axis we'll change its area and so the cross-sectional area will be a function of y, *i.e.* $A(y)$.

Hint

Determine a formula for the cross-sectional area in terms of y.

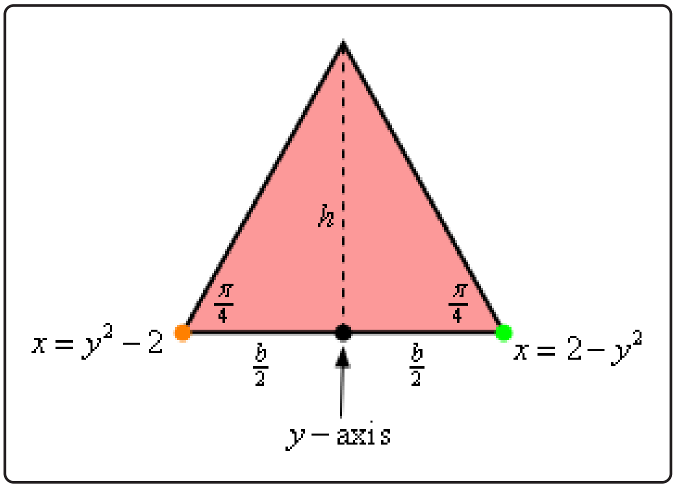
Step 2

While the sketches above are nice to get a feel for what the solid looks like, what we really need is just a sketch of the cross-section. So, here's a couple of sketches of the cross-sectional area.



The sketch on the top is really just the graph given in the problem statement that is included for a reference with the sketch on the bottom. The sketch on the bottom looks at the cross-section from directly above and is shown by the red line.

Let's get a quick sketch of just the cross-section and let's call the length of the base of triangle b and the height of the triangle h.



Now, along the bottom we've denoted the y-axis location in the cross-section with a black dot and the orange and green dots represent the left and right curves that define the left and right sides of the bottom of the solid. We can also see that, assuming the cross-section is placed at some y, the green dot must be a distance of $2 - y^2$ from the y-axis. Likewise, the orange dot must also be a distance of $-\left(y^2 - 2\right) = 2 - y^2$ from the y-axis (recall we want the distance to be positive here and so we add the minus sign to the function to get a positive distance).

Now, we can see that the base of the triangle is given by,

$$\frac{b}{2} = 2 - y^2 \qquad \Rightarrow \qquad b = 2\left(2 - y^2\right)$$

Likewise, the height can be found from basic right triangle trig.

$$\tan\left(\frac{\pi}{4}\right) = \frac{h}{{}^{b}/_{2}} \qquad \Rightarrow \qquad h = \frac{b}{2}\tan\left(\frac{\pi}{4}\right) = 2 - y^2$$

So, a formula for the area of the cross-section in terms of y is then,

$$A\left(y\right) = \frac{1}{2}bh = \left(2 - y^2\right)^2 = 4 - 4y^2 + y^4$$

Hint

All we need to do now is determine the volume itself.

Step 3

Finally, we need the volume itself. We know that the volume is found by evaluating the following integral.

$$V = \int_c^d A(y)\, dy$$

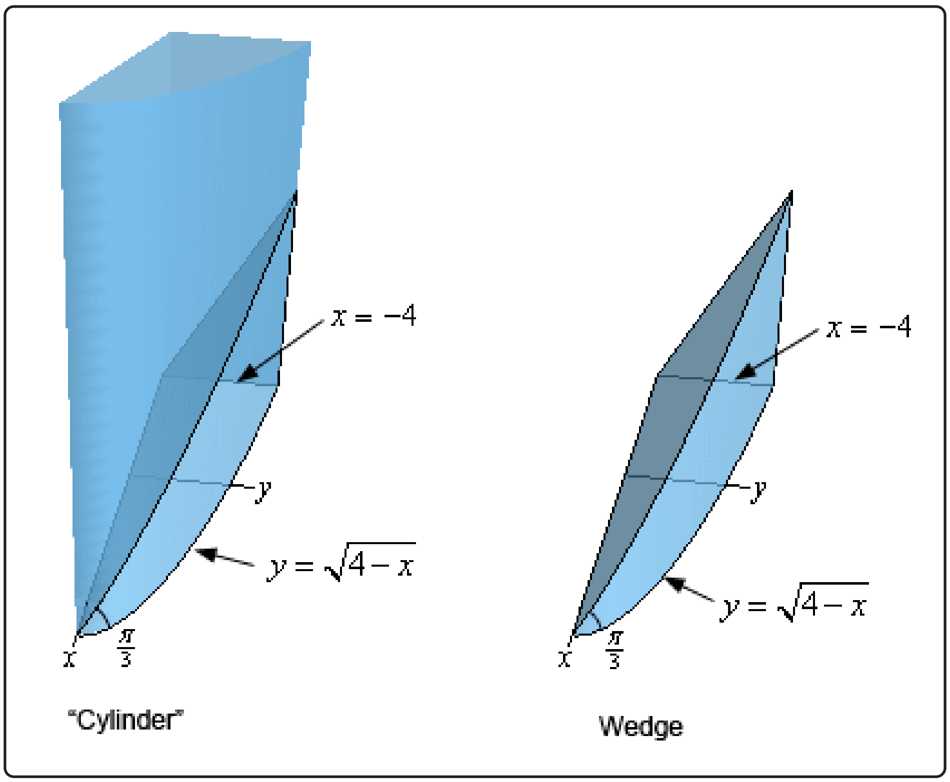
By setting $x = 0$ into either of the equations defining the left and right sides of the base of the solid (since they intersect at the y-axis) we can see that the "first" cross-section will occur at $y = -\sqrt{2}$ and the "last" cross-section will occur at $y = \sqrt{2}$ and so these are the limits for the integral.

The volume is then,

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} 4 - 4y^2 + y^4\, dy = \left(4y - \frac{4}{3}y^3 + \frac{1}{5}y^5\right)\Bigg|_{-\sqrt{2}}^{\sqrt{2}} = \boxed{\frac{64\sqrt{2}}{15}}$$

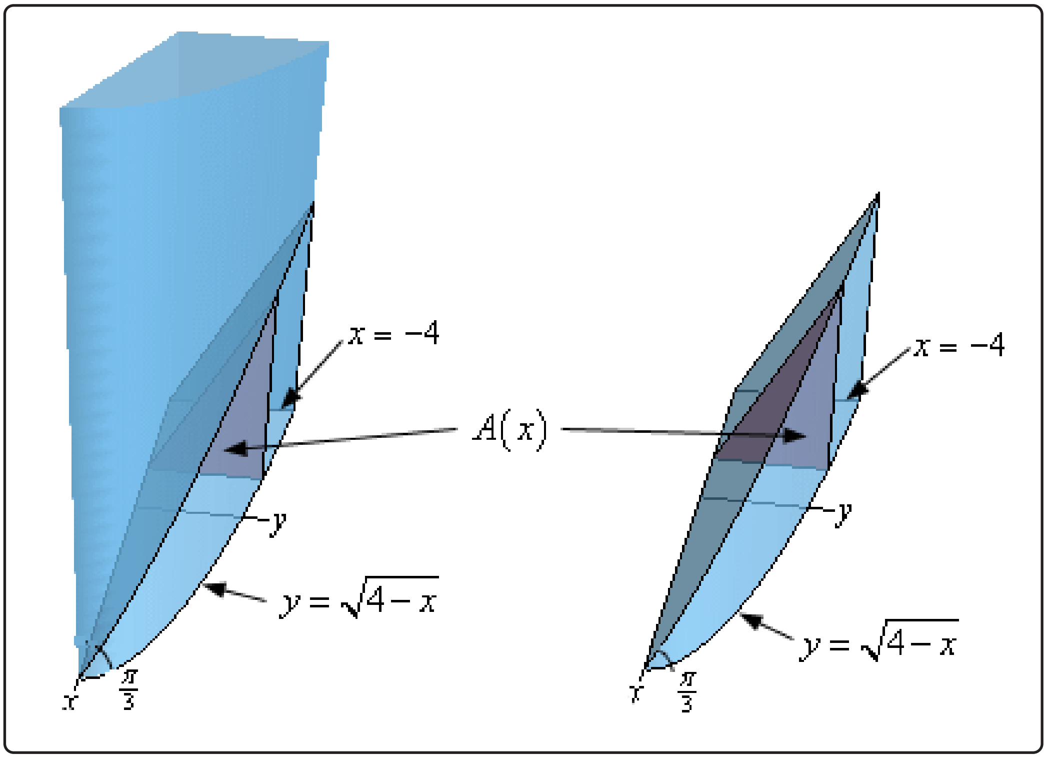
---

4. Find the volume of a wedge cut out of a "cylinder" whose base is the region bounded by $y = \sqrt{4-x}$, $x = -4$ and the x-axis. The angle between the top and bottom of the wedge is $\frac{\pi}{3}$. See the figure below for a sketch of the "cylinder" and the wedge (the positive x-axis and positive y-axis are shown in the sketch - they are just in a different orientation).



Step 1

While not strictly needed let's redo the sketch of the "cylinder" and wedge from the problem statement only this time let's also sketch in what the cross-section will look like.



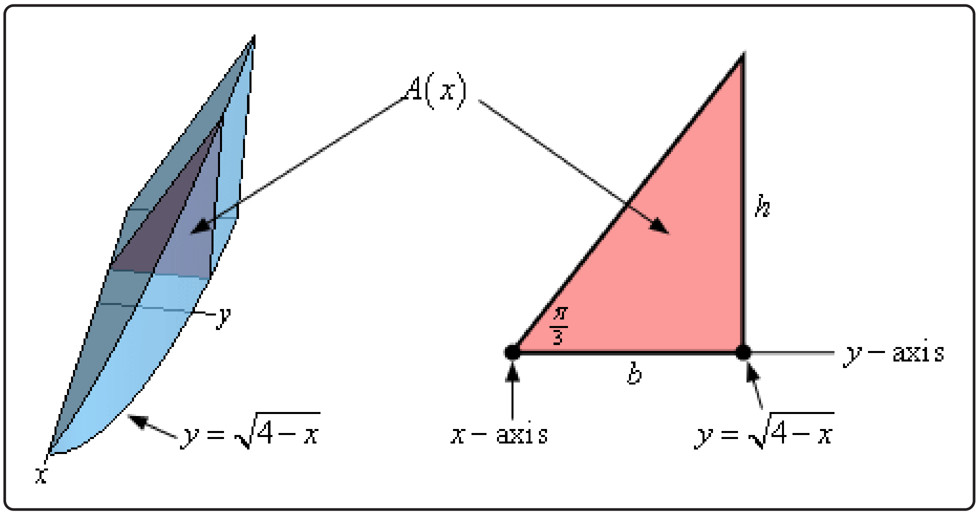
Because the cross-section is perpendicular to the x-axis as we move the cross-section along the x-axis we'll change its area and so the cross-sectional area will be a function of x, *i.e.* $A(x)$. Also note that as shown in the sketches the cross-section will be a right triangle.

Hint

Determine a formula for the cross-sectional area in terms of x.

Step 2

While the sketches above are nice to get a feel for what the solid and cross-sections look like, what we really need is just a sketch of just the cross-section. So, here are a couple of sketches of the cross-sectional area.



The sketch on the left is just pretty much the sketch we've seen before and is included to give us a reference point for the actual cross-section that is shown on the right.

As noted in the sketch on the right we'll call the base of the triangle b and the height of the triangle h. Also, the dot on the left side of the base represents where the x-axis is on the cross-section and the dot on the right side of the base represents the curve that defines the edge of the solid (and hence the wedge).

From this sketch it should then be pretty clear that the length of the base is simply the distance from the x-axis to the curve or,

$$b = \sqrt{4-x}$$

Likewise, the height can be found from basic right triangle trig.

$$\tan\left(\frac{\pi}{3}\right) = \frac{h}{b} \qquad \Rightarrow \qquad h = b\tan\left(\frac{\pi}{3}\right) = \sqrt{3}\sqrt{4-x}$$

So, a formula for the area of the cross-section in terms of x is then,

$$A(y) = \frac{1}{2}bh = \frac{\sqrt{3}}{2}\left(\sqrt{4-x}\right)^2 = \frac{\sqrt{3}}{2}(4-x)$$

Hint

All we need to do now is determine the volume itself.

Step 3

Finally, we need the volume itself. We know that the volume is found by evaluating the following integral.

$$V = \int_a^b A(x)\,dx$$

From the sketches in the problem statement or from Step 1 we can see that the "first" cross-section will occur at $x = -4$ (the back end of the "cylinder") and the "last" cross-section will occur at $x = 4$ (the front end of the "cylinder" where the curve intersects with the x-axis. These are then the limits for the integral.

The volume is then,

$$V = \int_{-4}^{4} \frac{\sqrt{3}}{2}(4 - x)\,dx = \frac{\sqrt{3}}{2}\left(4x - \frac{1}{2}x^2\right)\bigg|_{-4}^{4} = \boxed{16\sqrt{3}}$$

6.6 Work

1. A force of $F(x) = x^2 - \cos(3x) + 2$, x is in meters, acts on an object. What is the work required to move the object from $x = 3$ to $x = 7$?

Solution

There really isn't all that much to this problem. We are given the force function and limits for the integral ($x = 3$ and $x = 7$) and so all we need to do is write down the integral for the work and evaluate it.

$$W = \int_3^7 x^2 - \cos(3x) + 2\,dx$$
$$= \left(\frac{1}{3}x^3 - \frac{1}{3}\sin(3x) + 2x\right)\Big|_3^7 = \boxed{\frac{1}{3}(340 + \sin(9) - \sin(21)) = 113.1918}$$

---

2. A spring has a natural length of 18 inches and a force of 20 lbs is required to stretch and hold the spring to a length of 24 inches. What is the work required to stretch the spring from a length of 21 inches to a length of 26 inches?

Hint

What is the spring constant, k and the force function?

Step 1

Let's start off by finding the spring constant. We are told that a force of 20 lbs is needed to stretch the spring 24 in - 18 in = 6 in = 0.5 ft from its natural length. Then using Hooke's Law we have,

$$20 = k(0.5) \qquad \Rightarrow \qquad k = 40$$

Don't forget that we want the displacement in feet. Also, don't forget that the displacement needs to be the displacement from the natural length of the spring.

Again, using Hooke's Law we can see that the force function is,

$$F(x) = 40x$$

Step 2

For the limits of the integral we can see that we start with the spring at a length of 21 in - 18 in = 3 in or $\frac{1}{4}$ feet and we end with a length of 26 in - 18 in = 8 in or $\frac{2}{3}$ feet. These are then the limits of the integral (recall that we need the relative distance from the natural length for the limits).

The work is then,

$$W = \int_{\frac{1}{4}}^{\frac{2}{3}} 40x\,dx = 20x^2\Big|_{\frac{1}{4}}^{\frac{2}{3}} = \boxed{\frac{275}{36} = 7.6389\text{ ft - lbs}}$$

---

3. A cable with mass $\frac{1}{2}$ kg/meter is lifting a load of 150 kg that is initially at the bottom of a 50 meter shaft. How much work is required to lift the load ¼ of the way up the shaft?

Hint

What is the total mass of the chain and load at any point in the shaft? How does that relate to the force required to hold the chain and load at any point in the shaft?

Step 1

Let's start off with the convention that $x = 0$ defines the bottom of the shaft and $x = 50$ defines the top of the shaft. Therefore, x represents the distance that the load has been lifted. After lifting the load by x meters there will be $50 - x$ meters of the chain left in the shaft that needs to be lifted along with the load.

Therefore, after lifting the load x meters, the total mass of the chain left in the shaft as well as the load is,

$$\frac{1}{2}(50 - x) + 150\text{ kg} = 175 - \frac{1}{2}x\text{ kg}$$

We know that the force required to hold the chain and load at any point is just the total weight of the chain and load at that point. We also know that (because we are in the metric system) the weight of a given mass (in kg) is just then,

$$\text{Weight} = \text{mass} \times 9.8$$

where 9.8 is the gravitational acceleration.

The force required to hold the chain and load a distance of x meters above the bottom is then,

$$F(x) = (9.8)\left(175 - \frac{1}{2}x\right) = 1715 - 4.9x$$

Step 2

For the limits of the integral we can see that we start with the chain and load at the bottom of the shaft (*i.e.* at $x = 0$) and stop ¼ of the way up the shaft (*i.e.* at $x = 12.5$). These values are then the limits for the integral.

The work is then,

$$W = \int_0^{12.5} 1715 - 4.9x\,dx = \left(1715x - 2.45x^2\right)\Big|_0^{12.5} = \boxed{21,054.6875J}$$

---

4. A tank of water is 15 feet long and has a cross section in the shape of an equilateral triangle with sides 2 feet long (point of the triangle points directly down). The tank is filled with water to a depth of 9 inches. Determine the amount of work needed to pump all of the water to the top of the tank. Assume that the weight of the water is 62 lb/ft^3.

Hint

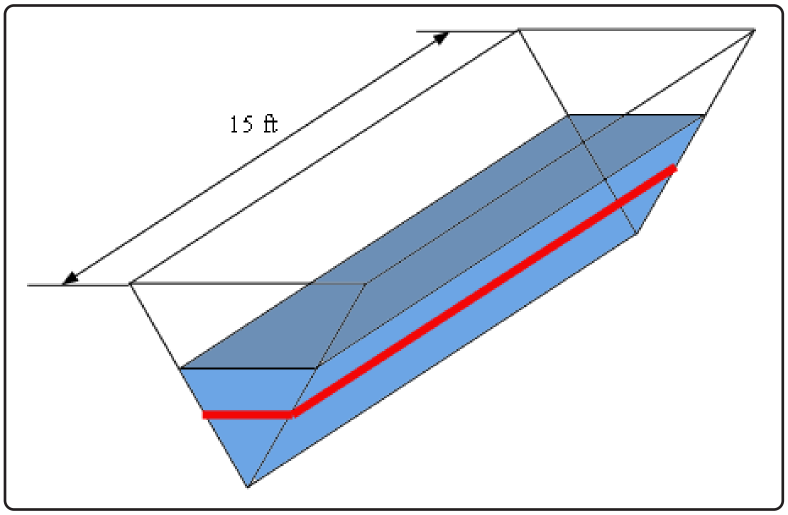
Get the basic problem set up. Determine all the known information and what you will need in order to work the problem. A sketch of at least the cross-section of the tank would probably be useful as well. Use the last example from this section as a general guide for this problem if you are having trouble. This problem will work in pretty much the same manner, although there will be some differences due to the obvious change in tank shape as well as the fact that we are using not using the Metric system for this problem.

Step 1

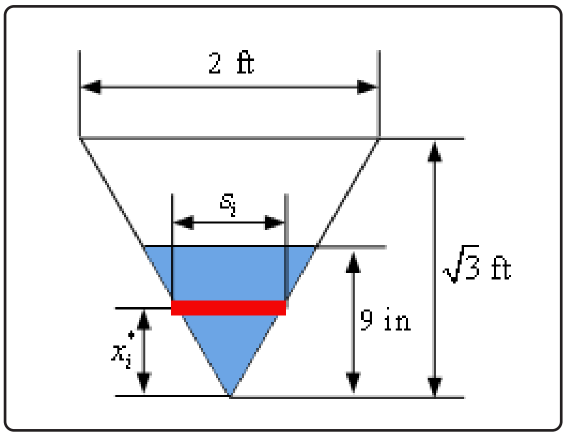
Okay, let's start off and define $x = 0$ to be the bottom point of the tank and the height of the water in the tank to be $x = \frac{9}{12} = \frac{3}{4}$ feet (because all the other quantities are in feet we converted this into feet as well). This means that we will be working in the interval $\left[0, \frac{3}{4}\right]$ for this problem.

We'll next divide the interval $\left[0, \frac{3}{4}\right]$ into n subintervals each of width Δx and we'll let x_i^* be any point in the i^{th} subinterval where $n = 1, 2, \ldots, n$. For each subinterval we can approximate the water in the tank corresponding to that subinterval as a box with length 15 ft, width s_i^* and height Δx.

Here is a quick sketch of the tank. The red strip represents the box we are using to approximate the water in the tank in the i^{th} subinterval.

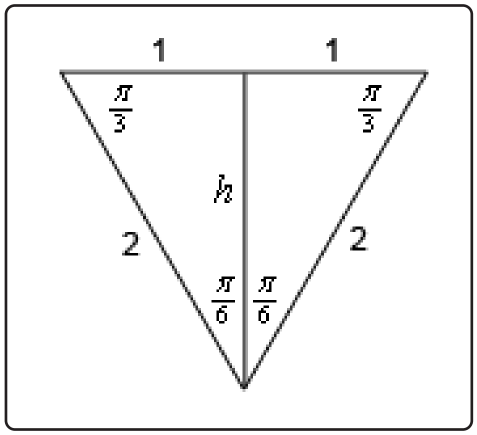


The sketch of the tank is nice and while it does help us to visualize the tank what we really need is a sketch of the tank from directly in front (*i.e.* a typical vertical equilateral triangular cross-section for the tank). Here is that sketch.



The red strip again represents the box we are using to approximate the water in the i^{th} subinterval. As noted in the problem statement the cross-section is an equilateral triangle and with sides of length 2 feet.

We included the height in the above sketch and this is easy to get using some basic right triangle trig. Here is yet another sketch of the cross-section.



Because the triangle is an equilateral triangle we know that each of the interior angles of the triangle must be $\frac{\pi}{3}$ and we're told the length of each side is 2. The height of the triangle is the line that bisects the triangle as shown. Each half of the triangle is then an identical right triangle and using any of the trig functions we can quickly determine the height of the triangle. We'll use cosine here.

$$\cos\left(\frac{\pi}{6}\right) = \frac{h}{2} \qquad \Rightarrow \qquad h = 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$$

Hint

What is the volume of the box of water we are using to approximate the volume of water in the i^{th} subinterval? Give the volume in terms of x_i^*.

Step 2

We'll next need the volume of the box of water we using to approximate the volume of water in the i^{th} subinterval (as represented by the red strip in the first two pictures from Step 1).

Our approximate volume is the volume of a box and so we know that the volume for the i^{th} subinterval would be,

$$V_i = (\text{length})\,(\text{width})\,(\text{height}) = (15)\,(s_i)\,(\Delta x) = 15\,s_i\,\Delta x$$

We will eventually need the volume to be in terms of x_i^* and luckily enough this is easy enough to do.

From the cross-section sketch with the red strip in Step 1 we see that we have two similar triangles (well actually we have three but we only need two of them). The two that we need are the triangle with width 2 and height $\sqrt{3}$ and the triangle whose width is s_i (*i.e.* the triangle whose top is the red strip) and whose height is x_i^*. Since these two triangles are similar we now the following two ratios must be equal.

$$\frac{s_i}{x_i^*} = \frac{2}{\sqrt{3}} \qquad \Rightarrow \qquad s_i = \frac{2}{\sqrt{3}} x_i^*$$

Plugging this into the volume formula above and we get,

$$V_i = \frac{30}{\sqrt{3}} x_i^* \, \Delta x$$

Hint

What is the approximate weight of the water in the i^{th} subinterval? Or in other words what is the approximate force needed to overcome the force of gravity acting on this volume of water? Note that because we are working with the Imperial system here the force in this case is just $F_i = \text{weight} \times V_i$.

Step 3

We next need to know how much force will be required to overcome the force of gravity that is acting on the water in the i^{th} subinterval. This will be approximately the forced needed to overcome the force of gravity acting on the volume of water we found in Step 2. Because we are working with the British system here the force is,

$$F_i = \text{weight} \times V_i \approx (62)\left(\frac{30}{\sqrt{3}} x_i^* \, \Delta x\right) = \frac{1860}{\sqrt{3}} x_i^* \, \Delta x$$

Hint

Approximately how much work is needed to raise the water in the i^{th} subinterval to the top of the tank?

Step 4

We will need the amount of work required to raise the volume of water in the i^{th} subinterval to the top of the tank, *i.e.* raise it a distance of $\sqrt{3}-x_i^*$. This is approximately,

$$W_i \approx F_i\left(\sqrt{3}-x_i^*\right) = \frac{1860}{\sqrt{3}}x_i^*\left(\sqrt{3}-x_i^*\right)\Delta x$$

Hint

Finally compute the total amount of work needed to pump all the to the top of the tank.

Step 5

The total amount of work to raise all the water to the top of the tank is the approximately the sum of all the W_i for $i = 1, 2, \ldots n$ or,

$$W \approx \sum_{i=1}^{n} \frac{1860}{\sqrt{3}}x_i^*\left(\sqrt{3}-x_i^*\right)\Delta x$$

The exact work required is then found by letting $n \to \infty$ or,

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} \frac{1860}{\sqrt{3}}x_i^*\left(\sqrt{3}-x_i^*\right)\Delta x$$

This however is just the definition of the following definite integral,

$$W = \int_0^{\frac{3}{4}} \frac{1860}{\sqrt{3}}x\left(\sqrt{3}-x\right)dx$$

The work required to pump all the water to the top of the tank is then,

$$W = \int_0^{\frac{3}{4}} \frac{1860}{\sqrt{3}}x\left(\sqrt{3}-x\right)dx = \frac{1860}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}x^2-\frac{1}{3}x^3\right)\Bigg|_0^{\frac{3}{4}} = \boxed{372.112 \text{ ft - lbs}}$$

7 Integration Techniques

By this point we've now looked at basic integration techniques. We've seen how to integrate most of the "basic" functions we're liable to run into : polynomials, roots, trig, exponential, logarithm and inverse trig functions to name a few. In addition, we've seen how to do basic u-substitutions allowing us to integrate some more complicated functions.

We've also taken a look at some basic applications of (definite) integrals. However, as was noted at the time, there are applications of (definite) integrals that will, on occasion, have integrals that need more than just a basic u-substitution. So, before we can take a look at those applications we'll need to first talk about some more involved integration techniques.

Before getting into the new techniques we first need to make it clear that in this chapter it is assumed at you are comfortable with basic integration, including u-substitutions. Many of the problems in this chapter will not have a lot, if any, discussion of the basic integration work under the assumption that you are comfortable enough with the basic work that discussion in simply not needed. In addition, we will usually, although not always, give the substitution that we're using for the u-substitution but we will generally not show the actual substitution work. Again, this is under the assumption that you are comfortable enough with basic u-substitutions that you can fill in the details if you need to.

The reason for skipping the discussion of the basic integration work and/or not showing the full substitution work is so we can concentrate our discussion on the particular method that we are covering in that particular section. This is not to "punish" you but simply to acknowledge that we only have so much time in which to discuss the material and just can't afford to spend a lot of time basically re-lecturing basic integration material. We realize that, for many of you, this is the start of your Calculus II course and so you may have had some time off and may well have some "rust" on your basic integration skills. This is a warning to start scraping that rust off. If you need do scrape some rust off you can check out the practice problems for some practice problems covering basic integration to refresh your memory on how basic integration works.

It is also very important for you to understand that most of the problems we'll be looking at in this chapter will involve u-substitutions in one way or another. In fact, many of the techniques in this chapter are really just substitutions. The only difference is that either they need a fair amount of work to get to the point where the substitutions can be used or they will involve substitutions used in ways that we've not seen to this point. So, again, if you have some rust on your u-substitution skills you'll need to get it scraped off so you can do the work in this chapter.

In addition, we will be doing indefinite integrals almost exclusively in most of the sections in this chapter. There are a few sections were we'll be doing some definite integrals but for the most part we'll keep the problems in most of the sections shorter by just doing indefinite integrals. It is assumed that if you were given a definite integral you could do the extra evaluation steps needed to finish the definite integral. Having said that, there are a few sections were definite integrals are done either because there are some subtleties that need to be dealt with for definite integrals or because the topic at hand, the last few sections in particular, involve only definite integrals.

So, with all that out of the way, here is a quick rundown of the new integration techniques we'll take a look at in this section.

Probably the most important technique, in this sense that it will be the most commonly seen technique out of this class, is integration by parts. This is the one new technique in this chapter that is not just u-substitutions done in new ways. Integration by Parts will involve u-substitutions at various steps the process on occasion but it will not be just a new way of doing a u-substitution.

As noted a lot of the techniques in this chapter are really just u-substitutions except they will need some manipulation of the integrand prior to actually doing the substitution. The techniques using this idea will include integrating some, but not all, products and quotients of trig functions, some integrands involving roots or quadratics that can't be done without manipulation of the integrand or "different" u-substitutions that we are used to. We'll also see how to use partial fractions to write some integrands involving rational expressions into a form that we can actually do the integral.

We'll also take a look at something called trig substitutions. This is probably the one technique that most find the most difficult, or at the least, the longest method. As we'll see a trig substitution is really a substitution but it is not a traditional u-substitution. However, having said that, if you understand how basic u-substitutions work it will help greatly when it comes to working with trig substitutions as the basic concepts are the same.

Next we'll be taking a look at a new kind of integral, Improper Integrals. This topic will address how to deal with definite integrals for which one or both of the limits of integration will be an infinity. In addition, we'll see how we can, on occasion, deal with discontinuities in the integrand (we'll focus on division by zero in the integrand).

We'll close our the section with a quick section on approximating the value of definite integrals.

We will leave this section with a warning. It is with this chapter that you will find that you can't just memorize your way through the class anymore. We will acknowledge that up to this point it is possible, for the most part, to just memorize your way through the class. You may not get the highest grades through just memorization as there are some topics that require a fair amount of understanding of the topic, but you can survive up to this point if your really good at memorization.

Integration by Parts is a really good example of this. While you will need to memorize/know the basic integration by parts formula simply memorizing that will not help you to actually use integration by parts on the problem. You will need to actually understand how integration by parts works and how to "assign" various portions of the integrand to the various portions of the integration parts formula.

Also while there are some basic formulas we can, and do on occasion, give for some of the methods there are also situations that just don't fit into those formulas and so again you'll really need to understand how to do those methods in order to work problems for which basic formulas just won't work. Or, again, you can't just memorize your way out of most the methods taught in this chapter. Memorization may allow you to get through the basic problems but will not help all that much with more complicated problems.

Finally, we also need to warn you about seeing "patterns" and just assuming that all the problems will fall into those patterns. Integration by Parts is, again, a good example of this. There are some "patterns" that seem to show up because a lot of the problems we do in that section do fall into the patterns. The problem is that there are also some problems for which the "patterns" simply don't work and yet they still require integration by parts. If you get so locked into "patterns" you'll find it all but impossible to do some problems because they simply don't fall into those patterns.

This is not to say that recognizing that patterns in always a bad thing. Patterns do, on occasion, show up and they can be useful to understand/know as a possible solution method. However, you also need to always remember that there are problems that just don't fit easily into the patterns. This is also a warning that will be valid in other chapters in a typical Calculus II course as well. Again, patterns aren't bad per se, you just need to be careful to not always assume that every problem will fall into the patterns.

The following sections are the practice problems, with solutions, for this material.

7.1 Integration by Parts

1. Evaluate $\displaystyle\int 4x\cos(2-3x)\,dx$.

Hint

Remember that we want to pick u and dv so that upon computing du and v and plugging everything into the Integration by Parts formula the new integral is one that we can do.

Step 1

The first step here is to pick u and dv. We want to choose u and dv so that when we compute du and v and plugging everything into the Integration by Parts formula the new integral we get is one that we can do.

With that in mind it looks like the following choices for u and dv should work for us.

$$u = 4x \qquad dv = \cos(2-3x)\,dx$$

Step 2

Next, we need to compute du (by differentiating u) and v (by integrating dv).

$$\begin{aligned} u &= 4x & &\to & du &= 4dx \\ dv &= \cos(2-3x)\,dx & &\to & v &= -\frac{1}{3}\sin(2-3x) \end{aligned}$$

Step 3

Plugging u, du, v and dv into the Integration by Parts formula gives,

$$\begin{aligned} \int 4x\cos(2-3x)\,dx &= (4x)\left(-\frac{1}{3}\sin(2-3x)\right) - \int -\frac{4}{3}\sin(2-3x)\,dx \\ &= -\frac{4}{3}x\sin(2-3x) + \frac{4}{3}\int \sin(2-3x)\,dx \end{aligned}$$

Step 4

Okay, the new integral we get is easily doable and so all we need to do to finish this problem out is do the integral.

$$\int 4x\cos(2-3x)\,dx = \boxed{-\frac{4}{3}x\sin(2-3x)+\frac{4}{9}\cos(2-3x)+c}$$

---

2. Evaluate $\int_{6}^{0}(2+5x)\,\mathbf{e}^{\frac{1}{3}x}\,dx$.

Hint

Remember that we want to pick u and dv so that upon computing du and v and plugging everything into the Integration by Parts formula the new integral is one that we can do. Also, don't forget that the limits on the integral won't have any effect on the choices of u and dv.

Step 1

The first step here is to pick u and dv. We want to choose u and dv so that when we compute du and v and plugging everything into the Integration by Parts formula the new integral we get is one that we can do.

With that in mind it looks like the following choices for u and dv should work for us.

$$u = 2+5x \qquad\qquad dv = \mathbf{e}^{\frac{1}{3}x}\,dx$$

Step 2

Next, we need to compute du (by differentiating u) and v (by integrating dv).

$$\begin{aligned} u &= 2+5x & \rightarrow && du &= 5dx \\ dv &= \mathbf{e}^{\frac{1}{3}x}\,dx & \rightarrow && v &= 3\mathbf{e}^{\frac{1}{3}x} \end{aligned}$$

Step 3

We can deal with the limits as we do the integral or we can just do the indefinite integral and then take care of the limits in the last step. We will be using the later way of dealing with the limits for this problem.

So, plugging u, du, v and dv into the Integration by Parts formula gives,

$$\int (2+5x)\,\mathbf{e}^{\frac{1}{3}x} = (2+5x)\left(3\mathbf{e}^{\frac{1}{3}x}\right) - \int 5\left(3\mathbf{e}^{\frac{1}{3}x}\right)dx = 3\mathbf{e}^{\frac{1}{3}x}(2+5x) - 15\int \mathbf{e}^{\frac{1}{3}x}\,dx$$

Step 4

Okay, the new integral we get is easily doable so let's evaluate it to get,

$$\int (2+5x)\,\mathbf{e}^{\frac{1}{3}x} = 3\mathbf{e}^{\frac{1}{3}x}(2+5x) - 45\mathbf{e}^{\frac{1}{3}x} + c = 15x\mathbf{e}^{\frac{1}{3}x} - 39\mathbf{e}^{\frac{1}{3}x} + c$$

Step 5

The final step is then to take care of the limits.

$$\int_6^0 (2+5x)\,\mathbf{e}^{\frac{1}{3}x}\,dx = \left(15x\mathbf{e}^{\frac{1}{3}x} - 39\mathbf{e}^{\frac{1}{3}x}\right)\Big|_6^0 = \boxed{-39 - 51\mathbf{e}^2 = -415.8419}$$

Do not get excited about the fact that the lower limit is larger than the upper limit. This can happen on occasion and in no way affects how the integral is evaluated.

3. Evaluate $\int (3t+t^2)\sin(2t)\,dt$.

Hint

Remember that we want to pick u and dv so that upon computing du and v and plugging everything into the Integration by Parts formula the new integral is one that we can do (or at least will be easier to deal with).

Step 1

The first step here is to pick u and dv. We want to choose u and dv so that when we compute du and v and plugging everything into the Integration by Parts formula the new integral we get is one that we can do or will at least be an integral that will be easier to deal with.

With that in mind it looks like the following choices for u and dv should work for us.

$$u = 3t + t^2 \qquad dv = \sin(2t)\ dt$$

Step 2

Next, we need to compute du (by differentiating u) and v (by integrating dv).

$$u = 3t + t^2 \quad \rightarrow \quad du = (3 + 2t)\,dt$$
$$dv = \sin(2t)\ dt \quad \rightarrow \quad v = -\frac{1}{2}\cos(2t)$$

Step 3

Plugging u, du, v and dv into the Integration by Parts formula gives,

$$\int (3t + t^2)\sin(2t)\ dt = -\frac{1}{2}(3t + t^2)\cos(2t) + \frac{1}{2}\int (3 + 2t)\cos(2t)\ dt$$

Step 4

Now, the new integral is still not one that we can do with only Calculus I techniques. However, it is one that we can do another integration by parts on and because the power on the t's have gone down by one we are heading in the right direction.

So, here are the choices for u and dv for the new integral.

$$u = 3 + 2t \quad \rightarrow \quad du = 2dt$$
$$dv = \cos(2t)\ dt \quad \rightarrow \quad v = \frac{1}{2}\sin(2t)$$

Step 5

Okay, all we need to do now is plug these new choices of u and dv into the new integral we got in Step 3 and finish the problem out.

$$\begin{aligned}\int \left(3t+t^2\right)\sin(2t)\,dt &= -\frac{1}{2}\left(3t+t^2\right)\cos(2t)+\frac{1}{2}\left[\frac{1}{2}(3+2t)\sin(2t)-\int \sin(2t)\,dt\right]\\ &= -\frac{1}{2}\left(3t+t^2\right)\cos(2t)+\frac{1}{2}\left[\frac{1}{2}(3+2t)\sin(2t)+\frac{1}{2}\cos(2t)\right]+c\\ &= \boxed{-\frac{1}{2}\left(3t+t^2\right)\cos(2t)+\frac{1}{4}(3+2t)\sin(2t)+\frac{1}{4}\cos(2t)+c}\end{aligned}$$

4. Evaluate $\displaystyle\int 6\tan^{-1}\left(\frac{8}{w}\right)dw$.

Hint

Be careful with your choices of u and dv here. If you think about it there is really only one way that the choice can be made here and have the problem be workable.

Step 1

The first step here is to pick u and dv.

Note that if we choose the inverse tangent for dv the only way to get v is to integrate dv and so we would need to know the answer to get the answer and so that won't work for us. Therefore, the only real choice for the inverse tangent is to let it be u.

So, here are our choices for u and dv.

$$u = 6\tan^{-1}\left(\frac{8}{w}\right) \qquad\qquad dv = dw$$

Don't forget the dw! The differential dw still needs to be put into the dv even though there is nothing else left in the integral.

Step 2

Next, we need to compute du (by differentiating u) and v (by integrating dv).

$$u = 6\tan^{-1}\left(\frac{8}{w}\right) \quad \to \quad du = 6\frac{-\frac{8}{w^2}}{1+\left(\frac{8}{w}\right)^2}dw = 6\frac{-\frac{8}{w^2}}{1+\frac{64}{w^2}}dw$$

$$dv = dw \quad \to \quad = w$$

Step 3

In order to complete this problem we'll need to do some rewrite on du as follows,

$$du = \frac{-48}{w^2+64}dw$$

Plugging u, du, v and dv into the Integration by Parts formula gives,

$$\int 6\tan^{-1}\left(\frac{8}{w}\right)dw = 6w\tan^{-1}\left(\frac{8}{w}\right) + 48\int \frac{w}{w^2+64}dw$$

Step 4

Okay, the new integral we get is easily doable (with the substitution $u = 64 + w^2$) and so all we need to do to finish this problem out is do the integral.

$$\int 6\tan^{-1}\left(\frac{8}{w}\right)dw = \boxed{6w\tan^{-1}\left(\frac{8}{w}\right) + 24\ln\left|w^2+64\right| + c}$$

5. Evaluate $\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)dz$.

Hint

This is one of the few integration by parts problems where either function can go on u and dv. Be careful however to not get locked into an endless cycle of integration by parts.

Step 1

The first step here is to pick u and dv.

In this case we can put the exponential in either the u or the dv and the cosine in the other. It is one of the few problems where the choice doesn't really matter.

For this problem well use the following choices for u and dv.

$$u = \cos\left(\frac{1}{4}z\right) \qquad\qquad dv = \mathbf{e}^{2z}\,dz$$

Step 2

Next, we need to compute du (by differentiating u) and v (by integrating dv).

$$u = \cos\left(\frac{1}{4}z\right) \quad \rightarrow \quad du = -\frac{1}{4}\sin\left(\frac{1}{4}z\right)\,dz$$

$$dv = \mathbf{e}^{2z}dz \quad \rightarrow \quad v = \frac{1}{2}\mathbf{e}^{2z}$$

Step 3

Plugging u, du, v and dv into the Integration by Parts formula gives,

$$\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)\,dz = \frac{1}{2}\mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right) + \frac{1}{8}\int \mathbf{e}^{2z}\sin\left(\frac{1}{4}z\right)\,dz$$

Step 4

We'll now need to do integration by parts again and to do this we'll use the following choices.

$$u = \sin\left(\frac{1}{4}z\right) \quad \rightarrow \quad du = \frac{1}{4}\cos\left(\frac{1}{4}z\right)\,dz$$

$$dv = \mathbf{e}^{2z}dz \quad \rightarrow \quad v = \frac{1}{2}\mathbf{e}^{2z}$$

Step 5

Plugging these into the integral from Step 3 gives,

$$\begin{aligned}\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)\,dz &= \frac{1}{2}\mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)+\frac{1}{8}\left[\frac{1}{2}\mathbf{e}^{2z}\sin\left(\frac{1}{4}z\right)-\frac{1}{8}\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)\,dz\right]\\ &= \frac{1}{2}\mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)+\frac{1}{16}\mathbf{e}^{2z}\sin\left(\frac{1}{4}z\right)-\frac{1}{64}\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)\,dz\end{aligned}$$

Step 6

To finish this problem all we need to do is some basic algebraic manipulation to get the identical integrals on the same side of the equal sign.

$$\begin{aligned}\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)\,dz &= \frac{1}{2}\mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)+\frac{1}{16}\mathbf{e}^{2z}\sin\left(\frac{1}{4}z\right)\\ &\qquad -\frac{1}{64}\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)\,dz\\ \int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)\,dz+\frac{1}{64}\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)\,dz &= \frac{1}{2}\mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)+\frac{1}{16}\mathbf{e}^{2z}\sin\left(\frac{1}{4}z\right)\\ \frac{65}{64}\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)\,dz &= \frac{1}{2}\mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)+\frac{1}{16}\mathbf{e}^{2z}\sin\left(\frac{1}{4}z\right)\end{aligned}$$

Finally, all we need to do is move the coefficient on the integral over to the right side.

$$\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)\,dz = \boxed{\frac{32}{65}\mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)+\frac{4}{65}\mathbf{e}^{2z}\sin\left(\frac{1}{4}z\right)+c}$$

6. Evaluate $\displaystyle\int_0^{\pi} x^2 \cos(4x)\, dx$.

Hint

Remember that we want to pick u and dv so that upon computing du and v and plugging everything into the Integration by Parts formula the new integral is one that we can do (or at least will be easier to deal with).

Also, don't forget that the limits on the integral won't have any effect on the choices of u and dv.

Step 1

The first step here is to pick u and dv. We want to choose u and dv so that when we compute du and v and plugging everything into the Integration by Parts formula the new integral we get is one that we can do or will at least be an integral that will be easier to deal with.

With that in mind it looks like the following choices for u and dv should work for us.

$$u = x^2 \qquad\qquad dv = \cos(4x)\, dx$$

Step 2

Next, we need to compute du (by differentiating u) and v (by integrating dv).

$$\begin{aligned} u &= x^2 & &\rightarrow & du &= 2x\, dx \\ dv &= \cos(4x)\, dx & &\rightarrow & v &= \frac{1}{4}\sin(4x) \end{aligned}$$

Step 3

We can deal with the limits as we do the integral or we can just do the indefinite integral and then take care of the limits in the last step. We will be using the later way of dealing with the limits for this problem.

So, plugging u, du, v and dv into the Integration by Parts formula gives,

$$\int x^2 \cos(4x)\, dx = \frac{1}{4}x^2 \sin(4x) - \frac{1}{2}\int x \sin(4x)\, dx$$

Step 4

Now, the new integral is still not one that we can do with only Calculus I techniques. However, it is one that we can do another integration by parts on and because the power on the x's have gone down by one we are heading in the right direction.

So, here are the choices for u and dv for the new integral.

$$\begin{aligned} u &= x & \rightarrow \quad & du = dx \\ dv &= \sin(4x)\,dx & \rightarrow \quad & v = -\frac{1}{4}\cos(4x) \end{aligned}$$

Step 5

Okay, all we need to do now is plug these new choices of u and dv into the new integral we got in Step 3 and evaluate the integral.

$$\begin{aligned} \int x^2\cos(4x)\,dx &= \frac{1}{4}x^2\sin(4x) - \frac{1}{2}\left[-\frac{1}{4}x\cos(4x) + \frac{1}{4}\int \cos(4x)\,dx\right] \\ &= \frac{1}{4}x^2\sin(4x) - \frac{1}{2}\left[-\frac{1}{4}x\cos(4x) + \frac{1}{16}\sin(4x)\right] + c \\ &= \frac{1}{4}x^2\sin(4x) + \frac{1}{8}x\cos(4x) - \frac{1}{32}\sin(4x) + c \end{aligned}$$

Step 6

The final step is then to take care of the limits.

$$\int_0^{\pi} x^2\cos(4x)\,dx = \left.\left(\frac{1}{4}x^2\sin(4x) + \frac{1}{8}x\cos(4x) - \frac{1}{32}\sin(4x)\right)\right|_0^{\pi} = \boxed{\frac{1}{8}\pi}$$

7. Evaluate $\int t^7 \sin\left(2t^4\right)\, dt$.

Hint

Be very careful with your choices of u and dv for this problem. It looks a lot like previous practice problems but it isn't!

Step 1

The first step here is to pick u and dv and, in this case, we'll need to be careful how we chose them.

If we follow the model of many of the examples/practice problems to this point it is tempting to let u be t^7 and to let dv be $\sin\left(2t^4\right)$.

However, this will lead to some real problems. To compute v we'd have to integrate the sine and because of the t^4 in the argument this is not possible. In order to integrate the sine we would have to have a t^3 in the integrand as well in order to a substitution as shown below,

$$\int t^3 \sin\left(2t^4\right)\, dt = \frac{1}{8}\int \sin\left(w\right)\, dw = -\frac{1}{8}\cos\left(2t^4\right) + c \qquad w = 2t^4$$

Now, this may seem like a problem, but in fact it's not a problem for this particular integral. Notice that we actually have 7 t's in the integral and there is no reason that we can't split them up as follows,

$$\int t^7 \sin\left(2t^4\right)\, dt = \int t^4\, t^3 \sin\left(2t^4\right)\, dt$$

After doing this we can now choose u and dv as follows,

$$u = t^4 \qquad\qquad dv = t^3 \sin\left(2t^4\right)\, dt$$

Step 2

Next, we need to compute du (by differentiating u) and v (by integrating dv).

$$u = t^4 \qquad \rightarrow \qquad du = 4t^3 dt$$

$$dv = t^3 \sin\left(2t^4\right)\, dt \qquad \rightarrow \qquad v = -\frac{1}{8}\cos\left(2t^4\right)$$

Step 3

Plugging u, du, v and dv into the Integration by Parts formula gives,

$$\int t^7 \sin\left(2t^4\right)\, dt = -\frac{1}{8}t^4\cos\left(2t^4\right) + \frac{1}{2}\int t^3\cos\left(2t^4\right)\, dt$$

Step 4

At this point, notice that the new integral just requires the same Calculus I substitution that we used to find v. So, all we need to do is evaluate the new integral and we'll be done.

$$\int t^7 \sin\left(2t^4\right)\, dt = \boxed{-\frac{1}{8}t^4\cos\left(2t^4\right) + \frac{1}{16}\sin\left(2t^4\right) + c}$$

Do not get so locked into patterns for these problems that you end up turning the patterns into "rules" on how certain kinds of problems work. Most of the easily seen patterns are also easily broken (as this problem has shown).

Because we (as instructors) tend to work a lot of "easy" problems initially they also tend to conform to the patterns that can be easily seen. This tends to lead students to the idea that the patterns will always work and then when they run into one where the pattern doesn't work they get in trouble. So, be careful!

Note as well that we're not saying that patterns don't exist and that it isn't useful to recognize them. You just need to be careful and understand that there will, on occasion, be problems where it will look like a pattern you recognize, but in fact will not quite fit the pattern and another approach will be needed to work the problem.

Alternative Solution

Note that there is an alternate solution to this problem. We could use the substitution $w = 2t^4$ as the first step as follows.

$$w = 2t^4 \qquad \to \qquad dw = 8t^3 dt \qquad \& \qquad t^4 = \frac{1}{2}w$$

$$\int t^7 \sin\left(2t^4\right)\, dt = \int t^4\, t^3 \sin\left(2t^4\right)\, dt = \int \left(\frac{1}{2}w\right)\left(\frac{1}{8}\right)\sin\left(w\right)\, dw = \int \frac{1}{16}w\sin\left(w\right)\, dw$$

We won't avoid integration by parts as we can see here, but it is somewhat easier to see it this time. Here is the rest of the work for this problem.

$$\begin{aligned} u &= \frac{1}{16}w & \to \qquad du &= \frac{1}{16}dw \\ dv &= \sin(w)\,dw & \to \qquad v &= -\cos(w) \end{aligned}$$

$$\int t^7 \sin\left(2t^4\right)\,dt = -\frac{1}{16}w\cos(w) + \frac{1}{16}\int \cos(w)\,dw = -\frac{1}{16}w\cos(w) + \frac{1}{16}\sin(w) + c$$

As the final step we just need to substitution back in for w.

$$\int t^7 \sin\left(2t^4\right)\,dt = -\frac{1}{8}t^4\cos\left(2t^4\right) + \frac{1}{16}\sin\left(2t^4\right) + c$$

8. Evaluate $\displaystyle\int y^6 \cos(3y)\,dy$.

Hint

Doing this with "standard" integration by parts would take a fair amount of time so maybe this would be a good candidate for the "table" method of integration by parts.

Step 1

Okay, with this problem doing the "standard" method of integration by parts (*i.e.* picking u and dv and using the formula) would take quite a bit of time. So, this looks like a good problem to use the table that we saw in the notes to shorten the process up.

Here is the table for this problem.

$$\begin{array}{rrc} y^6 & \cos(3y) & + \\ 6y^5 & \frac{1}{3}\sin(3y) & - \\ 30y^4 & -\frac{1}{9}\cos(3y) & + \\ 120y^3 & -\frac{1}{27}\sin(3y) & - \\ 360y^2 & \frac{1}{81}\cos(3y) & + \\ 720y & \frac{1}{243}\sin(3y) & - \\ 720 & -\frac{1}{729}\cos(3y) & + \\ 0 & -\frac{1}{2187}\sin(3y) & - \end{array}$$

Step 2

Here's the integral for this problem,

$$\begin{aligned}\int y^6 \cos(3y)\,dy &= \left(y^6\right)\left(\frac{1}{3}\sin(3y)\right) - \left(6y^5\right)\left(-\frac{1}{9}\cos(3y)\right) + \left(30y^4\right)\left(-\frac{1}{27}\sin(3y)\right)\\ &\quad - \left(120y^3\right)\left(\frac{1}{81}\cos(3y)\right) + \left(360y^2\right)\left(\frac{1}{243}\sin(3y)\right)\\ &\quad - \left(720y\right)\left(-\frac{1}{729}\cos(3y)\right) + \left(720\right)\left(-\frac{1}{2187}\sin(3y)\right) + c\\ &= \boxed{\begin{aligned}&\frac{1}{3}y^6\sin(3y) + \frac{2}{3}y^5\cos(3y) - \frac{10}{9}y^4\sin(3y) - \frac{40}{27}y^3\cos(3y)\\ &\quad + \frac{40}{27}y^2\sin(3y) + \frac{80}{81}y\cos(3y) - \frac{80}{243}\sin(3y) + c\end{aligned}}\end{aligned}$$

9. Evaluate $\displaystyle\int \left(4x^3 - 9x^2 + 7x + 3\right)\mathbf{e}^{-x}\,dx$.

Hint

Doing this with "standard" integration by parts would take a fair amount of time so maybe this would be a good candidate for the "table" method of integration by parts.

Step 1

Okay, with this problem doing the "standard" method of integration by parts (*i.e.* picking u and dv and using the formula) would take quite a bit of time. So, this looks like a good problem to use the table that we saw in the notes to shorten the process up.

Here is the table for this problem.

$$\begin{array}{rrc} 4x^3 - 9x^2 + 7x + 3 & \mathbf{e}^{-x} & + \\ 12x^2 - 18x + 7 & -\mathbf{e}^{-x} & - \\ 24x - 18 & \mathbf{e}^{-x} & + \\ 24 & -\mathbf{e}^{-x} & - \\ 0 & \mathbf{e}^{-x} & + \end{array}$$

Step 2

Here's the integral for this problem,

$$\begin{aligned}\int \left(4x^3 - 9x^2 + 7x + 3\right) \mathbf{e}^{-x}\, dx &= \left(4x^3 - 9x^2 + 7x + 3\right)\left(-\mathbf{e}^{-x}\right) \\ &\qquad - \left(12x^2 - 18x + 7\right)\left(\mathbf{e}^{-x}\right) \\ &\qquad + (24x - 18)\left(-\mathbf{e}^{-x}\right) - (24)\left(\mathbf{e}^{-x}\right) + c \\ &= -\mathbf{e}^{-x}\left(4x^3 - 9x^2 + 7x + 3\right) - \mathbf{e}^{-x}\left(12x^2 - 18x + 7\right) \\ &\qquad - \mathbf{e}^{-x}(24x - 18) - 24\mathbf{e}^{-x} + c \\ &= \boxed{-\mathbf{e}^{-x}\left(4x^3 + 3x^2 + 13x + 16\right) + c}\end{aligned}$$

7.2 Integrals Involving Trig Functions

1. Evaluate $\displaystyle\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)\,dx$.

Hint

Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1

The first thing to notice here is that the exponent on the sine is odd and so we can strip one of them out.

$$\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)\,dx = \int \sin^2\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)\sin\left(\frac{2}{3}x\right)\,dx$$

Step 2

Now we can use the trig identity $\sin^2(\theta)+\cos^2(\theta)=1$ to convert the remaining sines to cosines.

$$\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)\,dx = \int \left(1-\cos^2\left(\frac{2}{3}x\right)\right)\cos^4\left(\frac{2}{3}x\right)\sin\left(\frac{2}{3}x\right)\,dx$$

Step 3

We can now use the substitution $u=\cos\left(\frac{2}{3}x\right)$ to evaluate the integral.

$$\begin{aligned}\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)\,dx &= -\frac{3}{2}\int\left(1-u^2\right)u^4\,du\\ &= -\frac{3}{2}\int u^4-u^6\,du = -\frac{3}{2}\left(\frac{1}{5}u^5-\frac{1}{7}u^7\right)+c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4

Don't forget to substitute back in for u!

$$\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)\,dx = \boxed{\frac{3}{14}\cos^7\left(\frac{2}{3}x\right) - \frac{3}{10}\cos^5\left(\frac{2}{3}x\right) + c}$$

2. Evaluate $\int \sin^8(3z)\cos^5(3z)\,dz$.

Hint

Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1

The first thing to notice here is that the exponent on the cosine is odd and so we can strip one of them out.

$$\int \sin^8(3z)\cos^5(3z)\,dz = \int \sin^8(3z)\cos^4(3z)\cos(3z)\,dz$$

Step 2

Now we can use the trig identity $\sin^2(\theta) + \cos^2(\theta) = 1$ to convert the remaining cosines to sines.

$$\begin{aligned}\int \sin^8(3z)\cos^5(3z)\,dz &= \int \sin^8(3z)\left[\cos^2(3z)\right]^2\cos(3z)\,dz\\ &= \int \sin^8(3z)\left[1-\sin^2(3z)\right]^2\cos(3z)\,dz\end{aligned}$$

Step 3

We can now use the substitution $u = \sin(3z)$ to evaluate the integral.

$$\begin{aligned}\int \sin^8(3z)\cos^5(3z)\,dz &= \frac{1}{3}\int u^8\left[1-u^2\right]^2 du \\ &= \frac{1}{3}\int u^8 - 2u^{10} + u^{12}\,du = \frac{1}{3}\left(\frac{1}{9}u^9 - \frac{2}{11}u^{11} + \frac{1}{13}u^{13}\right) + c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4

Don't forget to substitute back in for u!

$$\int \sin^8(3z)\cos^5(3z)\,dz = \boxed{\frac{1}{27}\sin^9(3z) - \frac{2}{33}\sin^{11}(3z) + \frac{1}{39}\sin^{13}(3z) + c}$$

3. Evaluate $\displaystyle\int \cos^4(2t)\,dt$.

Hint

Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1

The first thing to notice here is that we only have even exponents and so we'll need to use half-angle and double-angle formulas to reduce this integral into one that we can do.

Also, do not get excited about the fact that we don't have any sines in the integrand. Sometimes we will not have both trig functions in the integrand. That doesn't mean that that we can't use the same techniques that we used in this section.

So, let's start this problem off as follows.

$$\int \cos^4(2t)\,dt = \int \left(\cos^2(2t)\right)^2 dt$$

Step 2

Now we can use the half-angle formula to get,

$$\int \cos^4(2t)\,dt = \int \left[\frac{1}{2}\left(1+\cos(4t)\right)\right]^2 dt = \int \frac{1}{4}\left(1+2\cos(4t)+\cos^2(4t)\right)\,dt$$

Step 3

We'll need to use the half-angle formula one more time on the third term to get,

$$\begin{aligned}\int \cos^4(2t)\,dt &= \frac{1}{4}\int 1+2\cos(4t)+\frac{1}{2}\left[1+\cos(8t)\right]\,dt\\ &= \frac{1}{4}\int \frac{3}{2}+2\cos(4t)+\frac{1}{2}\cos(8t)\,dt\end{aligned}$$

Step 4

Now all we have to do is evaluate the integral.

$$\begin{aligned}\int \cos^4(2t)\,dt &= \frac{1}{4}\left(\frac{3}{2}t+\frac{1}{2}\sin(4t)+\frac{1}{16}\sin(8t)\right)+c\\ &= \boxed{\frac{3}{8}t+\frac{1}{8}\sin(4t)+\frac{1}{64}\sin(8t)+c}\end{aligned}$$

4. Evaluate $\displaystyle\int_{\pi}^{2\pi} \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)\,dw$.

Hint

Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1

We have two options for dealing with the limits. We can deal with the limits as we do the integral or we can evaluate the indefinite integral and take care of the limits in the last step. We'll use the latter method of dealing with the limits for this problem.

In this case notice that both exponents are odd. This means that we can either strip out a cosine and convert the rest to sines or strip out a sine and convert the rest to cosines. Either are perfectly acceptable solutions. However, the exponent on the cosine is smaller and so there will be less conversion work if we strip out a cosine and convert the remaining cosines to sines.

Here is that work.

$$\begin{aligned}\int \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)\,dw &= \int \cos^2\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)\cos\left(\frac{1}{2}w\right)\,dw\\ &= \int \left(1-\sin^2\left(\frac{1}{2}w\right)\right)\sin^5\left(\frac{1}{2}w\right)\cos\left(\frac{1}{2}w\right)\,dw\end{aligned}$$

Step 2

We can now use the substitution $u = \sin\left(\frac{1}{2}w\right)$ to evaluate the integral.

$$\begin{aligned}\int \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)\,dw &= 2\int \left(1-u^2\right)u^5\,du\\ &= 2\int u^5 - u^7\,du = 2\left(\frac{1}{6}u^6 - \frac{1}{8}u^8\right) + c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 3

Don't forget to substitute back in for u!

$$\int \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)\,dw = \frac{1}{3}\sin^6\left(\frac{1}{2}w\right) - \frac{1}{4}\sin^8\left(\frac{1}{2}w\right) + c$$

Step 4

Now all we need to do is deal with the limits.

$$\int_{\pi}^{2\pi} \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)\,dw = \left(\frac{1}{3}\sin^6\left(\frac{1}{2}w\right) - \frac{1}{4}\sin^8\left(\frac{1}{2}w\right)\right)\Bigg|_{\pi}^{2\pi} = \boxed{-\frac{1}{12}}$$

Alternate Solution

As we noted above we could just have easily stripped out a sine and converted the rest to cosines if we'd wanted to. We'll not put that work in here, but here is the indefinite integral that you should have gotten had you done it that way.

$$\int \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)\,dw = -\frac{1}{2}\cos^4\left(\frac{1}{2}w\right) + \frac{2}{3}\cos^6\left(\frac{1}{2}w\right) - \frac{1}{4}\cos^8\left(\frac{1}{2}w\right) + c$$

Note as well that regardless of which approach we use to doing the indefinite integral the value of the definite integral will be the same.

5. Evaluate $\int \sec^6(3y)\tan^2(3y)\,dy$.

Hint

Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1

The first thing to notice here is that the exponent on the secant is even and so we can strip two of them out.

$$\int \sec^6(3y)\tan^2(3y)\,dy = \int \sec^4(3y)\tan^2(3y)\sec^2(3y)\,dy$$

Step 2

Now we can use the trig identity $\tan^2(\theta)+1=\sec^2(\theta)$ to convert the remaining secants to tangents.

$$\begin{aligned}\int \sec^6(3y)\tan^2(3y)\,dy &= \int \left[\sec^2(3y)\right]^2\tan^2(3y)\sec^2(3y)\,dy\\ &= \int \left[\tan^2(3y)+1\right]^2\tan^2(3y)\sec^2(3y)\,dy\end{aligned}$$

Step 3

We can now use the substitution $u=\tan(3y)$ to evaluate the integral.

$$\begin{aligned}\int \sec^6(3y)\tan^2(3y)\,dy &= \frac{1}{3}\int \left[u^2+1\right]^2u^2\,du\\ &= \frac{1}{3}\int u^6+2u^4+u^2\,du = \frac{1}{3}\left(\frac{1}{7}u^7+\frac{2}{5}u^5+\frac{1}{3}u^3\right)+c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4

Don't forget to substitute back in for u!

$$\int \sec^6(3y)\tan^2(3y)\,dy = \boxed{\frac{1}{21}\tan^7(3y) + \frac{2}{15}\tan^5(3y) + \frac{1}{9}\tan^3(3y) + c}$$

---

6. Evaluate $\displaystyle\int \tan^3(6x)\sec^{10}(6x)\,dx$.

Hint

Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1

The first thing to notice here is that the exponent on the tangent is odd and we've got a secant in the problems and so we can strip one of each of them out.

$$\int \tan^3(6x)\sec^{10}(6x)\,dx = \int \tan^2(6x)\sec^9(6x)\ \tan(6x)\sec(6x)\,dx$$

Step 2

Now we can use the trig identity $\tan^2(\theta)+1 = \sec^2(\theta)$ to convert the remaining tangents to secants.

$$\int \tan^3(6x)\sec^{10}(6x)\,dx = \int \left[\sec^2(6x) - 1\right]\sec^9(6x)\ \tan(6x)\sec(6x)\,dx$$

Note that because the exponent on the secant is even we could also have just stripped two of them out and converted the rest of them to tangents. However, that conversion process would have been significantly more work than the path that we chose here.

Step 3

We can now use the substitution $u = \sec(6x)$ to evaluate the integral.

$$\begin{aligned}\int \tan^3(6x)\sec^{10}(6x)\,dx &= \frac{1}{6}\int \left[u^2-1\right]u^9\,du \\ &= \frac{1}{6}\int u^{11}-u^9\,du = \frac{1}{6}\left(\frac{1}{12}u^{12}-\frac{1}{10}u^{10}\right)+c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4

Don't forget to substitute back in for u!

$$\int \tan^3(6x)\sec^{10}(6x)\,dx = \boxed{\frac{1}{72}\sec^{12}(6x)-\frac{1}{60}\sec^{10}(6x)+c}$$

7. Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^7(z)\sec^3(z)\,dz$.

Hint

Pay attention to the exponents and recall that for most of these kinds of problems you'll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1

We have two options for dealing with the limits. We can deal with the limits as we do the integral or we can evaluate the indefinite integral and take care of the limits in the last step. We'll use the latter method of dealing with the limits for this problem.

The first thing to notice here is that the exponent on the tangent is odd and we've got a secant in the problems and so we can strip one of each of them out and use the trig identity $\tan^2(\theta)+1=\sec^2(\theta)$ to convert the remaining tangents to secants.

$$\begin{aligned}\int \tan^7(z)\sec^3(z)\,dz &= \int \tan^6(z)\sec^2(z)\ \ \tan(z)\sec(z)\,dz\\ &= \int \left[\tan^2(z)\right]^3\sec^2(z)\ \ \tan(z)\sec(z)\,dz\\ &= \int \left[\sec^2(z)-1\right]^3\sec^2(z)\ \ \tan(z)\sec(z)\,dz\end{aligned}$$

Step 2

We can now use the substitution $u = \sec(z)$ to evaluate the integral.

$$\begin{aligned}\int \tan^7(z)\sec^3(z)\,dz &= \int \left[u^2-1\right]^3 u^2\,du\\ &= \int u^8-3u^6+3u^4-u^2\,du = \frac{1}{9}u^9-\frac{3}{7}u^7+\frac{3}{5}u^5-\frac{1}{3}u^3+c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 3

Don't forget to substitute back in for u!

$$\int \tan^7(z)\sec^3(z)\,dz = \frac{1}{9}\sec^9(z)-\frac{3}{7}\sec^7(z)+\frac{3}{5}\sec^5(z)-\frac{1}{3}\sec^3(z)+c$$

Step 4

Now all we need to do is deal with the limits.

$$\int_0^{\frac{\pi}{4}} \tan^7(z)\sec^3(z)\,dz = \left(\frac{1}{9}\sec^9(z) - \frac{3}{7}\sec^7(z) + \frac{3}{5}\sec^5(z) - \frac{1}{3}\sec^3(z)\right)\Bigg|_0^{\frac{\pi}{4}}$$
$$= \boxed{\frac{2}{315}\left(8+13\sqrt{2}\right) = 0.1675}$$

---

8. Evaluate $\displaystyle\int \cos(3t)\sin(8t)\,dt$.

Step 1

There really isn't all that much to this problem. All we have to do is use the formula given in this section for reducing a product of a sine and a cosine into a sum. Doing this gives,

$$\int \cos(3t)\sin(8t)\,dt = \int \frac{1}{2}\left[\sin(8t-3t)+\sin(8t+3t)\right]dt = \frac{1}{2}\int \sin(5t)+\sin(11t)\,dt$$

Make sure that you pay attention to the formula! The formula given in this section listed the sine first instead of the cosine. Make sure that you used the formula correctly!

Step 2

Now all we need to do is evaluate the integral.

$$\int \cos(3t)\sin(8t)\,dt = \frac{1}{2}\left(-\frac{1}{5}\cos(5t) - \frac{1}{11}\cos(11t)\right)+c$$
$$= \boxed{-\frac{1}{10}\cos(5t) - \frac{1}{22}\cos(11t)+c}$$

---

9. Evaluate $\displaystyle\int_1^3 \sin(8x)\sin(x)\,dx$.

Step 1

There really isn't all that much to this problem. All we have to do is use the formula given in this section for reducing a product of a sine and a cosine into a sum. Doing this gives,

$$\begin{aligned}\int_1^3 \sin(8x)\sin(x)\,dx &= \int_1^3 \frac{1}{2}\left[\cos(8x-x)-\cos(8x+x)\right]\,dx\\ &= \frac{1}{2}\int_1^3 \cos(7x)-\cos(9x)\,dx\end{aligned}$$

Step 2

Now all we need to do is evaluate the integral.

$$\begin{aligned}\int_1^3 \sin(8x)\sin(x)\,dx &= \frac{1}{2}\left[\frac{1}{7}\sin(7x)-\frac{1}{9}\sin(9x)\right]\Bigg|_1^3\\ &= \boxed{\frac{1}{14}\sin(21)-\frac{1}{18}\sin(27)-\frac{1}{14}\sin(7)+\frac{1}{18}\sin(9)}\\ &= \boxed{-0.0174}\end{aligned}$$

Make sure your calculator is set to radians if you computed a decimal answer!

---

10. Evaluate $\displaystyle\int \cot(10z)\csc^4(10z)\,dz$.

Hint

Even though no examples of products of cotangents and cosecants were done in the notes for this section you should know how to do them. Ask yourself how you would do the problem if it involved tangents and secants instead and you should be able to see how to do this problem as well.

Step 1

Other than the obvious difference in the actual functions there is no practical difference in how this problem and one that had tangents and secants would work. So, all we need to do is ask ourselves how this would work if it involved tangents and secants and we'll be able to work this on as well.

We can first notice here is that the exponent on the cotangent is odd and we've got a cosecant in the problems and so we can strip the (only) cotangent and one of the secants out.

$$\int \cot(10z)\csc^4(10z)\,dz = \int \csc^3(10z)\quad \cot(10z)\csc(10z)\,dz$$

Step 2

Normally we would use the trig identity $\cot^2(\theta)+1=\csc^2(\theta)$ to convert the remaining cotangents to cosecants. However, in this case there are no remaining cotangents to convert and so there really isn't anything to do at this point other than to use the substitution $u=\csc(10z)$ to evaluate the integral.

$$\int \cot(10z)\csc^4(10z)\,dz = -\frac{1}{10}\int u^3\,du = -\frac{1}{40}u^4+c$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 3

Don't forget to substitute back in for u!

$$\int \cot(10z)\csc^4(10z)\,dz = \boxed{-\frac{1}{40}\csc^4(10z)+c}$$

---

11. Evaluate $\int \csc^6\left(\frac{1}{4}w\right)\cot^4\left(\frac{1}{4}w\right)\,dw$.

Hint

Even though no examples of products of cotangents and cosecants were done in the notes for this section you should know how to do them. Ask yourself how you would do the problem if it involved tangents and secants instead and you should be able to see how to do this problem as well.

Step 1

Other than the obvious difference in the actual functions there is no practical difference in how this problem and one that had tangents and secants would work. So, all we need to do is ask ourselves how this would work if it involved tangents and secants and we'll be able to work this on as well.

We can first notice here is that the exponent on the cosecant is even and so we can strip out two of them.

$$\int \csc^6\left(\frac{1}{4}w\right)\cot^4\left(\frac{1}{4}w\right)\,dw = \int \csc^4\left(\frac{1}{4}w\right)\cot^4\left(\frac{1}{4}w\right)\,\csc^2\left(\frac{1}{4}w\right)\,dw$$

Step 2

Now we can use the trig identity $\cot^2(\theta)+1=\csc^2(\theta)$ to convert the remaining cosecants to cotangents.

$$\begin{aligned}\int \csc^6\left(\frac{1}{4}w\right)\cot^4\left(\frac{1}{4}w\right)\,dw &= \int \left[\csc^2\left(\frac{1}{4}w\right)\right]^2\cot^4\left(\frac{1}{4}w\right)\,\csc^2\left(\frac{1}{4}w\right)\,dw\\ &= \int \left[\cot^2\left(\frac{1}{4}w\right)+1\right]^2\cot^4\left(\frac{1}{4}w\right)\,\csc^2\left(\frac{1}{4}w\right)\,dw\end{aligned}$$

Step 3

Now we can use the substitution $u=\cot\left(\frac{1}{4}w\right)$ to evaluate the integral.

$$\int \csc^6\left(\frac{1}{4}w\right)\cot^4\left(\frac{1}{4}w\right)\,dw = -4\int \left[u^2+1\right]^2 u^4\,du$$
$$= -4\int u^8 + 2u^6 + u^4\,du = -4\left(\frac{1}{9}u^9 + \frac{2}{7}u^7 + \frac{1}{5}u^5\right) + c$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4

Don't forget to substitute back in for u!

$$\int \csc^6\left(\frac{1}{4}w\right)\cot^4\left(\frac{1}{4}w\right)\,dw = \boxed{-\frac{4}{9}\cot^9\left(\frac{1}{4}w\right) - \frac{8}{7}\cot^7\left(\frac{1}{4}w\right) - \frac{4}{5}\cot^5\left(\frac{1}{4}w\right) + c}$$

12. Evaluate $\displaystyle\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt$.

Hint

How would you do this problem if it were a product?

Step 1

If this were a product of secants and tangents we would know how to do it. The same ideas work here, except that we have to pay attention to only the numerator. We can't strip anything out of the denominator (in general) and expect it to work the same way. We can only strip things out of the numerator.

So, let's notice here is that the exponent on the secant is even and so we can strip out two of them.

$$\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt = \int \frac{\sec^2(2t)}{\tan^9(2t)}\sec^2(2t)\,dt$$

Step 2

Now we can use the trig identity $\tan^2(\theta) + 1 = \sec^2(\theta)$ to convert the remaining secants to tangents.

$$\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt = \int \frac{\tan^2(2t) + 1}{\tan^9(2t)}\sec^2(2t)\,dt$$

Step 3

Now we can use the substitution $u = \tan(2t)$ to evaluate the integral.

$$\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt = \frac{1}{2}\int \frac{u^2+1}{u^9}\,du = \frac{1}{2}\int u^{-7} + u^{-9}\,du = \frac{1}{2}\left[-\frac{1}{6}u^{-6} - \frac{1}{8}u^{-8}\right] + c$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4

Don't forget to substitute back in for u!

$$\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt = \boxed{-\frac{1}{12}\frac{1}{\tan^6(2t)} - \frac{1}{16}\frac{1}{\tan^8(2t)} + c = -\frac{1}{12}\cot^6(2t) - \frac{1}{16}\cot^8(2t) + c}$$

---

13. Evaluate $\displaystyle\int \frac{2 + 7\sin^3(z)}{\cos^2(z)}\,dz$.

Hint

How would you do this problem if it were a product?

Step 1

Because of the sum in the numerator it makes some sense (hopefully) to maybe split the integrand (and then the integral) up into two as follows.

$$\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz = \int \frac{2}{\cos^2(z)} + \frac{7\sin^3(z)}{\cos^2(z)}\,dz = \int \frac{2}{\cos^2(z)}\,dz + \int \frac{7\sin^3(z)}{\cos^2(z)}\,dz$$

Step 2

Now, the first integral looks difficult at first glance, but we can easily rewrite this in terms of secants at which point it becomes a really easy integral.

For the second integral again, think about how we would do that if it was a product instead of a quotient. In that case we would simply strip out a sine.

$$\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz = \int 2\sec^2(z)\;dz + 7\int \frac{\sin^2(z)}{\cos^2(z)}\sin(z)\;dz$$

Step 3

As noted above the first integral is now very easy (which we'll do in the next step) and for the second integral we can use the trig identity $\sin^2(\theta)+\cos^2(\theta)=1$ to convert the remaining sines in the second integral to cosines.

$$\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz = \int 2\sec^2(z)\;dz + 7\int \frac{1-\cos^2(z)}{\cos^2(z)}\sin(z)\;dz$$

Step 4

Now we can use the substitution $u=\cos(z)$ to evaluate the second integral. The first integral doesn't need any extra work.

$$\begin{aligned}\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz &= 2\tan(z) - 7\int \frac{1-u^2}{u^2}\,du \\ &= 2\tan(z) - 7\int u^{-2}-1\,du = 2\tan(z) - 7\left(-u^{-1}-u\right)+c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 5

Don't forget to substitute back in for u!

$$\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz$$
$$= \boxed{2\tan(z)+7\frac{1}{\cos(z)}+7\cos(z)+c = 2\tan(z)+7\sec(z)+7\cos(z)+c}$$

---

14. Evaluate $\displaystyle\int \left[9\sin^5(3x) - 2\cos^3(3x)\right]\csc^4(3x)\,dx$.

Hint

Since this has a mix of trig functions maybe the best option would be to first get it reduced down to just a couple that we know how to deal with.

Step 1

To get started on this problem we should first probably see if we can reduce the integrand down to just sines and cosines. This is easy enough to do simply by recalling the definition of cosecant in terms of sine.

$$\begin{aligned}\int \left[9\sin^5(3x) - 2\cos^3(3x)\right]\csc^4(3x)\,dx &= \int \left[9\sin^5(3x) - 2\cos^3(3x)\right]\frac{1}{\sin^4(3x)}\,dx \\ &= \int 9\sin(3x) - 2\frac{\cos^3(3x)}{\sin^4(3x)}\,dx\end{aligned}$$

Step 2

The first integral is simple enough to do without any extra work.

For the second integral again, think about how we would do that if it was a product instead of a quotient. In that case we would simply strip out a cosine.

$$\int \left[9\sin^5(3x) - 2\cos^3(3x)\right]\csc^4(3x)\,dx = \int 9\sin(3x) - 2\frac{\cos^2(3x)}{\sin^4(3x)}\cos(3x)\,dx$$

Step 3

For the second integral we can use the trig identity $\sin^2(\theta) + \cos^2(\theta) = 1$ to convert the remaining cosines to sines.

$$\int \left[9\sin^5(3x) - 2\cos^3(3x)\right]\csc^4(3x)\,dx$$

$$= \int 9\sin(3x)\,dx - 2\int \frac{1-\sin^2(3x)}{\sin^4(3x)}\cos(3x)\,dx$$

Step 4

Now we can use the substitution $u = \sin(3x)$ to evaluate the second integral. The first integral doesn't need any extra work.

$$\begin{aligned}\int \left[9\sin^5(3x) - 2\cos^3(3x)\right]\csc^4(3x)\,dx &= \int 9\sin(3x)\,dx - \frac{2}{3}\int \frac{1-u^2}{u^4}\,du \\ &= \int 9\sin(3x)\,dx - \frac{2}{3}\int u^{-4} - u^{-2}\,du \\ &= -3\cos(3x) - \frac{2}{3}\left(-\frac{1}{3}u^{-3} + u^{-1}\right) + c\end{aligned}$$

Note that we'll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 5

Don't forget to substitute back in for u!

$$\int \left[9\sin^5(3x) - 2\cos^3(3x)\right]\csc^4(3x)\,dx = -3\cos(3x) + \frac{2}{9}\frac{1}{\sin^3(3x)} - \frac{2}{3}\frac{1}{\sin(3x)} + c$$

$$= \boxed{-3\cos(3x) + \frac{2}{9}\csc^3(3x) - \frac{2}{3}\csc(3x) + c}$$

7.3 Trig Substitutions

1. Use a trig substitution to eliminate the root in $\sqrt{4-9z^2}$.

Hint

When determining which trig function to use for the substitution recall from the notes in this section that we will use one of three trig identities to convert the sum or difference under the root into a single trig function. Which trig identity is closest to the quantity under the root?

Step 1

The first step is to figure out which trig function to use for the substitution. To determine this notice that (ignoring the numbers) the quantity under the root looks similar to the identity,

$$1-\sin^2(\theta)=\cos^2(\theta)$$

So, it looks like sine is probably the correct trig function to use for the substitution. Now, we need to deal with the numbers on the two terms.

Hint

In order to actually use the identity from the first step we need to get the numbers in each term to be identical upon doing the substitution. So, what would the coefficient of the trig function need to be in order to convert the coefficient of the variable into the constant term once we've done the substitution?

Step 2

To get the coefficient on the trig function notice that we need to turn the 9 into a 4 once we've substituted the trig function in for z and squared the substitution out. With that in mind it looks like the substitution should be,

$$z=\frac{2}{3}\sin(\theta)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\sqrt{4-9z^2} = \sqrt{4-9\left(\frac{2}{3}\sin(\theta)\right)^2} = \sqrt{4-9\left(\frac{4}{9}\right)\sin^2(\theta)}$$
$$= \sqrt{4-4\sin^2(\theta)} = 2\sqrt{1-\sin^2(\theta)}$$
$$= 2\sqrt{\cos^2(\theta)} = \boxed{2\left|\cos(\theta)\right|}$$

Note that because we don't know the values of θ we can't determine if the cosine is positive or negative and so cannot get rid of the absolute value bars here.

---

2. Use a trig substitution to eliminate the root in $\sqrt{13+25x^2}$.

Hint

When determining which trig function to use for the substitution recall from the notes in this section that we will use one of three trig identities to convert the sum or difference under the root into a single trig function. Which trig identity is closest to the quantity under the root?

Step 1

The first step is to figure out which trig function to use for the substitution. To determine this notice that (ignoring the numbers) the quantity under the root looks similar to the identity,

$$1+\tan^2(\theta) = \sec^2(\theta)$$

So, it looks like tangent is probably the correct trig function to use for the substitution. Now, we need to deal with the numbers on the two terms.

Hint

In order to actually use the identity from the first step we need to get the numbers in each term to be identical upon doing the substitution. So, what would the coefficient of the trig function need to be in order to convert the coefficient of the variable into the constant term once we've done the substitution?

Step 2

To get the coefficient on the trig function notice that we need to turn the 25 into a 13 once we've substituted the trig function in for x and squared the substitution out. With that in mind it looks like the substitution should be,

$$x = \frac{\sqrt{13}}{5}\tan(\theta)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\begin{aligned}\sqrt{13+25x^2} &= \sqrt{13+25\left(\frac{\sqrt{13}}{5}\tan(\theta)\right)^2} = \sqrt{13+25\left(\frac{13}{25}\right)\tan^2(\theta)}\\ &= \sqrt{13+13\tan^2(\theta)} = \sqrt{13}\sqrt{1+\tan^2(\theta)}\\ &= \sqrt{13}\sqrt{\sec^2(\theta)} = \boxed{\sqrt{13}\,|\sec(\theta)|}\end{aligned}$$

Note that because we don't know the values of θ we can't determine if the secant is positive or negative and so cannot get rid of the absolute value bars here.

---

3. Use a trig substitution to eliminate the root in $(7t^2-3)^{\frac{5}{2}}$.

Hint

When determining which trig function to use for the substitution recall from the notes in this section that we will use one of three trig identities to convert the sum or difference under the root into a single trig function. Which trig identity is closest to the quantity under the root?

Step 1

First, notice that there really is a root here as the term can be written as,

$$(7t^2-3)^{\frac{5}{2}} = \left[(7t^2-3)^{\frac{1}{2}}\right]^5 = \left[\sqrt{7t^2-3}\right]^5$$

Now, we need to figure out which trig function to use for the substitution. To determine this notice that (ignoring the numbers) the quantity under the root looks similar to the identity,

$$\sec^2(\theta) - 1 = \tan^2(\theta)$$

So, it looks like secant is probably the correct trig function to use for the substitution. Now, we need to deal with the numbers on the two terms.

Hint

In order to actually use the identity from the first step we need to get the numbers in each term to be identical upon doing the substitution. So, what would the coefficient of the trig function need to be in order to convert the coefficient of the variable into the constant term once we've done the substitution?

Step 2

To get the coefficient on the trig function notice that we need to turn the 7 into a 3 once we've substituted the trig function in for t and squared the substitution out. With that in mind it looks like the substitution should be,

$$t = \frac{\sqrt{3}}{\sqrt{7}}\sec(\theta)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\begin{aligned}(7t^2 - 3)^{\frac{5}{2}} &= \left[\sqrt{7t^2 - 3}\right]^5 \\ &= \left[\sqrt{7\left(\frac{\sqrt{3}}{\sqrt{7}}\sec(\theta)\right)^2 - 3}\right]^5 = \left[\sqrt{7\left(\frac{3}{7}\right)\sec^2(\theta) - 3}\right]^5 \\ &= \left[\sqrt{3\sec^2(\theta) - 3}\right]^5 = \left[\sqrt{3}\sqrt{\sec^2(\theta) - 1}\right]^5 \\ &= \left[\sqrt{3}\sqrt{\tan^2(\theta)}\right]^5 = \boxed{3^{\frac{5}{2}}\left|\tan(\theta)\right|^5}\end{aligned}$$

Note that because we don't know the values of θ we can't determine if the tangent is positive or negative and so cannot get rid of the absolute value bars here.

4. Use a trig substitution to eliminate the root in $\sqrt{(w+3)^2 - 100}$.

Hint

Just because this looks a little different from the first couple of problems in this section doesn't mean that it works any differently. The term under the root still looks vaguely like one of three trig identities we need to use to convert the quantity under the root into a single trig function.

Step 1

Okay, first off we need to acknowledge that this does look a little bit different from the first few problems in this section. However, it isn't really all that different. We still have a difference between a squared term with a variable in it and a number. This looks similar to the following trig identity (ignoring the coefficients as usual).

$$\sec^2(\theta) - 1 = \tan^2(\theta)$$

So, secant is the trig function we'll need to use for the substitution here and we now need to deal with the numbers on the terms and get the substitution set up.

Hint

Dealing with the numbers in this case is no different than the first few problems in this section.

Step 2

Before dealing with the coefficient on the trig function let's notice that we'll be substituting in for $w+3$ in this case since that is the quantity that is being squared in the first term.

So, to get the coefficient on the trig function notice that we need to turn the 1 (*i.e.* the coefficient of the squared term) into a 100 once we've done the substitution. With that in mind it looks like the substitution should be,

$$w + 3 = 10\sec(\theta)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\sqrt{(w+3)^2-100}=\sqrt{(10\sec(\theta))^2-100}=\sqrt{100\sec^2(\theta)-100}=10\sqrt{\sec^2(\theta)-1}$$
$$=10\sqrt{\tan^2(\theta)}=\boxed{10\left|\tan(\theta)\right|}$$

Note that because we don't know the values of θ we can't determine if the tangent is positive or negative and so cannot get rid of the absolute value bars here.

5. Use a trig substitution to eliminate the root in $\sqrt{4(9t-5)^2+1}$.

Hint

Just because this looks a little different from the first couple of problems in this section doesn't mean that it works any differently. The term under the root still looks vaguely like one of three trig identities we need to use to convert the quantity under the root into a single trig function.

Step 1

Okay, first off we need to acknowledge that this does look a little bit different from the first few problems in this section. However, it isn't really all that different. We still have a sum of a squared term with a variable in it and a number. This looks similar to the following trig identity (ignoring the coefficients as usual).

$$\tan^2(\theta)+1=\sec^2(\theta)$$

So, tangent is the trig function we'll need to use for the substitution here and we now need to deal with the numbers on the terms and get the substitution set up.

Hint

Dealing with the numbers in this case is no different than the first few problems in this section.

Step 2

Before dealing with the coefficient on the trig function let's notice that we'll be substituting in for $9t - 5$ in this case since that is the quantity that is being squared in the first term.

So, to get the coefficient on the trig function notice that we need to turn the 4 (*i.e.* the coefficient of the squared term) into a 1 once we've done the substitution. With that in mind it looks like the substitution should be,

$$9t - 5 = \frac{1}{2}\tan(\theta)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\sqrt{4(9t-5)^2+1} = \sqrt{4\left(\frac{1}{2}\tan(\theta)\right)^2+1} = \sqrt{4\left(\frac{1}{4}\right)\tan^2(\theta)+1} = \sqrt{\tan^2(\theta)+1}$$
$$= \sqrt{\sec^2(\theta)} = \boxed{|\sec(\theta)|}$$

Note that because we don't know the values of θ we can't determine if the secant is positive or negative and so cannot get rid of the absolute value bars here.

6. Use a trig substitution to eliminate the root in $\sqrt{1-4z-2z^2}$.

Hint

This doesn't look much like a term that can use a trig substitution. So, the first step should probably be to some algebraic manipulation on the quantity under the root to make it look more like a problem that can use a trig substitution.

Step 1

We know that in order to do a trig substitution we really need a sum or difference of a term with a variable squared and a number. This clearly does not fit into that form. However, that doesn't mean that we can't do some algebraic manipulation on the quantity under the root to get into a form that we can do a trig substitution on.

Because the quantity under the root is a quadratic polynomial we know that we can complete the square on it to turn it into something like what we need for a trig substitution.

Here is the completing the square work.

$$\begin{aligned} 1-4z-2z^2 &= -2\left(z^2+2z-\frac{1}{2}\right) \\ &= -2\left(z^2+2z+1-1-\frac{1}{2}\right) \\ &= -2\left[(z+1)^2-\frac{3}{2}\right] \\ &= 3-2(z+1)^2 \end{aligned} \qquad \left[\frac{1}{2}(2)\right]^2 = [1]^2 = 1$$

So, after completing the square the term can be written as,

$$\sqrt{1-4z-2z^2} = \sqrt{3-2(z+1)^2}$$

Hint

At this point the problem works in the same manner as the previous problems in this section.

Step 2

So, in this case we see that we have a difference of a number and a squared term with a variable in it. This suggests that sine is the correct trig function to use for the substation.

Now, to get the coefficient on the trig function notice that we need to turn the 2 (*i.e.* the coefficient of the squared term) into a 3 once we've done the substitution. With that in mind it looks like the substitution should be,

$$z+1 = \frac{\sqrt{3}}{\sqrt{2}}\sin(\theta)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\sqrt{1-4z-2z^2} = \sqrt{3-2(z+1)^2} = \sqrt{3-2\left(\frac{\sqrt{3}}{\sqrt{2}}\sin(\theta)\right)^2}$$
$$= \sqrt{3-3\sin^2(\theta)} = \sqrt{3}\sqrt{\cos^2(\theta)} = \boxed{\sqrt{3}\left|\cos(\theta)\right|}$$

Note that because we don't know the values of θ we can't determine if the cosine is positive or negative and so cannot get rid of the absolute value bars here.

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7. Use a trig substitution to eliminate the root in $\left(x^2-8x+21\right)^{\frac{3}{2}}$.

Hint

This doesn't look much like a term that can use a trig substitution. So, the first step should probably be to some algebraic manipulation on the quantity under the root to make it look more like a problem that can use a trig substitution.

Step 1

We know that in order to do a trig substitution we really need a sum or difference of a term with a variable squared and a number. This clearly does not fit into that form. However, that doesn't mean that we can't do some algebraic manipulation on the quantity under the root to get into a form that we can do a trig substitution on.

Because the quantity under the root is a quadratic polynomial we know that we can complete the square on it to turn it into something like what we need for a trig substitution.

Here is the completing the square work.

$$x^2-8x+21 = x^2-8x+16-16+21 \qquad \left[\frac{1}{2}(-8)\right]^2 = [-4]^2 = 16$$
$$= (x-4)^2+5$$

So, after completing the square the term can be written as,

$$\left(x^2-8x+21\right)^{\frac{3}{2}} = \left((x-4)^2+5\right)^{\frac{3}{2}} = \left[\sqrt{(x-4)^2+5}\right]^3$$

Note that we also explicitly put the root into the problem as well.

Hint

At this point the problem works in the same manner as the previous problems in this section.

Step 2

So, in this case we see that we have a sum of a squared term with a variable in it and a number. This suggests that tangent is the correct trig function to use for the substation.

Now, to get the coefficient on the trig function notice that we need to turn the 1 (*i.e.* the coefficient of the squared term) into a 5 once we've done the substitution. With that in mind it looks like the substitution should be,

$$x - 4 = \sqrt{5}\tan(\theta)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\begin{aligned}\left(x^2 - 8x + 21\right)^{\frac{3}{2}} &= \left[\sqrt{(x-4)^2 + 5}\right]^3 = \left[\sqrt{\left(\sqrt{5}\tan(\theta)\right)^2 + 5}\right]^3 \\ &= \left[\sqrt{5\tan^2(\theta) + 5}\right]^3 = \left[\sqrt{5}\sqrt{\tan^2(\theta) + 1}\right]^3 \\ &= \left[\sqrt{5}\sqrt{\sec^2(\theta)}\right]^3 = \boxed{5^{\frac{3}{2}}\left|\sec(\theta)\right|^3}\end{aligned}$$

Note that because we don't know the values of θ we can't determine if the secant is positive or negative and so cannot get rid of the absolute value bars here.

8. Use a trig substitution to eliminate the root in $\sqrt{\mathbf{e}^{8x} - 9}$.

Hint

This doesn't look much like a term that can use a trig substitution. So, the first step should probably be to some algebraic manipulation on the quantity under the root to make it look more like a problem that can use a trig substitution.

Step 1

We know that in order to do a trig substitution we really need a sum or difference of a term with a variable squared and a number. Even though this doesn't look anything like the "normal" trig substitution problems it is actually pretty close to one. To see this all we need to do is rewrite the term under the root as follows.

$$\sqrt{\mathbf{e}^{8x} - 9} = \sqrt{\left(\mathbf{e}^{4x}\right)^2 - 9}$$

All we did here was take advantage of the basic exponent rules to make it clear that we really do have a difference here of a squared term containing a variable and a number.

Hint

At this point the problem works in the same manner as the previous problems in this section.

Step 2

The form of the quantity under the root suggests that secant is the correct trig function to use for the substation.

Now, to get the coefficient on the trig function notice that we need to turn the 1 (*i.e.* the coefficient of the squared term) into a 9 once we've done the substitution. With that in mind it looks like the substitution should be,

$$\mathbf{e}^{4x} = 3\sec\left(\theta\right)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\sqrt{\mathbf{e}^{8x}-9}=\sqrt{\left(3\sec\left(\theta\right)\right)^2-9}=\sqrt{9\sec^2\left(\theta\right)-9}$$
$$=3\sqrt{\sec^2\left(\theta\right)-1}=3\sqrt{\tan^2\left(\theta\right)}=\boxed{3\left|\tan\left(\theta\right)\right|}$$

Note that because we don't know the values of θ we can't determine if the tangent is positive or negative and so cannot get rid of the absolute value bars here.

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9. Use a trig substitution to evaluate $\displaystyle\int\frac{\sqrt{x^2+16}}{x^4}\,dx$.

Step 1

In this case it looks like we'll need the following trig substitution.

$$x=4\tan\left(\theta\right)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2

Let's first use the substitution to eliminate the root.

$$\sqrt{x^2+16}=\sqrt{16\tan^2\left(\theta\right)+16}=4\sqrt{\sec^2\left(\theta\right)}=4\left|\sec\left(\theta\right)\right|$$

Next, because we are doing an indefinite integral we will assume that the secant is positive and so we can drop the absolute value bars to get,

$$\sqrt{x^2+16}=4\sec\left(\theta\right)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$dx=4\sec^2\left(\theta\right)\,d\theta$$

Step 3

Now let's do the actual substitution.

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = \int \frac{4\sec(\theta)}{(4\tan(\theta))^4}\,4\sec^2(\theta)\,d\theta = \int \frac{\sec^3(\theta)}{16\tan^4(\theta)}\,d\theta$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4

We now need to evaluate the integral. In this case the integral looks to be a little difficult to do in terms of secants and tangents so let's convert the integrand to sines and cosines and see what we get. Doing this gives,

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = \frac{1}{16}\int \frac{\cos(\theta)}{\sin^4(\theta)}\,d\theta$$

This is a simple integral to evaluate so here is the integral evaluation.

$$\begin{aligned}\int \frac{\sqrt{x^2+16}}{x^4}\,dx &= \frac{1}{16}\int \frac{\cos(\theta)}{\sin^4(\theta)}\,d\theta \qquad\qquad u=\sin(\theta)\\ &= \frac{1}{16}\int u^{-4}\,du\\ &= -\frac{1}{48}u^{-3}+c = -\frac{1}{48}[\sin(\theta)]^{-3}+c = -\frac{1}{48}\csc^3(\theta)+c\end{aligned}$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Step 5

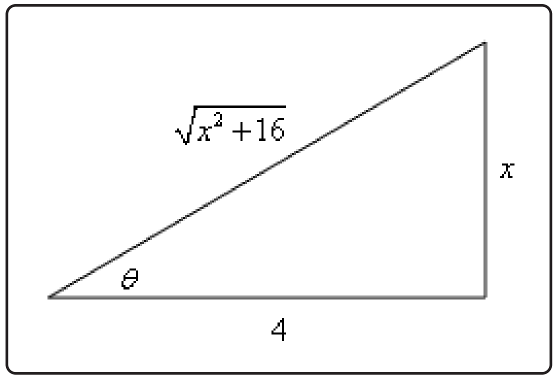
As the final step we just need to go back to x's. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\tan(\theta) = \frac{x}{4} \quad \left(= \frac{\text{opp}}{\text{adj}}\right)$$

From the right triangle we get,

$$\csc(\theta) = \frac{\sqrt{x^2+16}}{x}$$



The integral is then,

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = -\frac{1}{48}\left[\frac{\sqrt{x^2+16}}{x}\right]^3 + c = \boxed{-\frac{\left(x^2+16\right)^{\frac{3}{2}}}{48x^3} + c}$$

10. Use a trig substitution to evaluate $\int \sqrt{1-7w^2}\,dw$.

Step 1

In this case it looks like we'll need the following trig substitution.

$$w = \frac{1}{\sqrt{7}}\sin(\theta)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2

Let's first use the substitution to eliminate the root.

$$\sqrt{1-7w^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = |\cos(\theta)|$$

Next, because we are doing an indefinite integral we will assume that the cosine is positive and so we can drop the absolute value bars to get,

$$\sqrt{1-7w^2} = \cos(\theta)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$dw = \frac{1}{\sqrt{7}}\cos\left(\theta\right)\,d\theta$$

Step 3

Now let's do the actual substitution.

$$\int \sqrt{1-7w^2}\,dw = \int \cos\left(\theta\right)\left(\frac{1}{\sqrt{7}}\cos\left(\theta\right)\right)d\theta = \frac{1}{\sqrt{7}}\int \cos^2\left(\theta\right)\,d\theta$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4

We now need to evaluate the integral. Here is that work.

$$\int \sqrt{1-7w^2}\,dw = \frac{1}{\sqrt{7}}\int \frac{1}{2}\left[1+\cos\left(2\theta\right)\right]\,d\theta = \frac{1}{2\sqrt{7}}\left[\theta + \frac{1}{2}\sin\left(2\theta\right)\right] + c$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Step 5

As the final step we just need to go back to w's.

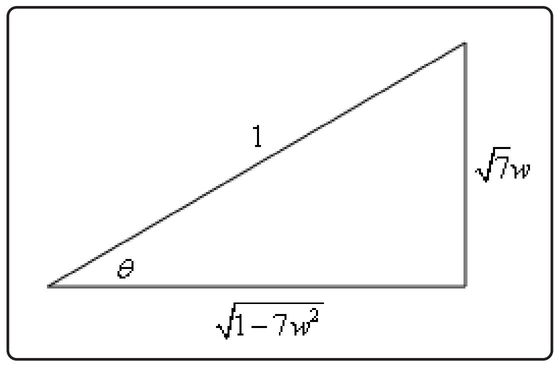
To eliminate the the first term (*i.e.* the θ) we can use any of the inverse trig functions. The easiest is to probably just use the original substitution and get a formula involving inverse sine but any of the six trig functions could be used if we wanted to. Using the substitution gives us,

$$\sin(\theta) = \sqrt{7}\,w \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(\sqrt{7}\,w\right)$$

Eliminating the $\sin(2\theta)$ requires a little more work. We can't just use a right triangle as we normally would because that would only give trig functions with an argument of θ and we have an argument of 2θ. However, we could use the double angle formula for sine to reduce this to trig functions with arguments of θ . Doing this gives,

$$\int \sqrt{1-7w^2}\,dw = \frac{1}{2\sqrt{7}}\left[\theta + \sin(\theta)\cos(\theta)\right] + c$$

We can now do the right triangle work.



From the substitution we have,

$$\sin(\theta) = \frac{\sqrt{7}w}{1} \quad \left(= \frac{\text{opp}}{\text{hyp}}\right)$$

From the right triangle we get,

$$\cos(\theta) = \sqrt{1-7w^2}$$

The integral is then,

$$\int \sqrt{1-7w^2}\,dw = \frac{1}{2\sqrt{7}}\left[\sin^{-1}\left(\sqrt{7}w\right) + \sqrt{7}\,w\sqrt{1-7w^2}\right] + c$$

11. Use a trig substitution to evaluate $\int t^3\left(3t^2-4\right)^{\frac{5}{2}}\,dt$.

Step 1

First, do not get excited about the exponent in the integrand. These types of problems work exactly the same as those with just a root (as opposed to this case in which we have a root to a power - you do agree that is what we have right?). So, in this case it looks like we'll need the following trig substitution.

$$t = \frac{2}{\sqrt{3}}\sec(\theta)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2

Let's first use the substitution to eliminate the root.

$$\left(3t^2-4\right)^{\frac{5}{2}}=\left[\sqrt{3t^2-4}\right]^5=\left[\sqrt{4\sec^2\left(\theta\right)-4}\right]^5=\left[2\sqrt{\tan^2\left(\theta\right)}\right]^5=32\left|\tan\left(\theta\right)\right|^5$$

Next, because we are doing an indefinite integral we will assume that the tangent is positive and so we can drop the absolute value bars to get,

$$\left(3t^2-4\right)^{\frac{5}{2}}=32\tan^5\left(\theta\right)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$dt=\frac{2}{\sqrt{3}}\sec\left(\theta\right)\tan\left(\theta\right)\,d\theta$$

Step 3

Now let's do the actual substitution.

$$\begin{aligned}\int t^3\left(3t^2-4\right)^{\frac{5}{2}}\,dt&=\int\left(\frac{2}{\sqrt{3}}\right)^3\sec^3\left(\theta\right)\left(32\tan^5\left(\theta\right)\right)\left(\frac{2}{\sqrt{3}}\sec\left(\theta\right)\tan\left(\theta\right)\right)\,d\theta\\&=\frac{512}{9}\int\sec^4\left(\theta\right)\tan^6\left(\theta\right)\,d\theta\end{aligned}$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4

We now need to evaluate the integral. Here is that work.

$$\begin{aligned}\int t^3\left(3t^2-4\right)^{\frac{5}{2}}dt &= \frac{512}{9}\int\left(\tan^2(\theta)+1\right)\tan^6(\theta)\ \sec^2(\theta)\ d\theta \qquad u=\tan(\theta)\\ &= \frac{512}{9}\int\left(u^2+1\right)u^6\,du = \frac{512}{9}\int u^8+u^6\,du\\ &= \frac{512}{9}\left[\frac{1}{9}\tan^9(\theta)+\frac{1}{7}\tan^7(\theta)\right]+c\end{aligned}$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Step 5

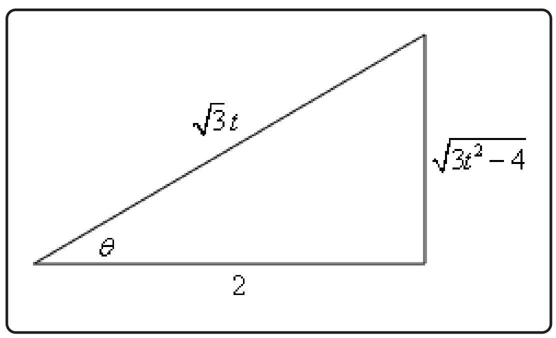
As the final step we just need to go back to t's. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\sec(\theta) = \frac{\sqrt{3}\,t}{2}\qquad\left(=\frac{\text{hyp}}{\text{adj}}\right)$$

From the right triangle we get,

$$\tan(\theta) = \frac{\sqrt{3t^2-4}}{2}$$



The integral is then,

$$\begin{aligned}\int t^3\left(3t^2-4\right)^{\frac{5}{2}}dt &= \frac{512}{9}\left[\frac{1}{9}\left(\frac{\sqrt{3t^2-4}}{2}\right)^9+\frac{1}{7}\left(\frac{\sqrt{3t^2-4}}{2}\right)^7\right]+c\\ &= \boxed{\frac{\left(3t^2-4\right)^{\frac{9}{2}}}{81}+\frac{4\left(3t^2-4\right)^{\frac{7}{2}}}{63}+c}\end{aligned}$$

12. Use a trig substitution to evaluate $\displaystyle\int_{-7}^{-5} \frac{2}{y^4\sqrt{y^2-25}}\,dy$.

Step 1

In this case it looks like we'll need the following trig substitution.

$$y = 5\sec(\theta)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2

Let's first use the substitution to eliminate the root.

$$\sqrt{y^2-25} = \sqrt{25\sec^2(\theta)-25} = 5\sqrt{\tan^2(\theta)} = 5\left|\tan(\theta)\right|$$

Step 3

Okay, in this case we have limits on y and so we can get limits on θ that will allow us to determine if tangent is positive or negative to allow us to eliminate the absolute value bars.

So, let's get some limits on θ.

$$\begin{aligned} y=-7: &\quad -7 = 5\sec(\theta) &\quad\rightarrow\quad& \sec(\theta) = -\frac{7}{5} &\quad\rightarrow\quad& \theta = \sec^{-1}\left(-\frac{7}{5}\right) = 2.3664 \\ y=-5: &\quad -5 = 5\sec(\theta) &\quad\rightarrow\quad& \sec(\theta) = -1 &\quad\rightarrow\quad& \theta = \pi \end{aligned}$$

So, θ's for this problem are in the range $2.3664 \le \theta \le \pi$ and these are in the second quadrant. In the second quadrant we know that tangent is negative and so we can drop the absolute value bars provided we add in a minus sign. This gives,

$$\sqrt{y^2-25} = -5\tan(\theta)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$dy = 5\sec(\theta)\tan(\theta)\,d\theta$$

Step 4

Now let's do the actual substitution.

$$\begin{aligned}\int_{-7}^{-5} \frac{2}{y^4\sqrt{y^2-25}}\,dy &= \int_{2.3664}^{\pi} \frac{2}{5^4\sec^4(\theta)\,(-5\tan(\theta))}\,(5\sec(\theta)\tan(\theta))\,d\theta \\ &= -\frac{2}{625}\int_{2.3664}^{\pi} \frac{1}{\sec^3(\theta)}\,d\theta\end{aligned}$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Also notice that upon doing the substation we replaced the y limits with the θ limits. This will help with a later step.

Step 5

We now need to evaluate the integral. In terms of secants this integral would be pretty difficult, however we a quick change to cosines we get the following integral.

$$\int_{-7}^{-5} \frac{2}{y^4\sqrt{y^2-25}}\,dy = -\frac{2}{625}\int_{2.3664}^{\pi} \cos^3(\theta)\,d\theta$$

This should be relatively simple to do so here is the integration work.

$$\begin{aligned}\int_{-7}^{-5} \frac{2}{y^4\sqrt{y^2-25}}\,dy &= -\frac{2}{625}\int_{2.3664}^{\pi} \left(1-\sin^2(\theta)\right)\cos(\theta)\,d\theta \qquad u=\sin(\theta) \\ &= -\frac{2}{625}\int_{\sin(2.3664)}^{\sin(\pi)} 1-u^2\,du \\ &= -\frac{2}{625}\left[u-\frac{1}{3}u^3\right]\Bigg|_{0.69986}^{0} = \boxed{0.001874}\end{aligned}$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Also, note that because we converted the limits at every substitution into limits for the "new" variable we did not need to do any back substitution work on our answer!

13. Use a trig substitution to evaluate $\displaystyle\int_1^4 2z^5\sqrt{2+9z^2}\,dz$.

Step 1

In this case it looks like we'll need the following trig substitution.

$$z = \frac{\sqrt{2}}{3}\tan\left(\theta\right)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2

Let's first use the substitution to eliminate the root.

$$\sqrt{2+9z^2} = \sqrt{2+2\tan^2\left(\theta\right)} = \sqrt{2}\sqrt{\sec^2\left(\theta\right)} = \sqrt{2}\left|\sec\left(\theta\right)\right|$$

Step 3

Okay, in this case we have limits on z and so we can get limits on θ that will allow us to determine if tangent is positive or negative to allow us to eliminate the absolute value bars.

So, let's get some limits on θ.

$$z=1: \quad 1 = \frac{\sqrt{2}}{3}\tan\left(\theta\right) \quad \to \quad \tan\left(\theta\right) = \frac{3}{\sqrt{2}} \quad \to \quad \theta = \tan^{-1}\left(\frac{3}{\sqrt{2}}\right) = 1.1303$$

$$z=4: \quad 4 = \frac{\sqrt{2}}{3}\tan\left(\theta\right) \quad \to \quad \tan\left(\theta\right) = \frac{12}{\sqrt{2}} \quad \to \quad \theta = \tan^{-1}\left(\frac{12}{\sqrt{2}}\right) = 1.4535$$

So, θ's for this problem are in the range $1.1303 \le \theta \le 1.4535$ and these are in the first quadrant. In the first quadrant we know that cosine, and hence secant, is positive and so we can just drop the absolute value bars. This gives,

$$\sqrt{2+9z^2} = \sqrt{2}\sec\left(\theta\right)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$dz = \frac{\sqrt{2}}{3}\sec^2(\theta)\,d\theta$$

Step 4

Now let's do the actual substitution.

$$\begin{aligned}\int_1^4 2z^5\sqrt{2+9z^2}\,dz &= \int_{1.1303}^{1.4535} 2\left(\frac{\sqrt{2}}{3}\right)^5 \tan^5(\theta)\left(\sqrt{2}\sec(\theta)\right)\left(\frac{\sqrt{2}}{3}\sec^2(\theta)\right)d\theta \\ &= \frac{16\sqrt{2}}{729}\int_{1.1303}^{1.4535}\tan^5(\theta)\sec^3(\theta)\,d\theta\end{aligned}$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Also notice that upon doing the substation we replaced the y limits with the θ limits. This will help with a later step.

Step 5

We now need to evaluate the integral. Here is that work.

$$\begin{aligned}\int_1^4 2z^5\sqrt{2+9z^2}\,dz &= \frac{16\sqrt{2}}{729}\int_{1.1303}^{1.4535}\left[\sec^2(\theta)-1\right]^2\sec^2(\theta)\ \tan(\theta)\sec(\theta)\,d\theta \\ &= \frac{16\sqrt{2}}{729}\int_{\sec(1.1303)}^{\sec(1.4535)}\left[u^2-1\right]^2u^2du \qquad\qquad u=\sec(\theta) \\ &= \frac{16\sqrt{2}}{729}\int_{2.3452}^{8.5440}u^6-2u^4+u^2du \\ &= \frac{16\sqrt{2}}{729}\left[\frac{1}{7}u^7-\frac{2}{5}u^5+\frac{1}{3}u^3\right]\Bigg|_{2.3452}^{8.5440} = \boxed{14178.20559}\end{aligned}$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Also, note that because we converted the limits at every substitution into limits for the "new" variable we did not need to do any back substitution work on our answer!

14. Use a trig substitution to evaluate $\displaystyle\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\,dx$.

Step 1

The first thing we'll need to do here is complete the square on the polynomial to get this into a form we can use a trig substitution on.

$$9x^2 - 36x + 37 = 9\left(x^2 - 4x + \frac{37}{9}\right) = 9\left(x^2 - 4x + 4 - 4 + \frac{37}{9}\right) = 9\left[(x-2)^2 + \frac{1}{9}\right]$$
$$= 9(x-2)^2 + 1$$

The integral is now,

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\,dx = \int \frac{1}{\sqrt{9(x-2)^2 + 1}}\,dx$$

Now we can proceed with the trig substitution.

Step 2

It looks like we'll need to the following trig substitution.

$$x - 2 = \frac{1}{3}\tan(\theta)$$

Next let's eliminate the root.

$$\sqrt{9(x-2)^2 + 1} = \sqrt{\tan(\theta)^2 + 1} = \sqrt{\sec^2(\theta)} = |\sec(\theta)|$$

Next, because we are doing an indefinite integral we will assume that the secant is positive and so we can drop the absolute value bars to get,

$$\sqrt{9(x-2)^2 + 1} = \sec(\theta)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$(1)\,dx = \frac{1}{3}\sec^2(\theta)\,d\theta \qquad \Rightarrow \qquad dx = \frac{1}{3}\sec^2(\theta)\,d\theta$$

Recall that all we really need to do here is compute the differential for both the right and left sides of the substitution.

Step 3

Now let's do the actual substitution.

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\,dx = \int \frac{1}{\sec(\theta)}\left(\frac{1}{3}\sec^2(\theta)\right)\,d\theta = \frac{1}{3}\int \sec(\theta)\,d\theta$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4

We now need to evaluate the integral. Here is that work.

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\,dx = \frac{1}{3}\ln\left|\sec(\theta) + \tan(\theta)\right| + c$$

Note that this was one of the few trig substitution integrals that didn't really require a lot of manipulation of trig functions to completely evaluate. All we had to really do here was use the fact that we determined the integral of $\sec(\theta)$ in the previous section and reuse that result here.

Step 5

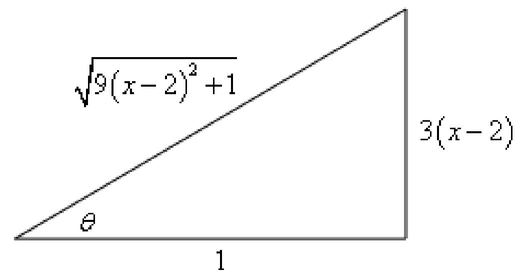
As the final step we just need to go back to x's. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\tan(\theta) = \frac{3(x-2)}{1} \qquad \left(= \frac{\text{opp}}{\text{adj}}\right)$$

From the right triangle we get,

$$\sec(\theta) = \sqrt{9(x-2)^2+1}$$



The integral is then,

$$\int \frac{1}{\sqrt{9x^2-36x+37}}\,dx = \boxed{\frac{1}{3}\ln\left|\sqrt{9(x-2)^2+1}+3(x-2)\right|+c}$$

15. Use a trig substitution to evaluate $\displaystyle\int \frac{(z+3)^5}{(40-6z-z^2)^{\frac{3}{2}}}\,dz$.

Step 1

The first thing we'll need to do here is complete the square on the polynomial to get this into a form we can use a trig substitution on.

$$\begin{aligned} 40-6z-z^2 &= -\left(z^2+6z-40\right) = -\left(z^2+6z+9-9-40\right) = -\left[(z+3)^2-49\right] \\ &= 49-(z+3)^2 \end{aligned}$$

The integral is now,

$$\int \frac{(z+3)^5}{(40-6z-z^2)^{\frac{3}{2}}}\,dz = \int \frac{(z+3)^5}{\left(49-(z+3)^2\right)^{\frac{3}{2}}}\,dz$$

Now we can proceed with the trig substitution.

Step 2

It looks like we'll need to the following trig substitution.

$$z+3 = 7\sin(\theta)$$

Next let's eliminate the root.

$$\left(49-(z+3)^2\right)^{\frac{3}{2}} = \left[\sqrt{49-(z+3)}\right]^3 = \left[\sqrt{49-49\sin^2(\theta)}\right]^3 = \left[7\sqrt{\cos^2(\theta)}\right]^3$$
$$= 343\left|\cos(\theta)\right|^3$$

Next, because we are doing an indefinite integral we will assume that the cosine is positive and so we can drop the absolute value bars to get,

$$\left(49-(z+3)^2\right)^{\frac{3}{2}} = 343\cos^3(\theta)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$(1)\,dz = 7\cos(\theta)\,d\theta \qquad \Rightarrow \qquad dz = 7\cos(\theta)\,d\theta$$

Recall that all we really need to do here is compute the differential for both the right and left sides of the substitution.

Step 3

Now let's do the actual substitution.

$$\int \frac{(z+3)^5}{(40-6z-z^2)^{\frac{3}{2}}}\,dz = \int \frac{16807\sin^5(\theta)}{343\cos^3(\theta)}\left(7\cos(\theta)\right)\,d\theta = 343\int \frac{\sin^5(\theta)}{\cos^2(\theta)}\,d\theta$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4

We now need to evaluate the integral. Here is that work.

$$\begin{aligned}\int \frac{(z+3)^5}{(40-6z-z^2)^{\frac{3}{2}}}\,dz &= 343\int \frac{\left[1-\cos^2(\theta)\right]^2}{\cos^2(\theta)}\sin(\theta)\,d\theta \qquad u=\cos(\theta)\\ &= -343\int \frac{\left[1-u^2\right]^2}{u^2}\,du = -343\int u^{-2}-2+u^2\,du\\ &= -343\left(-u^{-1}-2u+\frac{1}{3}u^3\right)+c\\ &= -343\left(-\frac{1}{\cos(\theta)}-2\cos(\theta)+\frac{1}{3}\cos^3(\theta)\right)+c\end{aligned}$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Step 5

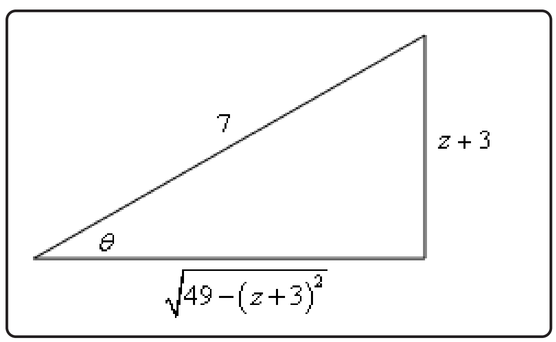
As the final step we just need to go back to z's. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\sin(\theta) = \frac{z+3}{7} \qquad \left(=\frac{\text{adj}}{\text{hyp}}\right)$$

From the right triangle we get,

$$\cos(\theta) = \frac{\sqrt{49-(z+3)^2}}{7}$$



The integral is then,

$$\int \frac{(z+3)^5}{(40-6z-z^2)^{\frac{3}{2}}}\,dz$$
$$= \boxed{\frac{2401}{\sqrt{49-(z+3)^2}} + 98\sqrt{49-(z+3)^2} - \frac{\left(49-(z+3)^2\right)^{\frac{3}{2}}}{3} + c}$$

16. Use a trig substitution to evaluate $\displaystyle\int \cos(x)\sqrt{9+25\sin^2(x)}\,dx$.

Step 1

Let's first rewrite the integral a little bit.

$$\int \cos(x)\sqrt{9+25\sin^2(x)}\,dx = \int \cos(x)\sqrt{9+25[\sin(x)]^2}\,dx$$

Step 2

With the integral written as it is in the first step we can now see that we do have a sum of a number and something squared under the root. We know from the problems done previously in this section that looks like a tangent substitution. So, let's use the following substitution.

$$\sin(x) = \frac{3}{5}\tan(\theta)$$

Do not get excited about the fact that we are substituting one trig function for another. That will happen on occasion with these kinds of problems. Note however, that we need to be careful and make sure that we also change the variable from x (*i.e.* the variable in the original trig function) into θ (*i.e.* the variable in the new trig function).

Next let's eliminate the root.

$$\sqrt{9+25[\sin(x)]^2} = \sqrt{9+25\left[\frac{3}{5}\tan(\theta)\right]^2} = \sqrt{9+9\tan^2(\theta)} = 3\sqrt{\sec^2(\theta)} = 3\left|\sec(\theta)\right|$$

Next, because we are doing an indefinite integral we will assume that the secant is positive and so we can drop the absolute value bars to get,

$$\sqrt{9+25[\sin(x)]^2} = 3\sec(\theta)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$\cos\left(x\right)\,dx = \frac{3}{5}\sec^2\left(\theta\right)\,d\theta$$

Recall that all we really need to do here is compute the differential for both the right and left sides of the substitution.

Step 3

Now let's do the actual substitution.

$$\begin{aligned}\int \cos\left(x\right)\sqrt{9+25\sin^2\left(x\right)}\,dx &= \int \sqrt{9+25[\sin\left(x\right)]^2}\ \ \cos\left(x\right)dx \\ &= \int \left(3\sec\left(\theta\right)\right)\left(\frac{3}{5}\sec^2\left(\theta\right)\right)d\theta = \frac{9}{5}\int \sec^3\left(\theta\right)d\theta\end{aligned}$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4

We now need to evaluate the integral. Here is that work.

$$\int \cos\left(x\right)\sqrt{9+25\sin^2\left(x\right)}\,dx = \frac{9}{10}\left[\sec\left(\theta\right)\tan\left(\theta\right)+\ln\left|\sec\left(\theta\right)+\tan\left(\theta\right)\right|\right]+c$$

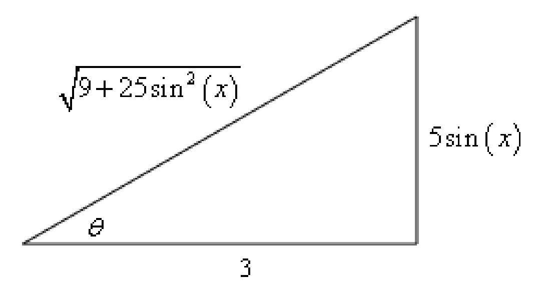
Note that this was one of the few trig substitution integrals that didn't really require a lot of manipulation of trig functions to completely evaluate. All we had to really do here was use the fact that we determined the integral of $\sec^3\left(\theta\right)$ in the previous section and reuse that result here.

Step 5

As the final step we just need to go back to x's. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\tan(\theta) = \frac{5\sin(x)}{3} \qquad \left(= \frac{\text{opp}}{\text{adj}}\right)$$



From the right triangle we get,

$$\sec(\theta) = \frac{\sqrt{9+25\sin^2(x)}}{3}$$

The integral is then,

$$\int \cos(x)\sqrt{9+25\sin^2(x)}\,dx$$

$$= \boxed{\frac{\sin(x)\sqrt{9+25\sin^2(x)}}{2} + \frac{9}{10}\ln\left|\frac{5\sin(x)+\sqrt{9+25\sin^2(x)}}{3}\right| + c}$$

7.4 Partial Fractions

1. Evaluate the integral $\displaystyle\int \frac{4}{x^2+5x-14}\,dx$.

Step 1

To get the problem started off we need the form of the partial fraction decomposition of the integrand. However, in order to get this, we'll need to factor the denominator.

$$\int \frac{4}{x^2+5x-14}\,dx = \int \frac{4}{(x+7)(x-2)}\,dx$$

The form of the partial fraction decomposition for the integrand is then,

$$\frac{4}{(x+7)(x-2)} = \frac{A}{x+7} + \frac{B}{x-2}$$

Step 2

Setting the numerators equal gives,

$$4 = A(x-2) + B(x+7)$$

Step 3

We can use the "trick" discussed in the notes to easily get the coefficients in this case so let's do that. Here is that work.

$$\begin{aligned} x &= 2 \quad : \quad 4 = 9B \\ x &= -7 : \quad 4 = -9A \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= -\frac{4}{9} \\ B &= \frac{4}{9} \end{aligned}$$

The partial fraction form of the integrand is then,

$$\frac{4}{(x+7)(x-2)} = \frac{-\frac{4}{9}}{x+7} + \frac{\frac{4}{9}}{x-2}$$

Step 4

We can now do the integral.

$$\int \frac{4}{(x+7)(x-2)}\,dx = \int \frac{-\frac{4}{9}}{x+7} + \frac{\frac{4}{9}}{x-2}\,dx = \boxed{\frac{4}{9}\ln|x-2| - \frac{4}{9}\ln|x+7| + c}$$

---

2. Evaluate the integral $\displaystyle\int \frac{8-3t}{10t^2+13t-3}\,dt$.

Step 1

To get the problem started off we need the form of the partial fraction decomposition of the integrand. However, in order to get this, we'll need to factor the denominator.

$$\int \frac{8-3t}{10t^2+13t-3}\,dt = \int \frac{8-3t}{(2t+3)(5t-1)}\,dt$$

The form of the partial fraction decomposition for the integrand is then,

$$\frac{8-3t}{10t^2+13t-3} = \frac{A}{2t+3} + \frac{B}{5t-1}$$

Step 2

Setting the numerators equal gives,

$$8-3t = A(5t-1) + B(2t+3)$$

Step 3

We can use the "trick" discussed in the notes to easily get the coefficients in this case so let's do that. Here is that work.

$$\begin{array}{lll} t = \frac{1}{5} \;: & \frac{37}{5} = \frac{17}{5}B & \qquad A = -\frac{25}{17} \\ & \qquad\qquad\Rightarrow & \\ t = -\frac{3}{2} : & \frac{25}{2} = -\frac{17}{2}A & \qquad B = \frac{37}{17} \end{array}$$

The partial fraction form of the integrand is then,

$$\frac{8-3t}{10t^2+13t-3} = \frac{-\frac{25}{17}}{2t+3} + \frac{\frac{37}{17}}{5t-1}$$

Step 4

We can now do the integral.

$$\int \frac{8-3t}{10t^2+13t-3}\,dt = \int \frac{-\frac{25}{17}}{2t+3} + \frac{\frac{37}{17}}{5t-1}\,dt = \boxed{\frac{37}{85}\ln|5t-1| - \frac{25}{34}\ln|2t+3| + c}$$

Hopefully you are getting good enough with integration that you can do some of these integrals in your head. Be careful however with both of these integrals. When doing these kinds of integrals in our head it is easy to forget about the substitutions that are technically required to do them and then miss the coefficients from the substitutions that need to show up in the answer.

---

3. Evaluate the integral $\displaystyle\int_{-1}^{0} \frac{w^2+7w}{(w+2)(w-1)(w-4)}\,dw$.

Step 1

In this case the denominator is already factored and so we can go straight to the form of the partial fraction decomposition for the integrand.

$$\frac{w^2+7w}{(w+2)(w-1)(w-4)} = \frac{A}{w+2} + \frac{B}{w-1} + \frac{C}{w-4}$$

Step 2

Setting the numerators equal gives,

$$w^2+7w = A(w-1)(w-4) + B(w+2)(w-4) + C(w+2)(w-1)$$

Step 3

We can use the "trick" discussed in the notes to easily get the coefficients in this case so let's do that. Here is that work.

$$\begin{array}{lll} w=1 \quad : & 8=-9B & \\ w=4 \quad : & 44=18C & \Rightarrow \\ w=-2: & -10=18A & \end{array} \qquad \begin{array}{l} A=-\frac{5}{9} \\ B=-\frac{8}{9} \\ C=\frac{22}{9} \end{array}$$

The partial fraction form of the integrand is then,

$$\frac{w^2+7w}{(w+2)(w-1)(w-4)}=\frac{-\frac{5}{9}}{w+2}-\frac{\frac{8}{9}}{w-1}+\frac{\frac{22}{9}}{w-4}$$

Step 4

We can now do the integral.

$$\begin{aligned} \int_{-1}^{0}\frac{w^2+7w}{(w+2)(w-1)(w-4)}\,dw &= \int_{-1}^{0}\frac{-\frac{5}{9}}{w+2}-\frac{\frac{8}{9}}{w-1}+\frac{\frac{22}{9}}{w-4}\,dw \\ &= \left(-\frac{5}{9}\ln|w+2|-\frac{8}{9}\ln|w-1|+\frac{22}{9}\ln|w-4|\right)\Big|_{-1}^{0} \\ &= \boxed{\frac{22}{9}\ln(4)+\frac{3}{9}\ln(2)-\frac{22}{9}\ln(5)} \\ &= \boxed{\frac{47}{9}\ln(2)-\frac{22}{9}\ln(5)} \end{aligned}$$

Note that we used a quick logarithm property to combine the first two logarithms into a single logarithm. You should probably review your logarithm properties if you don't recognize the one that we used. These kinds of property applications can really simplify your work on occasion if you know them!

4. Evaluate the integral $\displaystyle\int \frac{8}{3x^3+7x^2+4x}\,dx$.

Step 1

To get the problem started off we need the form of the partial fraction decomposition of the integrand. However, in order to get this, we'll need to factor the denominator.

$$\int \frac{8}{3x^3+7x^2+4x}\,dx = \int \frac{8}{x\,(3x+4)\,(x+1)}\,dx$$

The form of the partial fraction decomposition for the integrand is then,

$$\frac{8}{x\,(3x+4)\,(x+1)} = \frac{A}{x} + \frac{B}{3x+4} + \frac{C}{x+1}$$

Step 2

Setting the numerators equal gives,

$$8 = A\,(3x+4)\,(x+1) + Bx\,(x+1) + C\,x\,(3x+4)$$

Step 3

We can use the "trick" discussed in the notes to easily get the coefficients in this case so let's do that. Here is that work.

$$\begin{array}{lll} x = -\frac{4}{3} : \quad 8 = \frac{4}{9}B & & A = 2 \\ x = -1 : \quad 8 = -C & \Rightarrow & B = 18 \\ x = 0 : \quad 8 = 4A & & C = -8 \end{array}$$

The partial fraction form of the integrand is then,

$$\frac{8}{x\,(3x+4)\,(x+1)} = \frac{2}{x} + \frac{18}{3x+4} - \frac{8}{x+1}$$

Step 4

We can now do the integral.

$$\int \frac{8}{x\,(3x+4)\,(x+1)}\,dx = \int \frac{2}{x} + \frac{18}{3x+4} - \frac{8}{x+1}\,dx$$
$$= \boxed{2\ln|x| + 6\ln|3x+4| - 8\ln|x+1| + c}$$

Hopefully you are getting good enough with integration that you can do some of these integrals in your head. Be careful however with the second integral. When doing these kinds of integrals in our head it is easy to forget about the substitutions that are technically required to do them and then miss the coefficients from the substitutions that need to show up in the answer.

---

5. Evaluate the integral $\displaystyle\int_2^4 \frac{3z^2+1}{(z+1)\,(z-5)^2}\,dz$.

Step 1

In this case the denominator is already factored and so we can go straight to the form of the partial fraction decomposition for the integrand.

$$\frac{3z^2+1}{(z+1)\,(z-5)^2} = \frac{A}{z+1} + \frac{B}{z-5} + \frac{C}{(z-5)^2}$$

Step 2

Setting the numerators equal gives,

$$3z^2+1 = A(z-5)^2 + B\,(z+1)\,(z-5) + C\,(z+1)$$

Step 3

We can use the “trick” discussed in the notes to easily get two of the coefficients and then we can just pick another value of z to get the third so let’s do that. Here is that work.

$$\begin{aligned} z=-1:&\quad 4=36A & & A=\frac{1}{9} \\ z=5\quad:&\quad 76=6C & \Rightarrow\quad & B=\frac{26}{9} \\ z=0\quad:&\quad 1=25A-5B+C=\frac{139}{9}-5B & & C=\frac{38}{3} \end{aligned}$$

The partial fraction form of the integrand is then,

$$\frac{3z^2+1}{(z+1)(z-5)^2}=\frac{\frac{1}{9}}{z+1}+\frac{\frac{26}{9}}{z-5}+\frac{\frac{38}{3}}{(z-5)^2}$$

Step 4

We can now do the integral.

$$\begin{aligned}\int_2^4 \frac{3z^2+1}{(z+1)(z-5)^2}\,dz &= \int_2^4 \frac{\frac{1}{9}}{z+1}+\frac{\frac{26}{9}}{z-5}+\frac{\frac{38}{3}}{(z-5)^2}\,dz \\ &= \left.\left(\frac{1}{9}\ln|z+1|+\frac{26}{9}\ln|z-5|-\frac{\frac{38}{3}}{z-5}\right)\right|_2^4 \\ &= \boxed{\frac{1}{9}\ln(5)-\frac{27}{9}\ln(3)+\frac{76}{9}}\end{aligned}$$

---

6. Evaluate the integral $\displaystyle\int \frac{4x-11}{x^3-9x^2}\,dx$.

Step 1

To get the problem started off we need the form of the partial fraction decomposition of the integrand. However, in order to get this, we'll need to factor the denominator.

$$\int \frac{4x-11}{x^3-9x^2}\,dx=\int \frac{4x-11}{x^2(x-9)}\,dx$$

The form of the partial fraction decomposition for the integrand is then,

$$\frac{4x-11}{x^2(x-9)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-9}$$

Step 2

Setting the numerators equal gives,

$$4x - 11 = Ax\left(x-9\right) + B\left(x-9\right) + C\,x^2$$

Step 3

We can use the "trick" discussed in the notes to easily get two of the coefficients and then we can just pick another value of x to get the third so let's do that. Here is that work.

$$\begin{aligned} x = 0: &\quad -11 = -9B & & \qquad A = -\frac{25}{81} \\ x = 9: &\quad 25 = 81C & \Rightarrow & \qquad B = \frac{11}{9} \\ x = 1: &\quad -7 = -8A - 8B + C = -8A - \frac{767}{81} & & \qquad C = \frac{25}{81} \end{aligned}$$

The partial fraction form of the integrand is then,

$$\frac{4x-11}{x^2\left(x-9\right)} = \frac{-\frac{25}{81}}{x} + \frac{\frac{11}{9}}{x^2} + \frac{\frac{25}{81}}{x-9}$$

Step 4

We can now do the integral.

$$\int \frac{4x-11}{x^2\left(x-9\right)}\,dx = \int \frac{-\frac{25}{81}}{x} + \frac{\frac{11}{9}}{x^2} + \frac{\frac{25}{81}}{x-9}\,dx = \boxed{-\frac{25}{81}\ln\left|x\right| - \frac{\frac{11}{9}}{x} + \frac{25}{81}\ln\left|x-9\right| + c}$$

7. Evaluate the integral $\displaystyle\int \frac{z^2+2z+3}{(z-6)(z^2+4)}\,dz$.

Step 1

In this case the denominator is already factored and so we can go straight to the form of the partial fraction decomposition for the integrand.

$$\frac{z^2+2z+3}{(z-6)(z^2+4)} = \frac{A}{z-6} + \frac{Bz+C}{z^2+4}$$

Step 2

Setting the numerators equal gives,

$$z^2+2z+3 = A\left(z^2+4\right) + (Bz+C)(z-6) = (A+B)\,z^2 + (-6B+C)\,z + 4A - 6C$$

In this case the "trick" discussed in the notes won't work all that well for us and so we'll have to resort to multiplying everything out and collecting like terms as shown above.

Step 3

Now, setting the coefficients equal gives the following system.

$$\begin{array}{lcl} z^2: & A+B=1 & & A=\frac{51}{40} \\ z^1: & -6B+C=2 & \Rightarrow & B=-\frac{11}{40} \\ z^0: & 4A-6C=3 & & C=\frac{7}{20} \end{array}$$

The partial fraction form of the integrand is then,

$$\frac{z^2+2z+3}{(z-6)(z^2+4)} = \frac{\frac{51}{40}}{z-6} + \frac{-\frac{11}{40}z+\frac{7}{20}}{z^2+4}$$

Step 4

We can now do the integral.

$$\begin{aligned}\int \frac{z^2+2z+3}{(z-6)(z^2+4)}\,dz &= \int \frac{\frac{51}{40}}{z-6} + \frac{-\frac{11}{40}z+\frac{7}{20}}{z^2+4}\,dz \\ &= \int \frac{\frac{51}{40}}{z-6} + \frac{-\frac{11}{40}z}{z^2+4} + \frac{\frac{7}{20}}{z^2+4}\,dz \\ &= \boxed{\frac{51}{40}\ln|z-6| - \frac{11}{80}\ln\left|z^2+4\right| + \frac{7}{40}\tan^{-1}\left(\frac{z}{2}\right) + c}\end{aligned}$$

Note that the second integration needed the substitution $u = z^2 + 4$ while the third needed the formula provided in the notes.

---

8. Evaluate the integral $\displaystyle\int \frac{8+t+6t^2-12t^3}{(3t^2+4)(t^2+7)}\,dt$.

Step 1

In this case the denominator is already factored and so we can go straight to the form of the partial fraction decomposition for the integrand.

$$\frac{8+t+6t^2-12t^3}{(3t^2+4)(t^2+7)} = \frac{At+B}{3t^2+4} + \frac{Ct+D}{t^2+7}$$

Step 2

Setting the numerators equal gives,

$$\begin{aligned}8+t+6t^2-12t^3 &= (At+B)\left(t^2+7\right) + (Ct+D)\left(3t^2+4\right) \\ &= (A+3C)\,t^3 + (B+3D)\,t^2 + (7A+4C)\,t + 7B + 4D\end{aligned}$$

In this case the “trick” discussed in the notes won’t work all that well for us and so we’ll have to resort to multiplying everything out and collecting like terms as shown above.

Step 3

Now, setting the coefficients equal gives the following system.

$$\begin{array}{ll} t^3: & A+3C=-12 \\ t^2: & B+3D=6 \\ t^1: & 7A+4C=1 \\ t^0: & 7B+4D=8 \end{array} \quad \Rightarrow \quad \begin{array}{l} A=3 \\ B=0 \\ C=-5 \\ D=2 \end{array}$$

The partial fraction form of the integrand is then,

$$\frac{8+t+6t^2-12t^3}{(3t^2+4)(t^2+7)} = \frac{3t}{3t^2+4} + \frac{-5t+2}{t^2+7}$$

Step 4

We can now do the integral.

$$\begin{aligned} \int \frac{8+t+6t^2-12t^3}{(3t^2+4)(t^2+7)}\,dt &= \int \frac{3t}{3t^2+4} + \frac{-5t+2}{t^2+7}\,dt \\ &= \int \frac{3t}{3t^2+4} - \frac{5t}{t^2+7} + \frac{2}{t^2+7}\,dt \\ &= \boxed{\frac{1}{2}\ln\left|3t^2+4\right| - \frac{5}{2}\ln\left|t^2+7\right| + \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{t}{\sqrt{7}}\right) + c} \end{aligned}$$

Note that the first and second integrations needed the substitutions $u = 3t^2+4$ and $u = t^2+7$ respectively while the third needed the formula provided in the notes.

---

9. Evaluate the integral $\displaystyle\int \frac{6x^2-3x}{(x-2)(x+4)}\,dx$.

Hint

Pay attention to the degree of the numerator and denominator!

Step 1

Remember that we can only do partial fractions on a rational expression if the degree of the numerator is less than the degree of the denominator. In this case both the numerator and denominator are both degree 2. This can be easily seen if we multiply the denominator out.

$$\frac{6x^2 - 3x}{(x-2)(x+4)} = \frac{6x^2 - 3x}{x^2 + 2x - 8}$$

So, the first step is to do long division (we'll leave it up to you to check our Algebra skills for the long division) to get,

$$\frac{6x^2 - 3x}{(x-2)(x+4)} = 6 + \frac{48 - 15x}{(x-2)(x+4)}$$

Step 2

Now we can do the partial fractions on the second term. Here is the form of the partial fraction decomposition.

$$\frac{48 - 15x}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4}$$

Setting the numerators equal gives,

$$48 - 15x = A(x+4) + B(x-2)$$

Step 3

The "trick" will work here easily enough so here is that work.

$$\begin{aligned} x = -4: \quad & 108 = -6B \\ x = 2: \quad & 18 = 6A \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= 3 \\ B &= -18 \end{aligned}$$

The partial fraction form of the second term is then,

$$\frac{48 - 15x}{(x-2)(x+4)} = \frac{3}{x-2} - \frac{18}{x+4}$$

Step 4

We can now do the integral.

$$\int \frac{6x^2 - 3x}{(x-2)(x+4)}\,dx = \int 6 + \frac{3}{x-2} - \frac{18}{x+4}\,dx$$
$$= \boxed{6x + 3\ln|x-2| - 18\ln|x+4| + c}$$

---

10. Evaluate the integral $\displaystyle\int \frac{2+w^4}{w^3+9w}\,dw$.

Hint

Pay attention to the degree of the numerator and denominator!

Step 1

Remember that we can only do partial fractions on a rational expression if the degree of the numerator is less than the degree of the denominator. In this case the degree of the numerator is 4 and the degree of the denominator is 3.

So, the first step is to do long division (we'll leave it up to you to check our Algebra skills for the long division) to get,

$$\frac{2+w^4}{w^3+9w} = w + \frac{2-9w^2}{w\left(w^2+9\right)}$$

Step 2

Now we can do the partial fractions on the second term. Here is the form of the partial fraction decomposition.

$$\frac{2-9w^2}{w\left(w^2+9\right)} = \frac{A}{w} + \frac{Bw+C}{w^2+9}$$

Setting the numerators equal gives,

$$2 - 9w^2 = A\left(w^2+9\right) + w\left(Bw+C\right) = (A+B)\,w^2 + Cw + 9A$$

In this case the "trick" discussed in the notes won't work all that well for us and so we'll have to resort to multiplying everything out and collecting like terms as shown above.

Step 3

Now, setting the coefficients equal gives the following system.

$$\begin{array}{lrl} w^2: & A+B=-9 & \\ w^1: & C=0 & \Rightarrow \\ w^0: & 9A=2 & \end{array} \qquad \begin{array}{l} A=\dfrac{2}{9} \\ B=-\dfrac{83}{9} \\ C=0 \end{array}$$

The partial fraction form of the second term is then,

$$\frac{2-9w^2}{w\left(w^2+9\right)}=\frac{\frac{2}{9}}{w}-\frac{\frac{83}{9}w}{w^2+9}$$

Step 4

We can now do the integral.

$$\int\frac{2+w^4}{w^3+9w}\,dw=\int w+\frac{\frac{2}{9}}{w}-\frac{\frac{83}{9}w}{w^2+9}\,dw=\boxed{\frac{1}{2}w^2+\frac{2}{9}\ln|w|-\frac{83}{18}\ln\left|w^2+9\right|+c}$$

7.5 Integrals Involving Roots

1. Evaluate the integral $\displaystyle\int \frac{7}{2+\sqrt{x-4}}\,dx$.

Step 1

The substitution we'll use here is,

$$u = \sqrt{x-4}$$

Step 2

Now we need to get set up for the substitution. In other words, we need so solve for x and get dx.

$$x = u^2 + 4 \qquad \Rightarrow \qquad dx = 2u\,du$$

Step 3

Doing the substitution gives,

$$\int \frac{7}{2+\sqrt{x-4}}\,dx = \int \frac{7}{2+u}\,(2u)\,du = \int \frac{14u}{2+u}\,du$$

Step 4

This new integral can be done with the substitution $v = u + 2$. Doing this gives,

$$\int \frac{7}{2+\sqrt{x-4}}\,dx = \int \frac{14\,(v-2)}{v}\,dv = \int 14 - \frac{28}{v}\,dv = 14v - 28\ln|v| + c$$

Step 5

The last step is to now do all the back substitutions to get the final answer.

$$\begin{aligned}\int \frac{7}{2+\sqrt{x-4}}\,dx &= 14\left(u+2\right) - 28\ln\left|u+2\right| + c \\ &= \boxed{14\left(\sqrt{x-4}+2\right) - 28\ln\left|\sqrt{x-4}+2\right| + c}\end{aligned}$$

Note that we could have avoided the second substitution if we'd used $u = \sqrt{x-4}+2$ for the original substitution.

This often doesn't work, but in this case because the only extra term in the denominator was a constant it didn't change the differential work and so would work pretty easily for this problem.

---

2. Evaluate the integral $\displaystyle\int \frac{1}{w+2\sqrt{1-w}+2}\,dw$.

Step 1

The substitution we'll use here is,

$$u = \sqrt{1-w}$$

Step 2

Now we need to get set up for the substitution. In other words, we need so solve for w and get dw.

$$w = 1-u^2 \qquad \Rightarrow \qquad dw = -2u\,du$$

Step 3

Doing the substitution gives,

$$\int \frac{1}{w+2\sqrt{1-w}+2}\,dw = \int \frac{1}{1-u^2+2u+2}\left(-2u\right)du = \int \frac{2u}{u^2-2u-3}\,du$$

Step 4

This integral requires partial fractions to evaluate. Let's start with the form of the partial fraction decomposition.

$$\frac{2u}{(u+1)(u-3)} = \frac{A}{u+1} + \frac{B}{u-3}$$

Setting the coefficients equal gives,

$$2u = A(u-3) + B(u+1)$$

Using the "trick" to get the coefficients gives,

$$\begin{array}{lll} u=3: & 6=4B & \qquad A=\frac{1}{2} \\ & & \Rightarrow \\ u=-1: & -2=-4A & \qquad B=\frac{3}{2} \end{array}$$

The integral is then,

$$\int \frac{2u}{(u+1)(u-3)}\,du = \int \frac{\frac{1}{2}}{u+1} + \frac{\frac{3}{2}}{u-3}\,du = \frac{1}{2}\ln|u+1| + \frac{3}{2}\ln|u-3| + c$$

Step 5

The last step is to now do all the back substitutions to get the final answer.

$$\int \frac{1}{w+2\sqrt{1-w}+2}\,dw = \boxed{\frac{1}{2}\ln\left|\sqrt{1-w}+1\right| + \frac{3}{2}\ln\left|\sqrt{1-w}-3\right| + c}$$

---

3. Evaluate the integral $\displaystyle\int \frac{t-2}{t-3\sqrt{2t-4}+2}\,dt$.

Step 1

The substitution we'll use here is,

$$u = \sqrt{2t-4}$$

Step 2

Now we need to get set up for the substitution. In other words, we need so solve for t and get dt.

$$t = \frac{1}{2}u^2 + 2 \qquad \Rightarrow \qquad dt = u\,du$$

Step 3

Doing the substitution gives,

$$\int \frac{t-2}{t - 3\sqrt{2t-4}+2}\,dt = \int \frac{\frac{1}{2}u^2 + 2 - 2}{\frac{1}{2}u^2 + 2 - 3u + 2}\,(u)\,du = \int \frac{u^3}{u^2 - 6u + 8}\,du$$

Step 4

This integral requires partial fractions to evaluate.

However, we first need to do long division on the integrand since the degree of the numerator (3) is higher than the degree of the denominator (2). This gives,

$$\frac{u^3}{u^2 - 6u + 8} = u + 6 + \frac{28u - 48}{(u-2)(u-4)}$$

The form of the partial fraction decomposition on the third term is,

$$\frac{28u - 48}{(u-2)(u-4)} = \frac{A}{u-2} + \frac{B}{u-4}$$

Setting the coefficients equal gives,

$$28u - 48 = A(u-4) + B(u-2)$$

Using the "trick" to get the coefficients gives,

$$\begin{aligned} u = 4: &\quad 64 = 2B \\ u = 2: &\quad 8 = -2A \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= -4 \\ B &= 32 \end{aligned}$$

The integral is then,

$$\begin{aligned} \int \frac{u^3}{u^2 - 6u + 8}\,du &= \int u + 6 - \frac{4}{u-2} + \frac{32}{u-4}\,du \\ &= \frac{1}{2}u^2 + 6u - 4\ln|u-2| + 32\ln|u-4| + c \end{aligned}$$

Step 5

The last step is to now do all the back substitutions to get the final answer.

$$\int \frac{u^3}{u^2 - 6u + 8}\,du = \boxed{t - 2 + 6\sqrt{2t-4} - 4\ln\left|\sqrt{2t-4} - 2\right| + 32\ln\left|\sqrt{2t-4} - 4\right| + c}$$

7.6 Integrals Involving Quadratics

1. Evaluate the integral $\displaystyle\int \frac{7}{w^2+3w+3}\,dw$.

Step 1

The first thing to do is to complete the square (we'll leave it to you to verify the completing the square details) on the quadratic in the denominator.

$$\int \frac{7}{w^2+3w+3}\,dw = \int \frac{7}{\left(w+\frac{3}{2}\right)^2+\frac{3}{4}}\,dw$$

Step 2

From this we can see that the following substitution should work for us.

$$u = w + \frac{3}{2} \qquad \Rightarrow \qquad du = dw$$

Doing the substitution gives,

$$\int \frac{7}{w^2+3w+3}\,dw = \int \frac{7}{u^2+\frac{3}{4}}\,du$$

Step 3

This integral can be done with the formula given at the start of this section.

$$\int \frac{7}{w^2+3w+3}\,dw = \frac{14}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)+c = \boxed{\frac{14}{\sqrt{3}}\tan^{-1}\left(\frac{2w+3}{\sqrt{3}}\right)+c}$$

Don't forget to back substitute in for u!

2. Evaluate the integral $\displaystyle\int \frac{10x}{4x^2 - 8x + 9}\,dx$.

Step 1

The first thing to do is to complete the square (we'll leave it to you to verify the completing the square details) on the quadratic in the denominator.

$$\int \frac{10x}{4x^2 - 8x + 9}\,dx = \int \frac{10x}{4(x-1)^2 + 5}\,dx$$

Step 2

From this we can see that the following substitution should work for us.

$$u = x - 1 \qquad \Rightarrow \qquad du = dx \quad \& \quad x = u + 1$$

Doing the substitution gives,

$$\int \frac{10x}{4x^2 - 8x + 9}\,dx = \int \frac{10\,(u+1)}{4u^2 + 5}\,du$$

Step 3

We can quickly do this integral if we split it up as follows,

$$\int \frac{10x}{4x^2 - 8x + 9}\,dx = \int \frac{10u}{4u^2 + 5}\,du + \int \frac{10}{4u^2 + 5}\,du = \int \frac{10u}{4u^2 + 5}\,du + \frac{5}{2}\int \frac{1}{u^2 + \frac{5}{4}}\,du$$

After a quick rewrite of the second integral we can see that we can do the first with the substitution $v = 4u^2 + 5$ and the second is an inverse trig integral we can evaluate using the formula given at the start of the notes for this section.

$$\begin{aligned}\int \frac{10x}{4x^2 - 8x + 9}\,dx &= \frac{5}{4}\ln|v| + \frac{5}{2}\left(\frac{2}{\sqrt{5}}\right)\tan^{-1}\left(\frac{2u}{\sqrt{5}}\right) + c \\ &= \frac{5}{4}\ln\left|4u^2 + 5\right| + \sqrt{5}\tan^{-1}\left(\frac{2u}{\sqrt{5}}\right) + c \\ &= \boxed{\frac{5}{4}\ln\left|4(x-1)^2 + 5\right| + \sqrt{5}\tan^{-1}\left(\frac{2x-2}{\sqrt{5}}\right) + c}\end{aligned}$$

Don't forget to back substitute in for u!

3. Evaluate the integral $\displaystyle\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt$.

Step 1

The first thing to do is to complete the square (we'll leave it to you to verify the completing the square details) on the quadratic in the denominator.

$$\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt = \int \frac{2t+9}{\left((t-7)^2-3\right)^{\frac{5}{2}}}\,dt$$

Step 2

From this we can see that the following substitution should work for us.

$$u = t-7 \qquad \Rightarrow \qquad du = dt \quad \& \quad t = u+7$$

Doing the substitution gives,

$$\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt = \int \frac{2(u+7)+9}{\left(u^2-3\right)^{\frac{5}{2}}}\,du = \int \frac{2u+23}{\left(u^2-3\right)^{\frac{5}{2}}}\,du$$

Step 3

Next, we'll need to split the integral up as follows,

$$\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt = \int \frac{2u}{\left(u^2-3\right)^{\frac{5}{2}}}\,du + \int \frac{23}{\left(u^2-3\right)^{\frac{5}{2}}}\,du$$

The first integral can be done with the substitution $v = u^2-3$ and the second integral will require the trig substitution $u = \sqrt{3}\sec(\theta)$. Here is the substitution work.

$$\begin{aligned}\int \frac{2t+9}{(t^2-14t+46)^{\frac{5}{2}}}\,dt &= \int v^{-\frac{5}{2}}\,dv + \int \frac{23}{(3\sec^2(\theta)-3)^{\frac{5}{2}}}\left(\sqrt{3}\sec(\theta)\tan(\theta)\right)d\theta \\ &= \int v^{-\frac{5}{2}}\,dv + \int \frac{23\sqrt{3}\sec(\theta)\tan(\theta)}{\left(3\tan^2(\theta)\right)^{\frac{5}{2}}}\,d\theta \\ &= \int v^{-\frac{5}{2}}\,dv + \int \frac{23\sec(\theta)}{9\tan^4(\theta)}\,d\theta \\ &= \int v^{-\frac{5}{2}}\,dv + \frac{23}{9}\int \frac{\cos^3(\theta)}{\sin^4(\theta)}\,d\theta\end{aligned}$$

Now, for the second integral, don't forget the manipulations we often need to do so we can do these kinds of integrals. If you need some practice on these kinds of integrals go back to the practice problems for the second section of this chapter and work some of them.

Here is the rest of the integration process for this problem.

$$\begin{aligned}\int \frac{2t+9}{(t^2-14t+46)^{\frac{5}{2}}}\,dt &= \int v^{-\frac{5}{2}}\,dv + \frac{23}{9}\int \frac{1-\sin^2(\theta)}{\sin^4(\theta)}\cos(\theta)\,d\theta \qquad w=\sin(\theta) \\ &= \int v^{-\frac{5}{2}}\,dv + \frac{23}{9}\int w^{-4}-w^{-2}\,dw \\ &= -\frac{2}{3}v^{-\frac{3}{2}} + \frac{23}{9}\left[-\frac{1}{3}(\sin(\theta))^{-3} + (\sin(\theta))^{-1}\right] + c\end{aligned}$$

Step 4

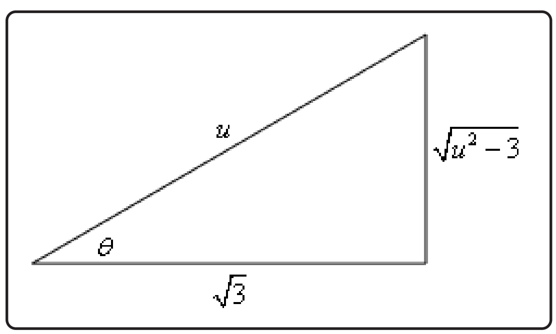
We now need to do quite a bit of back substitution to get the answer back into t's. Let's start with the result of the second integration. Converting the θ's back to u's will require a quick right triangle.

From the substitution we have,

$$\sec(\theta) = \frac{u}{\sqrt{3}} \qquad \left(=\frac{\text{hyp}}{\text{adj}}\right)$$

From the right triangle we get,

$$\sin(\theta) = \frac{\sqrt{u^2-3}}{u}$$



Plugging this into the integral above gives,

$$\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt = -\frac{2}{3\left(u^2-3\right)^{\frac{3}{2}}} - \frac{23u^3}{27\left(u^2-3\right)^{\frac{3}{2}}} + \frac{23u}{9\sqrt{u^2-3}} + c$$

Note that we also back substituted for the v in the first term as well and rewrote the first term a little. Finally, all we need to do is back substitute for the u.

$$\begin{aligned}\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt &= -\frac{2}{3\left((t-7)^2-3\right)^{\frac{3}{2}}} - \frac{23(t-7)^3}{27\left((t-7)^2-3\right)^{\frac{3}{2}}} + \frac{23\,(t-7)}{9\sqrt{(t-7)^2-3}} + c \\ &= \boxed{\frac{23\,(t-7)}{9\sqrt{(t-7)^2-3}} - \frac{18+23(t-7)^3}{27\left((t-7)^2-3\right)^{\frac{3}{2}}} + c}\end{aligned}$$

We'll leave this solution with a final note about these kinds of problems. They are often very long, messy and there are ample opportunities for mistakes so be careful with these and don't get into too much of a hurry when working them.

---

4. Evaluate the integral $\displaystyle\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz$.

Step 1

The first thing to do is to complete the square (we'll leave it to you to verify the completing the square details) on the quadratic in the denominator.

$$\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz = \int \frac{3z}{\left(3-2(z+1)^2\right)^2}\,dz$$

Step 2

From this we can see that the following substitution should work for us.

$$u = z+1 \qquad \Rightarrow \qquad du = dz \quad \& \quad z = u-1$$

Doing the substitution gives,

$$\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz = \int \frac{3\,(u-1)}{\left(3-2u^2\right)^2}\,du = \int \frac{3u-3}{\left(3-2u^2\right)^2}\,du$$

Step 3

Next, we'll need to split the integral up as follows,

$$\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz = \int \frac{3u}{\left(3-2u^2\right)^2}\,du - \int \frac{3}{\left(3-2u^2\right)^2}\,du$$

The first integral can be done with the substitution $v = 3-2u^2$ and the second integral will require the trig substitution $u = \frac{\sqrt{3}}{\sqrt{2}}\sin(\theta)$. Here is the substitution work.

$$\begin{aligned}\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz &= -\frac{3}{4}\int v^{-2}\,dv - \int \frac{3}{\left(3-3\sin^2(\theta)\right)^2}\left(\frac{\sqrt{3}}{\sqrt{2}}\cos(\theta)\right)\,d\theta \\ &= -\frac{3}{4}\int v^{-2}\,dv - \int \frac{3}{\left(3\cos^2(\theta)\right)^2}\left(\frac{\sqrt{3}}{\sqrt{2}}\cos(\theta)\right)\,d\theta \\ &= -\frac{3}{4}\int v^{-2}\,dv - \frac{1}{\sqrt{6}}\int \sec^3(\theta)\,d\theta\end{aligned}$$

The second integral for this problem comes down to an integral that was done in the notes for the second section of this chapter and so we'll just use the formula derived in that section to do this integral.

Here is the rest of the integration process for this problem.

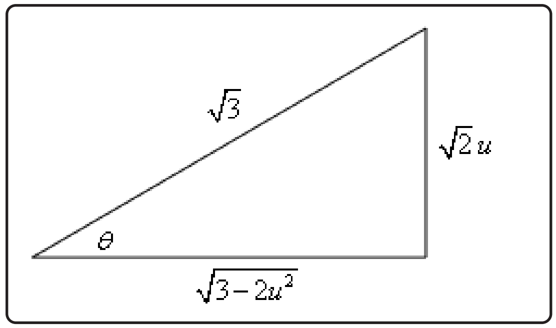
$$\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz = \frac{3}{4}v^{-1} - \frac{1}{2\sqrt{6}}\left[\sec(\theta)\tan(\theta) + \ln\left|\sec(\theta)+\tan(\theta)\right|\right] + c$$

Step 4

We now need to do quite a bit of back substitution to get the answer back into z's. Let's start with the result of the second integration. Converting the θ's back to u's will require a quick right triangle.

From the substitution we have,

$$\sin(\theta) = \frac{\sqrt{2}\,u}{\sqrt{3}} \qquad \left(= \frac{\text{opp}}{\text{hyp}}\right)$$



From the right triangle we get,

$$\tan(\theta) = \frac{\sqrt{2}\,u}{\sqrt{3-2u^2}} \quad \& \quad \sec(\theta) = \frac{\sqrt{3}}{\sqrt{3-2u^2}}$$

Plugging this into the integral above gives,

$$\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz = \frac{3}{4\left(3-2u^2\right)} - \frac{1}{2\sqrt{6}}\left[\frac{\sqrt{6}\,u}{3-2u^2} + \ln\left|\frac{\sqrt{3}+\sqrt{2}\,u}{\sqrt{3-2u^2}}\right|\right] + c$$

Note that we also back substituted for the v in the first term as well and rewrote the first term a little. Finally, all we need to do is back substitute for the u.

$$\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz$$

$$= \boxed{\frac{3}{4\left(3-2(z+1)^2\right)} - \frac{z+1}{6-4(z+1)^2} - \frac{1}{2\sqrt{6}}\ln\left|\frac{\sqrt{3}+\sqrt{2}\,(z+1)}{\sqrt{3-2(z+1)^2}}\right| + c}$$

We'll leave this solution with a final note about these kinds of problems. They are often very long, messy and there are ample opportunities for mistakes so be careful with these and don't get into too much of a hurry when working them.

7.7 Integration Strategy

Problems have not yet been written for this section.

I was finding it very difficult to come up with a good mix of new problems and decided my time was better spent writing problems for later sections rather than trying to come up with a sufficient number of problems for what is essentially a review section. I intend to come back at a later date when I have more time to devote to this section and add problems then.

7.8 Improper Integrals

1. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_0^{\infty} (1+2x)\,\mathbf{e}^{-x}\,dx$$

Hint

Don't forget that we can't do the integral as long as there is an infinity in one of the limits!

Step 1

First, we need to recall that we can't do the integral as long as there is an infinity in one of the limits. Therefore, we'll need to eliminate the infinity first as follows,

$$\int_0^{\infty} (1+2x)\,\mathbf{e}^{-x}\,dx = \lim_{t\to\infty}\int_0^{t} (1+2x)\,\mathbf{e}^{-x}\,dx$$

Note that this step really is needed for these integrals! For some integrals we can use basic logic and "evaluate" at infinity to get the answer. However, many of these kinds of improper integrals can't be done that way! This is the only way to make sure we can deal with the infinite limit in those cases.

So even if this ends up being one of the integrals in which we can "evaluate" at infinity we need to be in the habit of doing this for those that can't be done that way.

Step 2

Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we need to do integration by parts to evaluate this integral. Here is the integration work.

$$\begin{aligned} u &= 1+2x & \quad\rightarrow\quad & du = 2dx \\ dv &= \mathbf{e}^{-x}\,dx & \quad\rightarrow\quad & v = -\mathbf{e}^{-x} \end{aligned}$$

$$\begin{aligned} \int (1+2x)\,\mathbf{e}^{-x}\,dx &= -(1+2x)\,\mathbf{e}^{-x} + 2\int \mathbf{e}^{-x}\,dx \\ &= -(1+2x)\,\mathbf{e}^{-x} - 2\mathbf{e}^{-x} + c = -(3+2x)\,\mathbf{e}^{-x} + c \end{aligned}$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and so we held off dealing with them until the next step.

Step 3

Okay, now let's take care of the limits on the integral.

$$\int_0^\infty (1+2x)\,\mathbf{e}^{-x}\,dx = \lim_{t\to\infty}\left(-(3+2x)\,\mathbf{e}^{-x}\right)\Big|_0^t = \lim_{t\to\infty}\left(3-(3+2t)\,\mathbf{e}^{-t}\right)$$

Step 4

We now need to evaluate the limit in our answer from the previous step and note that, in this case, we really can't just "evaluate" at infinity! We need to do the limiting process here to make sure we get the correct answer.

We will need to do a quick L'Hospital's Rule on the second term to properly evaluate it. Here is the limit work.

$$\int_0^\infty (1+2x)\,\mathbf{e}^{-x}\,dx = \lim_{t\to\infty} 3 - \lim_{t\to\infty}\frac{3+2t}{\mathbf{e}^t} = 3 - \lim_{t\to\infty}\frac{2}{\mathbf{e}^t} = 3-0=3$$

Step 5

The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and was finite (*i.e.* not an infinity). Therefore, the integral **converges** and its value is **3**.

---

2. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{-\infty}^0 (1+2x)\,\mathbf{e}^{-x}\,dx$$

Hint

Don't forget that we can't do the integral as long as there is an infinity in one of the limits!

Step 1

First, we need to recall that we can't do the integral as long as there is an infinity in one of the limits. Therefore, we'll need to eliminate the infinity first as follows,

$$\int_{-\infty}^{0} (1+2x)\,\mathbf{e}^{-x}\,dx = \lim_{t\to-\infty}\int_{t}^{0} (1+2x)\,\mathbf{e}^{-x}\,dx$$

Note that this step really is needed for these integrals! For some integrals we can use basic logic and "evaluate" at infinity to get the answer. However, many of these kinds of improper integrals can't be done that way! This is the only way to make sure we can deal with the infinite limit in those cases.

So even if this ends up being one of the integrals in which we can "evaluate" at infinity we need to be in the habit of doing this for those that can't be done that way.

Step 2

Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we need to do integration by parts to evaluate this integral. Here is the integration work.

$$\begin{aligned} u &= 1+2x & &\to & du &= 2dx \\ dv &= \mathbf{e}^{-x}\,dx & &\to & v &= -\mathbf{e}^{-x} \end{aligned}$$

$$\begin{aligned} \int (1+2x)\,\mathbf{e}^{-x}\,dx &= -(1+2x)\,\mathbf{e}^{-x} + 2\int \mathbf{e}^{-x}\,dx \\ &= -(1+2x)\,\mathbf{e}^{-x} - 2\mathbf{e}^{-x} + c = -(3+2x)\,\mathbf{e}^{-x} + c \end{aligned}$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and so we held off dealing with them until the next step.

Step 3

Okay, now let's take care of the limits on the integral.

$$\int_{-\infty}^{0} (1+2x)\,\mathbf{e}^{-x}\,dx = \lim_{t\to\,-\infty}\left(-(3+2x)\,\mathbf{e}^{-x}\right)\Big|_{t}^{0} = \lim_{t\to-\infty}\left((3+2t)\,\mathbf{e}^{-t}-3\right)$$

Step 4

We now need to evaluate the limit in our answer from the previous step. In this case we can see that the first term will go to negative infinity since it is just a product of one factor that goes to negative infinity and another factor that goes to infinity. Therefore, the full limit will also be negative infinity since the constant second term won't affect the final value of the limit.

$$\int_{-\infty}^{0} (1+2x)\,\mathbf{e}^{-x}\,dx = \lim_{t\to-\infty}(3+2t)\,\mathbf{e}^{-t} - \lim_{t\to\infty} 3 = (-\infty)(\infty)-3 = -\infty-3 = -\infty$$

Step 5

The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and was negative infinity. Therefore, the integral **diverges**.

---

3. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{-5}^{1} \frac{1}{10+2z}\,dz$$

Hint

Don't forget that we can't do the integral as long as there is a division by zero in the integrand at some point in the interval of integration!

Step 1

First, notice that there is a division by zero issue (and hence a discontinuity) in the integrand at $z = -5$ and this is the lower limit of integration. We know that as long as that discontinuity is there we can't do the integral. Therefore, we'll need to eliminate the discontinuity first as follows,

$$\int_{-5}^{1} \frac{1}{10+2z}\,dz = \lim_{t \to -5^+} \int_{t}^{1} \frac{1}{10+2z}\,dz$$

Don't forget that the limits on these kinds of integrals must be one-sided limits. Because the interval of integration is $[-5, 1]$ we are only interested in the values of z that are greater than -5 and so we must use a right-hand limit to reflect that fact.

Step 2

Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calculus I substitution. Here is the integration work.

$$\int \frac{1}{10+2z}\,dz = \frac{1}{2}\ln|10+2z| + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and so we held off dealing with them until the next step.

Step 3

Okay, now let's take care of the limits on the integral.

$$\int_{-5}^{1} \frac{1}{10+2z}\,dz = \lim_{t \to -5^+} \left(\frac{1}{2}\ln|10+2z| \right)\Bigg|_{t}^{1} = \lim_{t \to -5^+} \left(\frac{1}{2}\ln|12| - \frac{1}{2}\ln|10+2t| \right)$$

Step 4

We now need to evaluate the limit in our answer from the previous step. Here is the limit work.

$$\int_{-5}^{1} \frac{1}{10+2z}\,dz = \lim_{t\to -5^{+}} \left(\frac{1}{2}\ln|12| - \frac{1}{2}\ln|10+2t|\right) = \frac{1}{2}\ln|12| + \infty = \infty$$

Step 5

The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and but was infinity. Therefore, the integral **diverges**.

4. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{1}^{2} \frac{4w}{\sqrt[3]{w^2-4}}\,dw$$

Hint

Don't forget that we can't do the integral as long as there is a division by zero in the integrand at some point in the interval of integration!

Step 1

First, notice that there is a division by zero issue (and hence a discontinuity) in the integrand at $w = 2$ and this is the upper limit of integration. We know that as long as that discontinuity is there we can't do the integral. Therefore, we'll need to eliminate the discontinuity first as follows,

$$\int_{1}^{2} \frac{4w}{\sqrt[3]{w^2-4}}\,dw = \lim_{t\to 2^{-}} \int_{1}^{t} \frac{4w}{\sqrt[3]{w^2-4}}\,dw$$

Don't forget that the limits on these kinds of integrals must be one-sided limits. Because the interval of integration is $[1,2]$ we are only interested in the values of t that are less than 2 and so we must use a left-hand limit to reflect that fact.

Step 2

Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calculus I substitution. Here is the integration work.

$$\int \frac{4w}{\sqrt[3]{w^2-4}}\,dw = 3\left(w^2-4\right)^{\frac{2}{3}} + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and so we held off dealing with them until the next step.

Step 3

Okay, now let's take care of the limits on the integral.

$$\int_1^2 \frac{4w}{\sqrt[3]{w^2-4}}\,dw = \lim_{t\to 2^-}\left(3\left(w^2-4\right)^{\frac{2}{3}}\right)\Big|_1^t = \lim_{t\to 2^-}\left(3\left(t^2-4\right)^{\frac{2}{3}} - 3(-3)^{\frac{2}{3}}\right)$$

Step 4

We now need to evaluate the limit in our answer from the previous step. Here is the limit work.

$$\int_1^2 \frac{4w}{\sqrt[3]{w^2-4}}\,dw = \lim_{t\to 2^-}\left(3\left(t^2-4\right)^{\frac{2}{3}} - 3(-3)^{\frac{2}{3}}\right) = -3(-3)^{\frac{2}{3}} = (-3)^{\frac{5}{3}}$$

Step 5

The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and was finite (*i.e.* not an infinity). Therefore, the integral **converges** and its value is $(-3)^{\frac{5}{3}}$.

5. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{-\infty}^{1} \sqrt{6-y}\,dy$$

Hint

Don't forget that we can't do the integral as long as there is an infinity in one of the limits!

Step 1

First, we need to recall that we can't do the integral as long as there is an infinity in one of the limits. Therefore, we'll need to eliminate the infinity first as follows,

$$\int_{-\infty}^{1} \sqrt{6-y}\,dy = \lim_{t\to-\infty} \int_{t}^{1} \sqrt{6-y}\,dy$$

Note that this step really is needed for these integrals! For some integrals we can use basic logic and "evaluate" at infinity to get the answer. However, many of these kinds of improper integrals can't be done that way! This is the only way to make sure we can deal with the infinite limit in those cases.

So even if this ends up being one of the integrals in which we can "evaluate" at infinity we need to be in the habit of doing this for those that can't be done that way.

Step 2

Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calculus I substitution. Here is the integration work.

$$\int \sqrt{6-y}\,dy = -\frac{2}{3}(6-y)^{\frac{3}{2}} + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and so we held off dealing with them until the next step.

Step 3

Okay, now let's take care of the limits on the integral.

$$\int \sqrt{6-y}\,dy = \lim_{t \to -\infty} \left(-\frac{2}{3}(6-y)^{\frac{3}{2}} \right)\Big|_t^1 = \lim_{t \to -\infty} \left(-\frac{2}{3}(5)^{\frac{3}{2}} + \frac{2}{3}(6-t)^{\frac{3}{2}} \right)$$

Step 4

We now need to evaluate the limit in our answer from the previous step. Here is the limit work.

$$\int \sqrt{6-y}\,dy = \lim_{t \to -\infty} \left(-\frac{2}{3}(5)^{\frac{3}{2}} + \frac{2}{3}(6-t)^{\frac{3}{2}} \right) = -\frac{2}{3}(5)^{\frac{3}{2}} + \infty = \infty$$

Step 5

The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and but was infinity. Therefore, the integral **diverges**.

---

6. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_2^{\infty} \frac{9}{(1-3z)^4}\,dz$$

Hint

Don't forget that we can't do the integral as long as there is an infinity in one of the limits!

Step 1

First, we need to recall that we can't do the integral as long as there is an infinity in one of the limits. Therefore, we'll need to eliminate the infinity first as follows,

$$\int_2^{\infty} \frac{9}{(1-3z)^4}\,dz = \lim_{t \to \infty} \int_2^{t} \frac{9}{(1-3z)^4}\,dz$$

Note that this step really is needed for these integrals! For some integrals we can use basic logic and "evaluate" at infinity to get the answer. However, many of these kinds of improper integrals can't be done that way! This is the only way to make sure we can deal with the infinite limit in those cases.

So even if this ends up being one of the integrals in which we can "evaluate" at infinity we need to be in the habit of doing this for those that can't be done that way.

Step 2

Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calculus I substitution. Here is the integration work.

$$\int \frac{9}{(1-3z)^4}\,dz = \frac{1}{(1-3z)^3} + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and so we held off dealing with them until the next step.

Step 3

Okay, now let's take care of the limits on the integral.

$$\int_2^\infty \frac{9}{(1-3z)^4}\,dz = \lim_{t\to\infty}\left(\frac{1}{(1-3z)^3}\right)\Bigg|_2^t = \lim_{t\to\infty}\left(\frac{1}{(1-3t)^3} - \left(-\frac{1}{125}\right)\right)$$

Step 4

We now need to evaluate the limit in our answer from the previous step. Here is the limit work

$$\int_2^\infty \frac{9}{(1-3z)^4}\,dz = \lim_{t\to\infty}\left(\frac{1}{(1-3t)^3} + \frac{1}{125}\right) = \frac{1}{125}$$

Step 5

The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and was finite (*i.e.* not an infinity). Therefore, the integral **converges** and its value is $\frac{1}{125}$.

---

7. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_0^4 \frac{x}{x^2-9}\,dx$$

Hint

Don't forget that we can't do the integral as long as there is a division by zero in the integrand at some point in the interval of integration! Also, do not just assume the division by zero will be at one of the limits of the integral.

Step 1

First, notice that there is a division by zero issue (and hence a discontinuity) in the integrand at $x = 3$ and note that this is between the limits of the integral. We know that as long as that discontinuity is there we can't do the integral.

However, recall from the notes in this section that we can only deal with discontinuities that if they occur at one of the limits of the integral. So, we'll need to break up the integral at $x = 3$.

$$\int_0^4 \frac{x}{x^2-9}\,dx = \int_0^3 \frac{x}{x^2-9}\,dx + \int_3^4 \frac{x}{x^2-9}\,dx$$

Remember as well, that we can only break up the integral like this provided both of the new integrals are convergent! If it turns out that even one of them is divergent then it will turn out that we couldn't have done this and the original integral will be divergent.

So, not worrying about if this was really possible to do or not, let's proceed with the problem. We can eliminate the discontinuity in each as follows,

$$\int_0^4 \frac{x}{x^2-9}\,dx = \lim_{t\to 3^-}\int_0^t \frac{x}{x^2-9}\,dx + \lim_{s\to 3^+}\int_s^4 \frac{x}{x^2-9}\,dx$$

Don't forget that the limits on these kinds of integrals must be one-sided limits.

Step 2

Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calc I substitution. Here is the integration work.

$$\int \frac{x}{x^2-9}\,dx = \frac{1}{2}\ln\left|x^2-9\right| + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and the integral for each was the same with only the limits being different so no reason to do the integral twice.

Step 3

Okay, now let's take care of the limits on the integral.

$$\begin{aligned}\int_0^4 \frac{x}{x^2-9}\,dx &= \lim_{t\to 3^-}\left(\frac{1}{2}\ln\left|x^2-9\right|\right)\Big|_0^t + \lim_{s\to 3^+}\left(\frac{1}{2}\ln\left|x^2-9\right|\right)\Big|_s^4 \\ &= \lim_{t\to 3^-}\left(\frac{1}{2}\ln\left|t^2-9\right| - \frac{1}{2}\ln(9)\right) + \lim_{s\to 3^+}\left(\frac{1}{2}\ln(7) - \frac{1}{2}\ln\left|s^2-9\right|\right)\end{aligned}$$

Step 4

We now need to evaluate the limits in our answer from the previous step. Here is the limit work.

$$\begin{aligned}\int_0^4 \frac{x}{x^2-9}\,dx &= \lim_{t\to 3^-}\left(\frac{1}{2}\ln\left|t^2-9\right| - \frac{1}{2}\ln(9)\right) + \lim_{s\to 3^+}\left(\frac{1}{2}\ln(7) - \frac{1}{2}\ln\left|s^2-9\right|\right) \\ &= \left[-\infty - \frac{1}{2}\ln(9)\right] + \left[\frac{1}{2}\ln(7) + \infty\right]\end{aligned}$$

Note that we put the answers for each limit in brackets to make it clear what each limit was. This will be important for the next step.

Step 5

The final step is to write down the answer!

Now, from the limit work in the previous step we see that,

$$\int_0^3 \frac{x}{x^2-9}\,dx = \lim_{t\to 3^-}\left(\frac{1}{2}\ln\left|t^2-9\right| - \frac{1}{2}\ln(9)\right) = \left[-\infty - \frac{1}{2}\ln(9)\right] = -\infty$$

$$\int_3^4 \frac{x}{x^2-9}\,dx = \lim_{s\to 3^+}\left(\frac{1}{2}\ln(7) - \frac{1}{2}\ln\left|s^2-9\right|\right) = \left[\frac{1}{2}\ln(7) + \infty\right] = \infty$$

Therefore, each of these integrals are divergent. This means that we were, in fact, not able to break up the integral as we did back in Step 1.

This in turn means that the integral **diverges**.

---

8. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{-\infty}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw$$

Hint

Don't forget that we can't do the integral as long as there is an infinity in one of the limits! Also, don't forget that infinities in both limits need as extra step to get set up.

Step 1

First, we need to recall that we can't do the integral as long as there is an infinity in one of the limits. Note as well that in this case we have infinities in both limits and so we'll need to split up the integral.

The integral can be split up at any point in this case and $w = 0$ seems like a good point to use for the split point. Splitting up the integral gives,

$$\int_{-\infty}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw = \int_{-\infty}^{0} \frac{6w^3}{\left(w^4+1\right)^2}\,dw + \int_{0}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw$$

Remember as well, that we can only break up the integral like this provided both of the new integrals are convergent! If it turns out that even one of them is divergent then it will turn out that we couldn't have done this and the original integral will be divergent.

So, not worrying about if this was really possible to do or not, let's proceed with the problem.

Now, we can eliminate the infinities as follows,

$$\int_{-\infty}^{\infty} \frac{6w^3}{(w^4+1)^2}\,dw = \lim_{t\to -\infty} \int_t^0 \frac{6w^3}{(w^4+1)^2}\,dw + \lim_{s\to\infty} \int_0^s \frac{6w^3}{(w^4+1)^2}\,dw$$

Step 2

Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calc I substitution. Here is the integration work.

$$\int \frac{6w^3}{(w^4+1)^2}\,dw = -\frac{3}{2}\frac{1}{w^4+1} + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and the integral for each was the same with only the limits being different so no reason to do the integral twice.

Step 3

Okay, now let's take care of the limits on the integral.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{6w^3}{(w^4+1)^2}\,dw &= \lim_{t\to -\infty} \left(-\frac{3}{2}\frac{1}{w^4+1}\right)\Bigg|_t^0 + \lim_{s\to\infty} \left(-\frac{3}{2}\frac{1}{w^4+1}\right)\Bigg|_0^s \\ &= \lim_{t\to -\infty} \left(-\frac{3}{2} + \frac{3}{2}\frac{1}{t^4+1}\right) + \lim_{s\to\infty} \left(-\frac{3}{2}\frac{1}{s^4+1} + \frac{3}{2}\right)\end{aligned}$$

Step 4

We now need to evaluate the limits in our answer from the previous step. Here is the limit work

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw &= \lim_{t\to\,-\infty}\left(-\frac{3}{2}+\frac{3}{2}\frac{1}{t^4+1}\right)+\lim_{s\to\infty}\left(-\frac{3}{2}\frac{1}{s^4+1}+\frac{3}{2}\right)\\ &= \qquad\left[-\frac{3}{2}\right] \qquad + \qquad \left[\frac{3}{2}\right]\end{aligned}$$

Note that we put the answers for each limit in brackets to make it clear what each limit was. This will be important for the next step.

Step 5

The final step is to write down the answer!

Now, from the limit work in the previous step we see that,

$$\int_{-\infty}^{0} \frac{6w^3}{(w^4+1)^2}\,dw = \lim_{t\to\,-\infty}\left(-\frac{3}{2}+\frac{3}{2}\frac{1}{t^4+1}\right) = -\frac{3}{2}$$

$$\int_{0}^{\infty} \frac{6w^3}{(w^4+1)^2}\,dw = \lim_{s\to\infty}\left(-\frac{3}{2}\frac{1}{s^4+1}+\frac{3}{2}\right) = \frac{3}{2}$$

Therefore, each of the integrals are convergent and have the values shown above. This means that we could in fact break up the integral as we did in Step 1. Also, the original integral is now,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{6w^3}{(w^4+1)^2}\,dw &= \int_{-\infty}^{0} \frac{6w^3}{(w^4+1)^2}\,dw + \int_{0}^{\infty} \frac{6w^3}{(w^4+1)^2}\,dw \\ &= \quad -\frac{3}{2} \quad + \quad \frac{3}{2} \\ &= 0\end{aligned}$$

Therefore, the integral **converges** and its value is **0**.

---

9. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{1}^{4} \frac{1}{x^2+x-6}\,dx$$

Hint

Don't forget that we can't do the integral as long as there is a division by zero in the integrand at some point in the interval of integration! Also, do not just assume the division by zero will be at one of the limits of the integral.

Step 1

First, notice that there is a division by zero issue (and hence a discontinuity) in the integrand at $x = 2$ and note that this is between the limits of the integral. We know that as long as that discontinuity is there we can't do the integral.

However, recall from the notes in this section that we can only deal with discontinuities that if they occur at one of the limits of the integral. So, we'll need to break up the integral at $x = 2$.

$$\int_1^4 \frac{1}{x^2+x-6}\,dx = \int_1^2 \frac{1}{(x+3)(x-2)}\,dx + \int_2^4 \frac{1}{(x+3)(x-2)}\,dx$$

Remember as well, that we can only break up the integral like this provided both of the new integrals are convergent! If it turns out that even one of them is divergent then it will turn out that we couldn't have done this and the original integral will be divergent.

So, not worrying about if this was really possible to do or not, let's proceed with the problem. We can eliminate the discontinuity in each as follows,

$$\int_1^4 \frac{1}{x^2+x-6}\,dx = \lim_{t\to 2^-}\int_1^t \frac{1}{(x+3)(x-2)}\,dx + \lim_{s\to 2^+}\int_s^4 \frac{1}{(x+3)(x-2)}\,dx$$

Don't forget that the limits on these kinds of integrals must be one-sided limits.

Step 2

Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we will need to do some partial fractions in order to the integral. Here is the partial fraction work.

$$\frac{1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} \qquad \Rightarrow \qquad 1 = A(x-2) + B(x+3)$$

$$\begin{array}{lll} x=2: \quad 1 = 5B & & A = -\frac{1}{5} \\ & \to & \\ x=-3: \quad 1 = -5A & & B = \frac{1}{5} \end{array}$$

The integration work is then,

$$\int \frac{1}{(x+3)(x-2)}\,dx = \int \frac{\frac{1}{5}}{x-2} - \frac{\frac{1}{5}}{x+3}\,dx = \frac{1}{5}\ln|x-2| - \frac{1}{5}\ln|x+3| + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and the integral for each was the same with only the limits being different so no reason to do the integral twice.

Step 3

Okay, now let's take care of the limits on the integral.

$$\begin{aligned}\int_1^4 \frac{1}{x^2+x-6}\,dx &= \lim_{t\to 2^-}\left(\frac{1}{5}\ln|x-2|-\frac{1}{5}\ln|x+3|\right)\Bigg|_1^t \\ &\qquad + \lim_{s\to 2^+}\left(\frac{1}{5}\ln|x-2|-\frac{1}{5}\ln|x+3|\right)\Bigg|_s^4 \\ &= \lim_{t\to 2^-}\left(\frac{1}{5}\ln|t-2|-\frac{1}{5}\ln|t+3|-\left(\frac{1}{5}\ln(1)-\frac{1}{5}\ln(4)\right)\right) \\ &\qquad + \lim_{s\to 2^+}\left(\frac{1}{5}\ln(2)-\frac{1}{5}\ln(7)-\left(\frac{1}{5}\ln|s-2|-\frac{1}{5}\ln|s+3|\right)\right)\end{aligned}$$

Step 4

We now need to evaluate the limits in our answer from the previous step. Here is the limit work.

$$\begin{aligned}\int_1^4 \frac{1}{x^2+x-6}\,dx &= \lim_{t\to 2^-}\left(\frac{1}{5}\ln|t-2|-\frac{1}{5}\ln|t+3|-\left(\frac{1}{5}\ln(1)-\frac{1}{5}\ln(4)\right)\right) \\ &\qquad + \lim_{s\to 2^+}\left(\frac{1}{5}\ln(2)-\frac{1}{5}\ln(7)-\left(\frac{1}{5}\ln|s-2|-\frac{1}{5}\ln|s+3|\right)\right) \\ &= \left[-\infty-\frac{1}{5}\ln(5)+\frac{1}{5}\ln(4)\right]+\left[\frac{1}{5}\ln(2)-\frac{1}{5}\ln(7)+\frac{1}{5}\ln(5)+\infty\right]\end{aligned}$$

Note that we put the answers for each limit in brackets to make it clear what each limit was. This will be important for the next step.

Step 5

The final step is to write down the answer!

Now, from the limit work in the previous step we see that,

$$\int_1^2 \frac{1}{(x+3)(x-2)}\,dx = \lim_{t\to 2^-}\left(\frac{1}{5}\ln|t-2|-\frac{1}{5}\ln|t+3|-\left(\frac{1}{5}\ln(1)-\frac{1}{5}\ln(4)\right)\right) = -\infty$$

$$\int_2^4 \frac{1}{(x+3)(x-2)}\,dx = \lim_{s\to 2^+}\left(\frac{1}{5}\ln(2)-\frac{1}{5}\ln(7)-\left(\frac{1}{5}\ln|s-2|-\frac{1}{5}\ln|s+3|\right)\right) = \infty$$

Therefore, each of these integrals are divergent. This means that we were, in fact, not able to break up the integral as we did back in Step 1.

This in turn means that the integral **diverges**.

---

10. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx$$

Hint

Be very careful with this problem as it is nothing like what we did in the notes. However, you should be able to take the material from the notes and use that to figure out how to do this problem.

Step 1

Now there is clearly an infinite limit here, but also notice that there is a discontinuity at $x = 0$ that we'll need to deal with.

Based on the material in the notes it should make sense that, provided both integrals converge, we should be able to split up the integral at any point. In this case let's split the integral up at $x = -1$. Doing this gives,

$$\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx = \int_{-\infty}^{-1} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx + \int_{-1}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx$$

Keep in mind that splitting up the integral like this can only be done if both of the integrals converge! If it turns out that even one of them is divergent then it will turn out that we couldn't have done this and the original integral will be divergent.

So, not worrying about if this was really possible to do or not let's proceed with the problem.

Now, we can eliminate the problems as follows,

$$\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx = \lim_{t \to -\infty} \int_{t}^{-1} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx + \lim_{s \to 0^-} \int_{-1}^{s} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx$$

Step 2

Next, let's do the integral. We'll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calc I substitution. Here is the integration work.

$$\int \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx = -\mathbf{e}^{\frac{1}{x}} + c$$

Note that we didn't do the definite integral here. The limits don't really affect how we do the integral and the integral for each was the same with only the limits being different so no reason to do the integral twice.

Step 3

Okay, now let's take care of the limits on the integral.

$$\begin{aligned}\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx &= \lim_{t\to\,-\infty}\left(-\mathbf{e}^{\frac{1}{x}}\right)\Big|_{t}^{-1} + \lim_{s\to 0^-}\left(-\mathbf{e}^{\frac{1}{x}}\right)\Big|_{-1}^{s} \\ &= \lim_{t\to\,-\infty}\left(-\mathbf{e}^{-1} + \mathbf{e}^{\frac{1}{t}}\right) + \lim_{s\to 0^-}\left(-\mathbf{e}^{\frac{1}{s}} + \mathbf{e}^{-1}\right)\end{aligned}$$

Step 4

We now need to evaluate the limits in our answer from the previous step. Here is the limit work

$$\begin{aligned}\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx &= \lim_{t\to\,-\infty}\left(-\mathbf{e}^{-1} + \mathbf{e}^{\frac{1}{t}}\right) + \lim_{s\to 0^-}\left(-\mathbf{e}^{\frac{1}{s}} + \mathbf{e}^{-1}\right) \\ &= \left[-\mathbf{e}^{-1} + \mathbf{e}^{0}\right] + \left[0 + \mathbf{e}^{-1}\right]\end{aligned}$$

Note that,

$$\lim_{s\to 0^-}\frac{1}{s} = -\infty$$

since we are doing a left-hand limit and so s will be negative. This in turn means that,

$$\lim_{s\to 0^-}\left(-\mathbf{e}^{\frac{1}{s}}\right) = 0$$

Step 5

The final step is to write down the answer!

Now, from the limit work in the previous step we see that,

$$\int_{-\infty}^{-1} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx = -\mathbf{e}^{-1} + 1 \qquad\qquad \int_{-1}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx = \mathbf{e}^{-1}$$

Therefore, each of the integrals are convergent and have the values shown above. This means that we could in fact break up the integral as we did in Step 1. Also, the original integral is now,

$$\begin{aligned}\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx &= \int_{-\infty}^{-1} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx + \int_{-1}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx \\ &= \quad -\mathbf{e}^{-1} + 1 \quad + \quad \mathbf{e}^{-1} \\ &= 1\end{aligned}$$

Therefore, the integral **converges** and its value is **1**.

7.9 Comparison Test for Improper Integrals

1. Use the Comparison Test to determine if the following integral converges or diverges.

$$\int_1^\infty \frac{1}{x^3+1}\,dx$$

Hint

Start off with a guess. Do you think this will converge or diverge?

Step 1

The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

The "+1" in the denominator does not really change the size of the denominator as x gets really large and so hopefully it makes sense that we can guess that this integral should behave like,

$$\int_1^\infty \frac{1}{x^3}\,dx$$

Then, by the fact from the previous section, we know that this integral converges since $p = 3 > 1$.

Therefore, we can guess that the integral,

$$\int_1^\infty \frac{1}{x^3+1}\,dx$$

will converge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we've guessed the integral converges we now know that it converges and that's all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we've guessed that the integral converges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn't correct.

All we've done is make a guess. Now we need to **prove** that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don't always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let's continue on with the problem.

Hint

Now that we've guessed the integral converges do we want a larger or smaller function that we know converges?

Step 2

Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral converges so if we find a larger function that we know converges the area analogy tells us that there would be a finite (*i.e.* not infinite) amount of area under the larger function.

Our function, which would be smaller, would then also have a finite amount of area under it. There is no way we can have an infinite amount of area inside of a finite amount of area!

Note that the opposite situation does us no good. If we find a smaller function that we know converges (and hence will have a finite amount of area under it) our function (which is now larger) can have either a larger finite amount of area or an infinite area under it.

In other words, if we find a smaller function that we know converges this will tell us nothing about our function. However, if we find a larger function that we know converges this will force our function to also converge.

Therefore we need to find a larger function that we know converges.

Step 3

Okay, now that we know we need to find a larger function that we know converges.

So, let's start with the function from the integral. It is a fraction and we know that we can make a fraction larger by making the denominator smaller. Also note that for $x > 1$ (which we can assume from the limits on the integral) we have,

$$x^3 + 1 > x^3$$

Therefore, we have,

$$\frac{1}{x^3+1} < \frac{1}{x^3}$$

since we replaced the denominator with something that we know is smaller.

Step 4

Finally, we know that,

$$\int_1^\infty \frac{1}{x^3}\,dx$$

converges. Then because the function in this integral is larger than the function in the original integral the Comparison Test tells us that,

$$\int_1^\infty \frac{1}{x^3+1}\,dx$$

must also **converge**.

---

2. Use the Comparison Test to determine if the following integral converges or diverges.

$$\int_3^\infty \frac{z^2}{z^3-1}\,dz$$

Hint

Start off with a guess. Do you think this will converge or diverge?

Step 1

The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

The “-1” in the denominator does not really change the size of the denominator as z gets really large and so hopefully it makes sense that we can guess that this integral should behave like,

$$\int_3^\infty \frac{z^2}{z^3}\,dz = \int_3^\infty \frac{1}{z}\,dz$$

Then, by the fact from the previous section, we know that this integral diverges since $p = 1 \le 1$.

Therefore, we can guess that the integral,

$$\int_3^\infty \frac{z^2}{z^3 - 1}\, dz$$

will diverge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we've guessed the integral diverges we now know that it diverges and that's all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we've guessed that the integral diverges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn't correct.

All we've done is make a guess. Now we need to **prove** that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don't always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let's continue on with the problem.

Hint

Now that we've guessed the integral diverges do we want a larger or smaller function that we know diverges?

Step 2

Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral diverges so if we find a smaller function that we know diverges the area analogy tells us that there would be an infinite amount of area under the smaller function.

Our function, which would be larger, would then also have an infinite amount of area under it. There is no way we can have an finite amount of area covering an infinite amount of area!

Note that the opposite situation does us no good. If we find a larger function that we know diverges (and hence will have a infinite amount of area under it) our function (which is now smaller) can have either a finite amount of area or an infinite area under it.

In other words, if we find a larger function that we know diverges this will tell us nothing about our function. However, if we find a smaller function that we know diverges this will force our function to also diverge.

Therefore we need to find a smaller function that we know diverges.

Step 3

Okay, now that we know we need to find a smaller function that we know diverges.

So, let's start with the function from the integral. It is a fraction and we know that we can make a fraction smaller by making the denominator larger. Also note that for $z > 3$ (which we can assume from the limits on the integral) we have,

$$z^3 - 1 < z^3$$

Therefore, we have,

$$\frac{z^2}{z^3 - 1} > \frac{z^2}{z^3} = \frac{1}{z}$$

since we replaced the denominator with something that we know is larger.

Step 4

Finally, we know that,

$$\int_3^\infty \frac{1}{z}\,dz$$

diverges. Then because the function in this integral is smaller than the function in the original integral the Comparison Test tells us that,

$$\int_3^\infty \frac{z^2}{z^3 - 1}\,dz$$

must also **diverge**.

3. Use the Comparison Test to determine if the following integral converges or diverges.

$$\int_4^\infty \frac{\mathbf{e}^{-y}}{y}\,dy$$

Hint

Start off with a guess. Do you think this will converge or diverge?

Step 1

The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

We need to be a little careful with the guess for this problem. We might be tempted to use the fact from the previous section to guess diverge since the exponent on the y in the denominator is $p = 1 \le 1$.

That would be incorrect however. Recall that the fact requires a constant in the numerator and we clearly do not have that in this case. In fact what we have in the numerator is $\mathbf{e}^{-y}$ and this goes to zero very fast as $y \to \infty$ and so there is a pretty good chance that this integral will in fact converge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we've guessed the integral converges we now know that it converges and that's all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we've guessed that the integral converges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn't correct.

All we've done is make a guess. Now we need to **prove** that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don't always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let's continue on with the problem.

Hint

Now that we've guessed the integral converges do we want a larger or smaller function that we know converges?

Step 2

Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral converges so if we find a larger function that we know converges the area analogy tells us that there would be a finite (*i.e.* not infinite) amount of area under the larger function.

Our function, which would be smaller, would then also have a finite amount of area under it. There is no way we can have an infinite amount of area inside of a finite amount of area!

Note that the opposite situation does us no good. If we find a smaller function that we know converges (and hence will have a finite amount of area under it) our function (which is now larger) can have either a larger finite amount of area or an infinite area under it.

In other words, if we find a smaller function that we know converges this will tell us nothing about our function. However, if we find a larger function that we know converges this will force our function to also converge.

Therefore we need to find a larger function that we know converges.

Step 3

Okay, now that we know we need to find a larger function that we know converges.

So, let's start with the function from the integral. It is a fraction and we know that we can make a fraction larger by making the denominator smaller. From the limits on the integral we can see that,

$$y > 4$$

Therefore, we have,

$$\frac{\mathbf{e}^{-y}}{y} < \frac{\mathbf{e}^{-y}}{4}$$

since we replaced the denominator with something that we know is smaller.

Step 4

Finally, we will need to prove that,

$$\int_4^\infty \mathbf{e}^{-y}\,dy$$

converges. However, after the previous section that shouldn't be too difficult. Here is that work.

$$\int_4^\infty \frac{1}{4}\mathbf{e}^{-y}\,dy = \lim_{t\to\infty}\int_4^t \frac{1}{4}\mathbf{e}^{-y}\,dy = \lim_{t\to\infty}\left(-\frac{1}{4}\mathbf{e}^{-y}\right)\bigg|_4^t = \lim_{t\to\infty}\left(-\frac{1}{4}\mathbf{e}^{-t}+\frac{1}{4}\mathbf{e}^{-4}\right) = \frac{1}{4}\mathbf{e}^{-4}$$

The limit existed and was finite and so we know that,

$$\int_4^\infty \frac{1}{4}\mathbf{e}^{-y}\,dy$$

converges.

Therefore, because the function in this integral is larger than the function in the original integral the Comparison Test tells us that,

$$\int_4^\infty \frac{\mathbf{e}^{-y}}{y}\,dy$$

must also **converge**.

---

4. Use the Comparison Test to determine if the following integral converges or diverges.

$$\int_1^\infty \frac{z-1}{z^4+2z^2}\,dz$$

Hint

Start off with a guess. Do you think this will converge or diverge?

Step 1

The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

Both the numerator and denominator of this function are polynomials and we know that as $z\to\infty$ the behavior of each of the polynomials will be the same as the behavior of the largest power of z. Therefore, it looks like this integral should behave like,

$$\int_1^\infty \frac{z}{z^4}\,dz = \int_1^\infty \frac{1}{z^3}\,dz$$

Then, by the fact from the previous section, we know that this integral converges since $p = 3 > 1$.

Therefore, we can guess that the integral,

$$\int_1^\infty \frac{z-1}{z^4+2z^2}\,dz$$

will converge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we've guessed the integral converges we now know that it converges and that's all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we've guessed that the integral converges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn't correct.

All we've done is make a guess. Now we need to **prove** that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don't always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let's continue on with the problem.

Hint

Now that we've guessed the integral converges do we want a larger or smaller function that we know converges?

Step 2

Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral converges so if we find a larger function that we know converges the area analogy tells us that there would be a finite (*i.e.* not infinite) amount of area under the larger function.

Our function, which would be smaller, would then also have a finite amount of area under it. There is no way we can have an infinite amount of area inside of a finite amount of area!

Note that the opposite situation does us no good. If we find a smaller function that we know converges (and hence will have a finite amount of area under it) our function (which is now larger) can have either a larger finite amount of area or an infinite area under it.

In other words, if we find a smaller function that we know converges this will tell us nothing about our function. However, if we find a larger function that we know converges this will force our function to also converge.

Therefore we need to find a larger function that we know converges.

Step 3

Okay, now that we know we need to find a larger function that we know converges.

So, let's start with the function from the integral. It is a fraction and we know that we can make a fraction larger by making numerator larger or the denominator smaller.

Note that for $z > 1$ (which we can assume from the limits on the integral) we have,

$$z - 1 < z$$

Therefore, we have,

$$\frac{z-1}{z^4 + 2z^2} < \frac{z}{z^4 + 2z^2} = \frac{1}{z^3 + 2z}$$

since we replaced the numerator with something that we know is larger.

Step 4

It is at this point that students again often make mistakes with this kind of problem. After doing one manipulation of the numerator or denominator they stop the manipulation and declare that the new function must converge (since that is what we want after all) and move on to the next problem.

Recall however that we must **know** that the new function converges and we've not gotten to a function yet that we know converges. To get to a function that we know converges we need to do one more manipulation of the function.

Again, note that for $z > 1$ we have,

$$z^3 + 2z > z^3$$

Therefore, we have,

$$\frac{1}{z^3 + 2z} < \frac{1}{z^3}$$

since we replaced the denominator with something that we know is smaller.

Step 5

Finally, putting the results of Steps 3 & 4 together we have,

$$\frac{z-1}{z^4 + 2z^2} < \frac{1}{z^3}$$

and we know that,

$$\int_1^\infty \frac{1}{z^3}\,dz$$

converges. Then because the function in this integral is larger than the function in the original integral the Comparison Test tells us that,

$$\int_1^\infty \frac{z-1}{z^4 + 2z^2}\,dz$$

must also **converge**.

---

5. Use the Comparison Test to determine if the following integral converges or diverges.

$$\int_6^\infty \frac{w^2 + 1}{w^3\left(\cos^2(w) + 1\right)}\,dw$$

Hint

Start off with a guess. Do you think this will converge or diverge?

Step 1

The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

The numerator of this function is a polynomial and we know that as $w \to \infty$ the behavior of polynomials will be the same as the behavior of the largest power of w. Also the cosine term in the denominator is bounded and never gets too large or small.

Therefore, it looks like this integral should behave like,

$$\int_6^\infty \frac{w^2}{w^3}\,dw = \int_6^\infty \frac{1}{w}\,dw$$

Then, by the fact from the previous section, we know that this integral diverges since $p = 1 \le 1$.

Therefore, we can guess that the integral,

$$\int_6^\infty \frac{w^2+1}{w^3\left(\cos^2(w)+1\right)}\,dw$$

will diverge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we've guessed the integral diverges we now know that it diverges and that's all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we've guessed that the integral diverges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn't correct.

All we've done is make a guess. Now we need to **prove** that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don't always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let's continue on with the problem.

Hint

Now that we've guessed the integral diverges do we want a larger or smaller function that we know diverges?

Step 2

Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral diverges so if we find a smaller function that we know diverges the area analogy tells us that there would be an infinite amount of area under the smaller function.

Our function, which would be larger, would then also have an infinite amount of area under it. There is no way we can have an finite amount of area covering an infinite amount of area!

Note that the opposite situation does us no good. If we find a larger function that we know diverges (and hence will have a infinite amount of area under it) our function (which is now smaller) can have either a finite amount of area or an infinite area under it.

In other words, if we find a larger function that we know diverges this will tell us nothing about our function. However, if we find a smaller function that we know diverges this will force our function to also diverge.

Therefore we need to find a smaller function that we know diverges.

Step 3

Okay, now that we know we need to find a smaller function that we know diverges.

So, let's start with the function from the integral. It is a fraction and we know that we can make a fraction smaller by making the numerator smaller or the denominator larger. Also note that for $w > 6$ (which we can assume from the limits on the integral) we have,

$$w^2 + 1 > w^2$$

Therefore, we have,

$$\frac{w^2+1}{w^3\left(\cos^2(w)+1\right)} > \frac{w^2}{w^3\left(\cos^2(w)+1\right)} = \frac{1}{w\left(\cos^2(w)+1\right)}$$

since we replaced the numerator with something that we know is smaller.

Step 4

It is at this point that students again often make mistakes with this kind of problem. After doing one manipulation of the numerator or denominator they stop the manipulation and declare that the new function must diverge (since that is what we want after all) and move on to the next problem.

Recall however that we must **know** that the new function diverges and we've not gotten to a function yet that we know diverges. To get to a function that we know diverges we need to do one more manipulation of the function.

For this step we know that $0 \le \cos^2(w) \le 1$ and so we will have,

$$\cos^2(w) + 1 < 1 + 1 = 2$$

Therefore, we have,

$$\frac{1}{w\left(\cos^2(w)+1\right)} > \frac{1}{w(2)} = \frac{1}{2w}$$

since we replaced the denominator with something that we know is larger.

Step 5

Finally, putting the results of Steps 3 & 4 together we have,

$$\frac{w^2+1}{w^3\left(\cos^2(w)+1\right)} > \frac{1}{2w}$$

and we know that,

$$\int_6^\infty \frac{1}{2w}\,dw = \frac{1}{2}\int_6^\infty \frac{1}{w}\,dw$$

diverges. Then because the function in this integral is smaller than the function in the original integral the Comparison Test tells us that,

$$\int_6^\infty \frac{w^2+1}{w^3\left(\cos^2(w)+1\right)}\,dw$$

must also **diverge**.

7.10 Approximating Definite Integrals

1. Using $n = 6$ approximate the value of $\displaystyle\int_1^7 \frac{1}{x^3+1}\,dx$ using

   (a) the Midpoint Rule,

   (b) the Trapezoid Rule, and

   (c) Simpson's Rule.

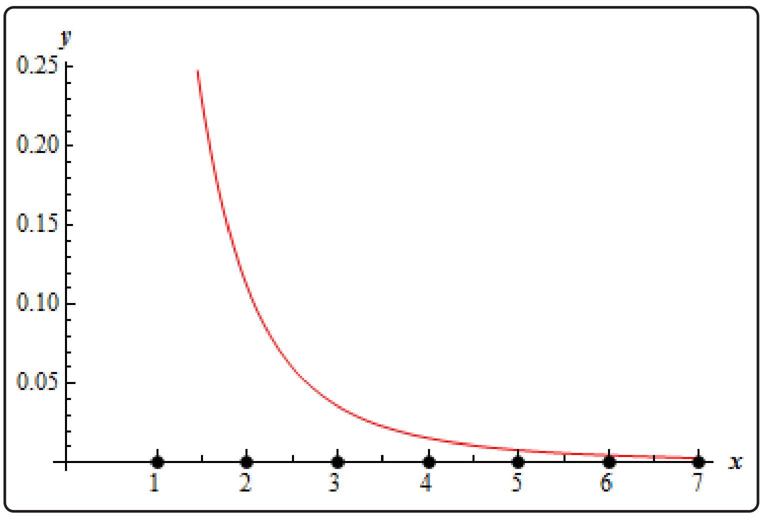
   Use at least 6 decimal places of accuracy for your work.

   Solutions

   (a) Midpoint Rule

Solution

While it's not really needed to do the problem here is a sketch of the graph.



We know that we need to divide the interval $[1, 7]$ into 6 subintervals each with width,

$$\Delta x = \frac{7-1}{6} = 1$$

The endpoints of each of these subintervals are represented by the dots on the x axis on the graph above.

The tick marks between each dot represents the midpoint of each of the subintervals. The x-values of the midpoints for each of the subintervals are then,

$$\frac{3}{2}, \quad \frac{5}{2}, \quad \frac{7}{2}, \quad \frac{9}{2}, \quad \frac{11}{2}, \quad \frac{13}{2}$$

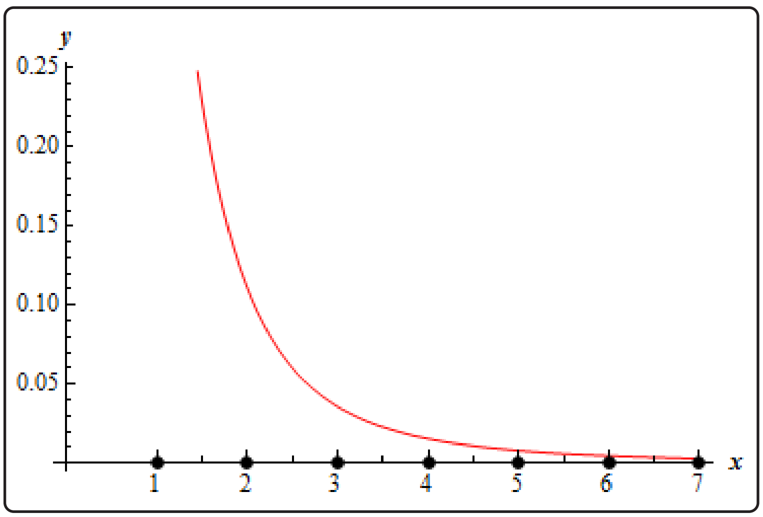
So, to use the Midpoint Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

$$\int_1^7 \frac{1}{x^3+1}\,dx \approx (1)\left[f\left(\frac{3}{2}\right)+f\left(\frac{5}{2}\right)+f\left(\frac{7}{2}\right)+f\left(\frac{9}{2}\right)+f\left(\frac{11}{2}\right)+f\left(\frac{13}{2}\right)\right]$$
$$= \boxed{0.33197137}$$

(b) Trapezoid Rule

Solution

From the Midpoint Rule work we know that the width of each subinterval is $\Delta x = 1$ and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.



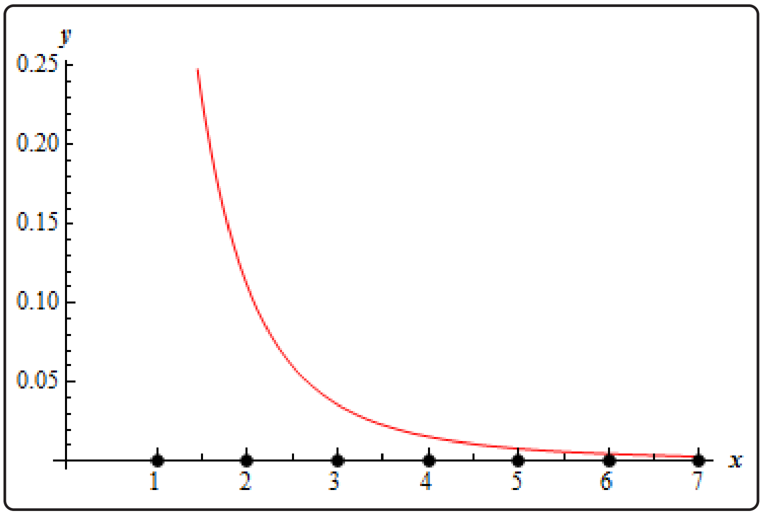
So, to use the Trapezoid Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

$$\int_1^7 \frac{1}{x^3+1}\,dx \approx \left(\frac{1}{2}\right)\left[f(1)+2f(2)+2f(3)+2f(4)+2f(5)+2f(6)+f(7)\right]$$
$$= \boxed{0.42620830}$$

(c) Simpson's Rule

Solution

From the Midpoint Rule work we know that the width of each subinterval is $\Delta x = 1$ and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.



As with the first two parts all we need to do is plug into the formula to use Simpson's Rule to approximate value of the integral. Doing this gives,

$$\int_1^7 \frac{1}{x^3+1}\,dx \approx \left(\frac{1}{3}\right)\left[f(1) + 4f(2) + 2f(3) + 4f(4) + 2f(5) + 4f(6) + f(7)\right]$$
$$= \boxed{0.37154155}$$

2. Using $n = 6$ approximate the value of $\int_{-1}^{2} \sqrt{\mathbf{e}^{-x^2} + 1}\,dx$ using

(a) the Midpoint Rule,

(b) the Trapezoid Rule, and

(c) Simpson's Rule.

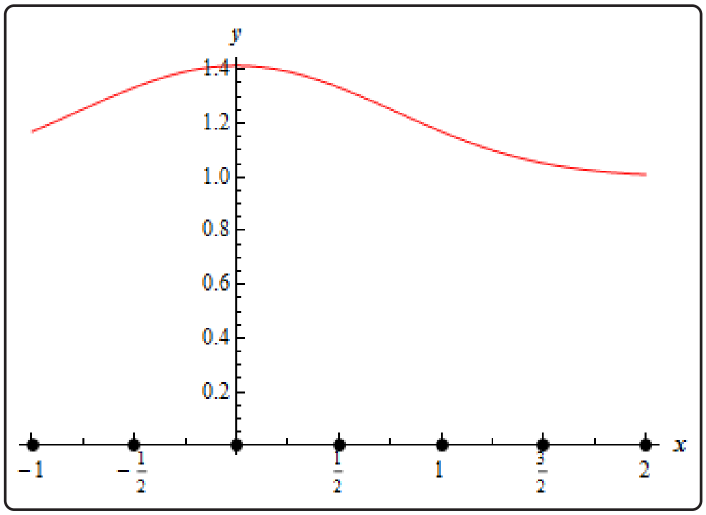
Use at least 6 decimal places of accuracy for your work.

Solutions

(a) Midpoint Rule

Solution

While it's not really needed to do the problem here is a sketch of the graph.



We know that we need to divide the interval $[-1, 2]$ into 6 subintervals each with width,

$$\Delta x = \frac{2-(-1)}{6} = \frac{1}{2}$$

The endpoints of each of these subintervals are represented by the dots on the x axis on the graph above.

The tick marks between each dot represents the midpoint of each of the subintervals. The x-values of the midpoints for each of the subintervals are then,

$$-\frac{3}{4}, \quad -\frac{1}{4}, \quad \frac{1}{4}, \quad \frac{3}{4}, \quad \frac{5}{4}, \quad \frac{7}{4}$$

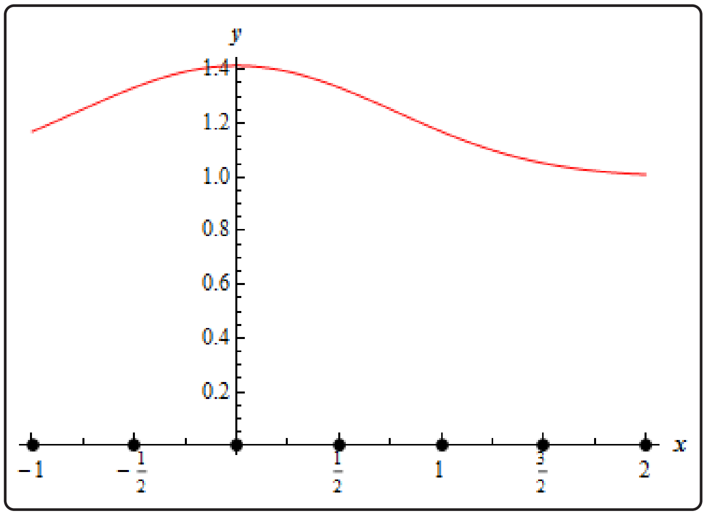
So, to use the Midpoint Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

$$\begin{aligned} \int_{-1}^{2} \sqrt{\mathbf{e}^{-x^2}+1}\,dx &\approx \left(\frac{1}{2}\right)\left[f\left(-\frac{3}{4}\right)+f\left(-\frac{1}{4}\right)+f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right. \\ &\qquad \left. +f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)\right] \\ &= \boxed{3.70700857} \end{aligned}$$

(b) Trapezoid Rule

Solution

From the Midpoint Rule work we know that the width of each subinterval is $\Delta x = \frac{1}{2}$ and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.



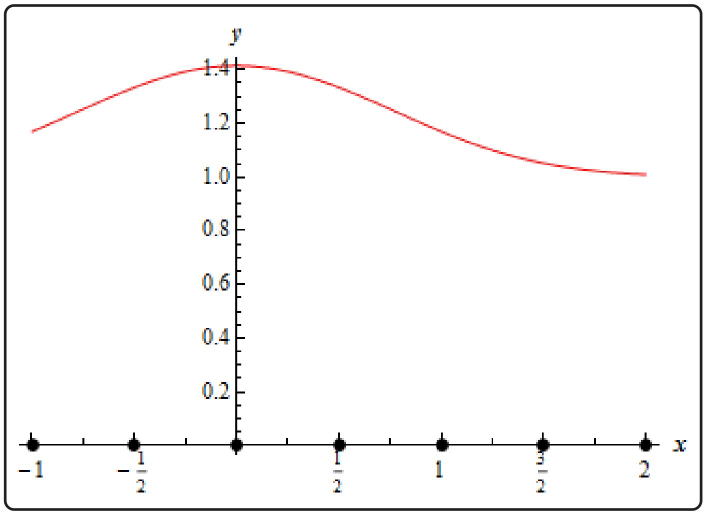
So, to use the Trapezoid Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

$$\int_{-1}^{2} \sqrt{\mathbf{e}^{-x^2}+1}\,dx \approx \left(\frac{1/2}{2}\right)\left[f(-1)+2f\left(-\frac{1}{2}\right)+2f(0)+2f\left(\frac{1}{2}\right)\right.$$
$$\left.+\,2f(1)+2f\left(\frac{3}{2}\right)+f(2)\right]$$
$$= \boxed{3.69596543}$$

(c) Simpson's Rule

Solution

From the Midpoint Rule work we know that the width of each subinterval is $\Delta x = \frac{1}{2}$ and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.



As with the first two parts all we need to do is plug into the formula to use Simpson's Rule to approximate value of the integral. Doing this gives,

$$\int_{-1}^{2} \sqrt{\mathbf{e}^{-x^2}+1}\,dx \approx \left(\frac{1/2}{3}\right)\left[f(-1)+4f\left(-\frac{1}{2}\right)+2f(0)+4f\left(\frac{1}{2}\right)+2f(1)+4f\left(\frac{3}{2}\right)+f(2)\right]$$

$$= \boxed{3.70358145}$$

3. Using $n = 8$ approximate the value of $\displaystyle\int_0^4 \cos\left(1+\sqrt{x}\right)\,dx$ using

(a) the Midpoint Rule,

(b) the Trapezoid Rule, and

(c) Simpson's Rule.

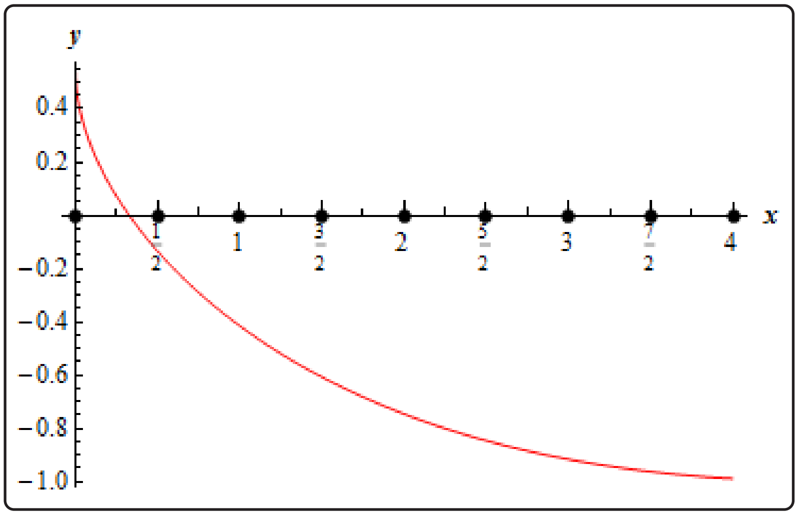
Use at least 6 decimal places of accuracy for your work.

Solutions

(a) Midpoint Rule

Solution

While it's not really needed to do the problem here is a sketch of the graph.



We know that we need to divide the interval $[0, 4]$ into 8 subintervals each with width,

$$\Delta x = \frac{4-0}{8} = \frac{1}{2}$$

The endpoints of each of these subintervals are represented by the dots on the x axis on the graph above.

The tick marks between each dot represents the midpoint of each of the subintervals. The x-values of the midpoints for each of the subintervals are then,

$$\frac{1}{4}, \quad \frac{3}{4}, \quad \frac{5}{4}, \quad \frac{7}{4}, \quad \frac{9}{4}, \quad \frac{11}{4}, \quad \frac{13}{4}, \quad \frac{15}{4}$$

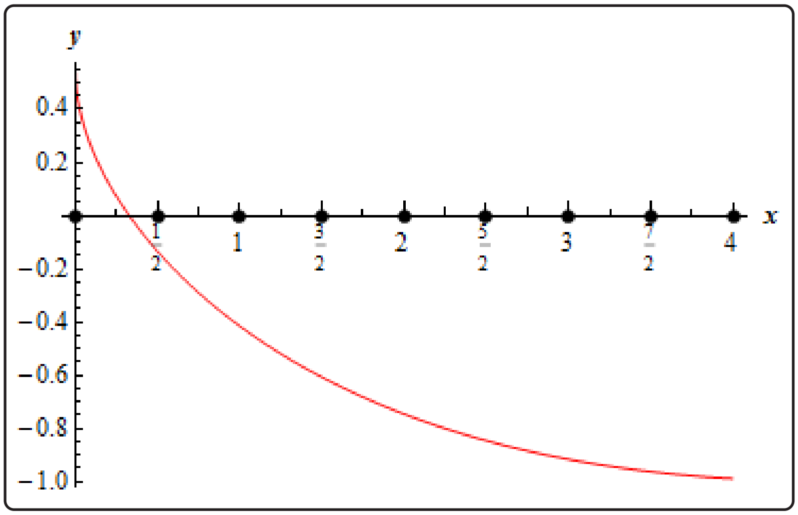
So, to use the Midpoint Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

$$\begin{aligned}\int_0^4 \cos\left(1+\sqrt{x}\right)\,dx &\approx \left(\frac{1}{2}\right)\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)\right.\\ &\qquad\left.+f\left(\frac{11}{4}\right)+f\left(\frac{13}{4}\right)+f\left(\frac{15}{4}\right)\right]\\ &= \boxed{-2.51625938}\end{aligned}$$

(b) Trapezoid Rule

Solution

From the Midpoint Rule work we know that the width of each subinterval is $\Delta x = \frac{1}{2}$ and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.



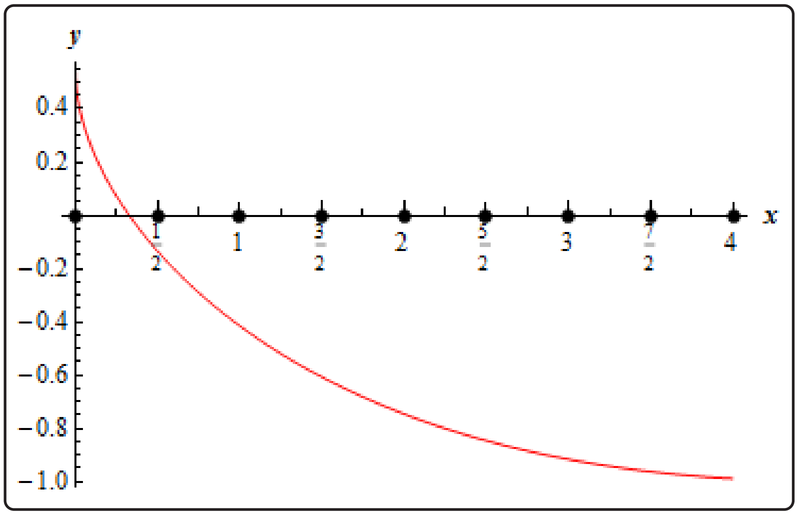
So, to use the Trapezoid Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

$$\begin{aligned}\int_0^4 \cos\left(1+\sqrt{x}\right)\,dx &\approx \left(\frac{1/2}{2}\right)\left[f(0)+2f\left(\frac{1}{2}\right)+2f(1)+2f\left(\frac{3}{2}\right)+2f(2)\right.\\ &\qquad\left.+2f\left(\frac{5}{2}\right)+2f(3)+2f\left(\frac{7}{2}\right)+f(4)\right]\\ &= \boxed{-2.43000475}\end{aligned}$$

(c) Simpson's Rule

Solution

From the Midpoint Rule work we know that the width of each subinterval is $\Delta x = \frac{1}{2}$ and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.



As with the first two parts all we need to do is plug into the formula to use Simpson's Rule to approximate value of the integral. Doing this gives,

$$\begin{aligned}\int_0^4 \cos\left(1+\sqrt{x}\right)\,dx &\approx \left(\frac{1/2}{3}\right)\left[f(0)+4f\left(\frac{1}{2}\right)+2f(1)+4f\left(\frac{3}{2}\right)+2f(2)\right.\\ &\qquad\left.+4f\left(\frac{5}{2}\right)+2f(3)+4f\left(\frac{7}{2}\right)+f(4)\right]\\ &=\boxed{-2.47160136}\end{aligned}$$

8 More Applications of Integrals

It is now time to take a look at some more applications of integrals. As noted the last time we looked at applications of integrals many, although, not all of these new applications in this chapter have a fairly high chance of needing some of the integration techniques from the last chapter.

The first application, Arc Length can be kept to only u-substitutions at the worst, but most of those problems tend to be very simple. Once we start moving into more complicated problems arc length problems tend to involve trig substitutions.

The next application, Surface Area tends to be u-substitutions but the notation used here is also used in the Arc Length section and so the surface area section is also here because of the shared notation.

Center of Mass and Probability are applications that will, in almost every case, involve integration by parts. In addition, the Probability section has the potential for improper integrals to show up.

The other application we'll be looking at in this chapter, Hydrostatic Pressure and Force, will typically involve fairly simple integrals that could have been done in the earlier chapter. The reason the topic is here is because we have to derive up the integral using the definition of the definite integral in every problem. In addition, more complicated problems could lead to much more complicated integrals. The integrals in this section are kept simple mostly to keep the derivation work simpler.

The following sections are the practice problems, with solutions, for this material.

8.1 Arc Length

1. Set up, but do not evaluate, an integral for the length of $y = \sqrt{x+2}$, $1 \le x \le 7$ using,

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$

Solutions

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$

Step 1

We'll need the derivative of the function first.

$$\frac{dy}{dx} = \frac{1}{2}(x+2)^{-\frac{1}{2}} = \frac{1}{2(x+2)^{\frac{1}{2}}}$$

Step 2

Plugging this into the formula for ds gives,

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \sqrt{1 + \left[\frac{1}{2(x+2)^{\frac{1}{2}}}\right]^2}\, dx = \sqrt{1 + \frac{1}{4\,(x+2)}}\, dx$$
$$= \sqrt{\frac{4x+9}{4\,(x+2)}}\, dx$$

Step 3

All we need to do now is set up the integral for the arc length. Also note that we have a dx in the formula for ds and so we know that we need x limits of integration which we've been given in the problem statement.

$$L = \int ds = \boxed{\int_1^7 \sqrt{\frac{4x+9}{4x+8}}\, dx}$$

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy$

Step 1

In this case we first need to solve the function for x so we can compute the derivative in the ds.

$$y = \sqrt{x+2} \qquad \to \qquad x = y^2 - 2$$

The derivative of this is,

$$\frac{dx}{dy} = 2y$$

Step 2

Plugging this into the formula for ds gives,

$$ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy = \sqrt{1 + [2y]^2}\,dy = \sqrt{1 + 4y^2}\,dy$$

Step 3

Next, note that the ds has a dy in it and so we'll need y limits of integration.

We are only given x limits in the problem statement. However, we can plug these into the function we were given in the problem statement to convert them to y limits. Doing this gives,

$$x = 1 : y = \sqrt{3} \qquad\qquad x = 7 : y = \sqrt{9} = 3$$

So, the corresponding y limits are : $\sqrt{3} \le y \le 3$.

Step 4

Finally, all we need to do is set up the integral.

$$L = \int ds = \boxed{\int_{\sqrt{3}}^{3} \sqrt{1 + 4y^2}\, dy}$$

---

2. Set up, but do not evaluate, an integral for the length of $x = \cos(y)$, $0 \le x \le \frac{1}{2}$ using,

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$

Solutions

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$

Step 1

In this case we first need to solve the function for y so we can compute the derivative in the ds.

$$x = \cos(y) \qquad \rightarrow \qquad y = \cos^{-1}(x) = \arccos(x)$$

Which notation you use for the inverse tangent is not important since it will be "disappearing" once we take the derivative.

Speaking of which, here is the derivative of the function.

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

Step 2

Plugging this into the formula for ds gives,

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \sqrt{1 + \left[-\frac{1}{\sqrt{1 - x^2}}\right]^2}\, dx = \sqrt{1 + \frac{1}{1 - x^2}}\, dx = \sqrt{\frac{2 - x^2}{1 - x^2}}\, dx$$

Step 3

All we need to do now is set up the integral for the arc length. Also note that we have a dx in the formula for ds and so we know that we need x limits of integration which we've been given in the problem statement.

$$L = \int ds = \boxed{\int_0^{\frac{1}{2}} \sqrt{\frac{2 - x^2}{1 - x^2}}\, dx}$$

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$

Step 1

We'll need the derivative of the function first.

$$\frac{dx}{dy} = -\sin(y)$$

Step 2

Plugging this into the formula for ds gives,

$$ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy = \sqrt{1 + [-\sin(y)]^2}\, dy = \sqrt{1 + \sin^2(y)}\, dy$$

Step 3

Next, note that the ds has a dy in it and so we'll need y limits of integration.

We are only given x limits in the problem statement. However, in part **(a)** we solved the function for y to get,

$$y = \cos^{-1}(x) = \arccos(x)$$

and all we need to do is plug x limits we were given into this to convert them to y

limits. Doing this gives,

$$x = 0: \quad y = \cos^{-1}(0) = \arccos(0) = \frac{\pi}{2}$$
$$x = \frac{1}{2}: \quad y = \cos^{-1}\left(\frac{1}{2}\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

So, the corresponding y limits are : $\frac{\pi}{3} \le y \le \frac{\pi}{2}$.

Note that we used both notations for the inverse cosine here but you only need to use the one you are comfortable with. Also, recall that we know that the range of the inverse cosine function is,

$$0 \le \cos^{-1}(x) \le \pi$$

Therefore, there is only one possible value of y that we can get out of each value of x.

Step 4

Finally, all we need to do is set up the integral.

$$L = \int ds = \boxed{\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{1 + \sin^2(y)}\, dy}$$

3. Determine the length of $y = 7(6+x)^{\frac{3}{2}}$, $189 \le y \le 875$.

Step 1

Since we are not told which ds to use we will have to decide which one to use. In this case the function is set up to use the ds in terms of x. Note as well that if we solve the function for x (which we'd need to do in order to use the ds that is in terms of y) we would still have a fractional exponent and the derivative will not work out as nice once we plug it into the ds formula.

So, let's take the derivative of the given function and plug into the ds formula.

$$\frac{dy}{dx} = \frac{21}{2}(6+x)^{\frac{1}{2}}$$

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\,dx = \sqrt{1 + \left[\frac{21}{2}(6+x)^{\frac{1}{2}}\right]^2}\,dx = \sqrt{1 + \frac{441}{4}(6+x)}\,dx$$
$$= \sqrt{\frac{2650}{4} + \frac{441}{4}x}\,dx = \frac{1}{2}\sqrt{2650 + 441x}\,dx$$

We did a little simplification that may or may not make the integration easier. That will probably depend upon the person doing the integration and just what they find the easiest to deal with. The point is there are several forms of the ds that we could use here. All will give the same answer.

Step 2

Next, we need to deal with the limits for the integral. The ds that we choose to use in the first step has a dx in it and that means that we'll need x limits for our integral. We, however, were given y limits in the problem statement. This means we'll need to convert those to x's before proceeding with the integral.

To do convert these all we need to do is plug them into the function we were given in the problem statement and solve for the corresponding x. Doing this gives,

$$\begin{aligned} y =& 189: \quad 189 = 7(6+x)^{\frac{3}{2}} \quad \rightarrow \quad 6 + x = 27^{\frac{2}{3}} = 9 \quad \rightarrow \quad x = 3 \\ y =& 875: \quad 875 = 7(6+x)^{\frac{3}{2}} \quad \rightarrow \quad 6 + x = 125^{\frac{2}{3}} = 25 \quad \rightarrow \quad x = 19 \end{aligned}$$

So, the corresponding ranges of x's is : $3 \le x \le 19$.

Step 3

The integral giving the arc length is then,

$$L = \int ds = \int_{3}^{19} \frac{1}{2}\sqrt{2650 + 441x}\,dx$$

Step 4

Finally, all we need to do is evaluate the integral. In this case all we need to do is use a quick Calc I substitution. We'll leave most of the integration details to you to verify.

The arc length of the curve is,

$$L = \int_{3}^{19} \frac{1}{2}\sqrt{2650 + 441x}\, dx = \frac{1}{1323}(2650 + 441x)^{\frac{3}{2}}\bigg|_{3}^{19}$$

$$= \boxed{\frac{1}{1323}\left(11029^{\frac{3}{2}} - 3973^{\frac{3}{2}}\right) = 686.1904}$$

4. Determine the length of $x = 4(3+y)^2$, $1 \le y \le 4$.

Step 1

Since we are not told which ds to use we will have to decide which one to use. In this case the function is set up to use the ds in terms of y.

If we were to solve the function for y (which we'd need to do in order to use the ds that is in terms of x) we would put a square root into the function and those can be difficult to deal with in arc length problems.

So, let's take the derivative of the given function and plug into the ds formula.

$$\frac{dx}{dy} = 8\,(3+y)$$

$$ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy = \sqrt{1 + [8\,(3+y)]^2}\, dy = \sqrt{1 + 64(3+y)^2}\, dy$$

Note that we did not square out the term under the root. Doing that would greatly complicate the integration process so we'll need to leave it as it is.

Step 2

In this case we don't need to anything special to get the limits for the integral. Our choice of ds contains a dy which means we need y limits for the integral and nicely enough that is what we were given in the problem statement.

So, the integral giving the arc length is,

$$L = \int ds = \int_{1}^{4} \sqrt{1 + 64(3+y)^2}\, dy$$

Step 3

Finally, all we need to do is evaluate the integral. In this case all we need to do is use a trig substitution. We'll not be putting a lot of explanation into the integration work so if you need a little refresher on trig substitutions you should go back to that section and work a few practice problems.

The substitution we'll need is,

$$3 + y = \frac{1}{8}\tan(\theta) \qquad \to \qquad dy = \frac{1}{8}\sec^2(\theta)\,d\theta$$

In order to properly deal with the square root we'll need to convert the y limits to θ limits. Here is that work.

$$\begin{aligned} y = 1: &\quad 4 = \frac{1}{8}\tan(\theta) \quad \to \quad \tan(\theta) = 32 \quad \to \quad \theta = \tan^{-1}(32) = 1.5396 \\ y = 4: &\quad 7 = \frac{1}{8}\tan(\theta) \quad \to \quad \tan(\theta) = 56 \quad \to \quad \theta = \tan^{-1}(56) = 1.5529 \end{aligned}$$

Now let's deal with the square root.

$$\sqrt{1 + 64(3+y)^2} = \sqrt{1 + 64\left(\frac{1}{8}\tan(\theta)\right)^2} = \sqrt{1 + \tan^2(\theta)} = \sqrt{\sec^2(\theta)} = |\sec(\theta)|$$

From the work above we know that θ is in the range $1.5396 \le \theta \le 1.5529$. This is in the first and fourth quadrants and cosine (and hence secant) is positive in this range. So,

$$\sqrt{1 + 64(3+y)^2} = \sec(\theta)$$

Putting all of this together gives,

$$L = \int_1^4 \sqrt{1 + 64(3+y)^2}\,dy = \frac{1}{8}\int_{1.5396}^{1.5529} \sec^3(\theta)\,d\theta$$

Evaluating the integral gives,

$$\begin{aligned} L = \int_1^4 \sqrt{1 + 64(3+y)^2}\,dy &= \frac{1}{16}\left(\tan(\theta)\sec(\theta) + \ln|\tan(\theta) + \sec(\theta)|\right)\Big|_{1.5396}^{1.5529} \\ &= \boxed{130.9570} \end{aligned}$$

Note that if you used more decimal places than four here (the standard number of decimal places that we tend to use for these problems) you may have gotten a slightly different answer. Using a computer to get an "exact" answer gives 132.03497085.

These kinds of different answers can be a real issues with these kinds of problems and illustrates the potential problems if you round numbers too much.

Of course, there is also the problem of often not knowing just how many decimal places are needed to get an "accurate" answer. In many cases 4 decimal places is sufficient but there are cases (such as this one) in which that is not enough. Often the best bet is to simply use as many decimal places as you can to have the best chance of getting an "accurate" answer.

8.2 Surface Area

1. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $x = \sqrt{y+5}$, $\sqrt{5} \le x \le 3$ about the y-axis using,

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$

Solutions

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$

Step 1

In this case we first need to solve the function for y so we can compute the derivative in the ds.

$$x = \sqrt{y+5} \qquad \to \qquad y = x^2 - 5$$

The derivative of this is,

$$\frac{dy}{dx} = 2x$$

Step 2

Plugging this into the formula for ds gives,

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \sqrt{1 + [2x]^2}\, dx = \sqrt{1 + 4x^2}\, dx$$

Step 3

Finally, all we need to do is set up the integral. Also note that we have a dx in the formula for ds and so we know that we need x limits of integration which we've been given in the problem statement.

$$SA = \int 2\pi x \, ds = \boxed{\int_{\sqrt{5}}^{3} 2\pi x \sqrt{1+4x^2} \, dx}$$

Be careful with the formula! Remember that the variable in the integral is always opposite the axis of rotation. In this case we rotated about the y-axis and so we needed an x in the integral.

As an aside, note that the ds we chose to use here is technically immaterial. Realistically however, one ds may be easier than the other to work with. Determining which might be easier comes with experience and in many cases simply trying both to see which is easier.

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2} \, dy$

Step 1

We'll need the derivative of the function first.

$$\frac{dx}{dy} = \frac{1}{2}(y+5)^{-\frac{1}{2}} = \frac{1}{2(y+5)^{\frac{1}{2}}}$$

Step 2

Plugging this into the formula for ds gives,

$$ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2} \, dy = \sqrt{1 + \left[\frac{1}{2(y+5)^{\frac{1}{2}}}\right]^2} \, dy = \sqrt{1 + \frac{1}{4(y+5)}} \, dy$$
$$= \sqrt{\frac{4y+21}{4(y+5)}} \, dy$$

Step 3

Next, note that the ds has a dy in it and so we'll need y limits of integration.

We are only given x limits in the problem statement. However, we can plug these into the function we derived in Step 1 of the first part to convert them to y limits. Doing this gives,

$$x = \sqrt{5} : y = 0 \qquad\qquad x = 3 : y = 4$$

So, the corresponding y limits are : $0 \le y \le 4$.

Step 4

Finally, all we need to do is set up the integral.

$$SA = \int 2\pi x ds = \int_0^4 2\pi x \sqrt{\frac{4y+21}{4(y+5)}}\,dy = \int_0^4 2\pi\sqrt{y+5}\sqrt{\frac{4y+21}{4(y+5)}}\,dy$$

$$= \int_0^4 2\pi\sqrt{y+5}\frac{\sqrt{4y+21}}{2\sqrt{y+5}}\,dy = \boxed{\int_0^4 \pi\sqrt{4y+21}\,dy}$$

Be careful with the formula! Remember that the variable in the integral is always opposite the axis of rotation. In this case we rotated about the y-axis and so we needed an x in the integral.

Note that with the ds we were told to use for this part we had a dy in the final integral and that means that all the variables in the integral need to be y's. This means that the x from the formula needs to be converted into y's as well. Luckily this is easy enough to do since we were given the formula for x in terms of y in the problem statement.

Finally, make sure you simplify these as much as possible as we did here. Had we not taken the square root of the numerator and denominator of the rational expression we would not have seen the cancelation that can happen there. Without that cancelation the integral would be much more difficult to do!

As an aside, note that the ds we chose to use here is technically immaterial. Realistically however, one ds may be easier than the other to work with. Determining which might be easier comes with experience and in many cases simply trying both to see which is easier.

2. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $y = \sin(2x)$, $0 \le x \le \frac{\pi}{8}$ about the x-axis using,

   (a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\,dx$

   (b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy$

Solutions

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\,dx$

Step 1

We'll need the derivative of the function first.

$$\frac{dy}{dx} = 2\cos(2x)$$

Step 2

Plugging this into the formula for ds gives,

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\,dx = \sqrt{1 + \left[2\cos(2x)\right]^2}\,dx = \sqrt{1 + 4\cos^2(2x)}\,dx$$

Step 3

Finally, all we need to do is set up the integral. Also note that we have a dx in the formula for ds and so we know that we need x limits of integration which we've been given in the problem statement.

$$SA = \int 2\pi y\,ds = \int_0^{\frac{\pi}{8}} 2\pi y\sqrt{1 + 4\cos^2(2x)}\,dx$$

$$= \boxed{\int_0^{\frac{\pi}{8}} 2\pi\sin(2x)\,\sqrt{1 + 4\cos^2(2x)}\,dx}$$

Be careful with the formula! Remember that the variable in the integral is always opposite the axis of rotation. In this case we rotated about the x-axis and so we needed a y in the integral.

Note that with the ds we were told to use for this part we had a dx in the final integral and that means that all the variables in the integral need to be x's. This means that the y from the formula needs to be converted into x's as well. Luckily this is easy enough to do since we were given the formula for y in terms of x in the problem statement.

As an aside, note that the ds we chose to use here is technically immaterial. Realistically however, one ds may be easier than the other to work with. Determining which might be easier comes with experience and in many cases simply trying both to see which is easier.

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy$

Step 1

In this case we first need to solve the function for x so we can compute the derivative in the ds.

$$y = \sin(2x) \qquad \to \qquad x = \frac{1}{2}\sin^{-1}(y)$$

The derivative of this is,

$$\frac{dx}{dy} = \frac{1}{2}\frac{1}{\sqrt{1-y^2}} = \frac{1}{2\sqrt{1-y^2}}$$

Step 2

Plugging this into the formula for ds gives,

$$ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy = \sqrt{1 + \left[\frac{1}{2\sqrt{1-y^2}}\right]^2}\,dy = \sqrt{1 + \frac{1}{4(1-y^2)}}\,dy$$
$$= \sqrt{\frac{5-4y^2}{4(1-y^2)}}\,dy$$

Step 3

Next, note that the ds has a dy in it and so we'll need y limits of integration.

We are only given x limits in the problem statement. However, we can plug these into the function we were given in the problem statement to convert them to y limits. Doing this gives,

$$x = 0 : y = \sin(0) = 0 \qquad\qquad x = \frac{\pi}{8} : y = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

So, the corresponding y limits are : $0 \le y \le \frac{\sqrt{2}}{2}$.

Step 4

Finally, all we need to do is set up the integral.

$$SA = \int 2\pi y\,ds = \boxed{\int_0^{\frac{\sqrt{2}}{2}} 2\pi y\sqrt{\frac{5-4y^2}{4-4y^2}}\,dy}$$

Be careful with the formula! Remember that the variable in the integral is always opposite the axis of rotation. In this case we rotated about the x-axis and so we needed an y in the integral.

Also note that the ds we chose to use is technically immaterial. Realistically one ds may be easier than the other to work with but technically either could be used.

---

3. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $y = x^3 + 4$, $1 \le x \le 5$ about the given axis. You can use either ds.

   (a) the x-axis

   (b) the y-axis

Solutions

(a) x-axis

Step 1

We are told that we can use either ds here and the function seems to be set up to use the following ds.

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\,dx$$

Note that we could use the other ds if we wanted to. However, that would require us to solve the equation for x in terms of y. That would, in turn, would give us fractional exponents that would make the derivatives and hence the integral potentially messier.

Therefore, we'll go with our first choice of ds.

Step 2

Now we'll need the derivative of the function.

$$\frac{dy}{dx} = 3x^2$$

Plugging this into the formula for our choice of ds gives,

$$ds = \sqrt{1 + [3x^2]^2}\,dx = \sqrt{1 + 9x^4}\,dx$$

Step 3

Finally, all we need to do is set up the integral. Also note that we have a dx in the formula for ds and so we know that we need x limits of integration which we've been given in the problem statement.

$$SA = \int 2\pi y\,ds = \int_1^5 2\pi y\,\sqrt{1+9x^4}\,dx = \boxed{\int_1^5 2\pi\left(x^3+4\right)\sqrt{1+9x^4}\,dx}$$

Be careful with the formula! Remember that the variable in the integral is always opposite the axis of rotation. In this case we rotated about the x-axis and so we needed a y in the integral.

Finally, with the ds we choose to use for this part we had a dx in the final integral and that means that all the variables in the integral need to be x's. This means that the y from the formula needs to be converted into x's as well. Luckily this is easy enough to do since we were given the formula for y in terms of x in the problem statement.

(b) y-axis

Step 1

We are told that we can use either ds here and the function seems to be set up to use the following ds for the same reasons we choose it in the first part.

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

Step 2

Now, as with the first part of this problem we'll need the derivative of the function and the ds. Here is that work again for reference purposes.

$$\frac{dy}{dx} = 3x^2 \qquad\qquad ds = \sqrt{1 + \left[3x^2\right]^2}\, dx = \sqrt{1 + 9x^4}\, dx$$

Step 3

Finally, all we need to do is set up the integral. Also note that we have a dx in the formula for ds and so we know that we need x limits of integration which we've been given in the problem statement.

$$SA = \int 2\pi x\, ds = \boxed{\int_1^5 2\pi x\, \sqrt{1 + 9x^4}\, dx}$$

Be careful with the formula! Remember that the variable in the integral is always opposite the axis of rotation. In this case we rotated about the y-axis and so we needed an x in the integral.

In this part, unlike the first part, we do not do any substitution for the x in front of the root. Our choice of ds for this part put a dx into the integral and this means we need x's the integral. Since the variable in front of the root was an x we don't need to do any substitution for the variable.

---

4. Find the surface area of the object obtained by rotating $y = 4 + 3x^2$, $1 \le x \le 2$ about the y-axis.

Step 1

The first step here is to decide on a ds to use for the problem. We can use either one, however the function is set up for,

$$ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\,dx$$

Using the other ds will put fractional exponents into the function and make the ds and integral potentially messier so we'll stick with this ds.

Step 2

Let's now set up the ds.

$$\frac{dy}{dx} = 6x \qquad \Rightarrow \qquad ds = \sqrt{1 + [6x]^2}\,dx = \sqrt{1 + 36x^2}\,dx$$

Step 3

The integral for the surface area is,

$$SA = \int 2\pi x\,ds = \int_1^2 2\pi x\,\sqrt{1 + 36x^2}\,dx$$

Note that because we are rotating the function about the y-axis for this problem we need an x in front of the root. Also note that because our choice of ds puts a dx in the integral we need x limits of integration which we were given in the problem statement.

Step 4

Finally, all we need to do is evaluate the integral. That requires a quick Calc I substitution. We'll leave most of the integration details to you to verify since you should be pretty good at Calc I substitutions by this point.

$$SA = \int_1^2 2\pi x\,\sqrt{1+36x^2}\,dx = \frac{\pi}{54}\left(1+36x^2\right)^{\frac{3}{2}}\Big|_1^2 = \boxed{\frac{\pi}{54}\left(145^{\frac{3}{2}} - 37^{\frac{3}{2}}\right) = 88.4864}$$

---

5. Find the surface area of the object obtained by rotating $y = \sin(2x)$, $0 \le x \le \frac{\pi}{8}$ about the x-axis.

Step 1

Note that we actually set this problem up in Part **(a)** of Problem 2. So, we'll just summarize the steps of the set up part of the problem here. If you need to see all the details please check out the work in Problem 2.

Here is ds for this problem.

$$\frac{dy}{dx} = 2\cos(2x) \qquad \Rightarrow \qquad ds = \sqrt{1+4\cos^2(2x)}\,dx$$

The integral for the surface area is,

$$SA = \int_0^{\frac{\pi}{8}} 2\pi\sin(2x)\,\sqrt{1+4\cos^2(2x)}\,dx$$

Step 2

In order to evaluate this integral we'll need the following trig substitution.

$$\cos(2x) = \frac{1}{2}\tan(\theta) \qquad \rightarrow \qquad -2\sin(2x)\,dx = \frac{1}{2}\sec^2(\theta)\,d\theta$$

$$\sqrt{1+4\cos^2(2x)} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)} = |\sec(\theta)|$$

In order to deal with the absolute value bars we'll need to convert the x limits to θ limits. Here's that work.

$$x = 0: \quad \cos(0) = 1 = \frac{1}{2}\tan(\theta) \qquad \rightarrow \qquad \theta = \tan^{-1}(2) = 1.1071$$

$$x = \frac{\pi}{8}: \quad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{1}{2}\tan(\theta) \qquad \rightarrow \qquad \theta = \tan^{-1}\left(\sqrt{2}\right) = 0.9553$$

The corresponding range of θ is $0.9553 \leq \theta \leq 1.1071$. This is in the first quadrant and secant is positive there. Therefore, we can drop the absolute value bars on the secant.

Step 3

Putting all the work from the previous step together gives,

$$SA = \int_{0}^{\frac{\pi}{8}} 2\pi \sin(2x)\,\sqrt{1 + 4\cos^2(2x)}\,dx = -\frac{\pi}{2}\int_{1.1071}^{0.9553} \sec^3(\theta)\;d\theta$$

Step 4

Using the formula for the integral of $\sec^3(\theta)$ we derived in the Integrals Involving Trig Functions we get,

$$\begin{aligned} SA &= -\frac{\pi}{2}\int_{1.1071}^{0.9553} \sec^3(\theta)\;d\theta = -\frac{\pi}{4}\left[\sec(\theta)\tan(\theta) + \ln\left|\sec(\theta) + \tan(\theta)\right|\right]\Bigg|_{1.1071}^{0.9553} \\ &= \boxed{1.8215} \end{aligned}$$

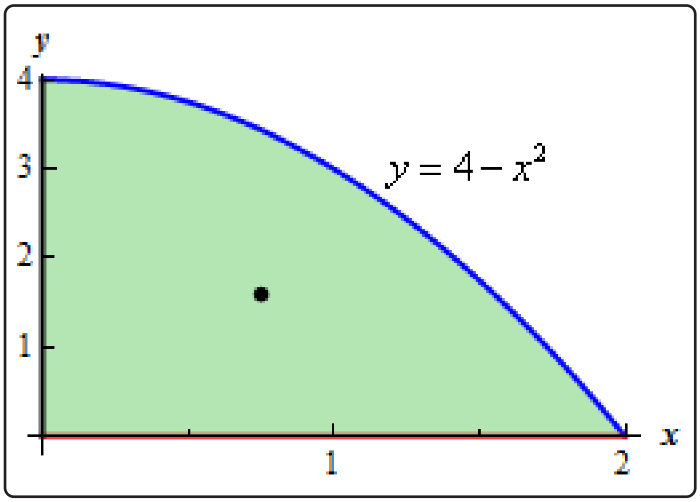
Note that depending upon the number of decimal places you used your answer may be slightly different from that give here. The "exact" answer, obtained by computer, is 1.8222.

8.3 Center Of Mass

1. Find the center of mass for the region bounded by $y = 4 - x^2$ that is in the first quadrant.

Step 1

Let's start out with a quick sketch of the region, with the center of mass indicated by the dot (the coordinates of this dot are of course to be determined in the final step.....).



We'll also need the area of this region so let's find that first.

$$A = \int_0^2 4 - x^2\,dx = \left(4x - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{16}{3}$$

Step 2

Next, we need to compute the two moments. We didn't include the density in the computations below because it will only cancel out in the final step.

$$M_x = \int_0^2 \frac{1}{2}\left(4 - x^2\right)^2 dx = \int_0^2 \frac{1}{2}\left(16 - 8x^2 + x^4\right)\,dx = \frac{1}{2}\left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^2 = \frac{128}{15}$$

$$M_y = \int_0^2 x\left(4 - x^2\right)\,dx = \int_0^2 4x - x^3\,dx = \left(2x^2 - \frac{1}{4}x^4\right)\Big|_0^2 = 4$$

Step 3

Finally, the coordinates of the center of mass is,

$$\overline{x} = \frac{M_y}{M} = \frac{\rho\,(4)}{\rho\left(\frac{16}{3}\right)} = \frac{3}{4} \qquad\qquad \overline{y} = \frac{M_x}{M} = \frac{\rho\left(\frac{128}{15}\right)}{\rho\left(\frac{16}{3}\right)} = \frac{8}{5}$$

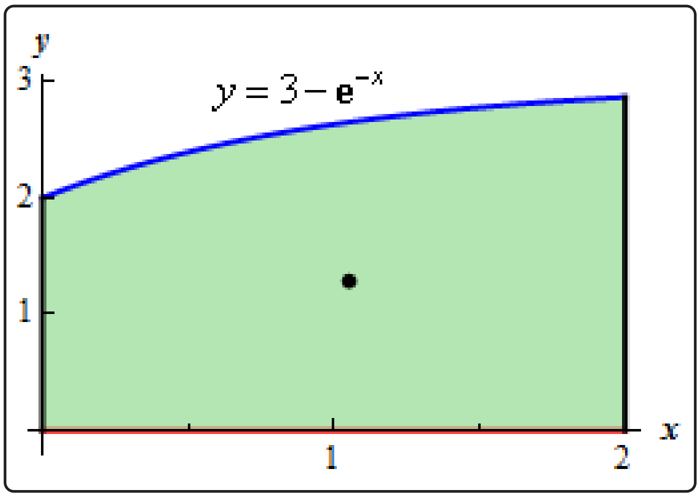
The center of mass is then : $\boxed{\left(\frac{3}{4}, \frac{8}{5}\right)}$.

---

2. Find the center of mass for the region bounded by $y = 3 - \mathbf{e}^{-x}$, the x-axis, $x = 2$ and the y-axis.

Step 1

Let's start out with a quick sketch of the region, with the center of mass indicated by the dot (the coordinates of this dot are of course to be determined in the final step.....).



We'll also need the area of this region so let's find that first.

$$A = \int_0^2 3 - \mathbf{e}^{-x}\,dx = \left(3x + \mathbf{e}^{-x}\right)\Big|_0^2 = 5 + \mathbf{e}^{-2}$$

Step 2

Next, we need to compute the two moments. We didn't include the density in the computations below because it will only cancel out in the final step.

$$M_x = \int_0^2 \frac{1}{2}\left(3 - \mathbf{e}^{-x}\right)^2 dx = \int_0^2 \frac{1}{2}\left(9 - 6\mathbf{e}^{-x} + \mathbf{e}^{-2x}\right)\,dx = \frac{1}{2}\left(9x + 6\mathbf{e}^{-x} - \frac{1}{2}\mathbf{e}^{-2x}\right)\Big|_0^2$$
$$= \frac{25}{4} + 3\mathbf{e}^{-2} - \frac{1}{4}\mathbf{e}^{-4}$$
$$M_y = \int_0^2 x\left(3 - \mathbf{e}^{-x}\right)\,dx = \int_0^2 3x - x\,\mathbf{e}^{-x}\,dx = \left(\frac{3}{2}x^2 + x\mathbf{e}^{-x} + \mathbf{e}^{-x}\right)\Big|_0^2 = 5 + 3\mathbf{e}^{-2}$$

For the second term in the M_y integration we used the following integration by parts.

$$\int x\,\mathbf{e}^{-x}\,dx \qquad u = x \quad du = dx \qquad dv = \mathbf{e}^{-x}dx \quad v = -\mathbf{e}^{-x}$$
$$\int x\,\mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} + \int \mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} - \mathbf{e}^{-x} = -\left(x\mathbf{e}^{-x} + \mathbf{e}^{-x}\right)$$

The minus sign here canceled with the minus sign that was in front of the term in the full integral.

Make sure you don't forget integration by parts! It is a fairly common integration technique for these kinds of problems.

Step 3

Finally, the coordinates of the center of mass is,

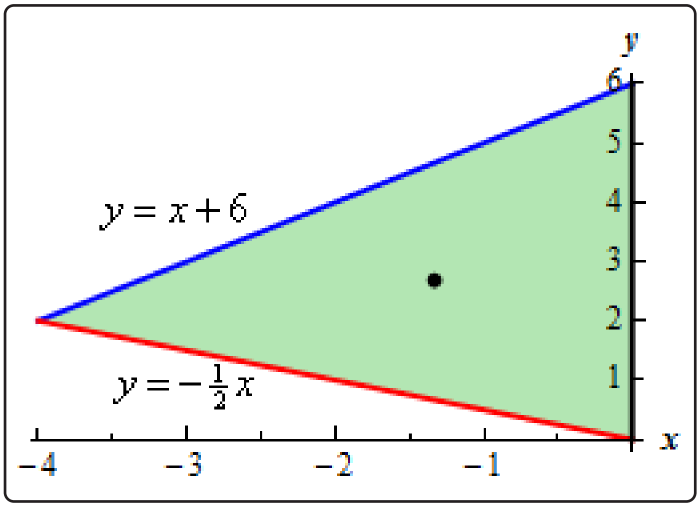
$$\overline{x} = \frac{M_y}{M} = \frac{\rho\left(5 + 3\mathbf{e}^{-2}\right)}{\rho\left(5 + \mathbf{e}^{-2}\right)} = 1.05271 \qquad \overline{y} = \frac{M_x}{M} = \frac{\rho\left(\frac{25}{4} + 3\mathbf{e}^{-2} - \frac{1}{4}\mathbf{e}^{-4}\right)}{\rho\left(5 + \mathbf{e}^{-2}\right)} = 1.29523$$

The center of mass is then : $\boxed{(1.05271,\ 1.29523)}$.

3. Find the center of mass for the triangle with vertices $(0,0)$, $(-4,2)$ and $(0,6)$.

Step 1

Let's start out with a quick sketch of the region, with the center of mass indicated by the dot (the coordinates of this dot are of course to be determined in the final step.....).



We'll leave it to you verify the equations of the upper and lower leg of the triangle.

We'll also need the area of this region so let's find that first.

$$A = \int_{-4}^{0} (x+6) - \left(-\frac{1}{2}x\right)\,dx = \int_{-4}^{0} \frac{3}{2}x + 6\,dx = \left(\frac{3}{4}x^2 + 6x\right)\Big|_{-4}^{0} = 12$$

Step 2

Next, we need to compute the two moments. We didn't include the density in the computations below because it will only cancel out in the final step.

$$M_x = \int_{-4}^{0} \frac{1}{2}\left[(x+6)^2 - \left(-\frac{1}{2}x\right)^2\right]dx = \int_{-4}^{0} \frac{3}{8}x^2 + 6x + 18\,dx$$

$$= \left(\frac{1}{8}x^3 + 3x^2 + 18x\right)\Big|_{-4}^{0} = 32$$

$$M_y = \int_{-4}^{0} x\left((x+6) - \left(-\frac{1}{2}x\right)\right)dx = \int_{-4}^{0} \frac{3}{2}x^2 + 6x\,dx = \left(\frac{1}{2}x^3 + 3x^2\right)\Big|_{-4}^{0} = -16$$

Step 3

Finally, the coordinates of the center of mass is,

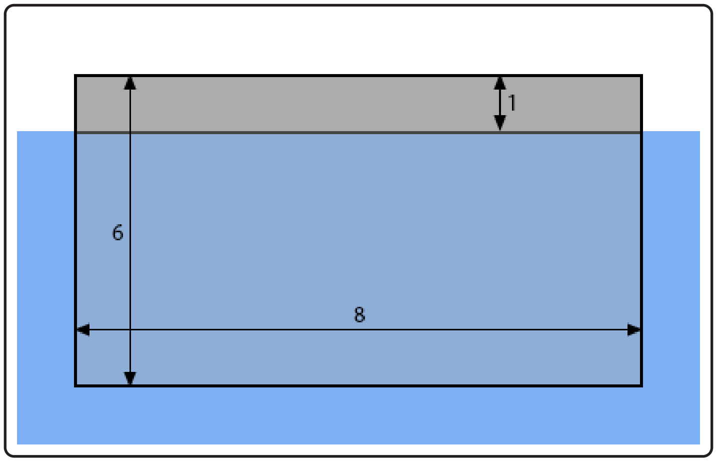
$$\overline{x} = \frac{M_y}{M} = \frac{\rho\,(-16)}{\rho\,(12)} = -\frac{4}{3} \qquad \overline{y} = \frac{M_x}{M} = \frac{\rho\,(32)}{\rho\,(12)} = \frac{8}{3}$$

The center of mass is then : $\boxed{\left(-\frac{4}{3}, \frac{8}{3}\right)}$.

8.4 Hydrostatic Force and Pressure

1. Find the hydrostatic force on the plate submerged in water as shown in the image below.

   Consider the top of the blue "box" to be the surface of the water in which the plate is submerged. Note as well that the dimensions in the image will not be perfectly to scale in order to better fit the plate in the image. The lengths given in the image are in meters.

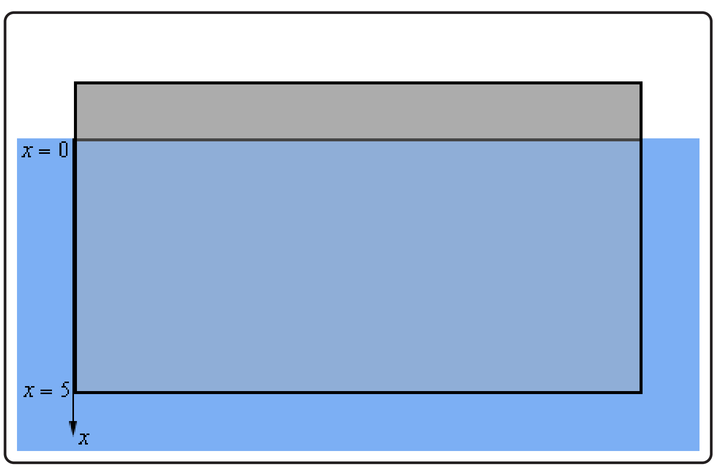


Hint

Start off by defining an "axis system" for the figure.

Step 1

The first thing we should do is define an axis system for the portion of the plate that is below the water.



Note that we started the x-axis at the surface of the water and by doing this x will give the depth of any point on the plate below the surface of the water. This in turn means that the bottom of the plate will be defined by $x = 5$.

It is always useful to define some kind of axis system for the plate to help with the rest of the problem. There are lots of ways to actually define the axis system and how we define them will in turn affect how we work the rest of the problem. There is nothing special about one definition over another but there is often an "easier" axis definition and by "easier' we mean is liable to make some portions of the rest of the work go a little easier.

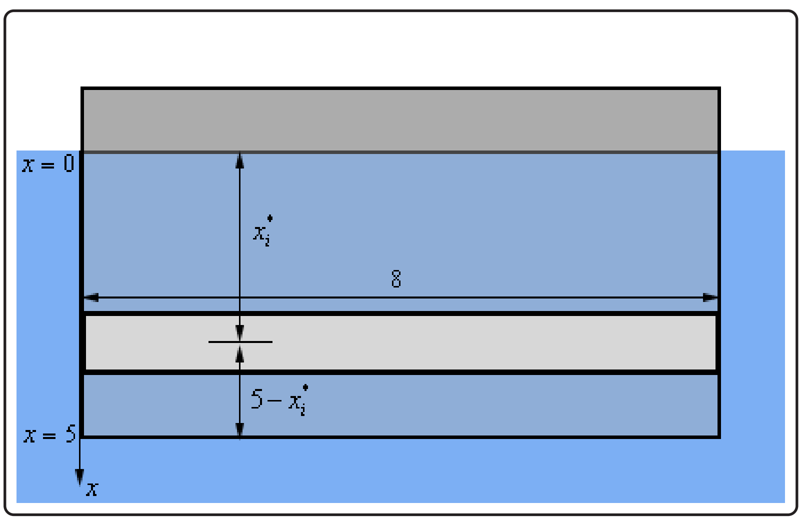
Hint

At this point it would probably be useful to break up the plate into horizontal strips and get a sketch of a representative strip.

Step 2

As we did in the notes we'll break up the portion of the plate that is below the surface of the water into n horizontal strips of width Δx and we'll let each strip be defined by the interval $[x_{i-1},\ x_i]$ with $i = 1, 2, 3, \ldots n$. Finally, we'll let x_i^* be any point that is in the interval and hence will be some point on the strip.

Below is yet another sketch of the plate only this time we've got a representative strip sketched on the plate. Note that the strip is "thicker" than the strip really should be but it will make it easier to see what the strip looks like and get all of the appropriate lengths clearly listed.



Now x_i^* is a point from the interval defining the strip and so, for sufficiently thin strips, it is safe to assume that the strip will be at the point x_i^* below the surface of the water as shown in the figure above. In other words, the strip is a distance of x_i^* below the surface of the water.

Also, because our plate is a rectangle we know that each strip will have a width of 8.

Hint

What is the hydrostatic pressure and force on the representative strip?

Step 3

We'll assume that the strip is sufficiently thin so the hydrostatic pressure on the strip will be constant and is given by,

$$P_i = \rho g d_i = (1000)(9.81)\,x_i^* = 9810x_i^*$$

This, in turn, means that the hydrostatic force on each strip is given by,

$$F_i = P_i A_i = (9810x_i^*)\left[(8)(\Delta x)\right] = 78480x_i^*\Delta x$$

Hint

How can we use the result from the previous step to approximate the total hydrostatic force on the plate and how can we modify that to get an expression for the actual hydrostatic force on the plate?

Step 4

We can now approximate the total hydrostatic force on plate as the sum off the force on each of the strips. Or,

$$F \approx \sum_{i=1}^{n} 78480x_i^*\Delta x$$

Now, we can get an expression for the actual hydrostatic force on the plate simply by letting n go to infinity.

Or in other words, we take the limit as follows,

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 78480x_i^* \Delta x$$

Hint

You do recall the definition of the definite integral don't you?

Step 5

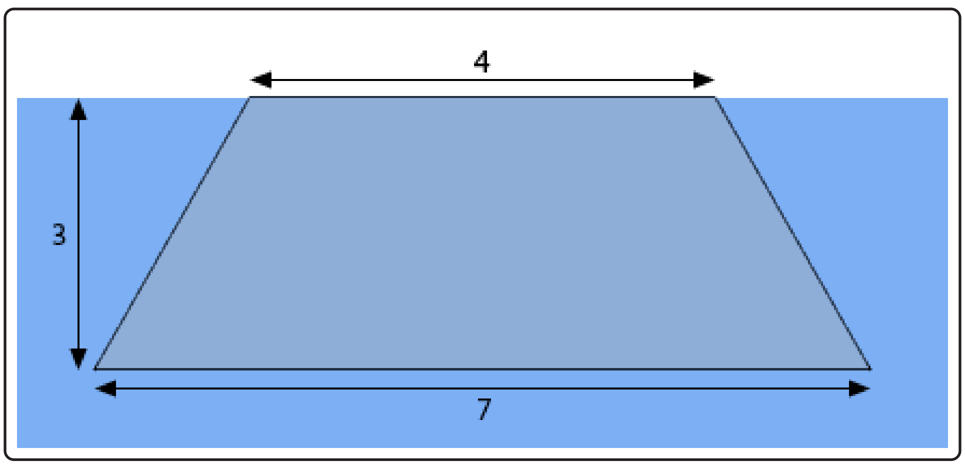
Finally, we know from the definition of the definite integral that this is nothing more than the following definite integral that we can easily compute.

$$F = \int_0^5 78480x \, dx = 39240x^2 \Big|_0^5 = \boxed{981,000N}$$

---

2. Find the hydrostatic force on the plate submerged in water as shown in the image below.

Consider the top of the blue "box" to be the surface of the water in which the plate is submerged. Note as well that the dimensions in the image will not be perfectly to scale in order to better fit the plate in the image. The lengths given in the image are in meters.



Hint

Start off by defining an "axis system" for the figure.

Step 1

The first thing we should do is define an axis system for the portion of the plate that is below the water.

Note that we started the x-axis at the surface of the water and by doing this x will give the depth of any point on the plate below the surface of the water. This in turn means that the bottom of the plate will be defined by $x = 3$.

It is always useful to define some kind of axis system for the plate to help with the rest of the problem. There are lots of ways to actually define the axis system and how we define them will in turn affect how we work the rest of the problem. There is nothing special about one definition over another but there is often an "easier" axis definition and by "easier' we mean is liable to make some portions of the rest of the work go a little easier.

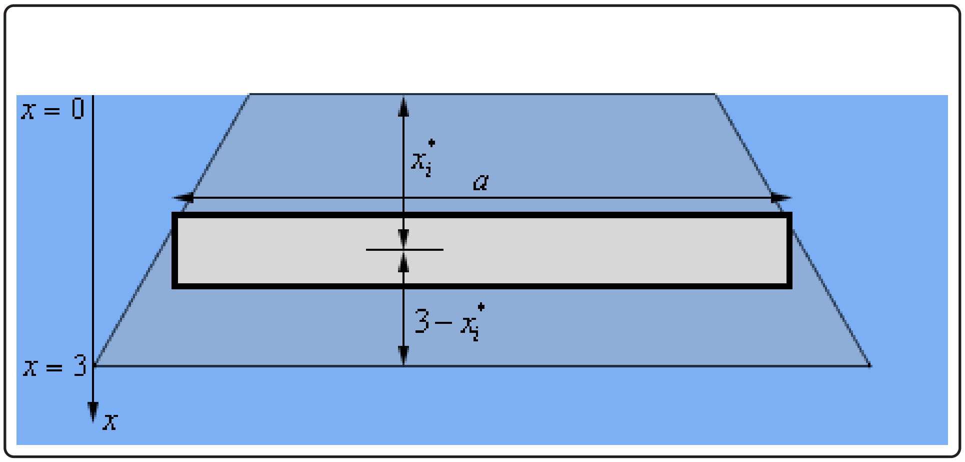
Hint

At this point it would probably be useful to break up the plate into horizontal strips and get a sketch of a representative strip.

Step 2

As we did in the notes we'll break up the plate into n horizontal strips of width Δx and we'll let each strip be defined by the interval $[x_{i-1},\ x_i]$ with $i = 1, 2, 3, \ldots n$. Finally, we'll let x_i^* be any point that is in the interval and hence will be some point on the strip.

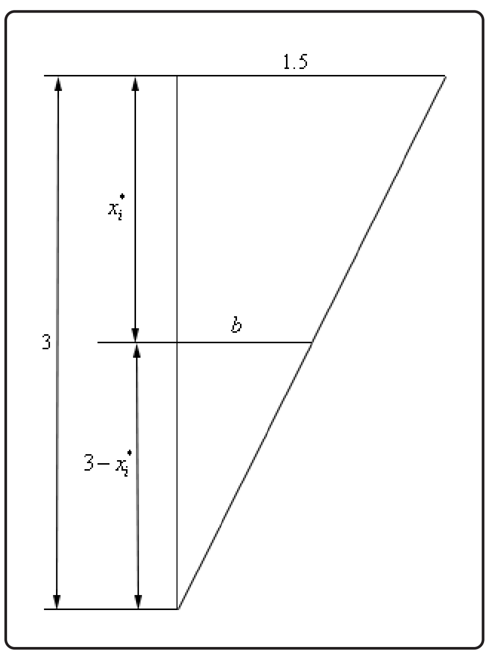
Below is yet another sketch of the plate only this time we've got a representative strip sketched on the plate. Note that the strip is "thicker" than the strip really should be but it will make it easier to see what the strip looks like and get all of the appropriate lengths clearly listed.



Now x_i^* is a point from the interval defining the strip and so, for sufficiently thin strips, it is safe to assume that the strip will be at the point x_i^* below the surface of the water as shown in the figure above. In other words, the strip is a distance of x_i^* below the surface of the water.

The width of each of the strips will be dependent on the depth of the strip and so temporarily let's just call the width a.

To determine the value of a for each strip let's consider the following set of similar triangles.



This is the triangle of "empty" space to the left of the plate. The overall height of the larger triangle is the same as the plate, namely 3. The overall width of the larger triangle is 1.5. We arrived at this number by noticing that the top of the plate was 3 meters shorter than the bottom and if we assume the top was perfectly centered over the bottom there must be 1.5 meters of "empty" space to either side of the top.

The top of the smaller triangle corresponds to the strip on the plate. We'll call the width of the smaller triangle b. and the height of the smaller triangle must be $3 - x_i^*$ for each strip.

Because the two triangles are similar triangles we have the following equation.

$$\frac{b}{3 - x_i^*} = \frac{1.5}{3} \qquad\qquad b = \frac{1}{2}\left(3 - x_i^*\right)$$

Note that while we looked only at the empty space to the left of the plate we'd get an almost identical triangle for the empty space to the right of the plate. The only exception would be that it would be a mirror image of this triangle.

Now, let's get back to the width of the strip in our picture of the plate. Assuming that the top is centered over the bottom of the plate we can see that we have to have,

$$a = 7 - 2b = 7 - 2\left(\frac{1}{2}\right)\left(3 - x_i^*\right) = 4 + x_i^*$$

Hint

What is the hydrostatic pressure and force on the representative strip?

Step 3

We'll assume that the strip is sufficiently thin so the hydrostatic pressure on the strip will be constant and is given by,

$$P_i = \rho g d_i = (1000)(9.81)\,x_i^* = 9810 x_i^*$$

This, in turn, means that the hydrostatic force on each strip is given by,

$$F_i = P_i A_i = \left(9810 x_i^*\right)\left[\left(4 + x_i^*\right)\left(\Delta\, x\right)\right] = 9810\left[4x_i^* + \left(x_i^*\right)^2\right]\Delta\, x$$

Hint

How can we use the result from the previous step to approximate the total hydrostatic force on the plate and how can we modify that to get an expression for the actual hydrostatic force on the plate?

Step 4

We can now approximate the total hydrostatic force on plate as the sum off the force on each of the strips. Or,

$$F \approx \sum_{i=1}^{n} 9810 \left[4x_i^* + (x_i^*)^2\right] \Delta x$$

Now, we can get an expression for the actual hydrostatic force on the plate simply by letting n go to infinity.

Or in other words, we take the limit as follows,

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 9810 \left[4x_i^* + (x_i^*)^2\right] \Delta x$$

Hint

You do recall the definition of the definite integral don't you?

Step 5

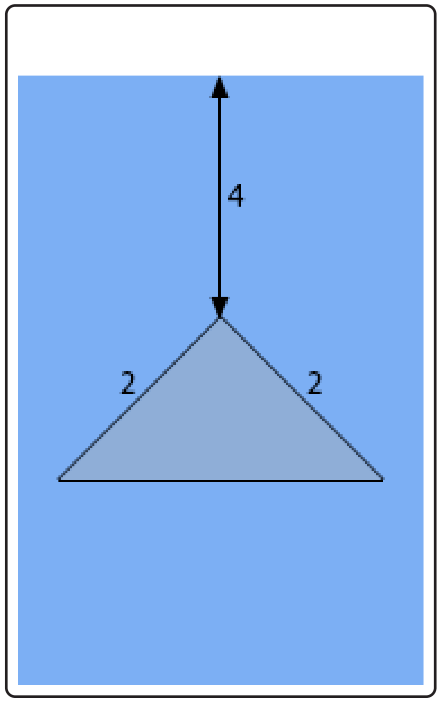
Finally, we know from the definition of the definite integral that this is nothing more than the following definite integral that we can easily compute.

$$F = \int_0^3 9810\left(4x + x^2\right) dx = 9810\left(2x^2 + \frac{1}{3}x^3\right)\Bigg|_0^3 = \boxed{264,870N}$$

---

3. Find the hydrostatic force on the plate submerged in water as shown in the image below. The plate in this case is the top half of a diamond formed from a square whose sides have a length of 2.

   Consider the top of the blue "box" to be the surface of the water in which the plate is submerged. Note as well that the dimensions in the image will not be perfectly to scale in order to better fit the plate in the image. The lengths given in the image are in meters.

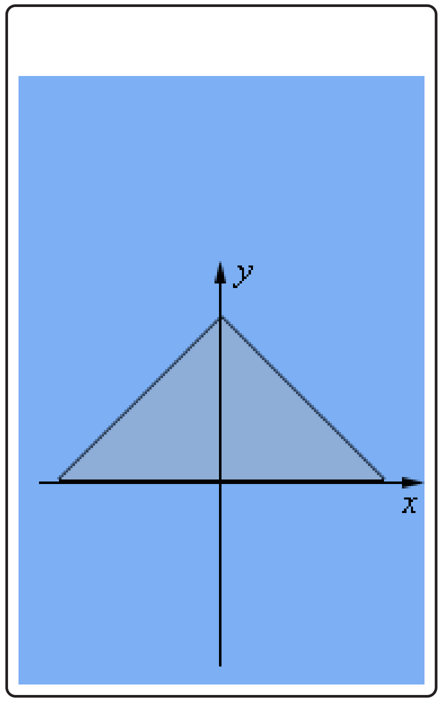


Hint

Start off by defining an "axis system" for the figure.

Step 1

The first thing we should do is define an axis system for the portion of the plate that is below the water.



In this case since we had the top half a diamond formed from a square it seemed convenient to center the axis system in the middle of the diamond.

It is always useful to define some kind of axis system for the plate to help with the rest of the problem. There are lots of ways to actually define the axis system and how we define them will in turn affect how we work the rest of the problem. There is nothing special about one definition over another but there is often an "easier" axis definition and by "easier' we mean is liable to make some portions of the rest of the work go a little easier.

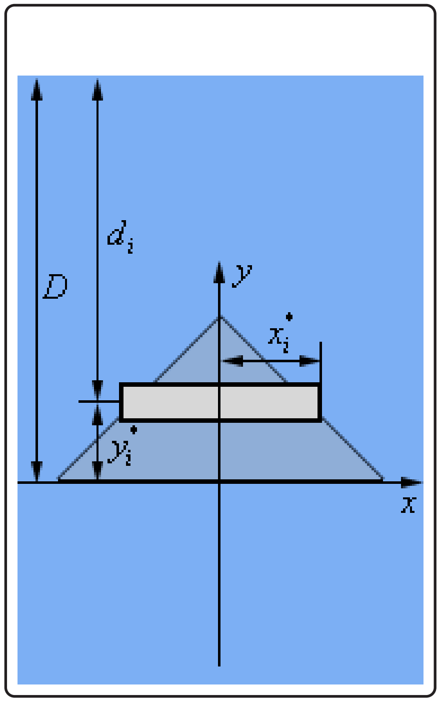
Hint

At this point it would probably be useful to break up the plate into horizontal strips and get a sketch of a representative strip.

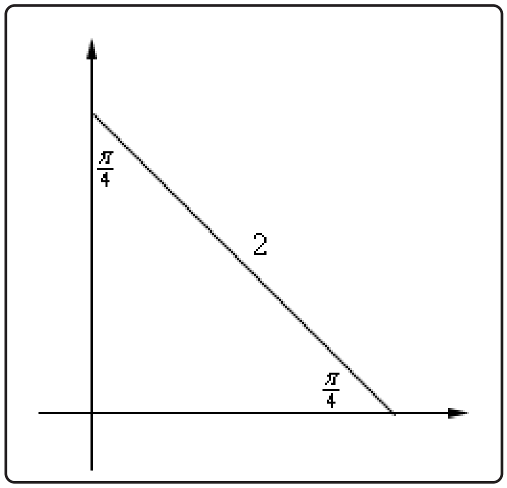
Step 2

As we did in the notes we'll break up the plate into n horizontal strips of width Δy and we'll let each strip be defined by the interval $[y_{i-1},\ y_i]$ with $i = 1, 2, 3, \ldots n$. Finally, we'll let y_i^* be any point that is in the interval and hence will be some point on the strip.

Below is yet another sketch of the plate only this time we've got a representative strip sketched on the plate. Note that the strip is "thicker" than the strip really should be but it will make it easier to see what the strip looks like and get all of the appropriate lengths clearly listed.



We've got a few quantities to determine at this point. To do that it would be convenient to have an equation for one of the sides of the plate. Let's take a look at the side in the first quadrant. Here is a quick sketch of that portion of the plate.



Now, as noted above we have the top half of a diamond formed from a square and so we can see that the "triangle" formed in the first quadrant by the plate must be an isosceles right triangle whose hypotenuse is 2 and the two interior angles other than the right angle must be $\frac{\pi}{4}$. Therefore, the bottom/left side of the triangle must be,

$$\text{side} = 2\cos\left(\frac{\pi}{4}\right) = \sqrt{2}$$

This means we know that the x and y-intercepts are $(\sqrt{2}, 0)$ and $(0, \sqrt{2})$ respectively and so the equation for the line representing the hypotenuse must be,

$$y = \sqrt{2} - x$$

Okay, let's get the various quantities in the figure.

We'll start with x_i^* . This we can get directly from the equation above by acknowledging that if we are at x_i^* then the y value must be y_i^*. In other words, plugging these into the equation and solving gives,

$$y_i^* = \sqrt{2} - x_i^* \qquad \to \qquad x_i^* = \sqrt{2} - y_i^*$$

Notice as well that the width of each strip in terms of y_i^* is then,

$$2x_i^* = 2\left(\sqrt{2} - y_i^*\right)$$

Next, let's get D. First we can see that D is the distance from the surface of the water to the x-axis in our figure. We know that the distance from the surface of the water to the top of the plate is 4 meters. Also, we found above that the top point of the plate is a distance of $\sqrt{2}$ above the x-axis. So, we then have,

$$D = 4 + \sqrt{2}$$

Finally, the depth of each strip below the surface of the water is,

$$d_i = D - y_i^* = 4 + \sqrt{2} - y_i^*$$

Hint

What is the hydrostatic pressure and force on the representative strip?

Step 3

We'll assume that the strip is sufficiently thin so the hydrostatic pressure on the strip will be constant and is given by,

$$P_i = \rho g d_i = (1000)\,(9.81)\left(4 + \sqrt{2} - y_i^*\right) = 9810\left(4 + \sqrt{2} - y_i^*\right)$$

This, in turn, means that the hydrostatic force on each strip is given by,

$$\begin{aligned} F_i = P_i A_i &= \left[9810\left(4 + \sqrt{2} - y_i^*\right)\right]\left[(2x_i^*)\,(\Delta\, y)\right] \\ &= \left[9810\left(4 + \sqrt{2} - y_i^*\right)\right]\left[2\left(\sqrt{2} - y_i^*\right)(\Delta\, y)\right] \\ &= 19620\left(4 + \sqrt{2} - y_i^*\right)\left(\sqrt{2} - y_i^*\right)(\Delta\, y) \end{aligned}$$

Hint

How can we use the result from the previous step to approximate the total hydrostatic force on the plate and how can we modify that to get an expression for the actual hydrostatic force on the plate?

Step 4

We can now approximate the total hydrostatic force on plate as the sum off the force on each of the strips. Or,

$$F \approx \sum_{i=1}^{n} 19620\left(4 + \sqrt{2} - y_i^*\right)\left(\sqrt{2} - y_i^*\right)(\Delta\, y)$$

Now, we can get an expression for the actual hydrostatic force on the plate simply by letting n go to infinity.

Or in other words, we take the limit as follows,

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} 19620 \left(4 + \sqrt{2} - y_i^*\right) \left(\sqrt{2} - y_i^*\right) (\Delta\, y)$$

Hint

You do recall the definition of the definite integral don't you?

Step 5

Finally, we know from the definition of the definite integral that this is nothing more than the following definite integral that we can easily compute.

$$\begin{aligned} F &= \int_0^{\sqrt{2}} 19620 \left(4 + \sqrt{2} - y\right) \left(\sqrt{2} - y\right) dy \\ &= \int_0^{\sqrt{2}} 19620 \left(y^2 - \left(4 + 2\sqrt{2}\right) y + 2 + 4\sqrt{2}\right) dy \\ &= 19620 \left(\frac{1}{3} y^3 - \left(2 + \sqrt{2}\right) y^2 + \left(2 + 4\sqrt{2}\right) y\right)\Bigg|_0^{\sqrt{2}} = \boxed{96,977.9N} \end{aligned}$$

8.5 Probability

1. Let,

$$f(x) = \begin{cases} \dfrac{3}{37760}x^2(20-x) & \text{if } 2 \le x \le 18 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $f(x)$ is a probability density function.

(b) Find $P(X \le 7)$.

(c) Find $P(X \ge 7)$.

(d) Find $P(3 \le X \le 14)$.

(e) Determine the mean value of X.

Solutions

(a) Show that $f(x)$ is a probability density function.

Solution

Okay, to show that this function is a probability density function we can first notice that in the range $2 \le x \le 18$ the function is positive and will be zero everywhere else and so the first condition is satisfied.

The main thing that we need to do here is show that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{2}^{18} \frac{3}{37760}x^2(20-x)\,dx \\ &= \frac{3}{37760}\int_{2}^{18} 20x^2 - x^3\,dx = \frac{3}{37760}\left(\frac{20}{3}x^3 - \frac{1}{4}x^4\right)\Big|_{2}^{18} = 1\end{aligned}$$

The integral is one and so this is in fact a probability density function.

(b) Find $P(X \le 7)$.

Solution

First note that because of our limits on x for which the function is not zero this is equivalent to $P(2 \le X \le 7)$. Here is the work for this problem.

$$P(X \le 7) = P(2 \le X \le 7) = \int_2^7 \frac{3}{37760} x^2 (20 - x)\, dx$$
$$= \frac{3}{37760}\left(\frac{20}{3}x^3 - \frac{1}{4}x^4\right)\Big|_2^7 = \boxed{0.130065}$$

Note that we made use of the fact that we've already done the indefinite integral itself in the first part. All we needed to do was change limits from that part to match the limits for this part.

(c) Find $P(X \ge 7)$.

Solution

First note that because of our limits on x for which the function is not zero this is equivalent to $P(7 \le X \le 18)$. Here is the work for this problem.

$$P(X \ge 7) = P(7 \le X \le 18) = \int_7^{18} \frac{3}{37760} x^2 (20 - x)\, dx$$
$$= \frac{3}{37760}\left(\frac{20}{3}x^3 - \frac{1}{4}x^4\right)\Big|_7^{18} = \boxed{0.869935}$$

Note that we made use of the fact that we've already done the indefinite integral itself in the first part. All we needed to do was change limits from that part to match the limits for this part.

(d) Find $P(3 \le X \le 14)$.

Solution

Not much to do here other than compute the integral.

$$P(3 \le X \le 14) = \int_3^{14} \frac{3}{37760} x^2 (20 - x)\, dx = \frac{3}{37760}\left(\frac{20}{3}x^3 - \frac{1}{4}x^4\right)\Big|_3^{14}$$
$$= \boxed{0.677668}$$

Note that we made use of the fact that we've already done the indefinite integral itself in the first part. All we needed to do was change limits from that part to match the limits for this part.

(e) Determine the mean value of X.

Solution

For this part all we need to do is compute the following integral.

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{2}^{18} x\left[\frac{3}{37760}x^2(20-x)\right]dx \\ &= \frac{3}{37760}\int_{2}^{18} 20x^3 - x^4\, dx = \frac{3}{37760}\left(5x^4 - \frac{1}{5}x^5\right)\Bigg|_{2}^{18} = 11.6705\end{aligned}$$

The mean value of X is then 11.6705.

---

2. For a brand of light bulb the probability density function of the life span of the light bulb is given by the function below, where t is in months.

$$f(t) = \begin{cases} 0.04\mathbf{e}^{-\frac{t}{25}} & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases}$$

(a) Verify that $f(t)$ is a probability density function.

(b) What is the probability that a light bulb will have a life span less than 8 months?

(c) What is the probability that a light bulb will have a life span more than 20 months?

(d) What is the probability that a light bulb will have a life span between 14 and 30 months?

(e) Determine the mean value of the life span of the light bulbs.

Solutions

(a) Verify that $f(t)$ is a probability density function.

Solution

Okay, to show that this function is a probability density function we can first notice that the exponential portion is always positive regardless of the value of t we plug in and the remainder of the function is always zero and so the first condition is satisfied.

The main thing that we need to do here is show that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} 0.04\mathbf{e}^{-\frac{t}{25}}\,dt = \lim_{n\to\infty}\int_{0}^{n} 0.04\mathbf{e}^{-\frac{t}{25}}\,dt$$
$$= \lim_{n\to\infty}\left(-\mathbf{e}^{-\frac{t}{25}}\right)\Big|_{0}^{n} = \lim_{n\to\infty}\left(-\mathbf{e}^{-\frac{n}{25}}+1\right) = 0+1 = 1$$

The integral is one and so this is in fact a probability density function.

For this integral do not forget to properly deal with the infinite limit! If you don't recall how to deal with these kinds of integrals go back to the Improper Integral section and do a quick review!

(b) What is the probability that a light bulb will have a life span less than 8 months?

Solution

What this problem is really asking us to compute is $P(X \le 8)$. Also, because of our limits on t for which the function is not zero this is equivalent to $P(0 \le X \le 8)$. Here is the work for this problem.

$$P(X \le 8) = P(0 \le X \le 8) = \int_{0}^{8} 0.04\mathbf{e}^{-\frac{t}{25}}\,dt = -\mathbf{e}^{-\frac{t}{25}}\Big|_{0}^{8} = \boxed{0.273851}$$

(c) What is the probability that a light bulb will have a life span more than 20 months?

Solution

What this problem is really asking us to compute is $P(X \ge 20)$. Here is the work for this problem.

$$P(X \ge 20) = \int_{20}^{\infty} 0.04\mathbf{e}^{-\frac{t}{25}}\,dt = \lim_{n\to\infty}\int_{20}^{n} 0.04\mathbf{e}^{-\frac{t}{25}}\,dt$$
$$= \lim_{n\to\infty}\left(-\mathbf{e}^{-\frac{t}{25}}\right)\Big|_{20}^{n} = \lim_{n\to\infty}\left(-\mathbf{e}^{-\frac{n}{25}}+\mathbf{e}^{-\frac{20}{25}}\right) = \boxed{0.449329}$$

For this integral do not forget to properly deal with the infinite limit! If you don't recall how to deal with these kinds of integrals go back to the Improper Integral section and do a quick review!

(d) What is the probability that a light bulb will have a life span between 14 and 30 months?

Solution

What this problem is really asking us to compute is $P(14 \le X \le 30)$. Here is the work for this problem.

$$P(14 \le X \le 30) = \int_{14}^{30} 0.04\mathbf{e}^{-\frac{t}{25}}\,dt = -\mathbf{e}^{-\frac{t}{25}}\Big|_{14}^{30} = \boxed{0.270015}$$

(e) Determine the mean value of the life span of the light bulbs.

Solution

For this part all we need to do is compute the following integral.

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} t\,f(t)\,dt = \int_0^{\infty} 0.04t\mathbf{e}^{-\frac{t}{25}}\,dt = \lim_{n\to\infty}\int_0^n 0.04t\mathbf{e}^{-\frac{t}{25}}\,dt \\ &= \lim_{n\to\infty}\left[-t\mathbf{e}^{-\frac{t}{25}} - 25\mathbf{e}^{-\frac{t}{25}}\right]\Big|_0^n = \lim_{n\to\infty}\left[-n\mathbf{e}^{-\frac{n}{25}} - 25\mathbf{e}^{-\frac{n}{25}} - (-25)\right] \\ &= \lim_{n\to\infty}\left[-\frac{n}{\mathbf{e}^{\frac{n}{25}}} - 25\mathbf{e}^{-\frac{n}{25}} + 25\right] = \lim_{n\to\infty}\left[-\frac{1}{\frac{1}{25}\mathbf{e}^{\frac{n}{25}}}\right] - 25(0) + 25 = \boxed{25}\end{aligned}$$

The mean value of the life span of the light bulbs is then 25 months.

We had to use integration by parts to do the integral. Here is that work if you need to see it.

$$u = 0.04t \quad du = 0.04dt \qquad dv = \mathbf{e}^{-\frac{t}{25}}dt \quad v = -25\mathbf{e}^{-\frac{t}{25}}$$

$$\int 0.04t\mathbf{e}^{-\frac{t}{25}}\,dt = -t\mathbf{e}^{-\frac{t}{25}} + \int \mathbf{e}^{-\frac{t}{25}}\,dt = -t\mathbf{e}^{-\frac{t}{25}} - 25\mathbf{e}^{-\frac{t}{25}}$$

Also, for the limit of the first term we used L'Hospital's Rule to do the limit.

---

3. Determine the value of c for which the function below will be a probability density function.

$$f(x) = \begin{cases} c\left(8x^3 - x^4\right) & \text{if } 0 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$$

Solution

This problem is actually easier than it might look like at first glance.

First, in order for the function to be a probability density function we know that the function must be positive or zero for all x. We can see that for $0 \le x \le 8$ we have $8x^3 - x^4 \ge 0$. Therefore, we need to require that whatever c is it must be a positive number.

To find c we'll use the fact that we must also have $\int_{-\infty}^{\infty} f(x)\, dx = 1$. So, let's compute this integral (with the c in the function) and see what we get.

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^8 c\left(8x^3 - x^4\right) dx = c\left(2x^4 - \frac{1}{5}x^5\right)\Big|_0^8 = \frac{8192}{5}c$$

So, we can see that in order for this integral to have a value of 1 (as it must in order for the function to be a probability density function) we must have,

$$\boxed{c = \frac{5}{8192}}$$

9 Parametric Equations and Polar Coordinates

We are now going to take a look at a couple of topics that are completely different from anything we've seen to this point. That does not mean, however, that We can just forget everything that we've seen to this point. As we will see before too long we will still need to be able to do a large part of the material (both Calculus I and Calculus II material) that we've looked at to this point.

The first major topic that we'll look at in this chapter will be that of Parametric Equations. Parametric Equations will allow us to work with and perform Calculus operations on equations that cannot be solved into the form $y = f(x)$ or $x = h(y)$ (assuming we are using x and y as our variables). Also, as we'll see we can write some equations that can be solved for y or x as a set of easier to work with parametric equations.

Once we've got an idea of what parametric equations are and how to sketch graphs of them we will revisit some of the Calculus topics we've looked at to this point. We'll take a look at how to use only parametric equations to get the equation of tangent lines, where the graph is increasing/decreasing and the concavity of the graph. In addition, we'll revisit the idea of using a definite integral to find the area between the graph of a set of parametric equation and the x-axis. We will close out the Calculus topics by discussing arc length and surface area for a set of parametric equations.

We will then move into the other major topic of this chapter, namely Polar Coordinates. Once we've defined polar coordinates and gotten comfortable with them we will, again, revisit the same Calculus topics we looked at in terms of parametric equations.

On the surface it will appear that polar coordinates has nothing in common with parametric equations. We will see however that several topics in Polar Coordinates can be easily done, in some way, if we first set them up in terms of parametric equations.

In addition, we should point out that the purpose of the topics in this chapter is in preparation for multi-variable Calculus (*i.e.* the material that is usually taught in Calculus III). As we will see when we get to that point there are a lot of topics that involve and/or require parametric equations. In addition, polar coordinates will pop up every so often so keep that in mind as we go through this stuff. It is easy sometimes to get the idea that the topics in this chapter don't have a lot of use but once we hit multi-variable Calculus they will start to pop up with some regularity.

The following sections are the practice problems, with solutions, for this material.

9.1 Parametric Equations and Curves

1. Eliminate the parameter for the following set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on x and y.

$$x = 4 - 2t \qquad y = 3 + 6t - 4t^2$$

Step 1

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we can do that by solving the x equation for t and plugging that into the y equation.

Doing that gives (we'll leave it to you to verify all the algebra bits...),

$$t = \frac{1}{2}(4 - x) \qquad \to \qquad y = 3 + 6\left[\frac{1}{2}(4 - x)\right] - 4\left[\frac{1}{2}(4 - x)\right]^2 = -x^2 + 5x - 1$$

Step 2

Okay, from this it looks like we have a parabola that opens downward. To sketch the graph of this we'll need the x-intercepts, y-intercept and most importantly the vertex.

For notational purposes let's define $f(x) = -x^2 + 5x - 1$.

The x-intercepts are then found by solving $f(x) = 0$. Doing this gives,

$$-x^2 + 5x - 1 = 0 \qquad \to \qquad x = \frac{-5 \pm \sqrt{(5)^2 - 4(-1)(-1)}}{2(-1)} = \frac{5 \pm \sqrt{21}}{2} = 0.2087, \;\; 4.7913$$

The y-intercept is : $(0, f(0)) = (0, -1)$.

Finally, the vertex is,

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(\frac{-5}{2(-1)}, f\left(\frac{5}{2}\right)\right) = \left(\frac{5}{2}, \frac{21}{4}\right)$$

Step 3

Before we sketch the graph of the parametric curve recall that all parametric curves have a direction of motion, *i.e.* the direction indicating increasing values of the parameter, t in this case.

There are several ways to get the direction of motion for the curve. One is to plug in values of t into the parametric equations to get some points that we can use to identify the direction of motion.

Here is a table of values for this set of parametric equations.

t	x	y
-1	6	-7
0	4	3
$\frac{3}{4}$	$\frac{5}{2}$	$\frac{21}{4}$
1	2	5
2	0	-1
3	-2	-15

Note that $t = \frac{3}{4}$ is the value of t that give the vertex of the parabola and is not an obvious value of t to use! In fact, this is a good example of why just using values of t to sketch the graph is such a bad way of getting the sketch of a parametric curve. It is often very difficult to determine a good set of t's to use.

For this table we first found the vertex t by using the fact that we actually knew the coordinates of the vertex (the x-coordinate for this example was the important one) as follows,

$$x = \frac{5}{2} \quad : \quad \frac{5}{2} = 4 - 2t \qquad \rightarrow \qquad t = \frac{3}{4}$$

Once this value of t was found we chose several values of t to either side for a good representation of t for our sketch.

Note that, for this case, we used the x-coordinates to find the value of the t that corresponds to the vertex because this equation was a linear equation and there would be only one solution for t. Had we used the y-coordinate we would have had to solve a quadratic (not hard to do of course) that would have resulted in two t's. The problem is that only one t gives the vertex for this problem and so we'd need to then check them in the x equation to determine the correct one. So, in this case we might as well just go with the x equation from the start.

Also note that there is an easier way (probably - it will depend on you of course) to determine direction of motion. Take a quick look at the x equation.

$$x = 4 - 2t$$

Because of the minus sign in front of the t we can see that as t increases x must decrease (we can verify with a quick derivative/Calculus I analysis if we want to). This means that the graph must be tracing out from right to left as the table of values above in the table also indicates.

Using a quick Calculus analysis of one, or both, of the parametric equations is often a better and easier method for determining the direction of motion for a parametric curve. For "simple" parametric equations we can often get the direction based on a quick glance at the parametric equations and it avoids having to pick "nice" values of t for a table.

Step 4

We could sketch the graph at this point, but let's first get any limits on x and y that might exist.

Because we have a parabola that opens downward and we've not restricted t's in any way we know that we'll get the whole parabola. This in turn means that we won't have any limits at all on x but y must satisfy $y \le \frac{21}{4}$ (remember the y-coordinate of the vertex?).

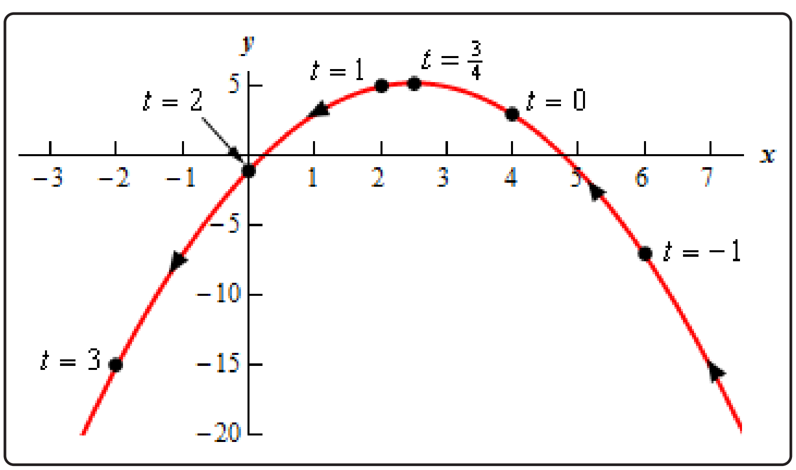
So, formally here are the limits on x and y.

$$-\infty < x < \infty \qquad\qquad y \le \frac{21}{4}$$

Note that having the limits on x and y will often help with the actual graphing step so it's often best to get them prior to sketching the graph. In this case they don't really help as we can sketch the graph of a parabola without these limits, but it's just good habit to be in so we did them first anyway.

Step 5

Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included the points from our table because we had them but we won't always include them as we are often only interested in the sketch itself and the direction of motion.

2. Eliminate the parameter for the following set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on x and y.

$$x = 4 - 2t \qquad y = 3 + 6t - 4t^2 \qquad 0 \le t \le 3$$

Step 1

Before we get started on this problem we should acknowledge that this problem is really just a restriction on the first problem (*i.e.* it is the same problem except we restricted the values of t to use). As such we could just go back to the first problem and modify the sketch to match the restricted values of t to get a quick solution and in general that is how a problem like this would work.

However, we're going to approach this solution as if this was its own problem because we won't always have the more general problem worked ahead of time. So, let's proceed with the problem assuming we haven't worked the first problem in this section.

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we can do that by solving the x equation for t and plugging that into the y equation.

Doing that gives (we'll leave it to you to verify all the algebra bits...),

$$t = \frac{1}{2}(4 - x) \qquad \to \qquad y = 3 + 6\left[\frac{1}{2}(4 - x)\right] - 4\left[\frac{1}{2}(4 - x)\right]^2 = -x^2 + 5x - 1$$

Step 2

Okay, from this it looks like we have a parabola that opens downward. To sketch the graph of this we'll need the x-intercepts, y-intercept and most importantly the vertex.

For notational purposes let's define $f(x) = -x^2 + 5x - 1$.

The x-intercepts are then found by solving $f(x) = 0$. Doing this gives,

$$-x^2 + 5x - 1 = 0 \qquad \to \qquad x = \frac{-5 \pm \sqrt{(5)^2 - 4(-1)(-1)}}{2(-1)} = \frac{5 \pm \sqrt{21}}{2} = 0.2087,\ \ 4.7913$$

The y-intercept is : $(0, f(0)) = (0, -1)$.

Finally, the vertex is,

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(\frac{-5}{2(-1)}, f\left(\frac{5}{2}\right)\right) = \left(\frac{5}{2}, \frac{21}{4}\right)$$

Step 3

Before we sketch the graph of the parametric curve recall that all parametric curves have a direction of motion, *i.e.* the direction indicating increasing values of the parameter, t in this case.

There are several ways to get the direction of motion for the curve. One is to plug in values of t into the parametric equations to get some points that we can use to identify the direction of motion.

Here is a table of values for this set of parametric equations. Also note that because we've restricted the value of t for this problem we need to keep that in mind as we chose values of t to use.

t	x	y
0	4	3
$\frac{3}{4}$	$\frac{5}{2}$	$\frac{21}{4}$
1	2	5
2	0	-1
3	-2	-15

Note that $t = \frac{3}{4}$ is the value of t that give the vertex of the parabola and is not an obvious value of t to use! In fact, this is a good example of why just using values of t to sketch the graph is such a bad way of getting the sketch of a parametric curve. It is often very difficult to determine a good set of t's to use.

For this table we first found the vertex t by using the fact that we actually knew the coordinates of the vertex (the x-coordinate for this example was the important one) as follows,

$$x = \frac{5}{2} \quad : \quad \frac{5}{2} = 4 - 2t \qquad \rightarrow \qquad t = \frac{3}{4}$$

Once this value of t was found we chose several values of t to either side for a good representation of t for our sketch.

Note that, for this case, we used the x-coordinates to find the value of the t that corresponds to the vertex because this equation was a linear equation and there would be only one solution for t. Had we used the y-coordinate we would have had to solve a quadratic (not hard to do of course) that would have resulted in two t's. The problem is that only one t gives the vertex for this problem and so we'd need to then check them in the x equation to determine the correct one. So, in this case we might as well just go with the x equation from the start.

Also note that there is an easier way (probably - it will depend on you of course) to determine direction of motion. Take a quick look at the x equation.

$$x = 4 - 2t$$

Because of the minus sign in front of the t we can see that as t increases x must decrease (we can verify with a quick derivative/Calculus I analysis if we want to). This means that the graph must be tracing out from right to left as the table of values above in the table also indicates.

Using a quick Calculus analysis of one, or both, of the parametric equations is often a better and easier method for determining the direction of motion for a parametric curve. For "simple" parametric equations we can often get the direction based on a quick glance at the parametric equations and it avoids having to pick "nice" values of t for a table.

Step 4

Let's now get the limits on x and y and note that we really do need these before we start sketching the curve!

In this case we have a parabola that opens downward and we could use that to get a general set of limits on x and y. However, for this problem we've also restricted the values of t that we're using and that will in turn restrict the values of x and y that we can use for the sketch of the graph.

As we discussed above we know that the graph will sketch out from right to left and so the rightmost value of x will come from $t = 0$, which is $x = 4$. Likewise, the leftmost value of y will come from $t = 3$, which is $x = -2$. So, from this we can see the limits on x must be $-2 \le x \le 4$.

For the limits on the y we've got be a little more careful. First, we know that the vertex occurs in the given range of t's and because the parabola opens downward the largest value of y we will have is $y = \frac{21}{4}$, *i.e.* the y-coordinate of the vertex. Also, because the parabola opens downward we know that the smallest value of y will have to be at one of the endpoints. So, for $t = 0$ we have $y = 3$ and for $t = 3$ we have $y = -15$. Therefore, the limits on y must be $-15 \le y \le \frac{21}{4}$.

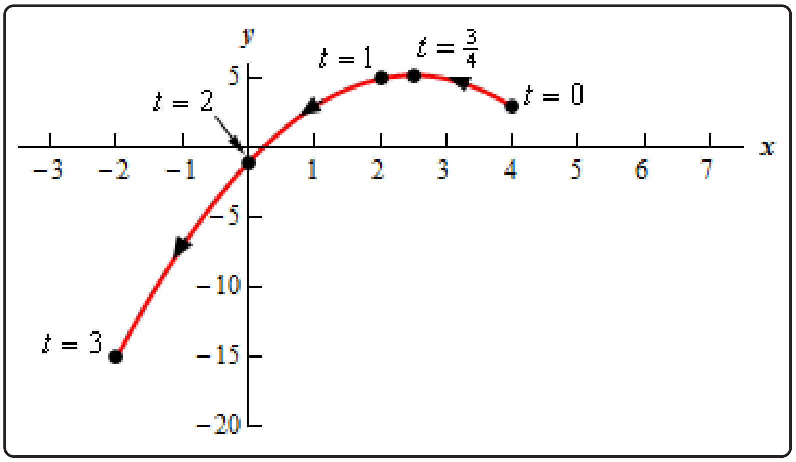
So, putting all this together here are the limits on x and y.

$$-2 < x < 4 \qquad -15 \le y \le \frac{21}{4}$$

Note that for this problem we must have these limits prior to the sketching step. Because we've restricted the values of t to use we will have limits on x and y (as we just discussed) and so we will only have a portion of the graph of the full parabola. Having these limits will allow us to get the sketch of the parametric curve.

Step 5

Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included the points from our table because we had them but we won't always include them as we are often only interested in the sketch itself and the direction of motion.

Also note that it is vitally important that we not extend the graph past the $t = 0$ and $t = 3$ points. If we extend the graph past these points we are implying that the graph will extend past them and of course it doesn't!

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3. Eliminate the parameter for the following set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on x and y.

$$x = \sqrt{t+1} \qquad y = \frac{1}{t+1} \qquad t > -1$$

Step 1

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations that is actually really easy to do if we notice the following.

$$x = \sqrt{t+1} \qquad \Rightarrow \qquad x^2 = t+1$$

With this we can quickly convert the y equation to,

$$y = \frac{1}{x^2}$$

Step 2

At this point we can get limits on x and y pretty quickly so let's do that.

First, we know that square roots always return positive values (or zero of course) and so from the x equation we see that we must have $x > 0$. Note as well that this must be a strict inequality because the inequality restricting the range of t's is also a strict inequality. In other words, because we aren't allowing $t = -1$ we will never get $x = 0$.

Speaking of which, you do see why we've restricted the t's don't you?

Now, from our restriction on t we know that $t + 1 > 0$ and so from the y parametric equation we can see that we also must have $y > 0$. This matches what we see from the equation without the parameter we found in Step 1.

So, putting all this together here are the limits on x and y.

$$x > 0 \qquad y > 0$$

Note that for this problem these limits are important (or at least the x limits are important). Because of the x limit we get from the parametric equation we can see that we won't have the full graph of the equation we found in the first step. All we will have is the portion that corresponds to $x > 0$.

Step 3

Before we sketch the graph of the parametric curve recall that all parametric curves have a direction of motion, *i.e.* the direction indicating increasing values of the parameter, t in this case.

There are several ways to get the direction of motion for the curve. One is to plug in values of t into the parametric equations to get some points that we can use to identify the direction of motion.

Here is a table of values for this set of parametric equations.

t	x	y
-0.95	0.2236	20
-0.75	0.5	4
0	1	1
2	$\sqrt{3}$	$\frac{1}{3}$

Note that there is an easier way (probably - it will depend on you of course) to determine direction of motion. Take a quick look at the x equation.

$$x = \sqrt{t+1}$$

Increasing the value of t will also cause $t + 1$ to increase and the square root will also increase (we can verify with a quick derivative/Calculus I analysis if we want to). This means that the graph must be tracing out from left to right as the table of values above in the table supports.

Likewise, we could use the y equation.

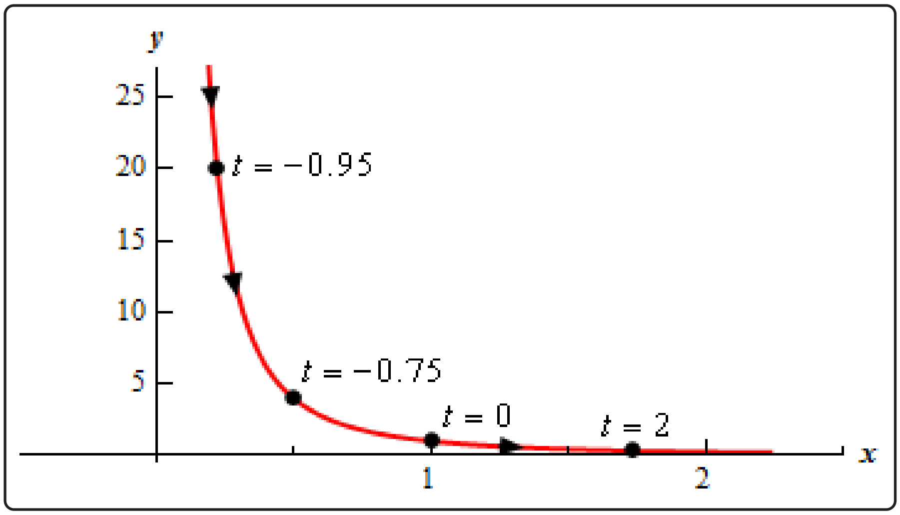
$$y = \frac{1}{t+1}$$

Again, we know that as t increases so does $t + 1$. Because the $t + 1$ is in the denominator we can further see that increasing this will cause the fraction, and hence y, to decrease. This means that the graph must be tracing out from top to bottom as both the x equation and table of values supports.

Using a quick Calculus analysis of one, or both, of the parametric equations is often a better and easier method for determining the direction of motion for a parametric curve. For "simple" parametric equations we can often get the direction based on a quick glance at the parametric equations and it avoids having to pick "nice" values of t for a table.

Step 4

Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included the points from our table because we had them but we won't always include them as we are often only interested in the sketch itself and the direction of motion.

4. Eliminate the parameter for the following set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on x and y.

$$x = 3\sin(t) \qquad y = -4\cos(t) \qquad 0 \le t \le 2\pi$$

Step 1

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we will make use of the well-known trig identity,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

We can solve each of the parametric equations for sine and cosine as follows,

$$\sin(t) = \frac{x}{3} \qquad \cos(t) = -\frac{y}{4}$$

Plugging these into the trig identity gives,

$$\left(-\frac{y}{4}\right)^2 + \left(\frac{x}{3}\right)^2 = 1 \qquad \Rightarrow \qquad \frac{x^2}{9} + \frac{y^2}{16} = 1$$

Therefore, the parametric curve will be some or all of the ellipse above.

We have to be careful when eliminating the parameter from a set of parametric equations. The graph of the resulting equation in only x and y may or may not be the graph of the parametric curve. Often, although not always, the parametric curve will only be a portion of the curve from the equation in terms of only x and y. Another situation that can happen is that the parametric curve will retrace some or all of the curve from the equation in terms of only x and y more than once.

The next few steps will help us to determine just how much of the ellipse we have and if it retraces the ellipse, or a portion of the ellipse, more than once.

Before we proceed with the rest of the problem let's fist note that there is really no set order for doing the steps. They can often be done in different orders and in some cases may actually be easier to do in different orders. The order we'll be following here is used simply because it is the order that I'm used to working them in. If you find a different order would be best for you then that is the order you should use.

Step 2

At this point we can get a good idea on what the limits on x and y are going to be so let's do that. Note that often we won't get the actual limits on x and y in this step. All we are really finding here is the largest possible range of limits for x and y. Having these can sometimes be useful for later steps and so we'll get them here.

We can use our knowledge of sine and cosine to get the following inequalities. Note as discussed above however that these may not be the limits on x and y we are after.

$$\begin{array}{cc} -1 \le \sin(t) \le 1 & -1 \le \cos(t) \le 1 \\ -3 \le 3\sin(t) \le 3 & 4 \ge -4\cos(t) \ge -4 \\ -3 \le x \le 3 & -4 \le y \le 4 \end{array}$$

Note that to find these limits in general we just start with the appropriate trig function and then build up the equation for x and y by first multiplying the trig function by any coefficient, if present, and then adding/subtracting any numbers that might be present (not needed in this case). This, in turn, gives us the largest possible set of limits for x and y. Just remember to be careful when multiplying an inequality by a negative number. Don't forget to flip the direction of the inequalities when doing this.

Now, at this point we need to be a little careful. What we've actually found here are the largest possible inequalities for the limits on x and y. This set of inequalities for the limits on x and y assume that the parametric curve will be completely traced out at least once for the range of t's we were given in the problem statement. It is always possible that the curve will not trace out a full trace in the given range of t's. In a later step we'll determine if the parametric curve does trace out a full trace and hence determine the actual limits on x and y.

Before we move onto the next step there are a couple of issues we should quickly discuss.

First, remember that when we talk about the parametric curve tracing out once we are not necessarily talking about the ellipse itself being fully traced out. The parametric curve will be at most the full ellipse and we haven't determined just yet how much of the ellipse the parametric curve will trace out. So, one trace of the parametric curve refers to the largest portion of the ellipse that the parametric curve can possibly trace out given no restrictions on t.

Second, if we can't completely determine the actual limits on x and y at this point why did we do them here? In part we did them here because we can and the answer to this step often does end up being the limits on x and y. Also, there are times where knowing the largest possible limits on x and/or y will be convenient for some of the later steps.

Finally, we can sometimes get these limits from the sketch of the parametric curve. However, there are some parametric equations that we can't easily get the sketch without doing this step. We'll eventually do some problems like that.

Step 3

Before we sketch the graph of the parametric curve recall that all parametric curves have a direction of motion, *i.e.* the direction indicating increasing values of the parameter, t in this case.

There are several ways to get the direction of motion for the curve. One is to plug in values of t into the parametric equations to get some points that we can use to identify the direction of motion.

Here is a table of values for this set of parametric equations. In this case we were also given a range of t's and we need to restrict the t's in our table to that range.

t	x	y
0	0	-4
$\frac{\pi}{2}$	3	0
π	0	4
$\frac{3\pi}{2}$	-3	0
2π	0	-4

Now, this table seems to suggest that the parametric equation will follow the ellipse in a counter clockwise rotation. It also seems to suggest that the ellipse will be traced out exactly once.

However, tables of values for parametric equations involving sine and/or cosine equations can be deceptive.

Because sine and cosine oscillate it is possible to choose "bad" values of t that suggest a single trace when in fact the curve is tracing out faster than we realize and it is in fact tracing out more than once. We'll need to do some extra analysis to verify if the ellipse traces out once or more than once.

Also, just because the table suggests a particular direction doesn't actually mean it is going in that direction. It could be moving in the opposite direction at a speed that just happens to match the points you got in the table. Go back to the notes and check out Example 5. Plug in the points we used in our table above and you'll get a set of points that suggest the curve is tracing out clockwise when in fact it is tracing out counter clockwise!

Note that because this is such a "bad" way of getting the direction of motion we put it in its own step so we could discuss it in detail. The actual method we'll be using is in the next step and we'll not be doing table work again unless it is absolutely required for some other part of the problem.

Step 4

As suggested in the previous step the table of values is not a good way to get direction of motion for parametric curves involving trig function so let's go through a much better way of determining the direction of motion. This method takes a little time to think things through but it will always get the correct direction if you take the time.

First, let's think about what happens if we start at $t = 0$ and increase t to $t = \pi$.

As we cover this range of t's we know that cosine starts at 1, decreasing through zero and finally stops at -1. So, that means that y will start at $y = -4$ (*i.e.* where cosine is 1), go through the x-axis (*i.e.* where cosine is zero) and finally stop at $y = 4$ (*i.e.* where cosine is -1). Now, this doesn't give us a direction of motion as all it really tells us that y increases and it could do this following the right side of the ellipse (*i.e.* counter clockwise) or it could do this following the left side of the ellipse (*i.e.* clockwise).

So, let's see what the behavior of sine in this range tells us. Starting at $t = 0$ we know that sine will be zero and so x will also be zero. As t increases to $t = \frac{\pi}{2}$ we know that sine increases from zero to one and so x will increase from zero to three. Finally, as we further increase t to $t = \pi$ sine will decrease from one back to zero and so x will also decrease from three to zero.

So, taking the x and y analysis above together we can see that at $t = 0$ the curve will start at the point $(0, -4)$. As we increase t to $t = \frac{\pi}{2}$ the curve will have to follow the ellipse with increasing x and y until it hits the point $(3, 0)$. The only way we can reach this second point and have the correct increasing behavior for both x and y is to move in a counter clockwise direction along the right half of the ellipse.

If we further increase t from $t = \frac{\pi}{2}$ to $t = \pi$ we can see that y must continue to increase but x now decreases until we get to the point $(0, 4)$ and again the only way we can reach this third point and have the required increasing/decreasing information for y/x respectively is to be moving in a counter clockwise direction along the right half.

We can do a similar analysis increasing t from $t = \pi$ to $t = 2\pi$ to see that we must still move in a counter clockwise direction that takes us through the point $(-3, 0)$ and then finally ending at the point $(0, -4)$.

So, from this analysis we can see that the curve must be tracing out in a counter clockwise direction.

This analysis seems complicated and maybe not so easy to do the first few times you see it. However, once you do it a couple of times you'll see that it's not quite as bad as it initially seems to be. Also, it really is the only way to guarantee that you've got the correct direction of motion for the curve when dealing with parametric equations involving sine and/or cosine.

If you had trouble visualizing how sine and cosine changed as we increased t you might want to do a quick sketch of the graphs of sine and cosine and you'll see right away that we were correct in our analysis of their behavior as we increased t.

Step 5

Okay, in the last step notice that we also showed that the curve will trace the ellipse out exactly once in the given range of t's. However, let's assume that we hadn't done the direction analysis yet and see if we can determine this without the direction analysis.

This is actually pretty simple to do, or at least simpler than the direction analysis. All it requires is that you know where sine and cosine are zero, 1 and -1. If you recall your unit circle it's always easy to know where sine and cosine have these values. We'll also be able to verify the ranges of x and y found in Step 2 were in fact the actual ranges for x and y.

Let's start with the "initial" point on the curve, *i.e.* the point at the left end of our range of t's, $t = 0$ in this case. Where you start this analysis is really dependent upon the set of parametric equations, the parametric curve and/or if there is a range of t's given. Good starting points are the "initial" point, one of the end points of the curve itself (if the curve does have endpoints) or $t = 0$. Sometimes one option will be better than the others and other times it won't matter.

In this case two of the options are the same point so it seems like a good point to use.

So, at $t = 0$ we are at the point $(0, -4)$. We know that the parametric curve is some or all of the ellipse we found in the first step. So, at this point let's assume it is the full ellipse and ask ourselves the following question. When do we get back to this point? Or, in other words, what is the next value of t after $t = 0$ (since that is the point we choose to start off with) are we back at the point $(0, -4)$?

Before doing this let's quickly note that if the parametric curve doesn't get back to this point we'll determine that in the following analysis and that will be useful in helping us to determine how much of the ellipse will get traced out by the parametric curve.

Okay let's back to the analysis. In order to be at the point $(0, -4)$ we know we must have $\sin(t) = 0$ (only way to get $x = 0$!) and we must have $\cos(t) = 1$ (only way to get $y = -4$!). For $t > 0$ we know that $\sin(t) = 0$ at $t = \pi, 2\pi, 3\pi, \ldots$ and likewise we know that $\cos(t) = 1$ at $t = 2\pi, 4\pi, 6\pi, \ldots$. The first value of t that is in both lists is $t = 2\pi$ and so this is the next value of t that will put us at that point.

This tells us several things. First, we found that the parametric equation will get back to the initial point and so it is possible for the parametric equation to trace out the full ellipse.

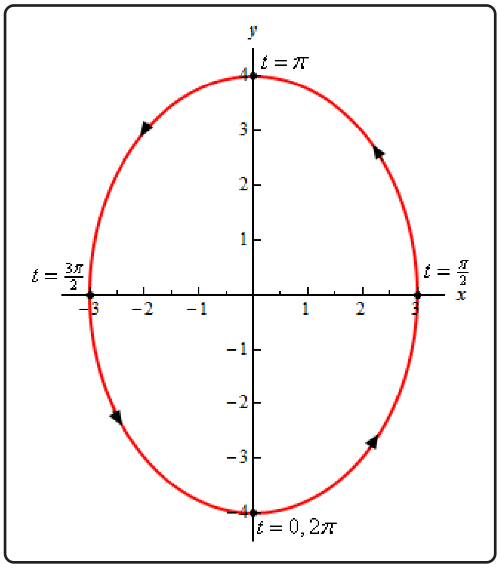
Secondly, we got back to the point $(0, -4)$ at the very last t from the range of t's we were given in the problem statement and so the parametric curve will trace out the ellipse exactly once for the given range of t's.

Finally, from this analysis we found the parametric curve traced out the full ellipse in the range of t's given in the problem statement and so we know now that the limits of x and y we found in Step 2 are in fact the actual limits on x and y for this curve.

As a final comment from this step let's note that this analysis in this step was a little easier than normal because the argument of the trig functions was just a t as opposed to say 2 t or $\frac{1}{3}t$ which does make the analysis a tiny bit more complicated. We'll see how to deal with these kinds of arguments in the next couple of problems.

Step 6

Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included the points from our table because we had them but we won't always include them as we are often only interested in the sketch itself and the direction of motion.

Also, because the problem asked for it here are the formal limits on x and y for this parametric curve.

$$-3 \le x \le 3 \qquad\qquad -4 \le y \le 4$$

As a final set of thoughts for this problem you really should go back and make sure you understand the processes we went through in Step 4 and Step 5. Those are often the best way of getting at the information we found in those steps. The processes can seem a little mysterious at first but once you've done a couple you'll find it isn't as bad as they might have first appeared.

Also, for the rest of the problems in this section we'll build a table of t values only if it is absolutely necessary for the problem. In other words, the process we used in Step 4 and 5 will be the processes we'll be using to get direction of motion for the parametric curve and to determine if the curve is traced out more than once or not.

You should also take a look at problems 5 and 6 in this section and contrast the number of traces of the curve with this problem. The only difference in the set of parametric equations in problems 4, 5 and 6 is the argument of the trig functions. After going through these three problems can you reach any conclusions on how the argument of the trig functions will affect the parametric curves for this type of parametric equations?

5. Eliminate the parameter for the following set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on x and y.

$$x = 3\sin(2t) \qquad y = -4\cos(2t) \qquad 0 \le t \le 2\pi$$

Step 1

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we will make use of the well-known trig identity,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

We can solve each of the parametric equations for sine and cosine as follows,

$$\sin(2t) = \frac{x}{3} \qquad \cos(2t) = -\frac{y}{4}$$

Plugging these into the trig identity (remember the identity holds as long as the argument of both trig functions, 2 t in this case, is the same) gives,

$$\left(-\frac{y}{4}\right)^2 + \left(\frac{x}{3}\right)^2 = 1 \qquad \Rightarrow \qquad \frac{x^2}{9} + \frac{y^2}{16} = 1$$

Therefore, the parametric curve will be some or all of the ellipse above.

We have to be careful when eliminating the parameter from a set of parametric equations. The graph of the resulting equation in only x and y may or may not be the graph of the parametric curve. Often, although not always, the parametric curve will only be a portion of the curve from the equation in terms of only x and y. Another situation that can happen is that the parametric curve will retrace some or all of the curve from the equation in terms of only x and y more than once.

This observation is especially important for this problem. The next few steps will help us to determine just how much of the ellipse we have and if it retraces the ellipse, or a portion of the ellipse, more than once.

Before we proceed with the rest of the problem let's fist note that there is really no set order for doing the steps. They can often be done in different orders and in some cases may actually be easier to do in different orders. The order we'll be following here is used simply because it is the order that I'm used to working them in. If you find a different order would be best for you then that is the order you should use.

Step 2

At this point we can get a good idea on what the limits on x and y are going to be so let's do that. Note that often we won't get the actual limits on x and y in this step. All we are really finding here is the largest possible range of limits for x and y. Having these can sometimes be useful for later steps and so we'll get them here.

We can use our knowledge of sine and cosine to determine the limits on x and y as follows,

$$\begin{array}{cc} -1 \le \sin(2t) \le 1 & -1 \le \cos(2t) \le 1 \\ -3 \le 3\sin(2t) \le 3 & 4 \ge -4\cos(2t) \ge -4 \\ -3 \le x \le 3 & -4 \le y \le 4 \end{array}$$

Note that to find these limits in general we just start with the appropriate trig function and then build up the equation for x and y by first multiplying the trig function by any coefficient, if present, and then adding/subtracting any numbers that might be present (not needed in this case). This, in turn, gives us the largest possible set of limits for x and y. Just remember to be careful when multiplying an inequality by a negative number. Don't forget to flip the direction of the inequalities when doing this.

Now, at this point we need to be a little careful. What we've actually found here are the largest possible inequalities for the limits on x and y. This set of inequalities for the limits on x and y assume that the parametric curve will be completely traced out at least once for the range of t's we were given in the problem statement. It is always possible that the curve will not trace out a full trace in the given range of t's. In a later step we'll determine if the parametric curve does trace out a full trace and hence determine the actual limits on x and y.

Before we move onto the next step there are a couple of issues we should quickly discuss.

First, remember that when we talk about the parametric curve tracing out once we are not necessarily talking about the ellipse itself being fully traced out. The parametric curve will be at most the full ellipse and we haven't determined just yet how much of the ellipse the parametric curve will trace out. So, one trace of the parametric curve refers to the largest portion of the ellipse that the parametric curve can possibly trace out given no restrictions on t.

Second, if we can't completely determine the actual limits on x and y at this point why did we do them here? In part we did them here because we can and the answer to this step often does end up being the limits on x and y. Also, there are times where knowing the largest possible limits on x and/or y will be convenient for some of the later steps.

Finally, we can sometimes get these limits from the sketch of the parametric curve. However, there are some parametric equations that we can't easily get the sketch without doing this step. We'll eventually do some problems like that.

Step 3

Before we sketch the graph of the parametric curve recall that all parametric curves have a direction of motion, *i.e.* the direction indicating increasing values of the parameter, t in this case.

In previous problems one method we looked at was to build a table of values for a sampling of t's in the range provided. However, as we discussed in Problem 4 of this section tables of values for parametric equations involving trig functions they can be deceptive and so we aren't going to use them to determine the direction of motion for this problem.

Also, as noted in the discussion in Problem 4 it also might help to have the graph of sine and cosine handy to look at since we'll be talking a lot about the behavior of sine/cosine as we increase the argument.

So for this problem we'll just do the analysis of the behavior of sine and cosine in the range of t's we were provided to determine the direction of motion. We'll be doing a quicker version of the analysis here than we did in Problem 4 so you might want to go back and check that problem out if you have trouble following everything we're going here.

Let's start at $t = 0$ since that is the first value of t in the range of t's we were given in the problem. This means we'll be starting the parametric curve at the point $(0, -4)$.

Now, what happens if we start to increase t? First, if we increase t then we also increase 2 t, the argument of the trig functions in the parametric equations. So, what does this mean for $\sin(2t)$ and $\cos(2t)$? Well initially, we know that $\sin(2t)$ will increase from zero to one and at the same time $\cos(2t)$ will also have to decrease from one to zero.

So, this means that x (given by $x = 3\sin(2t)$) will have to increase from 0 to 3. Likewise, it means that y (given by $y = -4\cos(2t)$) will have to increase from -4 to 0. For the y equation note that while the cosine is decreasing the minus sign on the coefficient means that y itself will actually be increasing.

Because this behavior for x and y must be happening at simultaneously we can see that the only possibility is for the parametric curve to start at $(0, -4)$ and as we increase the value of t we must move to the right in the counter clockwise direction until we reach the point $(3, 0)$.

Okay, we're now at the point $(3, 0)$, so $\sin(2t) = 1$ and $\cos(2t) = 0$. Let's continue to increase t. A further increase of t will force $\sin(2t)$ to decrease from 1 to 0 and at the same time $\cos(2t)$ will decrease from 0 to -1.

In terms of x and y this means that, at the same time, x will now decrease from 3 to 0 while y will continue to increase from 0 to 4 (again the minus sign on the y equation means y must increase as the cosine decreases from 0 to -1). So, we must be continuing to move in a counter clockwise direction until we reach the point $(0, 4)$.

For the remainder we'll go a little quicker in the analysis and just discuss the behavior of x and y and skip the discussion of the behavior of the sine and cosine.

Another increase in t will force x to decrease from 0 to -3 and at the same time y will have to also decrease from 4 to 0. The only way for this to happen simultaneously is to move along the ellipse starting that $(0, 4)$ in a counter clockwise motion until we reach $(-3, 0)$.

Continuing to increase t and we can see that, at the same time, x will increase from -3 to 0 and y will decrease from 0 to -4. Or, in other words we're moving along the ellipse in a counter clockwise motion from $(-3, 0)$ to $(0, -4)$.

At this point we've gotten back to the starting point and we got back to that point by always going in a counter clockwise direction and did not retrace any portion of the graph and so we can now safely say that the direction of motion for this curve will always counter clockwise.

We have to be very careful here to continue the analysis until we get back to the starting point and see just how we got back there. It is possible, as we'll see in later problems, for us to get back there by retracing back over the curve. This will have an effect on the direction of motion for the curve (*i.e.* the direction will change!). In this case however since we got back to the starting point without retracing any portion of the curve we know the direction will remain counter clockwise.

Step 4

Let's now think about how much of the ellipse is actually traced out or if the ellipse is traced out more than once for the range of t's we were given in the problem. We'll also be able to verify if the ranges of x and y we found in Step 2 are the correct ones or if we need to modify them (and we'll also determine just how to modify them if we need to).

Be careful to not draw any conclusions about how much of the ellipse is traced out from the analysis in the previous step. If we follow that analysis we see a full single trace of the ellipse. However, we didn't ever really mention any values of t with the exception of the starting value. Because of that we can't really use the analysis in the previous step to determine anything about how much of the ellipse we trace out or how many times we trace the ellipse out.

Let's go ahead and start this portion out at the same value of t we started with in the previous step. So, at $t = 0$ we are at the point $(0, -4)$. Now, when do we get back to this point? Or, in other words, what is the next value of t after $t = 0$ (since that is the point we choose to start off with) are we at the point $(0, -4)$?

In order to be at this point we know we must have $\sin(2t) = 0$ (only way to get $x = 0$!) and we must have $\cos(2t) = 1$ (only way to get $y = -4$!). Note the arguments of the sine and cosine! That is very important for this step.

Now, for $t > 0$ we know that $\sin(2t) = 0$ at $2t = \pi, 2\pi, 3\pi, \ldots$ and likewise we know that $\cos(2t) = 1$ at $2t = 2\pi, 4\pi, 6\pi, \ldots$. Again, note the arguments of sine and cosine here! Because we want $\sin(2t)$ and $\cos(2t)$ to have certain values we need to determine the values of 2 t we need to achieve the values of sine and cosine that we are looking for.

The first value of 2 t that is in both lists is $2t = 2\pi$. This now tells us the value of t we need to get back to the starting point. We just need to solve this for t!

$$2t = 2\pi \qquad \Rightarrow \qquad t = \pi$$

So, we will get back to the starting point, without retracing any portion of the ellipse, important in some later problems, when we reach $t = \pi$.

But this is in the middle of the range of t's we were given! So, just what does this mean for us? Well first of all, provided the argument of the sine/cosine is only in terms of t, as opposed to t^2 or $\sqrt{t}$ for example, the "net" range of t's for one trace will always be the same. So, we got one trace in the range of $0 \le t \le \pi$ and so the "net" range of t's here is $\pi - 0 = \pi$ and so any range of t's that span π will trace out the ellipse exactly once.

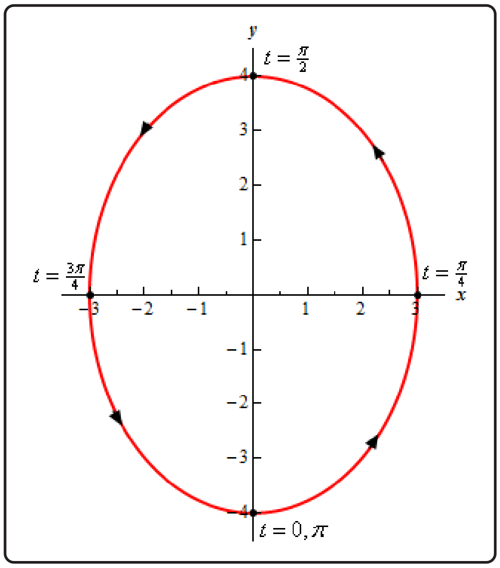
This means that the ellipse will also trace out exactly once in the range $\pi \le t \le 2\pi$. So, in this case, it looks like the ellipse will be traced out twice in the range $0 \le t \le 2\pi$.

This analysis also has shown us that the parametric curve traces out the full ellipse in the range of t's given in the problem statement (more than once in fact!) and so we know now that the limits of x and y we found in Step 2 are in fact the actual limits on x and y for this curve.

Before we leave this step we should note that once you get pretty good at the direction analysis we did in Step 3 you can combine the analysis Steps 3 and 4 into a single step to get both the direction and portion of the curve that is traced out. Initially however you might find them a little easier to do them separately.

Step 5

Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included a set of t's to illustrate a handful of points and their corresponding values of t's. For some practice you might want to follow the analysis from Step 4 to see if you can verify the values of t for the other three points on the graph. It would, of course, be easier to just plug them in to verify, but the practice of the process of the Step 4 analysis might be useful to you.

Also, because the problem asked for it here are the formal limits on x and y for this parametric curve.

$$-3 \le x \le 3 \qquad -4 \le y \le 4$$

You should also take a look at problems 4 and 6 in this section and contrast the number of traces of the curve with this problem. The only difference in the set of parametric equations in problems 4, 5 and 6 is the argument of the trig functions. After going through these three problems can you reach any conclusions on how the argument of the trig functions will affect the parametric curves for this type of parametric equations?

6. Eliminate the parameter for the following set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on x and y.

$$x = 3\sin\left(\frac{1}{3}t\right) \qquad y = -4\cos\left(\frac{1}{3}t\right) \qquad 0 \le t \le 2\pi$$

Step 1

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we will make use of the well-known trig identity,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

We can solve each of the parametric equations for sine and cosine as follows,

$$\sin\left(\frac{1}{3}t\right) = \frac{x}{3} \qquad \cos\left(\frac{1}{3}t\right) = -\frac{y}{4}$$

Plugging these into the trig identity (remember the identity holds as long as the argument of both trig functions, $\frac{1}{3}t$ in this case, is the same) gives,

$$\left(-\frac{y}{4}\right)^2 + \left(\frac{x}{3}\right)^2 = 1 \qquad \Rightarrow \qquad \frac{x^2}{9} + \frac{y^2}{16} = 1$$

Therefore, the parametric curve will be some or all of the ellipse above.

We have to be careful when eliminating the parameter from a set of parametric equations. The graph of the resulting equation in only x and y may or may not be the graph of the parametric curve. Often, although not always, the parametric curve will only be a portion of the curve from the equation in terms of only x and y. Another situation that can happen is that the parametric curve will retrace some or all of the curve from the equation in terms of only x and y more than once.

This observation is especially important for this problem. The next few steps will help us to determine just how much of the ellipse we have and if it retraces the ellipse, or a portion of the ellipse, more than once.

Before we proceed with the rest of the problem let's fist note that there is really no set order for doing the steps. They can often be done in different orders and in some cases may actually be easier to do in different orders. The order we'll be following here is used simply because it is the order that I'm used to working them in. If you find a different order would be best for you then that is the order you should use.

Step 2

At this point we can get a good idea on what the limits on x and y are going to be so let's do that. Note that often we won't get the actual limits on x and y in this step. All we are really finding here is the largest possible range of limits for x and y. Having these can sometimes be useful for later steps and so we'll get them here.

We can use our knowledge of sine and cosine to determine the limits on x and y as follows,

$$\begin{array}{cc} -1 \le \sin\left(\frac{1}{3}t\right) \le 1 & -1 \le \cos\left(\frac{1}{3}t\right) \le 1 \\ -3 \le 3\sin\left(\frac{1}{3}t\right) \le 3 & 4 \ge -4\cos\left(\frac{1}{3}t\right) \ge -4 \\ -3 \le x \le 3 & -4 \le y \le 4 \end{array}$$

Note that to find these limits in general we just start with the appropriate trig function and then build up the equation for x and y by first multiplying the trig function by any coefficient, if present, and then adding/subtracting any numbers that might be present (not needed in this case). This, in turn, gives us the largest possible set of limits for x and y. Just remember to be careful when multiplying an inequality by a negative number. Don't forget to flip the direction of the inequalities when doing this.

Now, at this point we need to be a little careful. What we've actually found here are the largest possible inequalities for the limits on x and y. This set of inequalities for the limits on x and y assume that the parametric curve will be completely traced out at least once for the range of t's we were given in the problem statement. It is always possible that the curve will not trace out a full trace in the given range of t's. In a later step we'll determine if the parametric curve does trace out a full trace and hence determine the actual limits on x and y.

Before we move onto the next step there are a couple of issues we should quickly discuss.

First, remember that when we talk about the parametric curve tracing out once we are not necessarily talking about the ellipse itself being fully traced out. The parametric curve will be at most the full ellipse and we haven't determined just yet how much of the ellipse the parametric curve will trace out. So, one trace of the parametric curve refers to the largest portion of the ellipse that the parametric curve can possibly trace out given no restrictions on t. This is especially important for this problem!

Second, if we can't completely determine the actual limits on x and y at this point why did we do them here? In part we did them here because we can and the answer to this step often does end up being the limits on x and y. Also, there are times where knowing the largest possible limits on x and/or y will be convenient for some of the later steps.

Finally, we can sometimes get these limits from the sketch of the parametric curve. However, there are some parametric equations that we can't easily get the sketch without doing this step. We'll eventually do some problems like that.

Step 3

Before we sketch the graph of the parametric curve recall that all parametric curves have a direction of motion, *i.e.* the direction indicating increasing values of the parameter, t in this case.

In previous problems one method we looked at was to build a table of values for a sampling of t's in the range provided. However, as we discussed in Problem 4 of this section tables of values for parametric equations involving trig functions they can be deceptive and so we aren't going to use them to determine the direction of motion for this problem.

Also, as noted in the discussion in Problem 4 it also might help to have the graph of sine and cosine handy to look at since we'll be talking a lot about the behavior of sine/cosine as we increase the argument.

So for this problem we'll just do the analysis of the behavior of sine and cosine in the range of t's we were provided to determine the direction of motion. We'll be doing a quicker version of the analysis here than we did in Problem 4 so you might want to go back and check that problem out if you have trouble following everything we're going here.

Let's start at $t = 0$ since that is the first value of t in the range of t's we were given in the problem. This means we'll be starting the parametric curve at the point $(0, -4)$.

Now, what happens if we start to increase t? First, if we increase t then we also increase $\frac{1}{3}t$, the argument of the trig functions in the parametric equations. So, what does this mean for $\sin\left(\frac{1}{3}t\right)$ and $\cos\left(\frac{1}{3}t\right)$? Well initially, we know that $\sin\left(\frac{1}{3}t\right)$ will increase from zero to one and at the same time $\cos\left(\frac{1}{3}t\right)$ will also have to decrease from one to zero.

So, this means that x (given by $x = 3\sin\left(\frac{1}{3}t\right)$) will have to increase from 0 to 3. Likewise, it means that y (given by $y = -4\cos\left(\frac{1}{3}t\right)$) will have to increase from -4 to 0. For the y equation note that while the cosine is decreasing the minus sign on the coefficient means that y itself will actually be increasing.

Because this behavior for the x and y must be happening at simultaneously we can see that the only possibility is for the parametric curve to start at $(0, -4)$ and as we increase the value of t we must move to the right in the counter clockwise direction until we reach the point $(3, 0)$.

Okay, we're now at the point $(3, 0)$, so $\sin\left(\frac{1}{3}t\right) = 1$ and $\cos\left(\frac{1}{3}t\right) = 0$. Let's continue to increase t. A further increase of t will force $\sin\left(\frac{1}{3}t\right)$ to decrease from 1 to 0 and at the same time $\cos\left(\frac{1}{3}t\right)$ will decrease from 0 to -1.

In terms of x and y this means that, at the same time, x will now decrease from 3 to 0 while y will continue to increase from 0 to 4 (again the minus sign on the y equation means y must increase as the cosine decreases from 0 to -1). So, we must be continuing to move in a counter clockwise direction until we reach the point $(0, 4)$.

For the remainder we'll go a little quicker in the analysis and just discuss the behavior of x and y and skip the discussion of the behavior of the sine and cosine.

Another increase in t will force x to decrease from 0 to -3 and at the same time y will have to also decrease from 4 to 0. The only way for this to happen simultaneously is to move along the ellipse starting that $(0,4)$ in a counter clockwise motion until we reach $(-3,0)$.

Continuing to increase t and we can see that, at the same time, x will increase from -3 to 0 and y will decrease from 0 to -4. Or, in other words we're moving along the ellipse in a counter clockwise motion from $(-3,0)$ to $(0,-4)$.

At this point we've gotten back to the starting point and we got back to that point by always going in a counter clockwise direction and did not retrace any portion of the graph and so we can now safely say that the direction of motion for this curve will always counter clockwise.

We have to be very careful here to continue the analysis until we get back to the starting point and see just how we got back there. It is possible, as we'll see in later problems, for us to get back there by retracing back over the curve. This will have an effect on the direction of motion for the curve (*i.e.* the direction will change!). In this case however since we got back to the starting point without retracing any portion of the curve we know the direction will remain counter clockwise.

Step 4

Let's now think about how much of the ellipse is actually traced out or if the ellipse is traced out more than once for the range of t's we were given in the problem. We'll also be able to verify if the ranges of x and y we found in Step 2 are the correct ones or if we need to modify them (and we'll also determine just how to modify them if we need to).

Be careful to not draw any conclusions about how much of the ellipse is traced out from the analysis in the previous step. If we follow that analysis we see a full single trace of the ellipse. However, we didn't ever really mention any values of t with the exception of the starting value. Because of that we can't really use the analysis in the previous step to determine anything about how much of the ellipse we trace out or how many times we trace the ellipse out.

Let's go ahead and start this portion out at the same value of t we started with in the previous step. So, at $t = 0$ we are at the point $(0,-4)$. Now, when do we get back to this point? Or, in other words, what is the next value of t after $t = 0$ (since that is the point we choose to start off with) are we at the point $(0,-4)$?

In order to be at this point we know we must have $\sin\left(\frac{1}{3}t\right) = 0$ (only way to get $x = 0$!) and we must have $\cos\left(\frac{1}{3}t\right) = 1$ (only way to get $y = -4$!). Note the arguments of the sine and cosine! That is very important for this step.

Now, for $t > 0$ we know that $\sin\left(\frac{1}{3}t\right) = 0$ at $\frac{1}{3}t = \pi, 2\pi, 3\pi, \ldots$ and likewise we know that $\cos\left(\frac{1}{3}t\right) = 1$ at $\frac{1}{3}t = 2\pi, 4\pi, 6\pi, \ldots$. Again, note the arguments of sine and cosine here!

Because we want sin $\left(\frac{1}{3}t\right)$ and cos $\left(\frac{1}{3}t\right)$ to have certain values we need to determine the values of $\frac{1}{3}t$ we need to achieve the values of sine and cosine that we are looking for.

The first value of $\frac{1}{3}t$ that is in both lists is $\frac{1}{3}t = 2\pi$. This now tells us the value of t we need to get back to the starting point. We just need to solve this for t!

$$\frac{1}{3}t = 2\pi \qquad \Rightarrow \qquad t = 6\pi$$

So, we will get back to the starting point, without retracing any portion of the ellipse, important in some later problems, when we reach $t = 6\pi$.

At this point we have a problem that we didn't have in the previous two problems. We get back to the point $(0,-4)$ at $t = 6\pi$ and this is outside the range of t's given in the problem statement, $0 \le t \le 2\pi$!

What this means for us is that the parametric curve will not trace out a full trace for the range of t's we were given for this problem. It also means that the range of limits for x and y from Step 2 are not the correct limits for x and y.

We know from the Step 3 analysis that the parametric curve will trace out in a counter clockwise direction and from the analysis in this step it won't trace out a full trace.

So, we know the parametric curve will start when $t = 0$ at $(0,-4)$ and will trace out in a counter clockwise direction until $t = 2\pi$ at which we will be at the point,

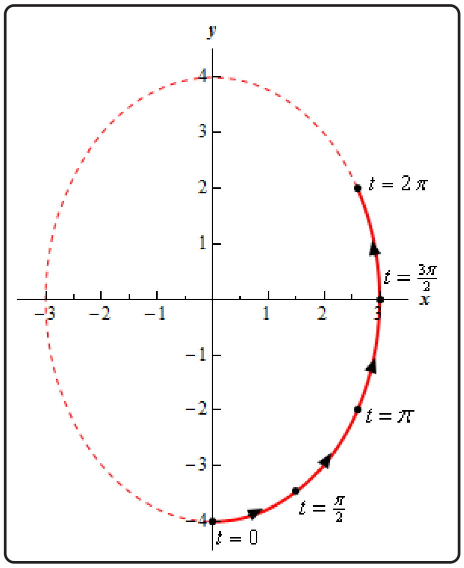
$$\left(3\sin\left(\frac{2\pi}{3}\right), -4\cos\left(\frac{2\pi}{3}\right)\right) = \left(\frac{3\sqrt{3}}{2}, 2\right)$$

This "ending" point is in the first quadrant and so we know that the curve has to have passed through $(3,0)$. This means that the limits on x are $0 \le x \le 3$. The limits on the y are simply those we get from the points $-4 \le y \le 2$.

Before we leave this step we should note that once you get pretty good at the direction analysis we did in Step 3 you can combine the analysis Steps 3 and 4 into a single step to get both the direction and portion of the curve that is traced out. Initially however you might find them a little easier to do them separately.

Step 5

Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included a set of t's to illustrate a handful of points and their corresponding values of t's. For some practice you might want to follow the analysis from Step 4 to see if you can verify the values of t for the other three points on the graph. It would, of course, be easier to just plug them in to verify, but the practice would of the Step 4 analysis might be useful to you.

Note as well that we included the full sketch of the ellipse as a dashed graph to help illustrate the portion of the ellipse that the parametric curve is actually covering.

Also, because the problem asked for it here are the formal limits on x and y for this parametric curve.

$$0 \le x \le 3 \qquad\qquad -4 \le y \le 2$$

You should also take a look at problems 4 and 5 in this section and contrast the number of traces of the curve with this problem. The only difference in the set of parametric equations in problems 4, 5 and 6 is the argument of the trig functions. After going through these three problems can you reach any conclusions on how the argument of the trig functions will affect the parametric curves for this type of parametric equations?

7. The path of a particle is given by the following set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

   (i) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

   (ii) Limits on x and y.

   (iii) A range of t's for a single trace of the parametric curve.

   (iv) The number of traces of the curve the particle makes if an overall range of t's is provided in the problem.

$$x = 3 - 2\cos(3t) \qquad y = 1 + 4\sin(3t)$$

Step 1

There's a lot of information we'll need to find to fully answer this problem. However, for most of it we can follow the same basic ordering of steps we used for the first few problems in this section. We will need however to do a little extra work along the way.

Also, because most of the work here is similar to the work we did in Problems 4 - 6 of this section we won't be putting in as much explanation to a lot of the work we're doing here. So, if you need some explanation for some of the work you should go back to those problems and check the corresponding steps.

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we will make use of the well-known trig identity,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

We can solve each of the parametric equations for sine and cosine as follows,

$$\cos(3t) = \frac{x-3}{-2} \qquad \sin(3t) = \frac{y-1}{4}$$

Plugging these into the trig identity gives,

$$\left(\frac{x-3}{-2}\right)^2 + \left(\frac{y-1}{4}\right)^2 = 1 \qquad \Rightarrow \qquad \frac{(x-3)^2}{4} + \frac{(y-1)^2}{16} = 1$$

Therefore, the parametric curve will be some or all of the graph of this ellipse.

Step 2

At this point let's get our first guess as to the limits on x and y. As noted in previous problems what we're really finding here is the largest possible ranges for x and y. In later steps we'll determine if this the actual set of limits on x and y or if we have smaller ranges.

We can use our knowledge of sine and cosine to determine the limits on x and y as follows,

$$\begin{array}{cc} -1 \le \cos(3t) \le 1 & -1 \le \sin(3t) \le 1 \\ 2 \ge -2\cos(3t) \ge -2 & -4 \le 4\sin(3t) \le 4 \\ 5 \ge 3 - 2\cos(3t) \ge 1 & -3 \le 1 + 4\sin(3t) \le 5 \\ 1 \le x \le 5 & -3 \le y \le 5 \end{array}$$

Remember that all we need to do is start with the appropriate trig function and then build up the equation for x and y by first multiplying the trig function by any coefficient, if present, and then adding/subtracting any numbers that might be present. We now have the largest possible set of limits for x and y.

This problem does not have a range of t's that might restrict how much of the parametric curve gets sketched out. This means that the parametric curve will be fully traced out.

Remember that when we talk about the parametric curve getting fully traced out this doesn't, in general, mean the full ellipse we found in Step 1 gets traced out by the parametric equation. All "fully traced out" means, in general, is that whatever portion of the ellipse that is described by the set of parametric curves will be completely traced out.

However, for this problem let's also note as well that the ranges for x and y we found above also correspond the maximum ranges for x and y we get from the equation of the ellipse we found in Step 1. This means that, for this problem, the ellipse will get fully traced out at least once by the parametric curve and so these are the full limits on x and y.

Step 3

Let's next get the direction of motion for the parametric curve.

Let's use $t = 0$ as a "starting" point for this analysis. At $t = 0$ we are at the point $(1, 1)$. If we increase t we can see that both x and y must increase until we get to the point $(3, 5)$. Increasing t further from this point will force x to continue to increase, but y will now start to decrease until we reach the point $(5, 1)$. Next, when we increase t further both x and y will decrease until we reach the point $(3, -3)$. Finally, increasing t even more we get x continuing to decrease while y starts to increase until we get back to $(1, 1)$, the point we "started" the analysis at.

We didn't put a lot of "explanation into this but if you think about the parametric equations and how sine/cosine behave as you increase t you should see what's going on. In the x equation we see that the coefficient of the cosine is negative and so if cosine increases x must decrease and if cosine decreases x must increase. For the y equation the coefficient of the sine is positive and so both y and sine will increase or decrease at the same time.

Okay, in all of the analysis above we must be moving in a clockwise direction. Also, note that because of the oscillating nature of sine and cosine once we reach back to the "starting" point the behavior will simply repeat itself. This in turn tells us that once we arrive back at the "starting" point we will continue to trace out the parametric curve in a clockwise direction.

Step 4

From the analysis in the last step we saw that without any range of t's restricting the parametric curve, which we don't have here, the parametric curve will completely trace out the ellipse that we found in Step 1.

Therefore, the next thing we should do is determine a range of t's that it will take to complete one trace of the parametric curve. Note that one trace of the parametric curve means that no portion of the parametric curve will ever be retraced. For this problem that means we trace out the ellipse exactly once.

So, as with the last step let's "start" at the point $(1,1)$, which corresponds to $t = 0$. So, the next question to ask is what value of $t > 0$ will we reach this point again.

In order to be at the point $(1,1)$ we need to require that $\cos(3t) = 1$ and $\sin(3t) = 0$. So, for $t > 0$ we know we'll have $\cos(3t) = 1$ if $3t = 2\pi, 4\pi, 6\pi, \ldots$ and we'll have $\sin(3t) = 0$ if $3t = \pi, 2\pi, 3\pi, \ldots$.

The first value of t that is in both of these lists is $3t = 2\pi$. So, we'll get back to the "starting" point at,

$$3t = 2\pi \qquad \Rightarrow \qquad t = \frac{2\pi}{3}$$

Therefore, one trace will be completed in the range,

$$0 \le t \le \frac{2\pi}{3}$$

Note that this is only one possible answer here. Any range of t's with a "net" range of $\frac{2\pi}{3}$ t's, with the endpoints of the t range corresponding to start/end points of the parametric equation, will work. So, for example, any of the following ranges of t's would also work.

$$-\frac{2\pi}{3} \le t \le 0 \qquad \frac{2\pi}{3} \le t \le \frac{4\pi}{3} \qquad -\frac{\pi}{3} \le t \le \frac{\pi}{3}$$

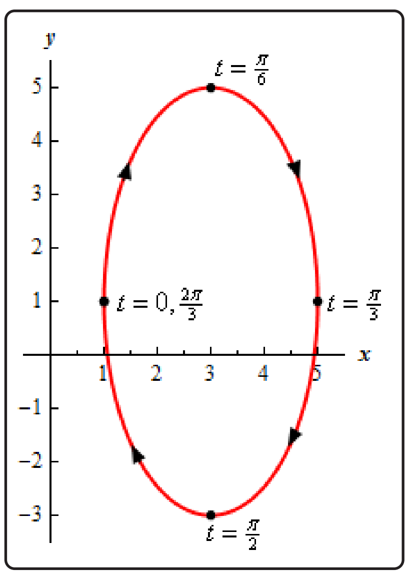
There are of course many other possible ranges of t's for a one trace. Note however, as the last example above shows, because the full ellipse is traced out, each range doesn't all need to start/end at the same place. The range we originally arrived at as well as the first two ranges above all start/end at $(1,1)$ while the third range above starts/ends at $(5,1)$.

Step 5

Now that we have a range of t's for one full trace of the parametric curve we could determine the number of traces the particle makes. However, because we weren't given an overall range of t's we can't do that for this problem.

Step 6

Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we included a set of t's to illustrate where the particle is at while tracing out of the curve. For some practice you might want to follow the analysis from Step 4 to see if you can verify the values of t for the other three points on the graph. It would, of course, be easier to just plug them in to verify, but the practice would of the Step 4 analysis might be useful to you.

Here is also the formal answers for all the rest of the information that problem asked for.

Range of x :	$1 \le x \le 5$
Range of y :	$-3 \le y \le 5$
Range of t for one trace :	$0 \le t \le \frac{2\pi}{3}$
Total number of traces :	n/a

8. The path of a particle is given by the following set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

   (i) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

   (ii) Limits on x and y.

   (iii) A range of t's for a single trace of the parametric curve.

   (iv) The number of traces of the curve the particle makes if an overall range of t's is provided in the problem.

$$x = 4\sin\left(\frac{1}{4}t\right) \qquad y = 1 - 2\cos^2\left(\frac{1}{4}t\right) \qquad -52\pi \le t \le 34\pi$$

Step 1

There's a lot of information we'll need to find to fully answer this problem. However, for most of it we can follow the same basic ordering of steps we used for the first few problems in this section. We will need however to do a little extra work along the way.

Also, because most of the work here is similar to the work we did in Problems 4 - 6 of this section we won't be putting in as much explanation to a lot of the work we're doing here. So, if you need some explanation for some of the work you should go back to those problems and check the corresponding steps.

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations we will make use of the well-known trig identity,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

We can solve each of the parametric equations for sine and cosine as follows,

$$\sin\left(\frac{1}{4}t\right) = \frac{x}{4} \qquad \cos^2\left(\frac{1}{4}t\right) = \frac{y-1}{-2}$$

Plugging these into the trig identity gives,

$$\frac{y-1}{-2} + \left(\frac{x}{4}\right)^2 = 1 \qquad \Rightarrow \qquad y = \frac{x^2}{8} - 1$$

Therefore, with a little algebraic manipulation, we see that the parametric curve will be some or all of the parabola above. Note that while many parametric equations involving sines and cosines are some or all of an ellipse they won't all be as this problem shows. Do not get so locked into ellipses when seeing sines/cosines that you always just assume the curve will be an ellipse.

Step 2

At this point let's get our first guess as to the limits on x and y. As noted in previous problems what we're really finding here is the largest possible ranges for x and y. In later steps we'll determine if this the actual set of limits on x and y or if we have smaller ranges.

We can use our knowledge of sine and cosine to determine the limits on x and y as follows,

$$\begin{array}{cc} -1 \le \sin\left(\frac{1}{4}t\right) \le 1 & -1 \le \cos\left(\frac{1}{4}t\right) \le 1 \\ -4 \le 4\sin\left(\frac{1}{4}t\right) \le 4 & 0 \le \cos^2\left(\frac{1}{4}t\right) \le 1 \\ -4 \le x \le 4 & 0 \ge -2\cos^2\left(\frac{1}{4}t\right) \ge -2 \\ & 1 \ge 1 - 2\cos^2\left(\frac{1}{4}t\right) \ge -1 \\ & -1 \le y \le 1 \end{array}$$

Remember that all we need to do is start with the appropriate trig function and then build up the equation for x and y by first multiplying the trig function by any coefficient, if present, and then adding/subtracting any numbers that might be present. We now have the largest possible set of limits for x and y.

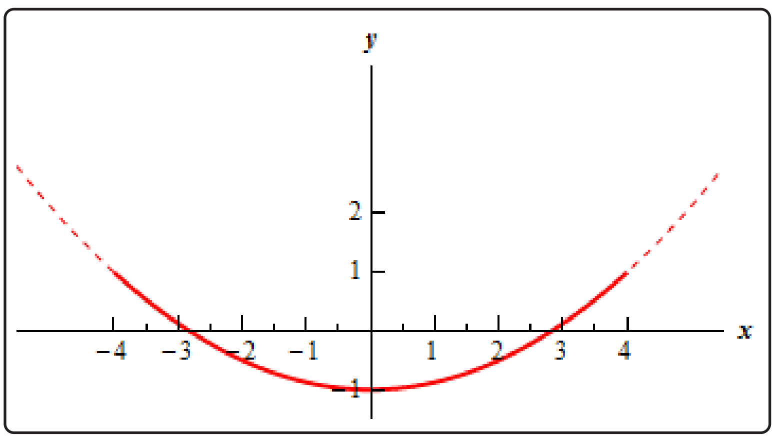
Now, at this point we need to be a little careful. As noted above what we've actually found here are the largest possible ranges for the limits on x and y. This set of inequalities for the limits on x and y assume that the parametric curve will be fully traced out at least once for the range of t's we were given in the problem statement. It is always possible that the parametric curve will not trace out a full trace in the given range of t's. In a later step we'll determine if the parametric curve does trace out a full trace and hence determine the actual limits on x and y.

Remember that when we talk about the parametric curve getting fully traced out this doesn't, in general, mean the full parabola we found in Step 1 gets traced out by the parametric equation. All "fully traced out" means, in general, is that whatever portion of the parabola that is described by the set of parametric curves will be completely traced out.

In fact, for this problem, we can see that the parabola from Step 1 will not get fully traced out by the particle regardless of any range of t's. The largest possible portion of the parabola that can be traced out by the particle is the portion that lies in the range of x and y given above. In a later step we'll determine if the largest possible portion of the parabola does get traced out or if the particle only traces out part of it.

Step 3

Let's next get the direction of motion for the parametric curve. For this analysis it might be useful to have a quick sketch of the largest possible parametric curve. So, here is a quick sketch of that.



The dashed line is the graph of the full parabola from Step 1 and the solid line is the portion that falls into our largest possible range of x and y we found in Step 2. As an aside here note that the two ranges are complimentary. In other words, if we sketch the graph only for the range of x we automatically get the range for y. Likewise, if we sketch the graph only for the range of y we automatically get the range for x. This is a good check for your graph. The x and y ranges should always match up!

When our parametric curve was an ellipse (the previous problem for example) no matter what point we started the analysis at the curve would eventually trace out around the ellipse and end up back at the starting point without ever going back over any portion of itself. The main issue we faced with the ellipse problem was we could rotate around the ellipse in a clockwise or a counter clockwise motion to do this and a careful analysis of the behavior of both the x and y parametric equations was required to determine just which direction we were going.

With a parabola for our parametric curve things work a lot differently. Let's suppose that we "started" at the right end point (this is just randomly picked for no other reason that I'm right handed so don't think there is anything special about this point!) and it doesn't matter what t we use to get to that point.

At this point we know that we are at $x = 4$ and in order for x to have that value we must also have $\sin\left(\frac{1}{4}t\right) = 1$. Now, as we increase t from this point (again it doesn't matter just what the value of t is) the only option for sine is for it to decrease until it has the value $\sin\left(\frac{1}{4}t\right) = -1$. This in turn means that if we start at the right end point we have no option but to proceed along the curve going to the left.

However, we don't just reach the left end point and then stop! Once we are at $\sin\left(\frac{1}{4}t\right) = -1$ if we further increase t we know that sine will also increase until it has the value $\sin\left(\frac{1}{4}t\right) = 1$ and so we must move back along the curve to the right until we are back at the right end point.

Unlike the ellipse however, the only way for this to happen is for the particle to go back over the parabola moving in a rightward direction. Remember that the particle moves to the right or left it must trace out a portion of the parabola that we found in Step 1! Any particle traveling along the path given by the set of parametric equations must follow the graph of the parabola and never leave it.

In other words, if we don't put any restrictions on t a particle on this parametric curve will simply oscillate left and right along the portion of the parabola sketched out above. In this case however we do have a range of t's so we'll need to determine a range of t's for one trace to fully know the direction of motion information of the particle on this path and we'll do that in the next step. With a restriction on the range of t's it is possible that the particle won't make a full trace or it might retrace some or all of the curve so we can't say anything definite about the direction of motion for the particle over the full range of t's until the next step when we determine a range of t's for one full trace of the curve.

Before we move on to the next step there is a quick topic we should address. We only used the x equation to do this analysis and never addressed the y-equation anywhere in the analysis. It doesn't really matter which one we use as both will give the same information.

Step 4

Now we need to determine a range of t's for one full trace of the parametric curve. It is important for this step to remember that one full trace of the parametric curve means that no portion of the parametric curve can be retraced.

Note that one full trace does not mean that we get back to the "starting" point. When we dealt with an ellipse in the previous problem that was one trace because we did not need to retrace any portion of the ellipse to get back to the starting point. However, as we saw in the previous step that for our parabola here we would have to retrace the full curve to get back to the starting point.

So, one full trace of the parametric curve means we move from the right end point to the left end point only or visa-versa and move from the left end point to the right end point. Which direction we move doesn't really matter here so let's get a range of t's that take us from the left end point to the right end point.

In all the previous problems we've used $t = 0$ as our "starting" point but that won't work for this problem because that actually corresponds to the vertex of the parabola. We want to start at the left end point so the first part of this process is actually determine a t that will put us at the left end point.

In order to be at the left end point, $(-4, 1)$, we need to require that $\sin\left(\frac{1}{4}t\right) = -1$ which occurs if $\frac{1}{4}t = \ldots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$. We also need to require that $\cos\left(\frac{1}{4}t\right) = 0$ which occurs if $\frac{1}{4}t = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$. There are going to be many numbers that are in both lists here so all we need to do is pick one and proceed. From the numbers that we've listed here we could use either $\frac{1}{4}t = -\frac{\pi}{2}$ or $\frac{1}{4}t = \frac{3\pi}{2}$. We'll use $\frac{1}{4}t = -\frac{\pi}{2}$, *i.e.* when $t = -2\pi$, simply because it is the first one that occurs in both lists. Therefore, we will be at the left end point when $t = -2\pi$.

Let's now move to the right end point, $(4, 1)$. In order to get the range of t's for one trace this means we'll need the next t with $t > -2\pi$ (which corresponds to $\frac{1}{4}t > -\frac{\pi}{2}$). To do this we need to require that $\sin\left(\frac{1}{4}t\right) = 1$ which occurs if $\frac{1}{4}t = \ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$ and we need to that $\cos\left(\frac{1}{4}t\right) = 0$ which occurs if $\frac{1}{4}t = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.

The first t that is in both of these lists with $\frac{1}{4}t > -\frac{\pi}{2}$ is then $\frac{1}{4}t = \frac{\pi}{2}$, *i.e.* when $t = 2\pi$. So, the first t after $t = -2\pi$ that puts us at the right end point is $t = 2\pi$. This means that a range of t's for one full trace of the parametric curve is then,

$$-2\pi \le t \le 2\pi$$

Note that this is only one possible answer here. Any range of t's with a "net" range of $2\pi - (-2\pi) = 4\pi$ t's, with the endpoints of the t range corresponding to start/end points of the parametric curve, will work. So, for example, any of the following ranges of t's would also work.

$$-6\pi \le t \le -2\pi \qquad 2\pi \le t \le 6\pi \qquad 6\pi \le t \le 10\pi$$

The direction of motion for each may be different range of t's of course. Some will trace out the curve moving from left to right while others will trace out the curve moving from right to left. Because the problem did not specify a particular direction any would work.

Note as well that the range $-2\pi \le t \le 2\pi$ falls completely inside the given range of t's specified in the problem and so we know that the particle will trace out the curve more than once over the full range of t's. Determining just how many times it traces over the curve will be determined in the next step.

Step 5

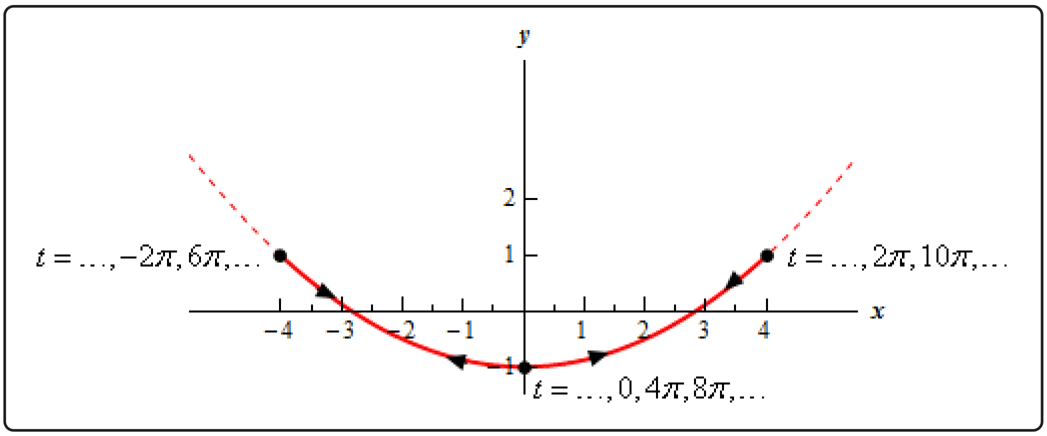
Now that we have a range of t's for one full trace of the parametric curve we can determine the number of traces the particle makes.

This is a really easy step. We know the total time the particle was traveling and we know how long it takes for a single trace. Therefore,

$$\text{Number Traces} = \frac{\text{Total Time Traveled}}{\text{Time for One Trace}} = \frac{34\pi - (-52\pi)}{2\pi - (-2\pi)} = \frac{86\pi}{4\pi} = \frac{43}{2} = 21.5 \text{ traces}$$

Step 6

Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we indicated the direction of motion by putting arrow heads going both directions in places on the curve. We also included a set of t's for a couple of points to illustrate where the particle is at while tracing out of the curve. The dashed line is the continuation of the parabola from Step 1 to illustrate that our parametric curve is only a part of the parabola.

Here is also the formal answers for all the rest of the information that problem asked for.

Range of x :	$-4 \le x \le 4$
Range of y :	$-1 \le y \le 1$
Range of t for one trace :	$-2\pi \le t \le 2\pi$
Total number of traces :	21.5

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9. The path of a particle is given by the following set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

   (i) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

   (ii) Limits on x and y.

   (iii) A range of t's for a single trace of the parametric curve.

   (iv) The number of traces of the curve the particle makes if an overall range of t's is provided in the problem.

$$x = \sqrt{4 + \cos\left(\frac{5}{2}t\right)} \qquad y = 1 + \frac{1}{3}\cos\left(\frac{5}{2}t\right) \qquad -48\pi \le t \le 2\pi$$

Step 1

There's a lot of information we'll need to find to fully answer this problem. However, for most of it we can follow the same basic ordering of steps we used for the first few problems in this section. We will need however to do a little extra work along the way.

Also, because most of the work here is similar to the work we did in Problems 4 - 6 of this section we won't be putting in as much explanation to a lot of the work we're doing here. So, if you need some explanation for some of the work you should go back to those problems and check the corresponding steps.

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations notice that we can quickly and easily eliminate the parameter simply by solving the y equation for cosine as follows,

$$\cos\left(\frac{5}{2}t\right) = 3y - 3$$

Plugging this into the cosine in the x equation gives,

$$x = \sqrt{4 + (3y - 3)} \qquad \Rightarrow \qquad x = \sqrt{1 + 3y}$$

So, the parametric curve will be some or all of the graph of this square root function.

Step 2

At this point let's get our first guess as to the limits on x and y. As noted in previous problems what we're really finding here is the largest possible ranges for x and y. In later steps we'll determine if this the actual set of limits on x and y or if we have smaller ranges.

We can use our knowledge of cosine to determine the limits on x and y as follows,

$$\begin{array}{cc} -1 \le \cos\left(\frac{5}{2}t\right) \le 1 & -1 \le \cos\left(\frac{5}{2}t\right) \le 1 \\ 3 \le 4 + \cos\left(\frac{5}{2}t\right) \le 5 & -\frac{1}{3} \le \frac{1}{3}\cos\left(\frac{5}{2}t\right) \le \frac{1}{3} \\ \sqrt{3} \le \sqrt{4 + \cos\left(\frac{5}{2}t\right)} \le \sqrt{5} & \frac{2}{3} \le 1 + \frac{1}{3}\cos\left(\frac{5}{2}t\right) \le \frac{4}{3} \\ \sqrt{3} \le x \le \sqrt{5} & \frac{2}{3} \le y \le \frac{4}{3} \end{array}$$

Remember that all we need to do is start with the cosine and then build up the equation for x and y by first multiplying the trig function by any coefficient, if present, and then adding/subtracting any numbers that might be present. We now have the largest possible set of limits for x and y.

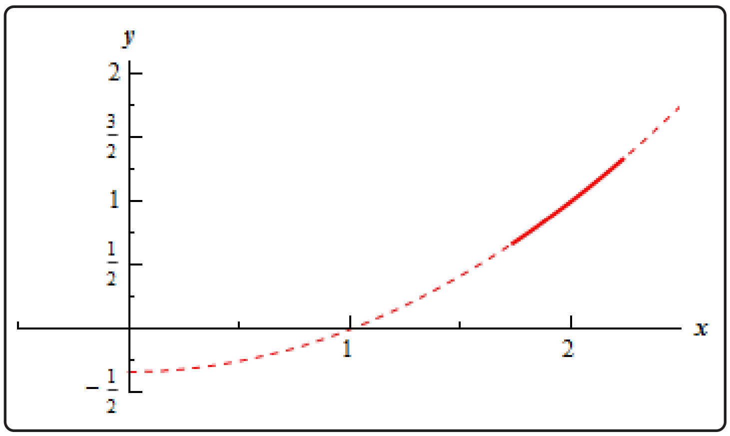
Now, at this point we need to be a little careful. As noted above what we've actually found here are the largest possible ranges for the limits on x and y. This set of inequalities for the limits on x and y assume that the parametric curve will be fully traced out at least once for the range of t's we were given in the problem statement. It is always possible that the parametric curve will not trace out a full trace in the given range of t's. In a later step we'll determine if the parametric curve does trace out a full trace and hence determine the actual limits on x and y.

Remember that when we talk about the parametric curve getting fully traced out this doesn't, in general, mean the full square root graph we found in Step 1 gets traced out by the parametric equation. All "fully traced out" means, in general, is that whatever portion of the square root graph that is described by the set of parametric curves will be completely traced out.

In fact, for this problem, we can see that the square root from Step 1 will not get fully traced out by the particle regardless of any range of t's. The largest possible portion of the square root graph that can be traced out by the particle is the portion that lies in the range of x and y given above. In a later step we'll determine if the largest possible portion of the square root graph does get traced out or if the particle only traces out part of it.

Step 3

Let's next get the direction of motion for the parametric curve. For this analysis it might be useful to have a quick sketch of the largest possible parametric curve. So, here is a quick sketch of that.



The dashed line is the graph of the full square root from Step 1 and the solid line is the portion that falls into our largest possible range of x and y we found in Step 2. As an aside here note that the two ranges are complimentary. In other words, if we sketch the graph only for the range of x we automatically get the range for y. Likewise, if we sketch the graph only for the range of y we automatically get the range for x. This is a good check for your graph. The x and y ranges should always match up!

Before moving on let's address the fact that is doesn't look like square root graphs that most of us are used to seeing. Keep in mind that the typical square root function that we're used to working at is in the form $y = \sqrt{x}$. Our equation for this problem however is in the form $x = \sqrt{y}$. If you think about it the graph of $x = \sqrt{y}$ is nothing more than the portion of the graph of $y = x^2$ corresponding to $x \geq 0$ (recall square roots only return positive or zero values!). Of course the function for this problem is not $x = \sqrt{y}$ but it is similar enough that the ideas discussed here are still valid just for a slightly different function.

Okay, let's get back to the problem.

This problem is going to be a lot like the previous problem in terms of direction of motion. First note that if we start at the lower left hand point we need to require that $\cos\left(\frac{5}{2}t\right) = -1$ since that is the only way for both x and y to have their minimal values (which puts us at the lower left-hand point)! It also doesn't matter what value of t we use at this point. All that matters is that we are at the lower left hand point.

If we now increase t (from whatever "starting" value we had) we know that cosine will need to increase from $\cos\left(\frac{5}{2}t\right) = -1$ until it reaches a value of $\cos\left(\frac{5}{2}t\right) = 1$. By looking at the parametric equations we can see that this will also force both x and y to increase until it reaches the upper right-hand point.

Now, the graph won't just stop here. Once cosine reaches a value of $\cos\left(\frac{5}{2}t\right) = 1$ we know that continuing to increase t will now cause cosine to decrease it reaches a value of $\cos\left(\frac{5}{2}t\right) = -1$. This in turn forces both x and y to decrease until it once again reaches the lower left-hand point.

In other words, if we don't put any restrictions on t a particle on this parametric curve will simply oscillate left and right along the portion of the square root sketched out above. In this case however we do have a range of t's so we'll need to determine a range of t's for one trace to fully know the direction of motion information of the particle on this path and we'll do that in the next step. With a restriction on the range of t's it is possible that the particle won't make a full trace or it might retrace some or all of the curve so we can't say anything definite about the direction of motion for the particle over the full range of t's until the next step when we determine a range of t's for one full trace of the curve.

Step 4

Now we need to determine a range of t's for one full trace of the parametric curve. It is important for this step to remember that one full trace of the parametric curve means that no portion of the parametric curve can be retraced.

Note that one full trace does not mean that we get back to the "starting" point. When we dealt with an ellipse in a previous problem that was one trace because we did not need to retrace any portion of the ellipse to get back to the starting point. However, as we saw in the previous step that for our square root here we would have to retrace the full curve to get back to the starting point.

So, one full trace of the parametric curve means we move from the right end point to the left end point only or visa-versa and move from the left end point to the right end point. Which direction we move doesn't really matter here so let's get a range of t's that take us from the left end point to the right end point.

In order to be at the left end point we need to require that $\cos\left(\frac{5}{2}t\right) = -1$ which occurs if $\frac{5}{2}t = \ldots, -3\pi, -\pi, \pi, 3\pi, \ldots$. Note as well that unlike the previous problems, which had both sine and cosine, this set of parametric equations has only cosine and so all we need to do here is look at this. Also, in order to be at the right end point we need to require that cos $\left(\frac{5}{2}t\right) = 1$ which occurs if $\frac{5}{2}t = \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$.

So, if we want to move from the left to right all we need to do is chose one from the list of t's corresponding to the left end point and then first t that comes that from the list corresponding to the right end point and we'll have a range of t's for one trace. To move from the right to left we just go the opposite direction, *i.e.* chose a t from the right end point list and then take the first t after that from the left end point list.

So, for this problem, since we said we were going to move from left to right, we'll use $\frac{5}{2}t = \pi$, which corresponds to $t = \frac{2}{5}\pi$, for the left end point. That in turn means that we'll need to use $\frac{5}{2}t = 2\pi$, which corresponds to $t = \frac{4}{5}\pi$, for the right end point. That means the range of t's for one trace is,

$$\frac{2}{5}\pi \le t \le \frac{4}{5}\pi$$

This is only one possible answer here. Any range of t's with a "net" range of $\frac{4}{5}\pi - \left(\frac{2}{5}\pi\right) = \frac{2}{5}\pi$ t's, with the endpoints of the t range corresponding to start/end points of the parametric curve, will work. So, for example, any of the following ranges of t's would also work.

$$-\frac{2}{5}\pi \le t \le 0 \qquad 0 \le t \le \frac{2}{5}\pi \qquad \frac{4}{5}\pi \le t \le \frac{6}{5}\pi$$

The direction of motion for each may be different range of t's of course. Some will trace out the curve moving from left to right while others will trace out the curve moving from right to left. Because the problem did not specify a particular direction any would work.

Note as well that the range $\frac{2}{5}\pi \le t \le \frac{4}{5}\pi$ falls completely inside the given range of t's specified in the problem and so we know that the particle will trace out the curve more than once over the full range of t's. Determining just how many times it traces over the curve will be determined in the next step.

Step 5

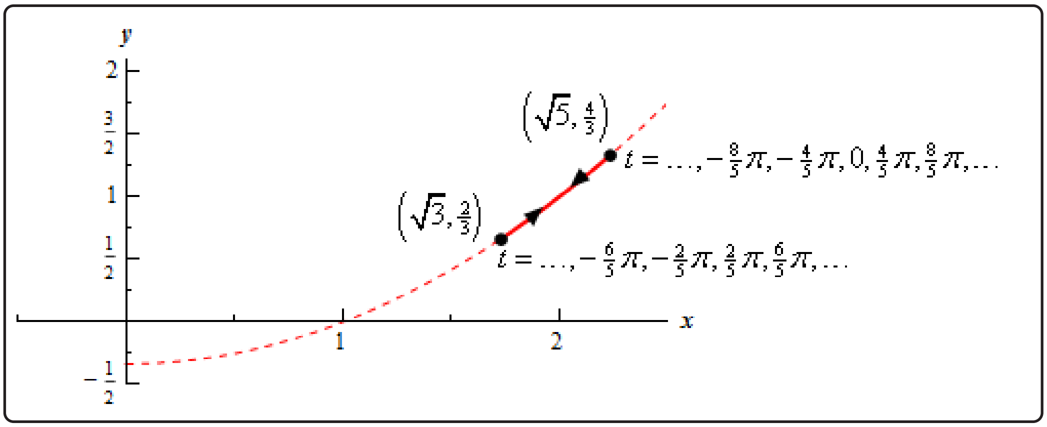
Now that we have a range of t's for one full trace of the parametric curve we can determine the number of traces the particle makes.

This is a really easy step. We know the total time the particle was traveling and we know how long it takes for a single trace. Therefore,

$$\text{Number Traces} = \frac{\text{Total Time Traveled}}{\text{Time for One Trace}} = \frac{2\pi - (-48\pi)}{\frac{4}{5}\pi - \left(\frac{2}{5}\pi\right)} = \frac{50\pi}{\frac{2}{5}\pi} = 125 \text{ traces}$$

Step 6

Finally, here is a sketch of the parametric curve for this set of parametric equations.



For this sketch we indicated the direction of motion by putting arrow heads going both directions in places on the curve. We also included a set of t's for a couple of points to illustrate where the particle is at while tracing out of the curve as well as coordinates for the end points since they aren't "nice" points.. The dashed line is the continuation of the square root from Step 1 to illustrate that our parametric curve is only a part of the square root.

Here is also the formal answers for all the rest of the information that problem asked for.

Range of x :	$\sqrt{3} \le x \le \sqrt{5}$
Range of y :	$\frac{2}{3} \le y \le \frac{4}{3}$
Range of t for one trace :	$\frac{2}{5}\pi \le t \le \frac{4}{5}\pi$
Total number of traces :	125

10. The path of a particle is given by the following set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

    (i) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

    (ii) Limits on x and y.

    (iii) A range of t's for a single trace of the parametric curve.

    (iv) The number of traces of the curve the particle makes if an overall range of t's is provided in the problem.

$$x = 2\mathbf{e}^{t} \qquad y = \cos\left(1 + \mathbf{e}^{3t}\right) \qquad 0 \le t \le \frac{3}{4}$$

Step 1

There's a lot of information we'll need to find to fully answer this problem. However, for most of it we can follow the same basic ordering of steps we used for the first few problems in this section. We will need however to do a little extra work along the way.

Also, because most of the work here is similar to the work we did in Problems 4 - 6 of this section we won't be putting in as much explanation to a lot of the work we're doing here. So, if you need some explanation for some of the work you should go back to those problems and check the corresponding steps.

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations let's first notice that we can solve the x equation for the exponential function as follows,

$$\mathbf{e}^{t} = \frac{1}{2}x$$

Now, just recall that $\mathbf{e}^{3t} = \left(\mathbf{e}^{t}\right)^{3}$ and so we can plug the above equation into the exponential in the y equation to get,

$$y = \cos\left(1 + \mathbf{e}^{3t}\right) = \cos\left(1 + \left(\mathbf{e}^{t}\right)^{3}\right) = \cos\left(1 + \left(\frac{1}{2}x\right)^{3}\right) = \cos\left(1 + \frac{1}{8}x^{3}\right)$$

So, the parametric curve will be some or all of the graph of this cosine function.

Step 2

At this point let's work on the limits for x and y. In this case, unlike most of the previous problems, things will work a little differently.

Let's start by noting that unlike sine and cosine functions we know $\mathbf{e}^{t}$ is always an increasing function (you can do some quick Calculus I work to verify this right?).

Why do we care about this? Well first the x equation is just a constant times $\mathbf{e}^{t}$ and we are given a range of t's for the problem. Next, the fact that $\mathbf{e}^{t}$ is an increasing function means that the x equation, $x = 2\mathbf{e}^{t}$, is also an increasing function (because the 2 is positive). Therefore, the smallest value of x will occur at the smallest value of t in the range of t's. Likewise, the largest value of x will occur at the largest value of t in the range of t's.

Therefore, the range of x for our parametric curve is,

$$2\mathbf{e}^0 \le x \le 2\mathbf{e}^{\frac{3}{4}} \qquad \Rightarrow \qquad 2 \le x \le 2\mathbf{e}^{\frac{3}{4}}$$

Unlike the previous problems where we usually needed to do a little more verification work we know at this point that this is the range of x's.

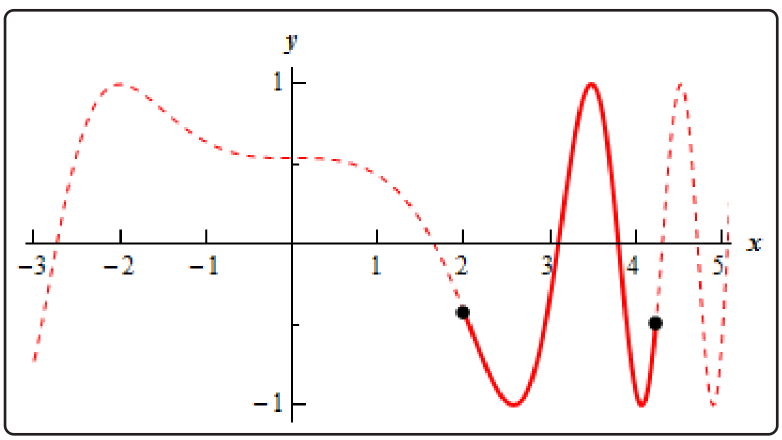
For the range of y's we will need to do a little work to get the correct range of y's but it won't be as much extra work as in previous problems and we can do it all in this step. First let's notice that because the y equation is in the form of $y = \cos(\cdots)$. The argument of the cosine doesn't matter for the first part of the work and so wasn't included here.

From the behavior of cosine we then know that the largest possible range of y would then be,

$$-1 \le y \le 1$$

Now, depending on just what values the argument of the cosine in the y equation takes over the give range of t's we may or may not cover this full range of values. We could do some work analyzing the argument of the cosine to figure that out if it does cover this full range. However, there is a really easy way to figure that if the full range is covered in this case.

Let's just sketch the graph and see what we get. Here is a quick sketch of the graph.



Given the "messy" nature of the argument of the cosine it's probably best to use some form of computational aid to get the graph. The dotted portion of the graph is full graph of the function on $-3 \le x \le 5$ without regards to the actual restriction on t. The solid portion of the graph is the portion that corresponds to the range of t's we were given in the problem.

From this graph we can see that the range of y's is in fact $-1 \le y \le 1$.

Before proceeding with the direction of motion let's note that we could also have just graphed the curve in many of the previous problems to determine if the work in this step was the actual range or not. We didn't do that because we could determine if these ranges were correct or not when we did the direction of motion and range of t's for one trace analysis (which we had to do anyway) and so didn't need to bother with a graph in this step for those problems.

Step 3

We now need to do the direction of motion for this curve but note that we actually found the direction of motion in the previous step.

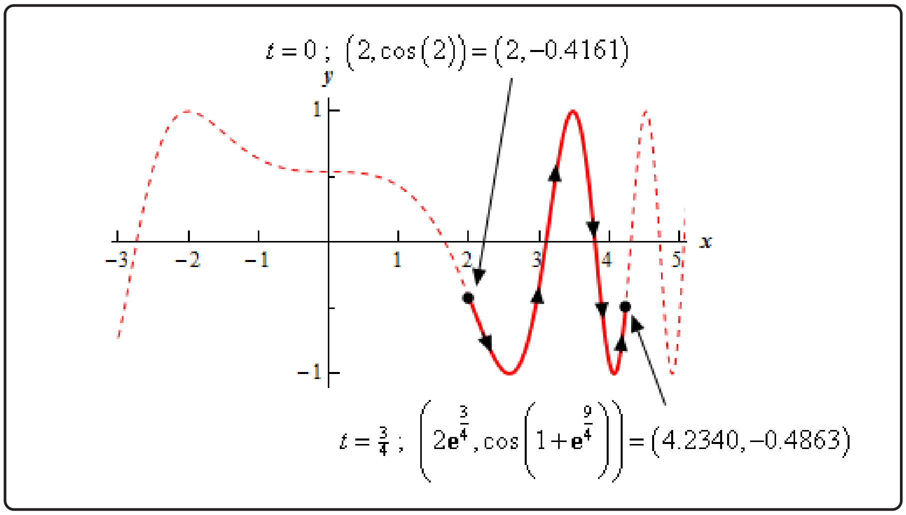
As noted in the previous step we know that $x = 2\mathbf{e}^t$ is an increasing function and so the x's must be increasing as t increases. Therefore, the equation must be moving from left to right as the curve is traced out over the given range of t's.

Also note that unlike the previous problems we know that no portion of the graph will be retraced. Again, we know the x equation is an increasing equation. If the curve were to retrace any portion we can see that the only way to do that would be to move back from right to left which would require x to decrease and that can't happen.

This means that we now know as well that the graph will trace out exactly once for the given range of t's, which in turn tells us that the given range of t's is also the range of t's for a single trace.

Step 4

Now that we have all the needed information we can do a formal sketch of the graph.



As with the graph above the dotted portion of the graph is full graph of the function on $-3 \le t \le 5$ without regards to the actual restriction on t. The solid portion of the graph is the portion that corresponds to the range of t's we were given in the problem. We also included the t value and coordinates of each end point for clarity although these are often not required for many problems.

Here is also the formal answers for all the rest of the information that problem asked for.

Range of x :	$2 \le x \le 2\mathbf{e}^{\frac{3}{4}}$
Range of y :	$-1 \le y \le 1$
Range of t for one trace :	$0 \le t \le \frac{3}{4}$
Total number of traces :	1

---

11. The path of a particle is given by the following set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

   (i) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

   (ii) Limits on x and y.

   (iii) A range of t's for a single trace of the parametric curve.

   (iv) The number of traces of the curve the particle makes if an overall range of t's is provided in the problem.

$$x = \frac{1}{2}\mathbf{e}^{-3t} \qquad\qquad y = \mathbf{e}^{-6t} + 2\mathbf{e}^{-3t} - 8$$

Step 1

There's a lot of information we'll need to find to fully answer this problem. However, for most of it we can follow the same basic ordering of steps we used for the first few problems in this section. We will need however to do a little extra work along the way.

Also, because most of the work here is similar to the work we did in Problems 4 - 6 of this section we won't be putting in as much explanation to a lot of the work we're doing here. So, if you need some explanation for some of the work you should go back to those problems and check the corresponding steps.

First, we'll eliminate the parameter from this set of parametric equations. For this particular set of parametric equations let's first notice that we can solve the x equation for the exponential function as follows,

$$\mathbf{e}^{-3t} = 2x$$

Now, just recall that $\mathbf{e}^{-6t} = \left(\mathbf{e}^{-3t}\right)^2$ and so we can plug the above equation into the exponential in the y equation to get,

$$y = \mathbf{e}^{-6t} + 2\mathbf{e}^{-3t} - 8 = \left(\mathbf{e}^{-3t}\right)^2 + 2\mathbf{e}^{-3t} - 8 = (2x)^2 + 2\,(2x) - 8 = 4x^2 + 4x - 8$$

So, the parametric curve will be some or all of the graph of this quadratic function.

Step 2

At this point let's work on the limits for x and y. In this case, unlike most of the previous problems, things will work a little differently.

Let's start by noting that unlike sine and cosine functions we know $\mathbf{e}^{-3t}$ is always a decreasing function as t increases (you can do some quick Calculus I work to verify this right?).

Why do we care about this? Well first the x equation is just a constant times $\mathbf{e}^{-3t}$ and so the fact that $\mathbf{e}^{-3t}$ is a decreasing function means that the x equation, $x = \frac{1}{2}\mathbf{e}^{-3t}$, is also a decreasing function (because the coefficient is positive).

Next, we aren't given a range of t's for this problem and so we can assume the largest possible range of t's. Therefore, we are safe in assuming a range of $-\infty < t < \infty$ for the t's.

Now, as we've already noted the know that the x equation is decreasing and so the largest value of x will occur at the left "end point" of the range. Likewise, the smallest value of x will occur at the right "end point" of the range. For this problem both "end points” of our range are in fact infinities so we can’t just plug in as we did in the previous problem. We can however take the following two limits.

$$\lim_{t \to -\infty} \left(\frac{1}{2} \mathbf{e}^{-3t} \right) = \infty \qquad\qquad \lim_{t \to \infty} \left(\frac{1}{2} \mathbf{e}^{-3t} \right) = 0$$

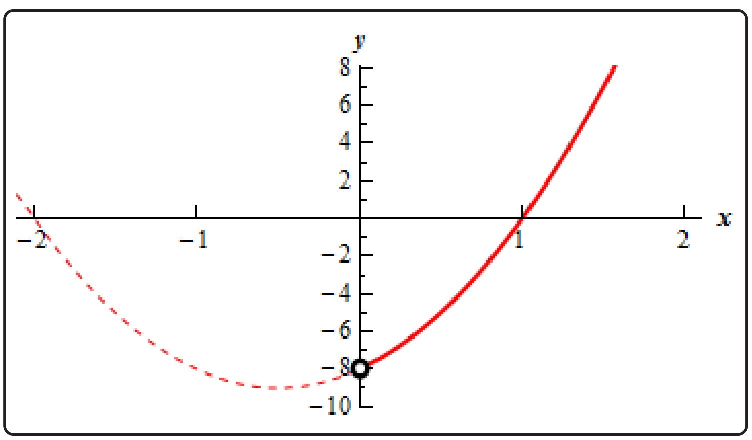
From this we can see that as we approach the left end point of the t range the value of x is going to infinity and as we approach the right end point of the t range the x value is going to zero. Note however that x can never actually be zero because x is still defined in terms of an exponential function (which can’t be zero). All the limit is telling us is that as we let $t \to \infty$ we will get $x \to 0$.

The range of x for our parametric curve is therefore,

$$0 < x < \infty$$

Again, be careful with the inequalities here! We know that x can be neither zero nor infinity so we must use strict inequalities for this range. This is something that we always need to be on the lookout for with variable ranges of parametric equations. Depending on the parametric equations sometimes the end points of the ranges will be strict inequalities (as with this problem) and for others they include the end points (as with the previous problems).

For the range of y’s we will need to do a little work to get the correct range of y’s but it won’t be as much extra work as in previous problems and we can do it all in this step. Let’s just sketch the graph and see what we get. Here is a quick sketch of the graph.



The dotted portion of the graph is full graph of the function on $-2 \le t \le 2$ without regards to the actual restriction on x. The solid portion of the graph is the portion of the graph that corresponds to the restriction on x that we found earlier in this step.

Note the “open dot” on the y-axis for the left end of the graph. This needs to be here to acknowledge that $x \neq 0$. We can also see that the y value at this point is $y = -8$ and again we can see that for the parametric curve we have $y \neq -8$.

Also, keep in mind that we know that $x \to \infty$ and so we also know that the y portion of the graph must also continue up to infinity to match the x behavior.

So, from this quick analysis of the graph we can see that the y range for the parametric curve must be,

$$-8 < y < \infty$$

Step 3

We now need to do the direction of motion for this curve but note that we actually found the direction of motion in the previous step.

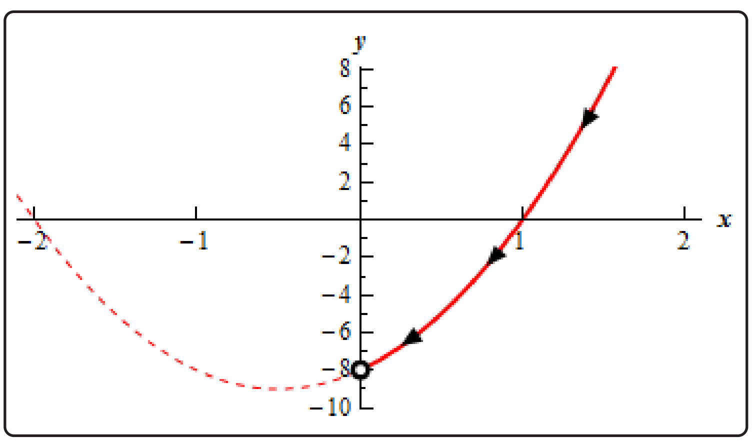
As noted in the previous step we know that $x = \frac{1}{2}\mathbf{e}^{-3t}$ is a decreasing function and so the x's must be decreasing as t increases. Therefore, the equation must be moving from right to left as the curve is traced out.

Also note that unlike most of the previous problems we know that no portion of the graph will be retraced. Again, we know the x equation is a decreasing equation. If the curve were to retrace any portion we can see that the only way to do that would be to move back from left to right which would require x to increase and that can't happen.

This means that we now know as well that the graph will trace out exactly once.

Step 4

Now that we have all the needed information we can do a formal sketch of the graph.



As with the graph above the dotted portion of the graph is full graph of the function on

$-2 \le t \le 2$ and the solid portion of the graph is the portion that corresponds to the restrictions x and y we found in Step 2.

Here is also the formal answers for all the rest of the information that problem asked for.

Range of x :	$0 \le x < \infty$
Range of y :	$-8 \le y < \infty$
Range of t for one trace :	$-\infty < t < \infty$
Total number of traces :	1

12. Write down a set of parametric equations for the following equation.

$$y = 3x^2 - \ln(4x+2)$$

Solution

There really isn't a lot to this problem. All we need to do is use the "formulas" right at the end of the notes for this section.

A set of parametric equations for the equation above are,

$$\begin{aligned} x &= t \\ y &= 3t^2 - \ln(4t+2) \end{aligned}$$

13. Write down a set of parametric equations for the following equation.

$$x^2 + y^2 = 36$$

The parametric curve resulting from the parametric equations should be at $(6,0)$ when $t = 0$ and the curve should have a counter clockwise rotation.

Solution

If we don't worry about the "starting" point (*i.e.* where the curve is at when $t = 0$) and we don't worry about the direction of motion we know from the notes that the following set of parametric equations will trace out a circle of radius 6 centered at the origin.

$$\begin{aligned} x &= 6\cos(t) \\ y &= 6\sin(t) \end{aligned}$$

All we need to do is verify if the extra requirements are met or not.

First, we can clearly see with a quick evaluation that when $t = 0$ we are at the point

$(6,0)$ as we need to be.

Next, we can either use our knowledge from the examples worked in the notes for this section or an analysis similar to some of the earlier problems in this section to verify that circles in this form will always trace out in a counter clockwise rotation.

In other words, the set of parametric equations give above is a set of parametric equations which will trace out the given circle with the given restrictions. So, formally the answer for this problem is,

$$\boxed{\begin{aligned} x &= 6\cos(t) \\ y &= 6\sin(t) \end{aligned}}$$

We'll leave this problem with a final note about the answer here. This is possibly the "simplest" answer we could give but it is completely possible that you may have come up with a different answer to this problem. There are almost always lots of different possible sets of parametric equations that will trace out a particular parametric curve according to some particular set of restrictions.

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14. Write down a set of parametric equations for the following equation.

$$\frac{x^2}{4} + \frac{y^2}{49} = 1$$

The parametric curve resulting from the parametric equations should be at $(0,-7)$ when $t=0$ and the curve should have a clockwise rotation.

Solution

If we don't worry about the "starting" point (*i.e.* where the curve is at when $t=0$) and we don't worry about the direction of motion we know from the notes that the following set of parametric equations will trace out the ellipse given by the equation above.

$$\begin{aligned} x &= 2\cos(t) \\ y &= 7\sin(t) \end{aligned}$$

The problem with this set of parametric equations is that when $t=0$ we are at the point $(2,0)$ which is not the point we are supposed to be at. Also, from our knowledge of the examples worked in the notes for this section or an analysis similar to some of the earlier problems in this section we can see that the parametric curve traced out by this set of equations will trace out in a counter clockwise rotation - again not what we need.

So, we need to come up with a different set of parametric equations that meets the requirements.

The first thing to acknowledge is that using sine and cosine will always be the easiest way to get a set of parametric equations for an ellipse. However, there is no reason at all to always use cosine for the x equation and sine for the y equation.

Knowing that we need $x = 0$ and $y = -7$ when $t = 0$ and using the fact that we know that $\sin(0) = 0$ and $\cos(0) = 1$ the following set of parametric equations will "start" at the correct point when $t = 0$.

$$x = -2\sin(t)$$
$$y = -7\cos(t)$$

All we need to do now is check if this will trace out the ellipse in a clockwise direction.

If we start at $t = 0$ and increase t until we reach $t = \frac{\pi}{2}$ we know that sine will increase from 0 to 1. This will in turn mean that x must decrease (don't forget the minus sign on the x equation) from 0 to -2.

Likewise, increasing t from $t = 0$ to $t = \frac{\pi}{2}$ we know that cosine will decrease from 1 to 0. This in turn means that y will increase (don't forget the minus sign on the y equation!) from -7 to 0.

The only way for both of these things to happen at the same time is for the curve to start at $(0, -7)$ when $t = 0$ and trace along the ellipse in a clockwise direction until we reach the point $(-2, 0)$ when $t = \frac{\pi}{2}$.

We could continue in this fashion further increasing t until it reaches $t = 2\pi$ (which will put us back at the "starting" point) and convince ourselves that the ellipse will continue to trace out in a clockwise direction.

Therefore, one possible set of parametric equations that we could use is,

$$\boxed{\begin{aligned} x &= -2\sin(t) \\ y &= -7\cos(t) \end{aligned}}$$

We'll leave this problem with a final note about the answer here. This is possibly the "simplest" answer we could give but it is completely possible that you may have come up with a different answer to this problem. There are almost always lots of different possible sets of parametric equations that will trace out a particular parametric curve according to some particular set of restrictions.

9.2 Tangents with Parametric Equations

1. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following set of parametric equations.

$$x = 4t^3 - t^2 + 7t \qquad y = t^4 - 6$$

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 12t^2 - 2t + 7 \qquad \frac{dy}{dt} = 4t^3$$

Step 2

The first derivative is then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \boxed{\frac{4t^3}{12t^2 - 2t + 7}}$$

Step 3

For the second derivative we'll now need,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{4t^3}{12t^2 - 2t + 7}\right) = \frac{\left(12t^2\right)\left(12t^2 - 2t + 7\right) - 4t^3\left(24t - 2\right)}{\left(12t^2 - 2t + 7\right)^2}$$

$$= \boxed{\frac{48t^4 - 16t^3 + 84t^2}{\left(12t^2 - 2t + 7\right)^2}}$$

Step 4

The second derivative is then,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{48t^4 - 16t^3 + 84t^2}{\left(12t^2 - 2t + 7\right)^2}}{12t^2 - 2t + 7} = \boxed{\frac{48t^4 - 16t^3 + 84t^2}{\left(12t^2 - 2t + 7\right)^3}}$$

2. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following set of parametric equations.

$$x = \mathbf{e}^{-7t} + 2 \qquad y = 6\mathbf{e}^{2t} + \mathbf{e}^{-3t} - 4t$$

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = -7\mathbf{e}^{-7t} \qquad \frac{dy}{dt} = 12\mathbf{e}^{2t} - 3\mathbf{e}^{-3t} - 4$$

Step 2

The first derivative is then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12\mathbf{e}^{2t} - 3\mathbf{e}^{-3t} - 4}{-7\mathbf{e}^{-7t}} = \boxed{-\frac{12}{7}\mathbf{e}^{9t} + \frac{3}{7}\mathbf{e}^{4t} + \frac{4}{7}\mathbf{e}^{7t}}$$

Step 3

For the second derivative we'll now need,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(-\frac{12}{7}\mathbf{e}^{9t} + \frac{3}{7}\mathbf{e}^{4t} + \frac{4}{7}\mathbf{e}^{7t}\right) = \boxed{-\frac{108}{7}\mathbf{e}^{9t} + \frac{12}{7}\mathbf{e}^{4t} + 4\mathbf{e}^{7t}}$$

Step 4

The second derivative is then,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\frac{108}{7}\mathbf{e}^{9t} + \frac{12}{7}\mathbf{e}^{4t} + 4\mathbf{e}^{7t}}{-7\mathbf{e}^{-7t}} = \boxed{\frac{108}{49}\mathbf{e}^{16t} - \frac{12}{49}\mathbf{e}^{11t} - \frac{4}{7}\mathbf{e}^{14t}}$$

3. Find the equation of the tangent line(s) to the following set of parametric equations at the given point.

$$x = 2\cos(3t) - 4\sin(3t) \qquad y = 3\tan(6t) \text{ at } t = \frac{\pi}{2}$$

Step 1

We'll need the first derivative for the set of parametric equations. We'll need the following derivatives,

$$\frac{dx}{dt} = -6\sin(3t) - 12\cos(3t) \qquad \frac{dy}{dt} = 18\sec^2(6t)$$

The first derivative is then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{18\sec^2(6t)}{-6\sin(3t) - 12\cos(3t)} = \frac{3\sec^2(6t)}{-\sin(3t) - 2\cos(3t)}$$

Step 2

The slope of the tangent line at $t = \frac{\pi}{2}$ is then,

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{3(-1)^2}{-(-1) - 2(0)} = 3$$

At $t = \frac{\pi}{2}$ the parametric curve is at the point,

$$\left.x\right|_{t=\frac{\pi}{2}} = 2(0) - 4(-1) = 4 \qquad \left.y\right|_{t=\frac{\pi}{2}} = 3(0) = 0 \qquad \Rightarrow \qquad (4, 0)$$

Step 3

The (only) tangent line for this problem is then,

$$y = 0 + 3(x - 4) \qquad \rightarrow \qquad \boxed{y = 3x - 12}$$

4. Find the equation of the tangent line(s) to the following set of parametric equations at the given point.

$$x = t^2 - 2t - 11 \qquad y = t(t-4)^3 - 3t^2(t-4)^2 + 7 \text{ at } (-3,7)$$

Step 1

We'll need the first derivative for the set of parametric equations. We'll need the following derivatives,

$$\frac{dx}{dt} = 2t - 2$$

$$\frac{dy}{dt} = (t-4)^3 + 3t(t-4)^2 - 6t(t-4)^2 - 6t^2(t-4) = (t-4)^3 - 3t(t-4)^2 - 6t^2(t-4)$$

The first derivative is then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(t-4)^3 - 3t(t-4)^2 - 6t^2(t-4)}{2t-2}$$

Hint

Don't forget that because the derivative we found above is in terms of t we need to determine the value(s) of t that put the parametric curve at the given point.

Step 2

Okay, the derivative we found above is in terms of t and we we'll need to next determine the value(s) of t that put the parametric curve at $(-3,7)$.

This is easy enough to do by setting the x and y coordinates equal to the known parametric equations and determining the value(s) of t that satisfy both equations.

Doing that gives,

$$\begin{aligned} -3 &= t^2 - 2t - 11 \\ 0 &= t^2 - 2t - 8 \\ 0 &= (t-4)(t+2) \qquad \rightarrow \qquad t = -2, t = 4 \end{aligned}$$

$$\begin{aligned} 7 &= t(t-4)^3 - 3t^2(t-4)^2 + 7 \\ 0 &= (t-4)^2\left[t(t-4) - 3t^2\right] \\ 0 &= (t-4)^2\left[-4t - 2t^2\right] \\ 0 &= -2t(t-4)^2[2+t] \qquad \rightarrow \qquad t = -2, t = 0, t = 4 \end{aligned}$$

We can see from this list that the parametric curve will be at $(-3,7)$ for $t=-2$ and $t=4$.

Step 3

From the previous step we can see that we will in fact have two tangent lines at the point. Here are the slopes for each tangent line.

The slope of the tangent line at $t=-2$ is,

$$m = \left.\frac{dy}{dx}\right|_{t=-2} = -24$$

and the slope of the tangent line at $t=4$ is,

$$m = \left.\frac{dy}{dx}\right|_{t=4} = 0$$

Step 4

The tangent line for $t=-2$ is then,

$$y = 7 - 24\left(x+3\right) \qquad \rightarrow \qquad \boxed{y = -24x - 65}$$

The tangent line for $t=4$ is then,

$$y = 7 - \left(0\right)\left(x+3\right) \qquad \rightarrow \qquad \boxed{y = 7}$$

Do not get excited about the second tangent line! It is just saying that the second tangent line is a horizontal line.

---

5. Find the values of t that will have horizontal or vertical tangent lines for the following set of parametric equations.

$$x = t^5 - 7t^4 - 3t^3 \qquad y = 2\cos(3t) + 4t$$

Step 1

We'll need the following derivatives for this problem.

$$\frac{dx}{dt} = 5t^4 - 28t^3 - 9t^2 \qquad\qquad \frac{dy}{dt} = -6\sin(3t) + 4$$

Step 2

We know that horizontal tangent lines will occur where $\frac{dy}{dt} = 0$, provided $\frac{dx}{dt} \neq 0$ at the same value of t.

So, to find the horizontal tangent lines we'll need to solve,

$$-6\sin(3t) + 4 = 0 \quad \rightarrow \quad \sin(3t) = \frac{2}{3} \quad \rightarrow \quad 3t = \sin^{-1}\left(\frac{2}{3}\right) = 0.7297$$

Also, a quick glance at a unit circle we can see that a second angle is,

$$3t = \pi - 0.7297 = 2.4119$$

All possible values of t that will give horizontal tangent lines are then,

$$\begin{aligned} 3t &= 0.7297 + 2\pi n \\ 3t &= 2.4119 + 2\pi n \end{aligned} \quad \rightarrow \quad \begin{aligned} t &= 0.2432 + \tfrac{2}{3}\pi n \\ t &= 0.8040 + \tfrac{2}{3}\pi n \end{aligned} \quad , \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Note that we don't officially know these do in fact give horizontal tangent lines until we also determine that $\frac{dx}{dt} \neq 0$ at these points. We'll be able to determine that after the next step.

Step 3

We know that vertical tangent lines will occur where $\frac{dx}{dt} = 0$, provided $\frac{dy}{dt} \neq 0$ at the same value of t.

So, to find the vertical tangent lines we'll need to solve,

$$5t^4 - 28t^3 - 9t^2 = 0$$

$$t^2\left(5t^2 - 28t - 9\right) = 0 \qquad \rightarrow \qquad \begin{array}{l} t = 0, \;\; t = \dfrac{28 \pm \sqrt{964}}{10} \\ t = 0, \;\; t = -0.3048, \;\; t = 5.9048 \end{array}$$

Step 4

From a quick inspection of the two lists of t values from Step 2 and Step 3 we can see there are no values in common between the two lists. Therefore, any values of t that gives $\frac{dy}{dt} = 0$ will not give $\frac{dx}{dt} = 0$ and visa-versa.

Therefore the values of t that gives horizontal tangent lines are,

$$\boxed{\begin{array}{l} t = 0.2432 + \frac{2}{3}\pi n \\ t = 0.8040 + \frac{2}{3}\pi n \end{array}, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots}$$

The values of t that gives vertical tangent lines are,

$$\boxed{t = 0, \;\; t = -0.3048, \;\; t = 5.9048}$$

9.3 Area with Parametric Equations

1. Determine the area of the region below the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once from right to left for the given range of t. You should only use the given parametric equations to determine the answer.

$$x = 4t^3 - t^2 \qquad y = t^4 + 2t^2 \qquad 1 \le t \le 3$$

Solution

There really isn't too much to this problem. Just recall that the formula from the notes assumes that $x = f(t)$ and $y = g(t)$. So, the area under the curve is,

$$\begin{aligned} A &= \int_1^3 \left(t^4 + 2t^2\right)\left(12t^2 - 2t\right)\,dt \\ &= \int_1^3 12t^6 - 2t^5 + 24t^4 - 4t^3\,dt \\ &= \left(\frac{12}{7}t^7 - \frac{1}{3}t^6 + \frac{24}{5}t^5 - t^4\right)\Bigg|_1^3 = \boxed{\frac{481568}{105} = 4586.3619} \end{aligned}$$

2. Determine the area of the region below the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once from right to left for the given range of t. You should only use the given parametric equations to determine the answer.

$$x = 3 - \cos^3(t) \qquad y = 4 + \sin(t) \qquad 0 \le t \le \pi$$

Solution

There really isn't too much to this problem. Just recall that the formula from the notes assumes that $x = f(t)$ and $y = g(t)$. So, the area under the curve is,

$$\begin{aligned}
A &= \int_0^{\pi} \left(4 + \sin(t)\right)\left(3\cos^2(t)\sin(t)\right)\,dt \\
&= \int_0^{\pi} 12\cos^2(t)\sin(t) + 3\cos^2(t)\sin^2(t)\,dt \\
&= \int_0^{\pi} 12\cos^2(t)\sin(t) + 3\left[\frac{1}{2}\sin(2t)\right]^2\,dt \\
&= \int_0^{\pi} 12\cos^2(t)\sin(t) + \frac{3}{4}\sin^2(2t)\,dt \\
&= \int_0^{\pi} 12\cos^2(t)\sin(t) + \frac{3}{8}\left(1 - \cos(4t)\right)\,dt \\
&= \left(-4\cos^3(t) + \frac{3}{8}t - \frac{3}{32}\sin(4t)\right)\Bigg|_0^{\pi} = \boxed{8 + \frac{3}{8}\pi}
\end{aligned}$$

You did recall how to do all the trig manipulations and trig integrals to do this integral correct? If not you should go back to the Integrals Involving Trig Functions section to do some review/problems.

9.4 Arc Length with Parametric Equations

1. Determine the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of t's.

$$x = 8t^{\frac{3}{2}} \qquad y = 3 + (8 - t)^{\frac{3}{2}} \qquad 0 \le t \le 4$$

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 12t^{\frac{1}{2}} \qquad\qquad \frac{dy}{dt} = -\frac{3}{2}(8-t)^{\frac{1}{2}}$$

Step 2

We'll need the ds for this problem.

$$ds = \sqrt{\left[12t^{\frac{1}{2}}\right]^2 + \left[-\frac{3}{2}(8-t)^{\frac{1}{2}}\right]^2}\,dt = \sqrt{144t + \frac{9}{4}(8-t)}\,dt = \sqrt{\frac{567}{4}t + 18}\,dt$$

Step 3

The integral for the arc length is then,

$$L = \int ds = \int_0^4 \sqrt{\frac{567}{4}t + 18}\,dt$$

Step 4

This is a simple integral to compute with a quick substitution. Here is the integral work,

$$L = \int_0^4 \sqrt{\frac{567}{4}t + 18}\,dt = \frac{4}{567}\left(\frac{2}{3}\right)\left(\frac{567}{4}t + 18\right)^{\frac{3}{2}}\Bigg|_0^4$$

$$= \boxed{\frac{8}{1701}\left(585^{\frac{3}{2}} - 18^{\frac{3}{2}}\right) = 66.1865}$$

2. Determine the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of t's.

$$x = 3t + 1 \qquad y = 4 - t^2 \qquad -2 \le t \le 0$$

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = -2t$$

Step 2

We'll need the ds for this problem.

$$ds = \sqrt{[3]^2 + [-2t]^2}\,dt = \sqrt{9 + 4t^2}\,dt$$

Step 3

The integral for the arc length is then,

$$L = \int ds = \int_{-2}^{0} \sqrt{9 + 4t^2}\,dt$$

Step 4

This integral will require a trig substitution (as will quite a few arc length integrals!).

Here is the trig substitution we'll need for this integral.

$$t = \frac{3}{2}\tan(\theta) \qquad dt = \frac{3}{2}\sec^2(\theta)\,d\theta$$

$$\sqrt{9 + 4t^2} = \sqrt{9 + 9\tan^2(\theta)} = 3\sqrt{1 + \tan^2(\theta)} = 3\sqrt{\sec^2(\theta)} = 3\left|\sec(\theta)\right|$$

To get rid of the absolute value on the secant will need to convert the limits into θ limits.

$$t = -2: \qquad -2 = \frac{3}{2}\tan(\theta) \quad \to \quad \tan(\theta) = -\frac{4}{3} \quad \to \quad \theta = \tan^{-1}\left(-\frac{4}{3}\right) = -0.9273$$

$$t = 0: \qquad 0 = \frac{3}{2}\tan(\theta) \quad \to \quad \tan(\theta) = 0 \quad \to \quad \theta = 0$$

Okay, the corresponding range of θ for this problem is $-0.9273 \le \theta \le 0$ (fourth quadrant) and in this range we know that secant is positive. Therefore the root becomes,

$$\sqrt{9+4t^2} = 3\sec(\theta)$$

The integral is then,

$$\begin{aligned} L = \int_{-2}^{0} \sqrt{9+4t^2}\,dt &= \int_{-0.9273}^{0} (3\sec(\theta))\left(\frac{3}{2}\sec^2(\theta)\right)\,d\theta \\ &= \int_{-0.9273}^{0} \frac{9}{2}\sec^3(\theta)\,d\theta \\ &= \frac{9}{4}\left[\sec(\theta)\tan(\theta) + ln\left|\sec(\theta)+\tan(\theta)\right|\right]\Big|_{-0.9273}^{0} = \boxed{7.4719} \end{aligned}$$

---

3. A particle travels along a path defined by the following set of parametric equations. Determine the total distance the particle travels and compare this to the length of the parametric curve itself.

$$x = 4\sin\left(\frac{1}{4}t\right) \qquad y = 1 - 2\cos^2\left(\frac{1}{4}t\right) \qquad -52\pi \le t \le 34\pi$$

Hint

Be very careful with this problem. Note the two quantities we are being asked to find, how they relate to each other and which of the two that we know how to compute from the material in this section.

Step 1

This is a problem that many students have issues with. First note that we are being asked to find both the total distance traveled by the particle AND the length of the curve. Also, recall that of these two quantities we only discussed how to determine the length of a curve in this section.

Therefore, let's concentrate on finding the length of the curve first, then we'll worry about the total distance traveled.

Step 2

To find the length we'll need the following two derivatives,

$$\frac{dx}{dt} = \cos\left(\frac{1}{4}t\right) \qquad\qquad \frac{dy}{dt} = \cos\left(\frac{1}{4}t\right)\sin\left(\frac{1}{4}t\right)$$

The ds for this problem is then,

$$\begin{aligned} ds &= \sqrt{\left[\cos\left(\frac{1}{4}t\right)\right]^2 + \left[\cos\left(\frac{1}{4}t\right)\sin\left(\frac{1}{4}t\right)\right]^2}\,dt \\ &= \sqrt{\cos^2\left(\frac{1}{4}t\right) + \cos^2\left(\frac{1}{4}t\right)\sin^2\left(\frac{1}{4}t\right)}\,dt \end{aligned}$$

Now, this is where many students run into issues with this problem. Many students use the following integral to determine the length of the curve.

$$\int_{-52\pi}^{34\pi} \sqrt{\cos^2\left(\frac{1}{4}t\right) + \cos^2\left(\frac{1}{4}t\right)\sin^2\left(\frac{1}{4}t\right)}\,dt$$

Can you see what is wrong with this integral?

Step 3

Remember from the discussion in this section that in order to use the arc length formula the curve can only trace out exactly once over the range of the limits in the integral.

Therefore, we can't even write down the formula that we did in the previous step until we first determine if the curve traces out exactly once in the given range of t's.

If it turns out that the curve traces out more than once in the given range of t's then the integral we wrote down in the previous step is simply wrong. We will then need to determine a range of t's for one trace so we can write down the proper integral for the length.

Luckily enough for us we actually did this in a practice problem in the Parametric Equations and Curves section. Examining this set of parametric equations was problem #8 from that section. From the solution to that problem we found a couple of pieces of information that will be needed for this problem.

First, we determined that the curve does trace out more than once in the given range of t's for the problem. In fact, we determined that the curve traced out 21.5 times over the given range of t's.

Secondly, and more importantly at this point, we also determined that the curve would trace out exactly once in the range of $-2\pi \le t \le 2\pi$. Note that we actually listed several possible ranges of t's for one trace and we can use any of them. This is simply the first one found and so it's the one we decided to use for this problem.

So, we can now see that the integral we wrote down in the previous step was in fact not correct and will not give the length of the curve.

Using the range of t's we found in the earlier problem we can see that the integral for the length is,

$$L = \int_{-2\pi}^{2\pi} \sqrt{\cos^2\left(\frac{1}{4}t\right) + \cos^2\left(\frac{1}{4}t\right)\sin^2\left(\frac{1}{4}t\right)}\,dt$$

Before proceeding with the next step we should address the fact that, in this case, we already had the information in hand to write down the proper integral for the length. In most cases this will not be the case and you will need to go back and do a shortened version of the analysis we did in the Parametric Equations and Curves section. We don't need all the information but to get the pertinent information we will need to go through most of the analysis.

If you need a refresher on how to do that analysis you should go back to that section and work through a few of the practice problems.

Step 4

Okay, let's get to work on evaluating the integral. At first glance this looks like a really unpleasant integral (and there's no ignoring the fact that it's not a super easy integral) however if we're careful it isn't as difficult as it might appear at first glance.

First, let's notice that we can do a little simplification as follows,

$$L = \int_{-2\pi}^{2\pi} \sqrt{\cos^2\left(\frac{1}{4}t\right)\left(1 + \sin^2\left(\frac{1}{4}t\right)\right)}\,dt = \int_{-2\pi}^{2\pi} \left|\cos\left(\frac{1}{4}t\right)\right| \sqrt{1 + \sin^2\left(\frac{1}{4}t\right)}\,dt$$

As always, be very careful with the absolute value bars! Depending on the range of t's we used for the integral it might not be possible to just drop them. So, the next thing let's do is determine how to deal with them.

We know that, for this integral, we used the following range of t's.

$$-2\pi \le t \le 2\pi$$

Now, notice that we don't just have a t in the argument for the trig functions in our integral. In fact what we have is $\frac{1}{4}t$. So, we can see that as t ranges over $-2\pi \le t \le 2\pi$ we will get the following range for $\frac{1}{4}t$ (just divide the above inequality by 4!),

$$-\frac{\pi}{2} \leq \frac{1}{4}t \leq \frac{\pi}{2}$$

We know from our trig knowledge that as long as $\frac{1}{4}t$ stays in this range, which it will for our integral, then $\cos\left(\frac{1}{4}t\right) \geq 0$ and so we can drop the absolute values from the cosine in the integral.

As a final word of warning here keep in mind that for other ranges of t's we might have had negative cosine in the range of t's and so we'd need to add in a minus sign to the integrand when we dropped the absolute value bars. Whether or not we need to do this will depend upon the range of t's we chose to use for one trace and so this quick analysis will need to be done for these kinds of integrals.

Step 5

Okay, at this point the integral for the length of the curve is now,

$$L = \int_{-2\pi}^{2\pi} \cos\left(\frac{1}{4}t\right) \sqrt{1 + \sin^2\left(\frac{1}{4}t\right)}\, dt$$

This still looks to be an unpleasant integral. However, in this case note that we can use the following simply substitution to convert it into a relatively easy integral.

$$u = \sin\left(\frac{1}{4}t\right) \qquad \rightarrow \qquad \sin^2\left(\frac{1}{4}t\right) = u^2 \qquad du = \frac{1}{4}\cos\left(\frac{1}{4}t\right)\, dt$$

$$t = -2\pi : \qquad u = \sin\left(-\frac{1}{2}\pi\right) = -1 \qquad\qquad t = 2\pi : \qquad u = \sin\left(\frac{1}{2}\pi\right) = 1$$

Under this substitution the integral of the length of the curve is then,

$$L = \int_{-1}^{1} 4\sqrt{1 + u^2}\, du$$

Step 6

Now, at this point we can see that we have a fairly simple trig substitution we'll need to evaluate to find the length of the curve. The trig substitution we'll need for this integral is,

$$t = \tan(\theta) \qquad dt = \sec^2(\theta)\,d\theta \qquad\qquad \sqrt{1+u^2} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)} = |\sec(\theta)|$$

To get rid of the absolute value on the secant will need to convert the limits into θ limits.

$$u = -1: \qquad -1 = \tan(\theta) \quad \rightarrow \quad \tan(\theta) = -1 \quad \rightarrow \quad \theta = -\frac{\pi}{4}$$
$$u = 1: \qquad 1 = \tan(\theta) \quad \rightarrow \quad \tan(\theta) = 1 \quad \rightarrow \quad \theta = \frac{\pi}{4}$$

Okay, the corresponding range of θ for this problem is $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ (first and fourth quadrant) and in this range we know that secant is positive. Therefore the root becomes,

$$\sqrt{1+u^2} = \sec(\theta)$$

The length of the curve is then,

$$L = \int_{-2\pi}^{2\pi} \sqrt{\cos^2\left(\frac{1}{4}t\right)\left(1+\sin^2\left(\frac{1}{4}t\right)\right)}\,dt = \int_{-1}^{1} 4\sqrt{1+u^2}\,du$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\sec^3(\theta)\,d\theta = 2\left[\sec(\theta)\tan(\theta) + ln\left|\sec(\theta)+\tan(\theta)\right|\right]\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \boxed{9.1824}$$

Step 7

Okay, that was quite a bit of work to get the length of the curve. We now need to recall that we were also asked to get the total distance traveled by the particle. This is actually quite easy to get now that we have the length of the curve.

To get the total distance traveled all we need to recall is that we noted in Step 3 above that we determined in problem #8 from the Parametric Equations and Curves section that the curve will trace out 21.5 times. Since we also know the length of a single trace of the curve we know that the total distance traveled by the particle must be,

$$\text{Total Distance Traveled} = (9.1824)\,(21.5) = \boxed{197.4216}$$

4. Set up, but do not evaluate, an integral that gives the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of t's.

$$x = 2 + t^2 \qquad y = \mathbf{e}^t \sin(2t) \qquad 0 \le t \le 3$$

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = \mathbf{e}^t \sin(2t) + 2\mathbf{e}^t \cos(2t)$$

Step 2

We'll need the ds for this problem.

$$ds = \sqrt{[2t]^2 + [\mathbf{e}^t \sin(2t) + 2\mathbf{e}^t \cos(2t)]^2}\,dt = \sqrt{4t^2 + [\mathbf{e}^t \sin(2t) + 2\mathbf{e}^t \cos(2t)]^2}\,dt$$

Step 3

The integral for the arc length is then,

$$L = \int ds = \int_0^3 \sqrt{4t^2 + [\mathbf{e}^t \sin(2t) + 2\mathbf{e}^t \cos(2t)]^2}\,dt$$

---

5. Set up, but do not evaluate, an integral that gives the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of t's.

$$x = \cos^3(2t) \qquad y = \sin\left(1 - t^2\right) \qquad -\frac{3}{2} \le t \le 0$$

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = -6\cos^2(2t)\sin(2t) \qquad \frac{dy}{dt} = -2t\cos\left(1 - t^2\right)$$

Step 2

We'll need the ds for this problem.

$$\begin{aligned} ds &= \sqrt{\left[-6\cos^2(2t)\sin(2t)\right]^2 + \left[-2t\cos\left(1-t^2\right)\right]^2}\,dt \\ &= \sqrt{36\cos^4(2t)\sin^2(2t) + 4t^2\cos^2\left(1-t^2\right)}\,dt \end{aligned}$$

Step 3

The integral for the arc length is then,

$$L = \int ds = \int_{-\frac{3}{2}}^{0} \sqrt{36\cos^4(2t)\sin^2(2t) + 4t^2\cos^2\left(1-t^2\right)}\,dt$$

9.5 Surface Area with Parametric Equations

1. Determine the surface area of the object obtained by rotating the parametric curve about the given axis. You may assume that the curve traces out exactly once for the given range of t's.

Rotate $x = 3 + 2t \qquad y = 9 - 3t \qquad 1 \le t \le 4$ about the y-axis.

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 2 \qquad\qquad \frac{dy}{dt} = -3$$

Step 2

We'll need the ds for this problem.

$$ds = \sqrt{[2]^2 + [-3]^2}\,dt = \sqrt{13}\,dt$$

Step 3

The integral for the surface area is then,

$$SA = \int 2\pi x\,ds = \int_1^4 2\pi\left(3+2t\right)\sqrt{13}\,dt = 2\pi\sqrt{13}\int_1^4 3 + 2t\,dt$$

Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about.

Step 4

This is a really simple integral...

$$SA = 2\pi\sqrt{13}\int_1^4 3 + 2t\,dt = 2\pi\sqrt{13}\left(3t + t^2\right)\Big|_1^4 = \boxed{48\pi\sqrt{13}}$$

2. Determine the surface area of the object obtained by rotating the parametric curve about the given axis. You may assume that the curve traces out exactly once for the given range of t's.

Rotate $x = 9 + 2t^2 \qquad y = 4t \qquad 0 \le t \le 2$ about the x-axis.

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 4t \qquad \frac{dy}{dt} = 4$$

Step 2

We'll need the ds for this problem.

$$ds = \sqrt{[4t]^2 + [4]^2}\,dt = \sqrt{16t^2 + 16}\,dt = 4\sqrt{t^2 + 1}\,dt$$

Note that we factored a 16 out of the root to make the rest of the work a little simpler to deal with.

Step 3

The integral for the surface area is then,

$$SA = \int 2\pi y\,ds = \int_0^2 2\pi\,(4t)\left(4\sqrt{t^2+1}\right)\,dt = 32\pi\int_0^2 t\sqrt{t^2+1}\,dt$$

Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about.

Step 4

This is a simple integral to compute with a quick substitution. Here is the integral work,

$$SA = 32\pi\int_0^2 t\sqrt{t^2+1}\,dt = \frac{32}{3}\pi\left(t^2+1\right)^{\frac{3}{2}}\Big|_0^2 = \boxed{\frac{32}{3}\pi\left(5^{\frac{3}{2}} - 1\right) = 341.1464}$$

3. Determine the surface area of the object obtained by rotating the parametric curve about the given axis. You may assume that the curve traces out exactly once for the given range of t's.

$$\text{Rotate } x = 3\cos(\pi t) \qquad y = 5t + 2 \qquad 0 \le t \le \frac{1}{2} \text{ about the } y\text{-axis.}$$

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = -3\pi\sin(\pi t) \qquad \frac{dy}{dt} = 5$$

Step 2

We'll need the ds for this problem.

$$ds = \sqrt{[-3\pi\sin(\pi t)]^2 + [5]^2}\,dt = \sqrt{9\pi^2\sin^2(\pi t) + 25}\,dt$$

Step 3

The integral for the surface area is then,

$$\begin{aligned} SA = \int 2\pi x\,ds &= \int_0^{\frac{1}{2}} 2\pi(3\cos(\pi t))\sqrt{9\pi^2\sin^2(\pi t) + 25}\,dt \\ &= 6\pi\int_0^{\frac{1}{2}} \cos(\pi t)\sqrt{9\pi^2\sin^2(\pi t) + 25}\,dt \end{aligned}$$

Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about.

Step 4

Okay, this is a particularly unpleasant looking integral but we need to be able to deal with these kinds of integrals on occasion. We'll be able to do quite a bit of simplification if we first use the following substitution.

$$u = \sin(\pi t) \quad \rightarrow \quad \sin^2(\pi t) = u^2 \qquad du = \pi\cos(\pi t)$$

$$t = 0: \quad u = \sin(0) = 0 \qquad\qquad t = \frac{1}{2}: \quad u = \sin\left(\frac{1}{2}\pi\right) = 1$$

With this substitution the integral becomes,

$$SA = 6\int_0^1 \sqrt{9\pi^2 u^2 + 25}\,du$$

Step 5

This integral can be evaluated with the following (somewhat messy...) trig substitution.

$$t = \frac{5}{3\pi}\tan(\theta) \qquad dt = \frac{5}{3\pi}\sec^2(\theta)\,d\theta$$

$$\sqrt{9\pi^2u^2 + 25} = \sqrt{25\tan^2(\theta) + 25} = 5\sqrt{\tan^2(\theta) + 1} = 5\sqrt{\sec^2(\theta)} = 5\left|\sec(\theta)\right|$$

To get rid of the absolute value on the secant will need to convert the limits into θ limits.

$$u = 0: \quad 0 = \frac{5}{3\pi}\tan(\theta) \quad \to \quad \tan(\theta) = 0 \quad \to \quad \theta = 0$$
$$u = 1: \quad 1 = \frac{5}{3\pi}\tan(\theta) \quad \to \quad \tan(\theta) = \frac{3\pi}{5} \quad \to \quad \theta = \tan^{-1}\left(\frac{3\pi}{5}\right) = 1.0830$$

Okay, the corresponding range of θ for this problem is $0 \le \theta \le 1.0830$ (first quadrant) and in this range we know that secant is positive. Therefore, the root becomes,

$$\sqrt{9\pi^2u^2 + 25} = 5\sec(\theta)$$

The surface area is then,

$$\begin{aligned} SA &= \int_0^{\frac{1}{2}} 2\pi\left(3\cos(\pi t)\right)\sqrt{9\pi^2\sin^2(\pi t) + 25}\,dt \\ &= 6\int_0^1 \sqrt{9\pi^2u^2 + 25}\,du \\ &= 6\int_0^{1.0830} \left(5\sec(\theta)\right)\left(\frac{5}{3\pi}\sec^2(\theta)\right)\,d\theta \\ &= 6\int_0^{1.0830} \frac{25}{3\pi}\sec^3(\theta)\,d\theta \\ &= \frac{25}{\pi}\left(\sec(\theta)\tan\theta + \ln\left|\sec(\theta) + \tan(\theta)\right|\right)\Big|_0^{1.0830} = \boxed{43.0705} \end{aligned}$$

This problem was a little messy but don't let that make you decide that you can't do these types of problems! They can be done and often can be simplified with some relatively simple substitutions.

4. Set up, but do not evaluate, an integral that gives the surface area of the object obtained by rotating the parametric curve about the given axis. You may assume that the curve traces out exactly once for the given range of t's.

Rotate $x = 1 + \ln\left(5 + t^2\right) \qquad y = 2t - 2t^2 \qquad 0 \le t \le 2$ about the x-axis

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = \frac{2t}{5+t^2} \qquad \frac{dy}{dt} = 2 - 4t$$

Step 2

We'll need the ds for this problem.

$$ds = \sqrt{\left[\frac{2t}{5+t^2}\right]^2 + \left[2-4t\right]^2}\,dt = \sqrt{\frac{4t^2}{\left(5+t^2\right)^2} + \left(2-4t\right)^2}\,dt$$

Step 3

The integral for the surface area is then,

$$SA = \int 2\pi y\,ds = \boxed{\int_0^2 2\pi\left(2t - 2t^2\right)\sqrt{\frac{4t^2}{\left(5+t^2\right)^2} + \left(2-4t\right)^2}\,dt}$$

Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about.

5. Set up, but do not evaluate, an integral that gives the surface area of the object obtained by rotating the parametric curve about the given axis. You may assume that the curve traces out exactly once for the given range of t's.

Rotate $x = 1 + 3t^2 \qquad y = \sin(2t)\cos\left(\frac{1}{4}t\right) \qquad 0 \le t \le \frac{1}{2}$ about the y-axis.

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 6t \qquad \frac{dy}{dt} = 2\cos(2t)\cos\left(\frac{1}{4}t\right) - \frac{1}{4}\sin(2t)\sin\left(\frac{1}{4}t\right)$$

Step 2

We'll need the ds for this problem.

$$ds = \sqrt{[6t]^2 + \left[2\cos(2t)\cos\left(\frac{1}{4}t\right) - \frac{1}{4}\sin(2t)\sin\left(\frac{1}{4}t\right)\right]^2}\,dt$$

Step 3

The integral for the surface area is then,

$$SA = \int 2\pi x\,ds$$

$$= \boxed{\int_0^{\frac{1}{2}} 2\pi\left(1 + 3t^2\right)\sqrt{36t^2 + \left(2\cos(2t)\cos\left(\frac{1}{4}t\right) - \frac{1}{4}\sin(2t)\sin\left(\frac{1}{4}t\right)\right)^2}\,dt}$$

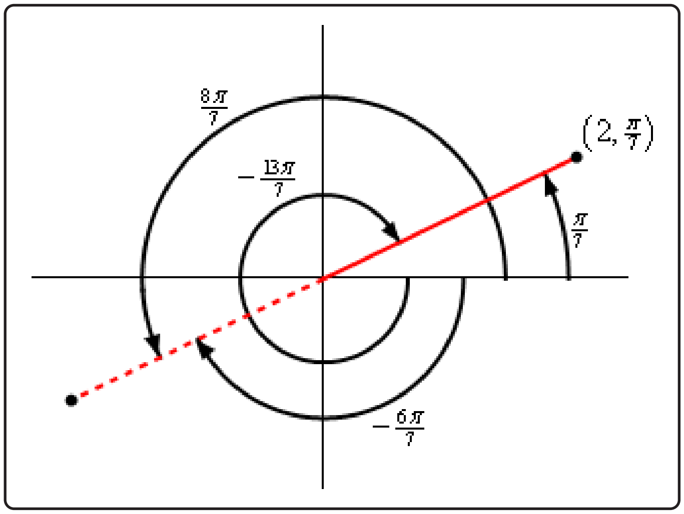
Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about.

9.6 Polar Coordinates

1. For the point with polar coordinates $\left(2, \frac{\pi}{7}\right)$ determine three different sets of coordinates for the same point all of which have angles different from $\frac{\pi}{7}$ and are in the range $-2\pi \le \theta \le 2\pi$.

Step 1

This problem is really as exercise in how well we understand the unit circle. Here is a quick sketch of the point and some angles.



We can see that the negative angle ending at the solid red line that is in the range specified in the problem statement is simply $\frac{\pi}{7} - 2\pi = -\frac{13\pi}{7}$.

If we extend the solid line into the third quadrant (*i.e.* the dashed red line) then the positive angle ending at the dashed red line is $\frac{\pi}{7} + \pi = \frac{8\pi}{7}$. Likewise, the negative angle ending at the dashed red line is $\frac{\pi}{7} - \pi = -\frac{6\pi}{7}$.

With these angles getting the other three points should be pretty simple.

Step 2

For the first "new" point we can use the negative angle that ends on the solid red line to get the point.

$$\boxed{\left(2, -\frac{13\pi}{7}\right)}$$

Step 3

For the remaining points recall that if we use a negative r then we go "backwards" from where the angle ends to get the point. So, if we use $r = -2$, any angle that ends on the dashed red line will go "backwards" into the first quadrant 2 units to get to the point.

This gives the remaining two points using both the positive and negative angle ending on the dashed red line,

$$\boxed{\left(-2, -\frac{6\pi}{7}\right)} \qquad \boxed{\left(-2, \frac{8\pi}{7}\right)}$$

---

2. The polar coordinates of a point are $(-5, 0.23)$. Determine the Cartesian coordinates for the point.

Solution

There really isn't too much to this problem. From the point we can see that we have $r = -5$ and $\theta = 0.23$ (in radians of course!). Once we have these all we need to is plug into the formulas from this section to get,

$$x = r\cos(\theta) = (-5)\cos(0.23) = -4.8683$$
$$y = r\sin(\theta) = (-5)\,sin\,(0.23) = -1.1399$$

So, the Cartesian coordinates for the point are then,

$$\boxed{(-4.8683,\ -1.1399)}$$

---

3. The Cartesian coordinate of a point are $(2, -6)$. Determine a set of polar coordinates for the point.

Step 1

Let's first determine r. That's always simple.

$$r = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (-6)^2} = \sqrt{40} = 2\sqrt{10}$$

Step 2

Next let's get θ. As we do this we need to remember that we actually have two possible values of which only one will work with the r we found in the first step.

Here are the two possible values of θ.

$$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-6}{2}\right) = -1.2490 \qquad \theta_2 = \theta_1 + \pi = 1.8926$$

So, we can see that $-\frac{\pi}{2} = -1.57 < \theta_1 = -1.2490 < 0$ and so θ_1 is in the fourth quadrant. Likewise, $\frac{\pi}{2} = 1.57 < \theta_2 = 1.8926 < \pi = 3.14$ and so θ_2 is in the second quadrant.

We can also see from the Cartesian coordinates of the point that our point must be in the fourth quadrant and so, for this problem, θ_1 is the correct value.

The polar coordinates of the point using the r from the first step and θ from this step is,

$$\boxed{\left(2\sqrt{10}, -1.2490\right)}$$

Note of course that there are many other sets of polar coordinates that are just as valid for this point. These are simply the set that we get from the formulas discussed in this section.

4. The Cartesian coordinate of a point are $(-8, 1)$. Determine a set of polar coordinates for the point.

Step 1

Let's first determine r. That's always simple.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-8)^2 + (1)^2} = \sqrt{65}$$

Step 2

Next let's get θ. As we do this we need to remember that we actually have two possible values of which only one will work with the r we found in the first step.

Here are the two possible values of θ.

$$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{-8}\right) = -0.1244 \qquad \theta_2 = \theta_1 + \pi = 3.0172$$

So, we can see that $-\frac{\pi}{2} = -1.57 < \theta_1 = -0.1244 < 0$ and so θ_1 is in the fourth quadrant. Likewise, $\frac{\pi}{2} = 1.57 < \theta_2 = 3.0172 < \pi = 3.14$ and so θ_2 is in the second quadrant.

We can also see from the Cartesian coordinates of the point that our point must be in the second quadrant and so, for this problem, θ_2 is the correct value.

The polar coordinates of the point using the r from the first step and θ from this step is,

$$\boxed{\left(\sqrt{65},\ \ 3.0172\right)}$$

Note of course that there are many other sets of polar coordinates that are just as valid for this point. These are simply the set that we get from the formulas discussed in this section.

5. Convert the following equation into an equation in terms of polar coordinates.

$$\frac{4x}{3x^2 + 3y^2} = 6 - xy$$

Solution

Basically, what we need to do here is to convert all the x's and y's into r's and θ's using the following formulas.

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad r^2 = x^2 + y^2$$

Don't forget about the last one! If it is possible to use this formula (and you can see where we'll use it in the case can't you?) it will save a lot of work!

First let's substitute in the equations as needed.

$$\frac{4\left(r\cos(\theta)\right)}{3r^2} = 6 - \left(r\cos(\theta)\right)\left(r\sin(\theta)\right)$$

Finally, as we need to do is take care of little simplification to get,

$$\boxed{\frac{4\cos(\theta)}{3r} = 6 - r^2\cos(\theta)\sin(\theta)}$$

6. Convert the following equation into an equation in terms of polar coordinates.

$$x^2 = \frac{4x}{y} - 3y^2 + 2$$

Solution

Basically, what we need to do here is to convert all the x's and y's into r's and θ's using the following formulas.

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad r^2 = x^2 + y^2$$

Don't forget about the last one! If it is possible to use this formula (which won't do us a lot of good in this problem) it will save a lot of work!

First let's substitute in the equations as needed.

$$(r\cos(\theta))^2 = \frac{4\,(r\cos(\theta))}{r\sin(\theta)} - 3(r\sin(\theta))^2 + 2$$

Finally, as we need to do is take care of little simplification to get,

$$\boxed{r^2\cos^2(\theta) = 4\cot(\theta) - 3r^2\sin^2(\theta) + 2}$$

7. Convert the following equation into an equation in terms of Cartesian coordinates.

$$6r^3\sin(\theta) = 4 - cos\theta$$

Solution

There is a variety of ways to work this problem. One way is to first multiply everything by r and then doing a little rearranging as follows,

$$6r^4\sin(\theta) = 4r - r\cos(\theta) \qquad \Rightarrow \qquad 6r^3\,(r\sin(\theta)) = 4r - r\cos(\theta)$$

We can now use the following formulas to finish this problem.

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad r = \sqrt{x^2 + y^2}$$

Here is the answer for this problem,

$$\boxed{6y\left[\sqrt{x^2+y^2}\right]^3 = 4\sqrt{x^2+y^2} - x}$$

8. Convert the following equation into an equation in terms of Cartesian coordinates.

$$\frac{2}{r} = \sin(\theta) - \sec(\theta)$$

Solution

There is a variety of ways to work this problem. One way is to first do the following rearranging/rewriting of the equation.

$$\frac{2}{r} = \sin(\theta) - \frac{1}{\cos(\theta)} \qquad \to \qquad \frac{2\cos(\theta)}{r} = \sin(\theta)\cos(\theta) - 1$$

At this point we can multiply everything by r^2 and do a little rearranging as follows,

$$2r\cos(\theta) = r^2\sin(\theta)\cos(\theta) - r^2 \qquad \to \qquad 2r\cos(\theta) = (r\sin(\theta))(r\cos(\theta)) - r^2$$

We can now use the following formulas to finish this problem.

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad r^2 = x^2 + y^2$$

Here is the answer for this problem,

$$\boxed{2x = yx - (x^2 + y^2)}$$

9. Sketch the graph of the following polar equation.

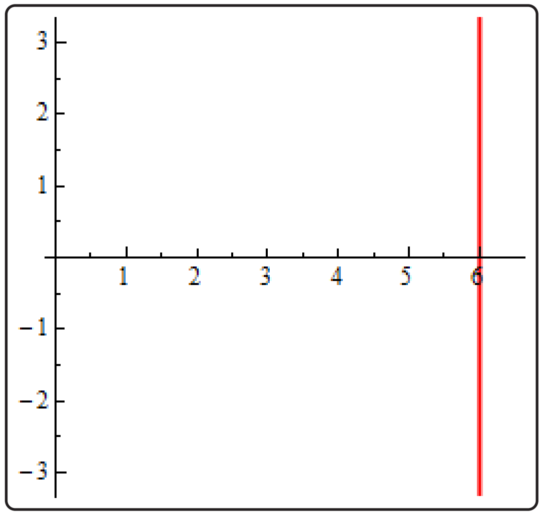
$$\cos(\theta) = \frac{6}{r}$$

Solution

Multiplying both sides by r gives,

$$r\cos(\theta) = 6$$

and we know from the notes on this section that this is simply the vertical line $x = 6$. So here is the graph of this function.



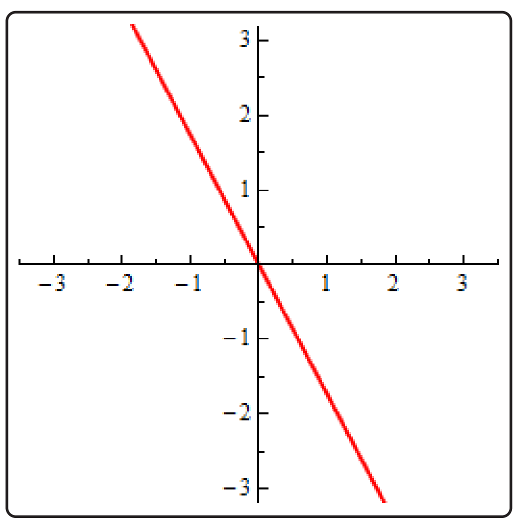
10. Sketch the graph of the following polar equation.

$$\theta = -\frac{\pi}{3}$$

Solution

We know from the notes on this section that this is simply the line that goes through the origin and has slope of $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$.

So here is the graph of this function.



11. Sketch the graph of the following polar equation.

$$r = -14\cos(\theta)$$

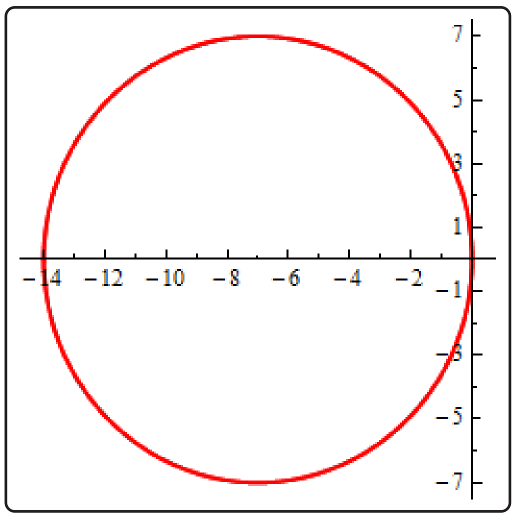
Solution

We can rewrite this as,

$$r = 2(-7)\cos(\theta)$$

and so we know from the notes on this section that this is simply the circle with radius 7 and center $(-7, 0)$.

So here is the graph of this function.



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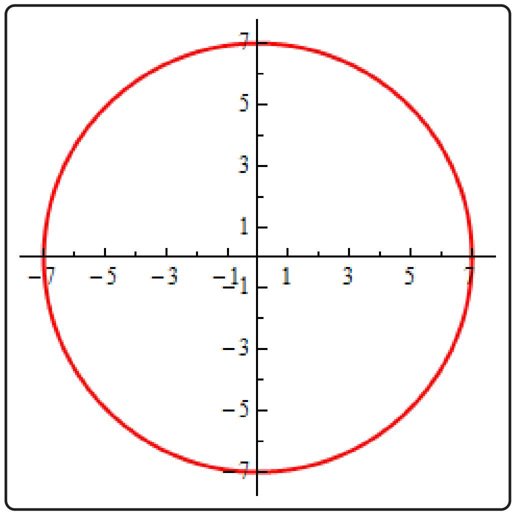
12. Sketch the graph of the following polar equation.

$$r = 7$$

Solution

We know from the notes on this section that this is simply the circle with radius 7 and centered at the origin.

So here is the graph of this function.



13. Sketch the graph of the following polar equation.

$$r = 9\sin(\theta)$$

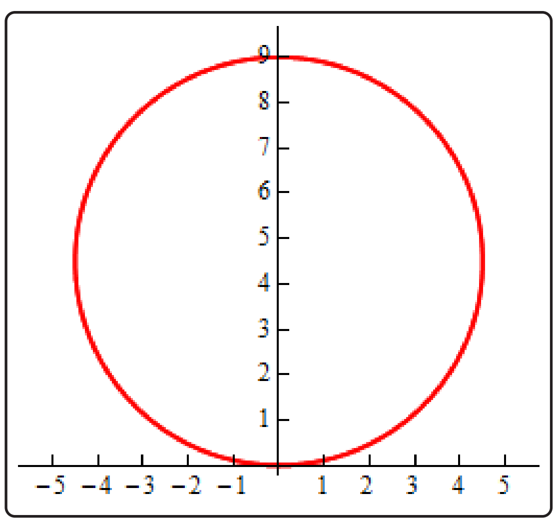
Solution

We can rewrite this as,

$$r = 2\left(\frac{9}{2}\right)\sin(\theta)$$

and so we know from the notes on this section that this is simply the circle with radius $\frac{9}{2}$ and center $\left(0, \frac{9}{2}\right)$.

So here is the graph of this function.



14. Sketch the graph of the following polar equation.

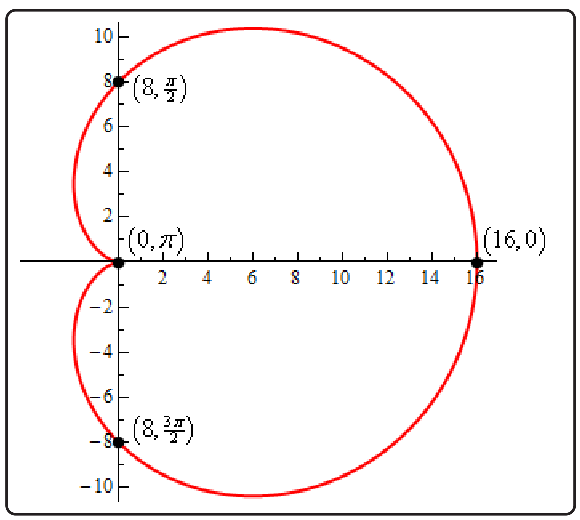
$$r = 8 + 8\cos(\theta)$$

Solution

We know from the notes on this section that this is a cardioid and so all we really need to get the graph is a quick chart of points.

θ	r	(r,θ)
0	16	$(16,0)$
$\frac{\pi}{2}$	8	$\left(8,\frac{\pi}{2}\right)$
π	0	$(0,\pi)$
$\frac{3\pi}{2}$	8	$\left(8,\frac{3\pi}{2}\right)$
2π	16	$(16,2\pi)$

So here is the graph of this function.



Be careful when plotting these points and remember the rules for graphing polar coordinates. The "tick marks" on the graph are really the Cartesian coordinate tick marks because those are the ones we are familiar with. Do not let them confuse you when you go to plot the polar points for our sketch.

15. Sketch the graph of the following polar equation.

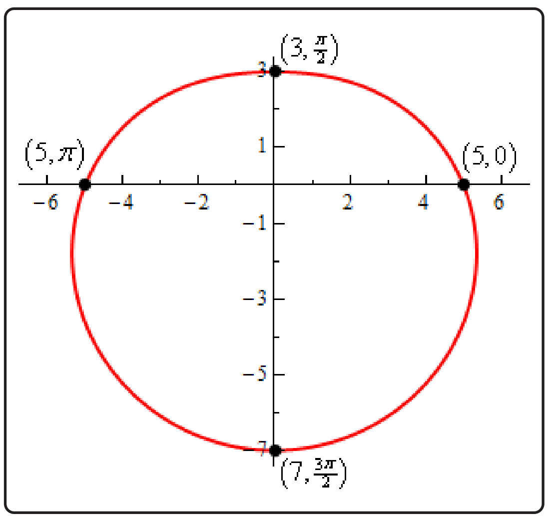
$$r = 5 - 2\sin(\theta)$$

Solution

We know from the notes on this section that this is a limacon without an inner loop and so all we really need to get the graph is a quick chart of points.

θ	r	(r, θ)
0	5	$(5, 0)$
$\frac{\pi}{2}$	3	$\left(3, \frac{\pi}{2}\right)$
π	5	$(5, \pi)$
$\frac{3\pi}{2}$	7	$\left(7, \frac{3\pi}{2}\right)$
2π	5	$(5, 2\pi)$

So here is the graph of this function.



Be careful when plotting these points and remember the rules for graphing polar coordinates. The "tick marks" on the graph are really the Cartesian coordinate tick marks because those are the ones we are familiar with. Do not let them confuse you when you go to plot the polar points for our sketch.

Also, many of these graphs are vaguely heart shaped although as this sketch has shown many do and this one is more circular than heart shaped.

16. Sketch the graph of the following polar equation.

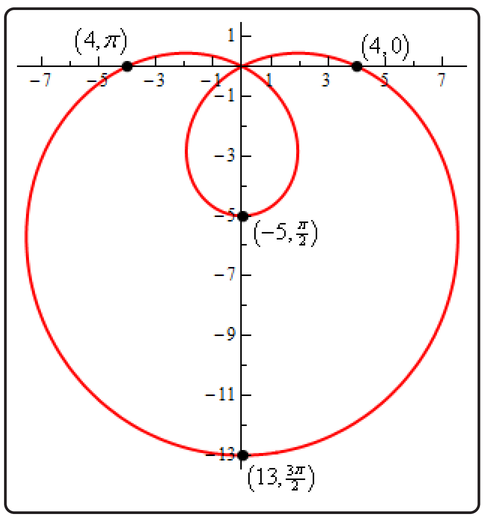
$$r = 4 - 9\sin(\theta)$$

Solution

We know from the notes on this section that this is a limacon with an inner loop and so all we really need to get the graph is a quick chart of points.

θ	r	(r,θ)
0	4	$(4,0)$
$\frac{\pi}{2}$	-5	$\left(-5,\frac{\pi}{2}\right)$
π	4	$(4,\pi)$
$\frac{3\pi}{2}$	13	$\left(13,\frac{3\pi}{2}\right)$
2π	4	$(4,2\pi)$

So here is the graph of this function.



Be careful when plotting these points and remember the rules for graphing polar coordinates. The "tick marks" on the graph are really the Cartesian coordinate tick marks because those are the ones we are familiar with. Do not let them confuse you when you go to plot the polar points for our sketch.

9.7 Tangents with Polar Coordinates

1. Find the tangent line to $r = \sin(4\theta)\cos(\theta)$ at $\theta = \frac{\pi}{6}$.

Step 1

First, we'll need to following derivative,

$$\frac{dr}{d\theta} = 4\cos(4\theta)\cos(\theta) - \sin(4\theta)\sin(\theta)$$

Step 2

Next using the formula from the notes on this section we have,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos\theta - r\sin(\theta)} \\ &= \frac{(4\cos(4\theta)\cos(\theta) - \sin(4\theta)\sin(\theta))\sin(\theta) + (\sin(4\theta)\cos(\theta))\cos(\theta)}{(4\cos(4\theta)\cos(\theta) - \sin(4\theta)\sin(\theta))\,cos\theta - (\sin(4\theta)\cos(\theta))\sin(\theta)}\end{aligned}$$

This is a very messy derivative (these often are) and, at least in this case, there isn't a lot of simplification that we can do...

Step 3

Next, we'll need to evaluate both the derivative from the previous step as well as r at $\theta = \frac{\pi}{6}$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{1}{3\sqrt{3}} \qquad\qquad r\big|_{\theta=\frac{\pi}{6}} = \frac{3}{4}$$

You can see why we need both of these right?

Step 4

Last, we need the x and y coordinate that we'll be at when $\theta = \frac{\pi}{6}$. These values are easy enough to find given that we know what r is at this point and we also know the polar to Cartesian coordinate conversion formulas. So,

$$x = r\cos(\theta) = \frac{3}{4}\cos\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8}$$
$$y = r\sin(\theta) = \frac{3}{4}\sin\left(\frac{\pi}{6}\right) = \frac{3}{8}$$

Of course, we also have the slope of the tangent line since it is just the value of the derivative we computed in the previous step.

Step 5

The tangent line is then,

$$\boxed{y = \frac{3}{8} + \frac{1}{3\sqrt{3}}\left(x - \frac{3\sqrt{3}}{8}\right) = \frac{1}{3\sqrt{3}}x + \frac{1}{4}}$$

2. Find the tangent line to $r = \theta - \cos(\theta)$ at $\theta = \frac{3\pi}{4}$.

Step 1

First, we'll need to following derivative,

$$\frac{dr}{d\theta} = 1 + \sin(\theta)$$

Step 2

Next using the formula from the notes on this section we have,

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos\theta - r\sin(\theta)}$$

$$= \frac{(1+\sin(\theta))\sin(\theta) + (\theta - \cos(\theta))\cos(\theta)}{(1+\sin(\theta))\,cos\theta - (\theta - \cos(\theta))\sin(\theta)}$$

This is a somewhat messy derivative (these often are) and, at least in this case, there isn't a lot of simplification that we can do...

Step 3

Next, we'll need to evaluate both the derivative from the previous step as well as r at $\theta = \frac{3\pi}{4}$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{3\pi}{4}} = 0.2843 \qquad\qquad r|_{\theta=\frac{3\pi}{4}} = 3.0633$$

You can see why we need both of these right?

Step 4

Last, we need the x and y coordinate that we'll be at when $\theta = \frac{3\pi}{4}$. These values are easy enough to find given that we know what r is at this point and we also know the polar to Cartesian coordinate conversion formulas. So,

$$x = r\cos(\theta) = 3.0633\cos\left(\frac{3\pi}{4}\right) = -2.1661$$
$$y = r\sin(\theta) = 3.0633\sin\left(\frac{3\pi}{4}\right) = 2.1661$$

Of course, we also have the slope of the tangent line since it is just the value of the derivative we computed in the previous step.

Step 5

The tangent line is then,

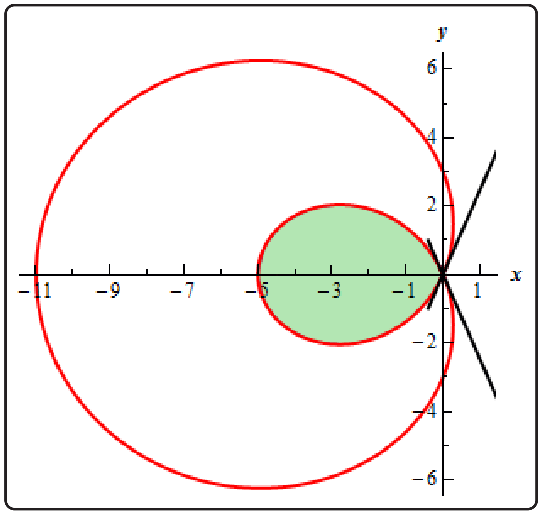
$$\boxed{y = 2.1661 + 0.2843\,(x + 2.1661) = 0.2843x + 2.7819}$$

9.8 Area with Polar Coordinates

1. Find the area inside the inner loop of $r = 3 - 8\cos(\theta)$.

Step 1

First, here is a quick sketch of the graph of the region we are interested in.



Step 2

Now, we'll need to determine the values of θ that the graph goes through the origin (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graph goes through the origin we know that we must have $r = 0$. So,

$$3 - 8\cos(\theta) = 0$$

$$\cos(\theta) = \frac{3}{8} \qquad \Rightarrow \qquad \theta = \cos^{-1}\left(\frac{3}{8}\right) = 1.1864$$

This is the angle in the first quadrant where the graph goes through the origin.

We next need the angle in the fourth quadrant. We need to be a little careful with this second angle. We need to always remember that the limits on the integral we'll eventually be computing must go from smaller to larger value. Also, as the angle moves from the smaller to larger value they must trace out the boundary curve of the region we are interested in.

From a quick sketch of a unit circle we can quickly get two possible values for the angle in the fourth quadrant.

$$\theta = 2\pi - 1.1864 = 5.0968 \qquad\qquad \theta = -1.1864$$

Depending upon the problem we are being asked to do either of these could be the one we need. However, in this case we can see that if we use the first angle (*i.e.* the positive angle) we actually end up tracing out the outer portion of the curve and that isn't what we want here. However, if we use the second angle (*i.e.* the negative angle) we will trace out the inner loop as we move from this angle to the angle in the first quadrant.

So, for this particular problem, we need to use $\theta = -1.1864$.

The ranges of θ for this problem is then $-1.1864 \le \theta \le 1.1864$.

Step 3

The area of the inner loop is then,

$$\begin{aligned} A &= \int_{-1.1864}^{1.1864} \frac{1}{2}(3 - 8\cos(\theta))^2 \, d\theta \\ &= \frac{1}{2}\int_{-1.1864}^{1.1864} 9 - 48\cos(\theta) + 64cos^2(\theta)\, d\theta \\ &= \frac{1}{2}\int_{-1.1864}^{1.1864} 9 - 48\cos(\theta) + 32\left(1 + \cos(2\theta)\right)\, d\theta \\ &= \frac{1}{2}\int_{-1.1864}^{1.1864} 41 - 48\cos(\theta) + 32\cos(2\theta)\, d\theta \\ &= \frac{1}{2}\left(41\theta - 48\sin(\theta) + 16\sin(2\theta)\right)\Big|_{-1.1864}^{1.1864} = \boxed{15.2695} \end{aligned}$$

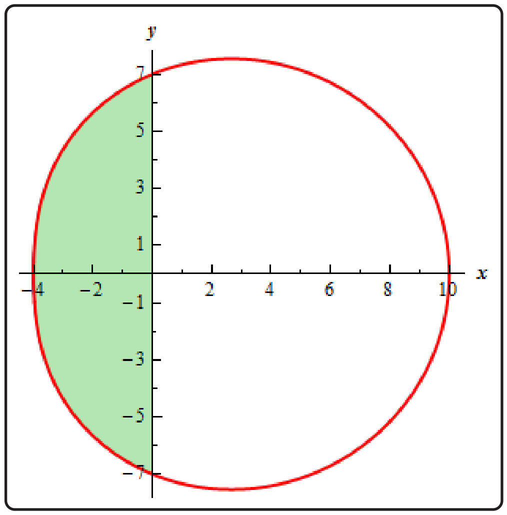
Make sure you can do the trig manipulations required to do these integrals. Most of the integrals in this section will involve this kind of manipulation. If you don't recall how to do them go back and take a look at the Integrals Involving Trig Functions section.

---

2. Find the area inside the graph of $r = 7 + 3\cos(\theta)$ and to the left of the y-axis.

Step 1

First, here is a quick sketch of the graph of the region we are interested in.



Step 2

For this problem there isn't too much difficulty in getting the limits for the problem. We will need to use the limits $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ to trace out the portion of the graph to the left of the y-axis.

Remember that it is important to trace out the portion of the curve defining the area we are interested in as the θ's increase from the smaller to larger value.

Step 3

The area is then,

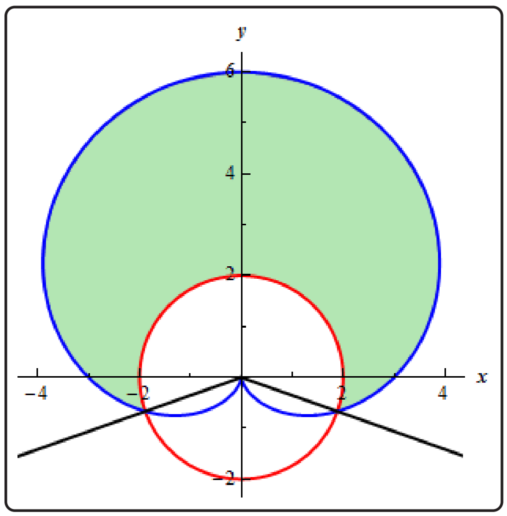
$$\begin{aligned} A &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}(7+3\cos(\theta))^2\,d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 49+42\cos(\theta)+9cos^2(\theta)\,d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 49+42\cos(\theta)+\frac{9}{2}(1+\cos(2\theta))\,d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{107}{2}+42\cos(\theta)+\frac{9}{2}\cos(2\theta)\,d\theta \\ &= \frac{1}{2}\left(\frac{107}{2}\theta+42\sin(\theta)+\frac{9}{4}\sin(2\theta)\right)\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \boxed{42.0376} \end{aligned}$$

Make sure you can do the trig manipulations required to do these integrals. Most of the integrals in this section will involve this kind of manipulation. If you don't recall how to do them go back and take a look at the Integrals Involving Trig Functions section.

3. Find the area that is inside $r = 3 + 3\sin(\theta)$ and outside $r = 2$.

Step 1

First, here is a quick sketch of the graph of the region we are interested in.



Step 2

Now, we'll need to determine the values of θ where the graphs intersect (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graphs intersect we know they must have the same value of r. So,

$$3 + 3\sin(\theta) = 2$$

$$\sin(\theta) = -\frac{1}{3} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(-\frac{1}{3}\right) = -0.3398$$

This is the angle in the fourth quadrant where the graphs intersect.

From a quick sketch of a unit circle we can quickly get the angle in the third quadrant where the two graphs intersect.

$$\theta = \pi + 0.3398 = 3.4814$$

The ranges of θ for this problem is then $-0.3398 \le \theta \le 3.4814$.

Step 3

From the graph we can see that $r = 3 + 3\sin(\theta)$ is the “outer” graph for this region and $r = 2$ is the “inner” graph.

The area is then,

$$\begin{aligned} A &= \int_{-0.3398}^{3.4814} \frac{1}{2}\left[(3+3\sin(\theta))^2 - (2)^2\right] d\theta \\ &= \frac{1}{2}\int_{-0.3398}^{3.4814} 5 + 18\sin(\theta) + 9\sin^2(\theta)\, d\theta \\ &= \frac{1}{2}\int_{-0.3398}^{3.4814} 5 + 18\sin(\theta) + \frac{9}{2}(1-\cos(2\theta))\, d\theta \\ &= \frac{1}{2}\int_{-0.3398}^{3.4814} \frac{19}{2} + 18\sin(\theta) - \frac{9}{2}\cos(2\theta)\, d\theta \\ &= \frac{1}{2}\left(\frac{19}{2}\theta - 18\cos(\theta) - \frac{9}{4}\sin(2\theta)\right)\Bigg|_{-0.3398}^{3.4814} = \boxed{33.7074} \end{aligned}$$

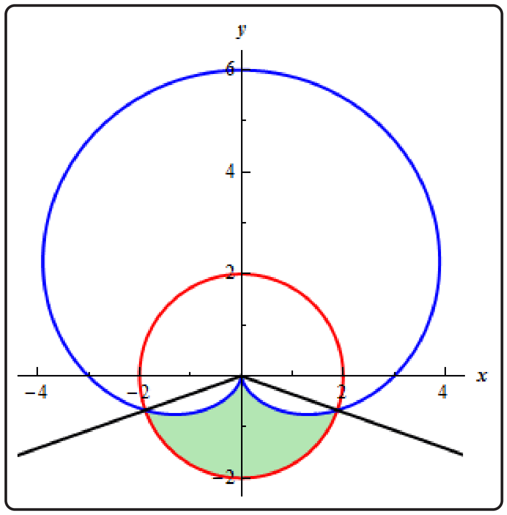
Make sure you can do the trig manipulations required to do these integrals. Most of the integrals in this section will involve this kind of manipulation. If you don’t recall how to do them go back and take a look at the Integrals Involving Trig Functions section.

---

4. Find the area that is inside $r = 2$ and outside $r = 3 + 3\sin(\theta)$.

Step 1

First, here is a quick sketch of the graph of the region we are interested in.



Step 2

Now, we'll need to determine the values of θ where the graphs intersect (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graphs intersect we know they must have the same value of r. So,

$$3 + 3\sin(\theta) = 2$$

$$\sin(\theta) = -\frac{1}{3} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(-\frac{1}{3}\right) = -0.3398$$

This is one possible value for the angle in the fourth quadrant where the graphs intersect.

From a quick sketch of a unit circle we can quickly get the angle in the third quadrant where the two graphs intersect.

$$\theta = \pi + 0.3398 = 3.4814$$

Now, we'll have a problem if we use these two angles for our area integral. Recall that the angles must go from smaller to larger values and as they do that they must trace out the boundary curves of the enclosed area. These two clearly will not do that. In fact, they trace out the area from the previous problem.

To fix this problem it is probably easiest to use a quick sketch of a unit circle to see that another value for the angle in the fourth quadrant is,

$$\theta = 2\pi - 0.3398 = 5.9434$$

Using this angle along with the angle we already have in the third quadrant will trace out the area we are interested in.

Therefore, the ranges of θ for this problem is then $3.4814 \le \theta \le 5.9434$.

Step 3

From the graph we can see that $r = 2$ is the "outer" graph for this region and $r = 3 + 3\sin(\theta)$ is the "inner" graph.

The area is then,

$$\begin{aligned} A &= \int_{3.4814}^{5.9434} \frac{1}{2}\left[(2)^2 - (3+3\sin(\theta))^2\right]\,d\theta \\ &= \frac{1}{2}\int_{3.4814}^{5.9434} -5 - 18\sin(\theta) - 9\sin^2(\theta)\,d\theta \\ &= \frac{1}{2}\int_{3.4814}^{5.9434} -5 - 18\sin(\theta) - \frac{9}{2}(1-\cos(2\theta))\,d\theta \\ &= \frac{1}{2}\int_{3.4814}^{5.9434} -\frac{19}{2} - 18\sin(\theta) + \frac{9}{2}\cos(2\theta)\,d\theta \\ &= \frac{1}{2}\left(-\frac{19}{2}\theta + 18\cos(\theta) + \frac{9}{4}\sin(2\theta)\right)\Bigg|_{3.4814}^{5.9434} = \boxed{3.8622} \end{aligned}$$

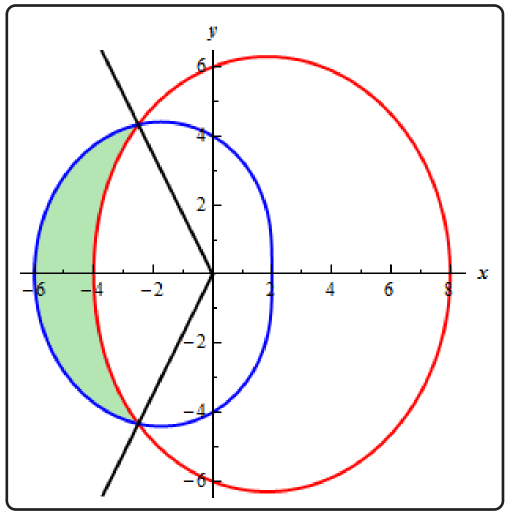
Do not get too excited about all the minus signs in the second step above. Just because all the terms have minus signs in front of them does not mean that we should get a negative value from out integral!

---

5. Find the area that is inside $r = 4 - 2\cos(\theta)$ and outside $r = 6 + 2\cos(\theta)$.

Step 1

First, here is a quick sketch of the graph of the region we are interested in.



Step 2

Now, we'll need to determine the values of θ where the graphs intersect (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graphs intersect we know they must have the same value of r. So,

$$6 + 2\cos(\theta) = 4 - 2\cos(\theta)$$
$$\cos(\theta) = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

This is the value for the angle in the second quadrant where the graphs intersect.

From a quick sketch of a unit circle we can quickly see two possible values for the angle in the third quadrant where the two graphs intersect.

$$\theta = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3} \qquad\qquad \theta = -\frac{2\pi}{3}$$

Now, we need to recall that the angles must go from smaller to larger values and as they do that they must trace out the boundary curves of the enclosed area. Keeping this in mind and we can see that we'll need to use the positive angle for this problem. If we used the negative angle we'd be tracing out the "right" portions of our curves and we need to trace out the "left" portions of our curves.

Therefore, the ranges of θ for this problem is then $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$.

Step 3

From the graph we can see that $r = 4 - 2\cos(\theta)$ is the "outer" graph for this region and $r = 6 + 2\cos(\theta)$ is the "inner" graph.

The area then,

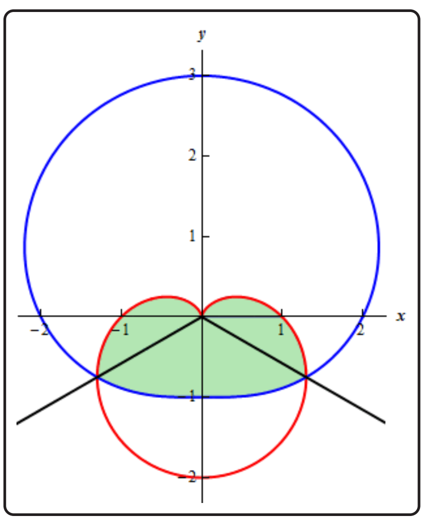
$$\begin{aligned} A &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left[(4 - 2\cos(\theta))^2 - (6 + 2\cos(\theta))^2\right] d\theta \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} -10 - 20\cos(\theta)\, d\theta \\ &= \left(-10\theta - 20\sin(\theta)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = \boxed{13.6971} \end{aligned}$$

Do not get too excited about all the minus signs in the integral. Just because all the terms have minus signs in front of them does not mean that we should get a negative value from the integral!

6. Find the area that is inside both $r = 1 - \sin(\theta)$ and $r = 2 + \sin(\theta)$.

Step 1

First, here is a quick sketch of the graph of the region we are interested in.



Step 2

Now, we'll need to determine the values of θ where the graphs intersect (indicated by the black lines on the graph in the previous step).

There are easy enough to find. Because they are where the graphs intersect we know they must have the same value of r. So,

$$2 + \sin(\theta) = 1 - \sin(\theta)$$

$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

This is one possible value for the angle in the fourth quadrant where the graphs intersect. From a quick sketch of a unit circle we can see that a second possible value for this angle is,

$$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

From a quick sketch of a unit circle we can quickly get a value for the angle in the third quadrant where the two graphs intersect.

$$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

Okay. We now have a real problem. Recall that the angles must go from smaller to larger values and as they do that they must trace out the boundary curves of the enclosed area.

If we use $-\frac{\pi}{6} \le \theta \le \frac{7\pi}{6}$ we actually end up tracing out the large "open" or unshaded region that lies above the shaded region.

Likewise, if we use $\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$ we actually end up tracing out the smaller "open" or unshaded region that lies below the shaded region.

In other words, we can't find the shaded area simply by using the formula from this section.

Step 3

As we saw in the previous step we can't just compute an integral in order to get the area of the shaded region. However, that doesn't mean that we can't find the area of the shaded region. We just need to work a little harder at it for this problem.

To find the area of the shaded area we can notice that the shaded area is really nothing more than the remainder of the area inside $r = 2 + \sin(\theta)$ once we take out the portion that is also outside $r = 1 - \sin(\theta)$.

Another way to look at is that the shaded area is simply the remainder of the area inside $r = 1 - \sin(\theta)$ once we take out the portion that is also outside $r = 2 + \sin(\theta)$.

We can use either of these ideas to find the area of the shaded region. We'll use the first one for no other reason that it was the first one listed.

If we knew the total area that is inside $r = 2 + \sin(\theta)$ (which we can find with a simple integral) and if we also knew the area that is inside $r = 2 + \sin(\theta)$ and outside $r = 1 - \sin(\theta)$ then the shaded area is nothing more than the difference between these two areas.

Step 4

Okay, let's start this off by getting the total area that is inside $r = 2 + \sin(\theta)$. This can be found by evaluating the following integral.

$$\begin{aligned}
A &= \int_0^{2\pi} \frac{1}{2}(2+\sin(\theta))^2\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 4 + 4\sin(\theta) + \sin^2(\theta)\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 4 + 4\sin(\theta) + \frac{1}{2}(1-\cos(2\theta))\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \frac{9}{2} + 4\sin(\theta) - \frac{1}{2}\cos(2\theta)\,d\theta \\
&= \frac{1}{2}\left(\frac{9}{2}\theta - 4\cos(\theta) - \frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{2\pi} = \boxed{\frac{9\pi}{2}}
\end{aligned}$$

Note that we need to do a full "revolution" to get all the area inside $r = 2 + \sin(\theta)$ and so we used the range $0 \le \theta \le 2\pi$ for this integral.

Step 5

Now, the area that is inside $r = 2 + \sin(\theta)$ and outside $r = 1 - \sin(\theta)$ is,

$$\begin{aligned} A &= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left[(2+\sin(\theta))^2 - (1-\sin(\theta))^2\right] d\theta \\ &= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 3 + 6\sin(\theta)\, d\theta \\ &= \frac{1}{2}(3\theta - 6\cos(\theta))\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} = \boxed{2\pi + 3\sqrt{3}} \end{aligned}$$

Note that we found the limits for this region in Step 2.

Step 6

Finally, the shaded area is simply,

$$A = \frac{9\pi}{2} - \left(2\pi + 3\sqrt{3}\right) = \boxed{\frac{5}{2}\pi - 3\sqrt{3} = 2.6578}$$

9.9 Arc Length with Polar Coordinates

1. Determine the length of the following polar curve. You may assume that the curve traces out exactly once for the given range of θ.

$$r = -4\sin(\theta),\ 0 \le \theta \le \pi$$

Step 1

The first thing we'll need here is the following derivative.

$$\frac{dr}{d\theta} = -4\cos(\theta)$$

Step 2

We'll need the ds for this problem.

$$\begin{aligned} ds &= \sqrt{[-4\sin(\theta)]^2 + [-4\cos(\theta)]^2}\, d\theta \\ &= \sqrt{16\sin^2(\theta) + 16\cos^2(\theta)}\, d\theta = 4\sqrt{\sin^2(\theta) + \cos^2(\theta)}\, d\theta = 4d\theta \end{aligned}$$

Step 3

The integral for the arc length is then,

$$L = \int ds = \int_0^{\pi} 4\, d\theta$$

Step 4

This is a really simple integral to compute. Here is the integral work,

$$L = \int_0^{\pi} 4\, d\theta = 4\theta\Big|_0^{\pi} = \boxed{4\pi}$$

2. Set up, but do not evaluate, and integral that gives the length of the following polar curve. You may assume that the curve traces out exactly once for the given range of θ.

$$r = \theta\cos(\theta), \;\; 0 \le \theta \le \pi$$

Step 1

The first thing we'll need here is the following derivative.

$$\frac{dr}{d\theta} = \cos(\theta) - \theta\sin(\theta)$$

Step 2

We'll need the ds for this problem.

$$ds = \sqrt{[\theta\cos(\theta)]^2 + [\cos(\theta) - \theta\sin(\theta)]^2}\, d\theta$$

Step 3

The integral for the arc length is then,

$$\boxed{L = \int_0^{\pi} \sqrt{[\theta\cos(\theta)]^2 + [\cos(\theta) - \theta\sin(\theta)]^2}\, d\theta}$$

---

3. Set up, but do not evaluate, and integral that gives the length of the following polar curve. You may assume that the curve traces out exactly once for the given range of θ.

$$r = \cos(2\theta) + \sin(3\theta), \;\; 0 \le \theta \le 2\pi$$

Step 1

The first thing we'll need here is the following derivative.

$$\frac{dr}{d\theta} = -2\sin(2\theta) + 3\cos(3\theta)$$

Step 2

We'll need the ds for this problem.

$$ds = \sqrt{\left[\cos\left(2\theta\right) + \sin\left(3\theta\right)\right]^2 + \left[-2\sin\left(2\theta\right) + 3\cos\left(3\theta\right)\right]^2}\,d\theta$$

Step 3

The integral for the arc length is then,

$$\boxed{L = \int_0^{2\pi} \sqrt{\left[\cos\left(2\theta\right) + \sin\left(3\theta\right)\right]^2 + \left[-2\sin\left(2\theta\right) + 3\cos\left(3\theta\right)\right]^2}\,d\theta}$$

9.10 Surface Area with Polar Coordinates

1. Set up, but do not evaluate, an integral that gives the surface area of the curve rotated about the given axis. You may assume that the curve traces out exactly once for the given range of θ.

$r = 5 - 4\sin(\theta)$, $0 \le \theta \le \pi$ rotated about the x-axis

Step 1

The first thing we'll need here is the following derivative.

$$\frac{dr}{d\theta} = -4\cos(\theta)$$

Step 2

We'll need the ds for this problem.

$$\begin{aligned} ds &= \sqrt{[5 - 4\sin(\theta)]^2 + [-4\cos(\theta)]^2}\,d\theta \\ &= \sqrt{25 - 40\sin(\theta) + 16\sin^2(\theta) + 16\cos^2(\theta)}\,d\theta = \sqrt{41 - 40\sin(\theta)}\,d\theta \end{aligned}$$

Step 3

The integral for the surface area is then,

$$SA = \int 2\pi y\,ds = \boxed{\int_0^{\pi} 2\pi\,(5 - 4\sin(\theta))\sin(\theta)\sqrt{41 - 40\sin(\theta)}\,d\theta}$$

Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about. Also, do not forget to substitute the polar conversion formula for y!

2. Set up, but do not evaluate, an integral that gives the surface area of the curve rotated about the given axis. You may assume that the curve traces out exactly once for the given range of θ.

$$r = \cos^2(\theta),\ -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \text{ rotated about the } y\text{-axis.}$$

Step 1

The first thing we'll need here is the following derivative.

$$\frac{dr}{d\theta} = -2\cos(\theta)\sin(\theta)$$

Step 2

We'll need the ds for this problem.

$$ds = \sqrt{\left[cos^2\theta\right]^2 + \left[-2\cos(\theta)\sin(\theta)\right]^2}\,d\theta = \sqrt{\cos^4(\theta) + 4\cos^2(\theta)\sin^2(\theta)}\,d\theta$$

Step 3

The integral for the surface area is then,

$$SA = \int 2\pi x\,ds = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2\pi\left(\cos^2(\theta)\right)\cos(\theta)\sqrt{\cos^4(\theta) + 4\cos^2(\theta)\sin^2(\theta)}\,d\theta$$

$$= \boxed{\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2\pi\cos^3(\theta)\sqrt{\cos^4(\theta) + 4\cos^2(\theta)\sin^2(\theta)}\,d\theta}$$

Remember to be careful with the formula for the surface area! The formula used is dependent upon the axis we are rotating about. Also, do not forget to substitute the polar conversion formula for x!

9.11 Arc Length and Surface Area Revisited

Problems have not yet been written for this section and probably won't be to be honest since this is just a summary section.

10 Series and Sequences

Once again, as with the last chapter, we are going to be looking at a completely different topic in this chapter. The only material from previous chapters that will be needed here will be the ability to compute limits at infinity (we'll do a fair amount of these), compute the rare derivatives and compute the occasional integral. The integrals will, generally, be fairly simple and needing u substitutions every once in a while although we will see the occasional integral requiring integration by parts or partial fractions. So, basically, the material in this chapter doesn't really rely all that much on previous material.

Series is one of those topics that many students don't find all that useful. To be honest, many students will never see series outside of their calculus class. However, series do play an important role in the field of ordinary differential equations and without series large portions of the field of partial differential equations would not be possible.

In other words, series is an important topic even if you won't ever see any of the applications. Most of the applications are beyond the scope of most Calculus courses and tend to occur in classes that many students don't take. So, as you go through this material keep in mind that these do have applications even if we won't really be covering many of them in this class.

The first topic we'll be looking at in this chapter is that of a sequence. We'll define just what we mean by a sequence and look at some basic topics and concepts that we'll need to work with them.

The other topic will be that of (infinite) series. In fact, we will spend the vast majority of this chapter deal with series. We can't, however, fully discuss series without understanding sequences and hence the reason for discussing sequences first. We will define just what an infinite series is and what it means for a series to converge or diverge. The majority of this chapter will then be spent discussing a variety of methods for testing whether or not a series will converge or diverge.

We'll close out the chapter with a discussion of power series and Taylor series as well as a couple of quick applications of series that we can easily discuss without needing any extra knowledge (as is needed for most applications of series).

The following sections are the practice problems, with solutions, for this material.

10.1 Sequences

1. List the first 5 terms of the following sequence.

$$\left\{ \frac{4n}{n^2 - 7} \right\}_{n=0}^{\infty}$$

Solution

There really isn't all that much to this problem. All we need to do is, starting at $n = 0$, plug in the first five values of n into the formula for the sequence terms. Doing that gives,

$$\begin{aligned}
n = 0: &\qquad \frac{4\,(0)}{(0)^2 - 7} = 0 \\
n = 1: &\qquad \frac{4\,(1)}{(1)^2 - 7} = \frac{4}{-6} = -\frac{2}{3} \\
n = 2: &\qquad \frac{4\,(2)}{(2)^2 - 7} = \frac{8}{-3} = -\frac{8}{3} \\
n = 3: &\qquad \frac{4\,(3)}{(3)^2 - 7} = \frac{12}{2} = 6 \\
n = 4: &\qquad \frac{4\,(4)}{(4)^2 - 7} = \frac{16}{9}
\end{aligned}$$

So, the first five terms of the sequence are,

$$\boxed{\left\{ 0,\ -\frac{2}{3},\ -\frac{8}{3},\ 6,\ \frac{16}{9},\ \ldots \right\}}$$

Note that we put the formal answer inside the braces to make sure that we don't forget that we are dealing with a sequence and we made sure and included the "..." at the end to reminder ourselves that there are more terms to this sequence that just the five that we listed out here.

2. List the first 5 terms of the following sequence.

$$\left\{ \frac{(-1)^{n+1}}{2n + (-3)^n} \right\}_{n=2}^{\infty}$$

Solution

There really isn't all that much to this problem. All we need to do is, starting at $n = 2$, plug in the first five values of n into the formula for the sequence terms. Doing that gives,

$$\begin{aligned}
n = 2: &\qquad \frac{(-1)^{2+1}}{2(2) + (-3)^2} = \frac{-1}{13} = -\frac{1}{13}\\
n = 3: &\qquad \frac{(-1)^{3+1}}{2(3) + (-3)^3} = \frac{1}{-21} = -\frac{1}{21}\\
n = 4: &\qquad \frac{(-1)^{4+1}}{2(4) + (-3)^4} = \frac{-1}{89} = -\frac{1}{89}\\
n = 5: &\qquad \frac{(-1)^{5+1}}{2(5) + (-3)^5} = \frac{1}{-233} = -\frac{1}{233}\\
n = 6: &\qquad \frac{(-1)^{6+1}}{2(6) + (-3)^6} = \frac{-1}{741} = -\frac{1}{741}
\end{aligned}$$

So, the first five terms of the sequence are,

$$\boxed{\left\{ -\frac{1}{13}, -\frac{1}{21}, -\frac{1}{89}, -\frac{1}{233}, -\frac{1}{741}, \ldots \right\}}$$

Note that we put the formal answer inside the braces to make sure that we don't forget that we are dealing with a sequence and we made sure and included the "..." at the end to reminder ourselves that there are more terms to this sequence that just the five that we listed out here.

3. Determine if the given sequence converges or diverges. If it converges what is its limit

$$\left\{\frac{n^2-7n+3}{1+10n-4n^2}\right\}_{n=3}^{\infty}$$

Step 1

To answer this all we need is the following limit of the sequence terms.

$$\lim_{n\to\infty}\frac{n^2-7n+3}{1+10n-4n^2}=-\frac{1}{4}$$

You do recall how to take limits at infinity right? If not you should go back into the Calculus I material do some refreshing on limits at infinity as well at L'Hospital's rule.

Step 2

We can see that the limit of the terms existed and was a finite number and so we know that the sequence **converges** and its limit is $-\frac{1}{4}$.

---

4. Determine if the given sequence converges or diverges. If it converges what is its limit

$$\left\{\frac{(-1)^{n-2}n^2}{4+n^3}\right\}_{n=0}^{\infty}$$

Step 1

To answer this all we need is the following limit of the sequence terms.

$$\lim_{n\to\infty}\frac{(-1)^{n-2}n^2}{4+n^3}$$

However, because of the $(-1)^{n-2}$ we can't compute this limit using our knowledge of computing limits from Calculus I.

Step 2

Recall however, that we had a nice Fact in the notes from this section that had us computing not the limit above but instead computing the limit of the absolute value of the sequence terms.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n-2}n^2}{4+n^3}\right| = \lim_{n\to\infty}\frac{n^2}{4+n^3} = 0$$

This is a limit that we can compute because the absolute value got rid of the alternating sign, *i.e.* the $(-1)^{n+2}$.

Step 3

Now, by the Fact from class we know that because the limit of the absolute value of the sequence terms was zero (and recall that to use that fact the limit MUST be zero!) we also know the following limit.

$$\lim_{n\to\infty}\frac{(-1)^{n-2}n^2}{4+n^3} = 0$$

Step 4

We can see that the limit of the terms existed and was a finite number and so we know that the sequence **converges** and its limit is zero.

---

5. Determine if the given sequence converges or diverges. If it converges what is its limit

$$\left\{\frac{\mathbf{e}^{5n}}{3-\mathbf{e}^{2n}}\right\}_{n=1}^{\infty}$$

Step 1

To answer this all we need is the following limit of the sequence terms.

$$\lim_{n\to\infty}\frac{\mathbf{e}^{5n}}{3-\mathbf{e}^{2n}} = \lim_{n\to\infty}\frac{5\mathbf{e}^{5n}}{-2\mathbf{e}^{2n}} = \lim_{n\to\infty}\frac{5}{-2}\mathbf{e}^{3n} = -\infty$$

You do recall how to use L'Hospital's rule to compute limits at infinity right? If not you should go back into the Calculus I material do some refreshing.

Step 2

We can see that the limit of the terms existed and but was infinite and so we know that the sequence **diverges**.

---

6. Determine if the given sequence converges or diverges. If it converges what is its limit

$$\left\{\frac{\ln(n+2)}{\ln(1+4n)}\right\}_{n=1}^{\infty}$$

Step 1

To answer this all we need is the following limit of the sequence terms.

$$\lim_{n\to\infty}\frac{\ln(n+2)}{\ln(1+4n)} = \lim_{n\to\infty}\frac{{}^{1}/_{n+2}}{{}^{4}/_{1+4n}} = \lim_{n\to\infty}\frac{1+4n}{4(n+2)} = 1$$

You do recall how to use L'Hospital's rule to compute limits at infinity right? If not, you should go back into the Calculus I material do some refreshing.

Step 2

We can see that the limit of the terms existed and was a finite number and so we know that the sequence **converges** and its limit is one.

10.2 More on Sequences

1. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{\frac{1}{4n}\right\}_{n=1}^{\infty}$$

Hint

There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. This is one of those sequences that it doesn't matter which set of information you find first and both sets should be fairly easy to determine the answers without a lot of work.

Step 1

For this problem let's get the bounded information first as that seems to be pretty simple.

First note that because both the numerator and denominator are positive then the quotient is also positive and so we can see that the sequence must be **bounded below by zero**.

Next let's note that because we are starting with $n = 1$ the denominator will always be $4n \geq 4 > 1$ and so we can also see that the sequence must be **bounded above by one**. Note that, in this case, this not the "best" upper bound for the sequence but the problem didn't ask for that. For this sequence we'll be able to get a better one once we have the increasing/decreasing information.

Because the sequence is bounded above and bounded below the sequence is also **bounded**.

Step 2

For the increasing/decreasing information we can see that, for our range of $n \geq 1$, we have,

$$4n < 4(n+1)$$

and so,

$$\frac{1}{4n} > \frac{1}{4(n+1)}$$

If we define $a_n = \frac{1}{4n}$ this in turn tells us that $a_n > a_{n+1}$ for all $n \geq 1$ and so the sequence is **decreasing** and hence **monotonic**.

Note that because we have now determined that the sequence is decreasing we can see that the "best" upper bound would be the first term of the sequence or, $\frac{1}{4}$.

2. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{n(-1)^{n+2}\right\}_{n=0}^{\infty}$$

Hint

There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. This is one of those sequences that it doesn't matter which set of information you find first and both sets should be fairly easy to determine the answers without a lot of work.

Step 1

For this problem let's get the increasing/decreasing information first as that seems to be pretty simple and will help at least a little bit with the bounded information.

In this case let's just write out the first few terms of the sequence.

$$\left\{n(-1)^{n+2}\right\}_{n=0}^{\infty} = \{0,\ -1,\ 2,\ -3,\ 4,\ -5,\ 6,\ -7,\ldots\}$$

Just from the first three terms we can see that this sequence is **not an increasing sequence** and it is **not a decreasing sequence** and therefore is **not monotonic**.

Step 2

Now let's see what bounded information we can get.

From the first few terms of the sequence we listed out above we can see that each successive term will get larger and change signs. Therefore, there cannot be an upper or a lower bound for the sequence. No matter what value we would try to use for an upper or a lower bound all we would need to do is take n large enough and we would eventually get a sequence term that would go past the proposed bound.

Therefore, this sequence is **not bounded**.

3. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{3^{-n}\right\}_{n=0}^{\infty}$$

Hint

There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. For this sequence it might be a little easier to find the bounds (if any exist) if you first have the increasing/decreasing information.

Step 1

For this problem let's get the increasing/decreasing information first as that seems to be pretty simple and will help at least a little bit with the bounded information.

We' all agree that, for our range of $n \geq 0$, we have,

$$n < n + 1$$

This in turn gives,

$$3^{-n} = \frac{1}{3^n} > \frac{1}{3^{n+1}} = 3^{-(n+1)}$$

So, if we define $a_n = 3^{-n}$ we have $a_n > a_{n+1}$ for all $n \geq 0$ and so the sequence is **decreasing** and hence is also **monotonic**.

Step 2

Now let's see what bounded information we can get.

First, it is hopefully obvious that all the terms are positive and so the sequence is **bounded below by zero**.

Next, we saw in the first step that the sequence was decreasing and so the first term will be the largest term and so the sequence is **bounded above by** $3^{-(0)} = 1$ (*i.e.* the $n = 0$ sequence term).

Therefore, because this sequence is bounded below and bounded above the sequence is **bounded**.

---

4. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{\frac{2n^2-1}{n}\right\}_{n=2}^{\infty}$$

Hint

There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. For this sequence having the increasing/decreasing information will probably make the determining the bounds (if any exist) somewhat easier.

Step 1

For this problem let's get the increasing/decreasing information first.

For Problems 1 - 3 in this section it was easy enough to just ask what happens if we increase n to determine the increasing/decreasing information for this problem. However, in this case, increasing n will increase both the numerator and denominator and so it would be somewhat more difficult to do that analysis here.

Therefore, we will resort to some quick Calculus I to determine increasing/decreasing information. We can define the following function and take its derivative.

$$f(x)=\frac{2x^2-1}{x} \qquad \Rightarrow \qquad f'(x)=\frac{2x^2+1}{x^2}$$

We can clearly see that the derivative will always be positive for $x \neq 0$ and so the function is increasing for $x \neq 0$. Therefore, because the function values are the same as the sequence values when x is an integer we can see that the sequence, which starts at $n=2$, must also be **increasing** and hence it is also **monotonic**.

Step 2

Now let's see what bounded information we can get.

First, it is hopefully obvious that all the terms are positive for our range of $n \geq 2$ and so the sequence is **bounded below by zero**. We could also use the fact that the sequence is increasing the first term would have to be the smallest term in the sequence and so a better lower bound would be the first sequence term which is $\frac{7}{2}$. Either would work for this problem.

Now let's see what we can determine about an upper bound (provided it has one of course...).

We know that the function is increasing but that doesn't mean there is no upper bound. Take a look at Problems 1 and 3 above. Each of those were decreasing sequences and yet they had a lower bound. Do not make the mistake of assuming that an increasing sequence will not have an upper bound or a decreasing sequence will not have a lower bound. Sometimes they will and sometimes they won't.

For this sequence we'll need to approach any potential upper bound a little differently than the previous problems. Let's first compute the following limit of the terms,

$$\lim_{n\to\infty} \frac{2n^2-1}{n} = \lim_{n\to\infty} \left(2n - \frac{1}{n}\right) = \infty$$

Since the limit of the terms is infinity we can see that the terms will increase without bound. Therefore, in this case, there really is **no upper bound** for this sequence. Please remember the warning above however! Just because this increasing sequence had no upper bound does not mean that every increasing sequence will not have an upper bound.

Finally, because this sequence is bounded below but not bounded above the sequence is **not bounded**.

5. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{\frac{4-n}{2n+3}\right\}_{n=1}^{\infty}$$

Hint

There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. For this sequence having the increasing/decreasing information will probably make the determining the bounds (if any exist) somewhat easier.

Step 1

For this problem let's get the increasing/decreasing information first.

For Problems 1 - 3 in this section it was easy enough to just ask what happens if we increase n to determine the increasing/decreasing information for this problem. However, in this case, increasing n will increase both the numerator and denominator and so it would be somewhat more difficult to do that analysis here.

Therefore, we will resort to some quick Calculus I to determine increasing/decreasing information. We can define the following function and take its derivative.

$$f\left(x\right)=\frac{4-x}{2x+3} \qquad \Rightarrow \qquad f'\left(x\right)=\frac{-11}{\left(2x+3\right)^{2}}$$

We can clearly see that the derivative will always be negative for $x\neq-\frac{3}{2}$ and so the function is decreasing for $x\neq-\frac{3}{2}$. Therefore, because the function values are the same as the sequence values when x is an integer we can see that the sequence, which starts at $n=1$, must also be **decreasing** and hence it is also **monotonic**.

Step 2

Now let's see what bounded information we can get.

First, because the sequence is decreasing we can see that the first term of the sequence will be the largest and hence will also be an upper bound for the sequence. So, the sequence is **bounded above** by $\frac{3}{5}$ (*i.e.* the $n=1$ sequence term).

Next let's look for the lower bound (if it exists). For this problem let's first take a quick look at the limit of the sequence terms. In this case the limit of the sequence terms is,

$$\lim_{n\to\infty}\frac{4-n}{2n+3}=-\frac{1}{2}$$

Recall what this limit tells us about the behavior of our sequence terms. The limit means that as $n\to\infty$ the sequence terms must be getting closer and closer to $-\frac{1}{2}$.

Now, for a second, let's suppose that that $-\frac{1}{2}$ is not a lower bound for the sequence terms and let's also keep in mind that we've already determined that the sequence is decreasing (means that each successive term must be smaller than (*i.e.* below) the previous one...).

So, if $-\frac{1}{2}$ is not a lower bound then we know that somewhere there must be sequence terms below (or smaller than) $-\frac{1}{2}$. However, because we also know that terms must be getting closer and closer to $-\frac{1}{2}$ and we've now assumed there are terms below $-\frac{1}{2}$ the only way for that to happen at this point is for at least a few sequence terms to increase up towards $-\frac{1}{2}$ (remember we've assumed there are terms below this!). That can't happen however because we know the sequence is a decreasing sequence.

Okay, what was the point of all this? Well recall that we got to this apparent contradiction to the decreasing nature of the sequence by first assuming that $-\frac{1}{2}$ was not a lower bound. Since making this assumption led us to something that can't possibly be true that in turn means that $-\frac{1}{2}$ must in fact be a lower bound since we've shown that sequence terms simply cannot go below this value!

Therefore, the sequence is **bounded below** by $-\frac{1}{2}$.

Finally, because this sequence is both bounded above and bounded below the sequence is **bounded**.

Before leaving this problem a quick word of caution. The limit of a sequence is not guaranteed to be a bound (upper or lower) for a sequence. It will only be a bound under certain circumstances and so we can't simply compute the limit and assume it will be a bound for every sequence! Can you see a condition that will allow the limit to be a bound?

---

6. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{\frac{-n}{n^2+25}\right\}_{n=2}^{\infty}$$

Hint

There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. For this sequence having the increasing/decreasing information will probably make the determining the bounds (if any exist) somewhat easier.

Step 1

For this problem let's get the increasing/decreasing information first.

For Problems 1 - 3 in this section it was easy enough to just ask what happens if we increase n to determine the increasing/decreasing information for this problem. However, in this case, increasing n will increase both the numerator and denominator and so it would be somewhat more difficult to do that analysis here.

Therefore, we will resort to some quick Calculus I to determine increasing/decreasing information. We can define the following function and take its derivative.

$$f(x)=\frac{-x}{x^2+25} \qquad \Rightarrow \qquad f'(x)=\frac{x^2-25}{\left(x^2+25\right)^2}$$

Hopefully, it's fairly clear that the critical points of the function are $x=\pm 5$. We'll leave it to you to draw a quick number line or sign chart to verify that the function will be decreasing in the range $2 \le x < 5$ and increasing in the range $x > 5$. Note that we just looked at the ranges of x that correspond to the ranges of n for our sequence here.

Now, because the function values are the same as the sequence values when x is an integer we can see that the sequence, which starts at $n=2$, has terms that increase and terms that decrease and hence the sequence is **not an increasing sequence** and the sequence is **not a decreasing sequence**. That also means that the sequence is **not monotonic**.

Step 2

Now let's see what bounded information we can get.

In this case, unlike many of the previous problems in this section, we don't have a monotonic sequence. However, we can still use the increasing/decreasing information above to help us out with the bounds.

First, we know that the sequence is decreasing in the range $2 \le n < 5$ and increasing in the range $n > 5$. From our Calculus I knowledge we know that this means $n = 5$ must be a minimum of the sequence terms and hence the sequence is **bounded below** by $\frac{-5}{50} = -\frac{1}{10}$ (*i.e.* the $n = 5$ sequence term).

Next let's look for the upper bound (if it exists). For this problem let's first take a quick look at the limit of the sequence terms. In this case the limit of the sequence terms is,

$$\lim_{n \to \infty} \frac{-n}{n^2 + 25} = 0$$

Recall what this limit tells us about the behavior of our sequence terms. The limit means that as $n \to \infty$ the sequence terms must be getting closer and closer to zero.

Now, for a second, let's look at just the portion of the sequence with $n > 5$ and let's further suppose that zero is not an upper bound for the sequence terms with $n > 5$. Let's also keep in mind that we've already determined that the sequence is increasing for $n > 5$ (means that each successive term must be larger than (*i.e.* above) the previous one...).

So, if zero is not an upper bound (for $n > 5$) then we know that somewhere there must be sequence terms with $n > 5$ above (or larger than) zero. So, we know that terms must be getting closer and closer to zero and we've now assumed there are terms above zero. Therefore the only way for the terms to approach the limit of zero is for at least a few sequence terms with $n > 5$ to decrease down towards zero (remember we've assumed there are terms above this!). That can't happen however because we know that for $n > 5$ the sequence is increasing.

Okay, what was the point of all this? Well recall that we got to this apparent contradiction to the increasing nature of the sequence for $n > 5$ by first assuming that zero was not an upper bound for the portion of the sequence with $n > 5$. Since making this assumption led us to something that can't possibly be true that in turn means that zero must in fact be an upper bound for the portion of the sequence with $n > 5$ since we've shown that sequence terms simply cannot go above this value!

Note that we've not yet actually shown that zero in an upper bound for the sequence and in fact it might not actually be an upper bound. However, what we have shown is that it is an upper bound for the vast majority of the sequence, *i.e.* for the portion of the sequence with $n > 5$.

All we need to do to finish the upper bound portion of this problem off is check what the first few terms of the sequence are doing. There are several ways to do this. One is to just compute the remaining initial terms of the sequence to see if they are above or below zero. For this sequence that isn't too bad as there are only 4 terms ($n = 2, 3, 4, 5$). However, if there'd been several hundred terms that wouldn't be so easy so let's take a look at another approach that will always be easy to do in this case because we have the increasing/decreasing information for this initial portion of the sequence.

Let's simply note that for the first part of this sequence we've already shown that the sequence is decreasing. Therefore, the very first sequence term of $-\frac{2}{29}$ (*i.e.* the $n = 2$ sequence term) will be the largest term for this initial bit of the sequence that is decreasing. This term is clearly less than zero and so zero will also be larger than all the remaining terms in the initial decreasing portion of the sequence and hence the sequence is **bounded above by zero**.

Finally, because this sequence is both bounded above and bounded below the sequence is **bounded**.

Before leaving this problem a couple of quick words of caution.

First, the limit of a sequence is not guaranteed to be a bound (upper or lower) for a sequence so be careful to not just always assume that the limit is an upper/lower bound for a sequence.

Second, as this problem has shown determining the bounds of a sequence can sometimes be a fairly involved process that involves quite a bit of work and lots of various pieces of knowledge about the other behavior of the sequence.

10.3 Series - The Basics

1. Perform an index shift so that the following series starts at $n = 3$.

$$\sum_{n=1}^{\infty} \left(n2^n - 3^{1-n}\right)$$

Solution

There really isn't all that much to this problem. Just remember that, in this case, we'll need to increase the initial value of the index by two so it will start at $n = 3$ and this means all the n's in the series terms will need to decrease by the same amount (two in this case...).

Doing this gives the following series.

$$\sum_{n=1}^{\infty} \left(n2^n - 3^{1-n}\right) = \sum_{n=3}^{\infty} \left((n-2)\,2^{n-2} - 3^{1-(n-2)}\right) = \boxed{\sum_{n=3}^{\infty} \left((n-2)\,2^{n-2} - 3^{3-n}\right)}$$

Be careful with parenthesis, exponents, coefficients and negative signs when "shifting" the n's in the series terms. When replacing n with $n-2$ make sure to add in parenthesis where needed to preserve coefficients and minus signs.

2. Perform an index shift so that the following series starts at $n = 3$.

$$\sum_{n=7}^{\infty} \frac{4-n}{n^2+1}$$

Solution

There really isn't all that much to this problem. Just remember that, in this case, we'll need to decrease the initial value of the index by four so it will start at $n = 3$ and this means all the n's in the series terms will need to increase by the same amount (four in this case...).

Doing this gives the following series.

$$\sum_{n=7}^{\infty} \frac{4-n}{n^2+1} = \sum_{n=3}^{\infty} \frac{4-(n+4)}{(n+4)^2+1} = \boxed{\sum_{n=3}^{\infty} \frac{-n}{(n+4)^2+1}}$$

Be careful with parenthesis, exponents, coefficients and negative signs when "shifting" the n's in the series terms. When replacing n with $n+4$ make sure to add in parenthesis where needed to preserve coefficients and minus signs.

3. Perform an index shift so that the following series starts at $n = 3$.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-3}(n+2)}{5^{1+2n}}$$

Solution

There really isn't all that much to this problem. Just remember that, in this case, we'll need to increase the initial value of the index by one so it will start at $n = 3$ and this means all the n's in the series terms will need to decrease by the same amount (one in this case...).

Doing this gives the following series.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-3}(n+2)}{5^{1+2n}} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1-3}(n-1+2)}{5^{1+2(n-1)}} = \boxed{\sum_{n=3}^{\infty} \frac{(-1)^{n-4}(n+1)}{5^{2n-1}}}$$

Be careful with parenthesis, exponents, coefficients and negative signs when "shifting" the n's in the series terms. When replacing n with $n-1$ make sure to add in parenthesis where needed to preserve coefficients and minus signs.

4. Strip out the first 3 terms from the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^{-n}}{n^2+1}$.

Solution

Remember that when we say we are going to "strip out" terms from a series we aren't really getting rid of them. All we are doing is writing the first few terms of the series as a summation in front of the series.

So, for this series stripping out the first three terms gives,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^{-n}}{n^2+1} &= \frac{2^{-1}}{1^2+1} + \frac{2^{-2}}{2^2+1} + \frac{2^{-3}}{3^2+1} + \sum_{n=4}^{\infty} \frac{2^{-n}}{n^2+1} \\ &= \frac{1}{4} + \frac{1}{20} + \frac{1}{80} + \sum_{n=4}^{\infty} \frac{2^{-n}}{n^2+1} = \boxed{\frac{5}{16} + \sum_{n=4}^{\infty} \frac{2^{-n}}{n^2+1}}\end{aligned}$$

This first step isn't really all that necessary but was included here to make it clear that we were plugging in $n = 1$, $n = 2$ and $n = 3$ (*i.e.* the first three values of n) into the general series term. Also, don't forget to change the starting value of n to reflect the fact that we've "stripped out" the first three values of n or terms.

---

5. Given that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^3+1} = 1.6865$ determine the value of $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3+1}$.

Step 1

First notice that if we strip out the first two terms from the series that starts at $n = 0$ the result will involve a series that starts at $n = 2$.

Doing this gives,

$$\sum_{n=0}^{\infty} \frac{1}{n^3+1} = \frac{1}{0^3+1} + \frac{1}{1^3+1} + \sum_{n=2}^{\infty} \frac{1}{n^3+1} = \frac{3}{2} + \sum_{n=2}^{\infty} \frac{1}{n^3+1}$$

Step 2

Now, for this situation we are given the value of the series that starts at $n = 0$ and are asked to determine the value of the series that starts at $n = 2$. To do this all we need to do is plug in the known value of the series that starts at $n = 0$ into the "equation" above and "solve" for the value of the series that starts at $n = 2$.

This gives,

$$1.6865 = \frac{3}{2} + \sum_{n=2}^{\infty} \frac{1}{n^3+1} \qquad \Rightarrow \qquad \sum_{n=2}^{\infty} \frac{1}{n^3+1} = 1.6865 - \frac{3}{2} = \boxed{0.1865}$$

10.4 Convergence & Divergence of Series

1. Compute the first 3 terms in the sequence of partial sums for the following series.

$$\sum_{n=1}^{\infty} n\,2^n$$

Solution

Remember that n^{th} term in the sequence of partial sums is just the sum of the first n terms of the series. So, computing the first three terms in the sequence of partial sums is pretty simple to do.

Here is the work for this problem.

$$\boxed{\begin{aligned} s_1 &= (1)\,2^1 = 2 \\ s_2 &= (1)\,2^1 + (2)\,2^2 = 10 \\ s_3 &= (1)\,2^1 + (2)\,2^2 + (3)\,2^3 = 34 \end{aligned}}$$

---

2. Compute the first 3 terms in the sequence of partial sums for the following series.

$$\sum_{n=3}^{\infty} \frac{2n}{n+2}$$

Solution

Remember that n^{th} term in the sequence of partial sums is just the sum of the first n terms of the series. So, computing the first three terms in the sequence of partial sums is pretty simple to do.

Here is the work for this problem.

$$\boxed{\begin{aligned} s_3 &= \frac{2\,(3)}{3+2} = \frac{6}{5} \\ s_4 &= \frac{2\,(3)}{3+2} + \frac{2\,(4)}{4+2} = \frac{38}{15} \\ s_5 &= \frac{2\,(3)}{3+2} + \frac{2\,(4)}{4+2} + \frac{2\,(5)}{5+2} = \frac{416}{105} \end{aligned}}$$

---

3. Assume that the n^{th} term in the sequence of partial sums for the series $\displaystyle\sum_{n=0}^{\infty} a_n$ is given below. Determine if the series $\displaystyle\sum_{n=0}^{\infty} a_n$ is convergent or divergent. If the series is convergent determine the value of the series.

$$s_n = \frac{5+8n^2}{2-7n^2}$$

Solution

There really isn't all that much that we need to do here other than to recall,

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n$$

So, to determine if the series converges or diverges, all we need to do is compute the limit of the sequence of the partial sums. The limit of the sequence of partial sums is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{5+8n^2}{2-7n^2} = -\frac{8}{7}$$

Now, we can see that this limit exists and is finite (*i.e.* is not plus/minus infinity). Therefore, we now know that the series, $\sum_{n=0}^{\infty} a_n$, **converges** and its value is,

$$\sum_{n=0}^{\infty} a_n = -\frac{8}{7}$$

If you are unfamiliar with limits at infinity then you really need to go back to the Calculus I material and do some review of limits at infinity and L'Hospital's Rule as we will be doing quite a bit of these kinds of limits off and on over the next few sections.

4. Assume that the n^{th} term in the sequence of partial sums for the series $\sum_{n=0}^{\infty} a_n$ is given below. Determine if the series $\sum_{n=0}^{\infty} a_n$ is convergent or divergent. If the series is convergent determine the value of the series.

$$s_n = \frac{n^2}{5+2n}$$

Solution

There really isn't all that much that we need to do here other than to recall,

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n$$

So, to determine if the series converges or diverges, all we need to do is compute the limit of the sequence of the partial sums. The limit of the sequence of partial sums is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n^2}{5+2n} = \infty$$

Now, we can see that this limit exists and is infinite. Therefore, we now know that the series, $\sum_{n=0}^{\infty} a_n$, **diverges**.

If you are unfamiliar with limits at infinity then you really need to go back to the Calculus I material and do some review of limits at infinity and L'Hospital's Rule as we will be doing quite a bit of these kinds of limits off and on over the next few sections.

---

5. Show that the following series is divergent.

$$\sum_{n=0}^{\infty} \frac{3n\,\mathbf{e}^n}{n^2+1}$$

Solution

First let's note that we're being asked to show that the series is divergent. We are not being asked to determine **if** the series is divergent. At this point we really only know of two ways to actually show this.

The first option is to show that the limit of the sequence of partial sums either doesn't exist or is infinite. The problem with this approach is that for many series determining the general formula for the n^{th} term of the sequence of partial sums is very difficult if not outright impossible to do. That is true for this series and so that is not really a viable option for this problem.

Luckily enough for us there is actually an easier option to simply show that a series is divergent. All we need to do is use the Divergence Test.

The limit of the series terms is,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n\,\mathbf{e}^n}{n^2+1} = \infty \neq 0$$

The limit of the series terms is not zero and so by the Divergence Test we know that the series in this problem is **divergent**.

---

6. Show that the following series is divergent.

$$\sum_{n=5}^{\infty} \frac{6+8n+9n^2}{3+2n+n^2}$$

Solution

First let's note that we're being asked to show that the series is divergent. We are not being asked to determine **if** the series is divergent. At this point we really only know of two ways to actually show this.

The first option is to show that the limit of the sequence of partial sums either doesn't exist or is infinite. The problem with this approach is that for many series determining the general formula for the n^{th} term of the sequence of partial sums is very difficult if not outright impossible to do. That is true for this series and so that is not really a viable option for this problem.

Luckily enough for us there is actually an easier option to simply show that a series is divergent. All we need to do is use the Divergence Test.

The limit of the series terms is,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{6+8n+9n^2}{3+2n+n^2} = 9 \neq 0$$

The limit of the series terms is not zero and so by the Divergence Test we know that the series in this problem is **divergent**.

10.5 Special Series

1. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=0}^{\infty} 3^{2+n}\, 2^{1-3n}$$

Step 1

Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a geometric series.

Step 2

Let's also notice that the initial value of the index is $n = 0$ and so we can put this into the form,

$$\sum_{n=0}^{\infty} a\, r^n$$

At that point we'll be able to determine if it converges or diverges and the value of the series if it does happen to converge.

In this case it's pretty simple to put the series into the form above so here is that work.

$$\sum_{n=0}^{\infty} 3^{2+n}\, 2^{1-3n} = \sum_{n=0}^{\infty} 3^2 3^n\, 2^1 2^{-3n} = \sum_{n=0}^{\infty} (9)\,(2)\,\frac{3^n}{2^{3n}} = \sum_{n=0}^{\infty} 18\frac{3^n}{8^n} = \sum_{n=0}^{\infty} 18\left(\frac{3}{8}\right)^n$$

Make sure you properly deal with any negative exponents that might happen to be in the terms!

Also recall that all the exponents must be simply n and can't be $3n$ or anything else. So, for this problem, we'll need to use basic exponent rules to write $2^{3n} = \left(2^3\right)^n = 8^n$.

Step 3

With the series in "proper" form we can see that $a = 18$ and $r = \frac{3}{8}$. Therefore, because we can clearly see that $|r| = \frac{3}{8} < 1$, the series will **converge** and its value is,

$$\sum_{n=0}^{\infty} 3^{2+n}\, 2^{1-3n} = \sum_{n=0}^{\infty} 18\left(\frac{3}{8}\right)^n = \frac{18}{1-\frac{3}{8}} = \boxed{\frac{144}{5}}$$

2. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty} \frac{5}{6n}$$

Step 1

Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a harmonic series.

Step 2

So because this is a harmonic series we know that it will **diverge**.

3. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty} \frac{(-6)^{3-n}}{8^{2-n}}$$

Step 1

Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a geometric series.

Step 2

Let's also notice that the initial value of the index is $n = 1$ and so we can put this into the form,

$$\sum_{n=1}^{\infty} a\, r^{n-1}$$

At that point we'll be able to determine if it converges or diverges and the value of the series if it does happen to converge.

So, let's get started on the work to put the series into the form above. First, let's get take care of the fact that both the n's in the exponents are negative and they should be positive. Converting to positive n's gives,

$$\sum_{n=1}^{\infty} \frac{(-6)^{3-n}}{8^{2-n}} = \sum_{n=1}^{\infty} \frac{8^{n-2}}{(-6)^{n-3}}$$

Note that how you chose to deal with the 3 and the 2 in the respective exponents is up to you. You can either do it the way we did here or strip them out and then move the terms to the numerator or denominator.

As noted above we need the two exponents to be $n-1$. This is an easy "fix" if we note that using basic exponent properties we can write each term as follows,

$$8^{n-2} = 8^{n-1}8^{-1} \qquad\qquad (-6)^{n-3} = (-6)^{n-1}(-6)^{-2}$$

With these two rewrites the series becomes,

$$\sum_{n=1}^{\infty} \frac{(-6)^{3-n}}{8^{2-n}} = \sum_{n=1}^{\infty} \frac{8^{n-1}8^{-1}}{(-6)^{n-1}(-6)^{-2}} = \sum_{n=1}^{\infty} \frac{(-6)^2}{8^1}\left(\frac{8}{-6}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{9}{2}\left(-\frac{4}{3}\right)^{n-1}$$

Step 3

With the series in "proper" form we can see that $a = \frac{9}{2}$ and $r = -\frac{4}{3}$. Therefore, because we can clearly see that $|r| = \left|-\frac{4}{3}\right| = \frac{4}{3} > 1$, the series will **diverge.**

4. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty} \frac{3}{n^2 + 7n + 12}$$

Step 1

Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is not a geometric or harmonic series. That only leaves telescoping as a possibility.

Step 2

Now, we need to be careful here. There is no way to actually identify the series as a telescoping series at this point. We are only hoping that it is a telescoping series.

Therefore, the first real step here is to perform partial fractions on the series term to see what we get. Here is the partial fraction work for the series term.

$$\frac{3}{n^2+7n+12} = \frac{3}{(n+3)(n+4)} = \frac{A}{n+3} + \frac{B}{n+4} \quad \to \quad 3 = A(n+4) + B(n+3)$$

$$\begin{array}{lll} n=-3 & 3=A & \\ & & \to \\ n=-4 & 3=-B & \end{array} \quad \begin{array}{l} A=3 \\ B=-3 \end{array}$$

The series term in partial fraction form is then,

$$\frac{3}{n^2+7n+12} = \frac{3}{n+3} - \frac{3}{n+4}$$

Step 3

The partial sums for this series are then,

$$s_n = \sum_{i=1}^{n} \left[\frac{3}{i+3} - \frac{3}{i+4}\right]$$

Step 4

Expanding the partial sums from the previous step give,

$$\begin{aligned} s_n = \sum_{i=1}^{n} \left[\frac{3}{i+3} - \frac{3}{i+4}\right] &= \left[\frac{3}{4} - \cancel{\frac{3}{5}}\right] + \left[\cancel{\frac{3}{5}} - \cancel{\frac{3}{6}}\right] + \left[\cancel{\frac{3}{6}} - \cancel{\frac{3}{7}}\right] + \cdots \\ &\quad + \left[\cancel{\frac{3}{n+1}} - \cancel{\frac{3}{n+2}}\right] + \left[\cancel{\frac{3}{n+2}} - \cancel{\frac{3}{n+3}}\right] + \left[\cancel{\frac{3}{n+3}} - \frac{3}{n+4}\right] \\ &= \frac{3}{4} - \frac{3}{n+4} \end{aligned}$$

It is important when doing this expanding to expand out from both the initial and final values of i and to expand out until all the parts of a series term cancel. Once that has been done it is safe to assume that the cancelling will continue until we get near the end of the expansion.

Note that at this point we now know that the series was a telescoping series since we got all the "interior" terms to cancel out.

Step 5

At this point all we need to do is look at the limit of the partial sums to get,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[\frac{3}{4} - \frac{3}{n+4}\right] = \frac{3}{4}$$

Step 6

The limit of the partial sums exists and is a finite number (*i.e.* not infinity) and so we can see that the series **converges** and its value is,

$$\sum_{n=1}^{\infty} \frac{3}{n^2+7n+12} = \frac{3}{4}$$

---

5. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^{n-2}}$$

Step 1

Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a geometric series.

Step 2

Let's also notice that the initial value of the index is $n = 1$ and so we can put this into the form,

$$\sum_{n=1}^{\infty} a\,r^{n-1}$$

At that point we'll be able to determine if it converges or diverges and the value of the series if it does happen to converge.

As noted above we need the two exponents to be $n-1$. This is an easy "fix" if we note that using basic exponent properties we can write each term as follows,

$$5^{n+1} = 5^{n-1}5^2 \qquad\qquad 7^{n-2} = 7^{n-1}7^{-1}$$

With these two rewrites the series becomes,

$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^{n-2}} = \sum_{n=1}^{\infty} \frac{5^{n-1}5^2}{7^{n-1}7^{-1}} = \sum_{n=1}^{\infty} (25)(7)\frac{5^{n-1}}{7^{n-1}} = \sum_{n=1}^{\infty} 175\left(\frac{5}{7}\right)^{n-1}$$

Step 3

With the series in "proper" form we can see that $a = 175$ and $r = \frac{5}{7}$. Therefore, because we can clearly see that $|r| = \frac{5}{7} < 1$, the series will **converge** and its value is,

$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^{n-2}} = \sum_{n=1}^{\infty} 175\left(\frac{5}{7}\right)^{n-1} = \frac{175}{1-\frac{5}{7}} = \boxed{\frac{1225}{2}}$$

6. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=2}^{\infty} \frac{5^{n+1}}{7^{n-2}}$$

Step 1

Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a geometric series.

Step 2

Now, while we have correctly identified this as a geometric series it doesn't start at either of the two standard starting values of n, *i.e.* $n = 0$ or $n = 1$.

This won't stop us from determining if the series converges or diverges because that only depends on the value of r which we can determine regardless of the starting value of n with enough work. However, if the series does converge we won't be able to use the formula for determining the value of the series as that also needs the value of a and that does require the series to start at one of the two standard starting values.

We have two options for taking care of this problem. One is to use an index shift to convert this into a series that starts at one of the standard starting values of n. In most cases this is probably the only real option.

However, in this case let's notice that this series is almost identical to the series from the previous problem. The only difference is that this series starts at $n = 2$ while the series in the previous problem starts at $n = 1$. This means that we can use the results of the previous problem to greatly reduce the amount of work needed here.

Step 3

We know that the series in the previous problem converged and since we're only changing the starting value of n that will not affect the convergence of the series. Therefore, the series in this problem will also **converge**.

Since we also know that the value of the series in the previous series is $\frac{1225}{2}$ we can find the value of the series in this problem. All we need to do is strip out one term from the series in the previous problem to get,

$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^{n-2}} = \frac{5^2}{7^{-1}} + \sum_{n=2}^{\infty} \frac{5^{n+1}}{7^{n-2}}$$

Then using the value we found in the previous problem can get the value of the series from this problem as follows,

$$\frac{1225}{2} = 175 + \sum_{n=2}^{\infty} \frac{5^{n+1}}{7^{n-2}} \qquad \Rightarrow \qquad \sum_{n=2}^{\infty} \frac{5^{n+1}}{7^{n-2}} = \frac{1225}{2} - 175 = \boxed{\frac{875}{2}}$$

On a quick side note if you did chose to do an index shift here are the two series (for each possible starting value of n) that you should have gotten.

$$\sum_{n=2}^{\infty} \frac{5^{n+1}}{7^{n-2}} = \sum_{n=1}^{\infty} \frac{5^{n+2}}{7^{n-1}} = \sum_{n=0}^{\infty} \frac{5^{n+3}}{7^{n}}$$

Both of the last two are in the "standard" form and can be used to arrive at the same result as above.

7. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=4}^{\infty} \frac{10}{n^2 - 4n + 3}$$

Step 1

Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is not a geometric or harmonic series. That only leaves telescoping as a possibility.

Step 2

Now, we need to be careful here. There is no way to actually identify the series as a telescoping series at this point. We are only hoping that it is a telescoping series.

Therefore, the first real step here is to perform partial fractions on the series term to see what we get. Here is the partial fraction work for the series term.

$$\frac{10}{n^2 - 4n + 3} = \frac{10}{(n-1)(n-3)} = \frac{A}{n-1} + \frac{B}{n-3} \qquad \to \qquad 10 = A(n-3) + B(n-1)$$

$$\begin{array}{lll} n = 1 \qquad & 10 = -2A & \qquad \to \qquad A = -5 \\ n = 3 \qquad & 10 = 2B & \qquad\qquad\quad\; B = 5 \end{array}$$

The series term in partial fraction form is then,

$$\frac{10}{n^2 - 4n + 3} = \frac{5}{n-3} - \frac{5}{n-1}$$

Step 3

The partial sums for this series are then,

$$s_n = \sum_{i=4}^{n} \left[\frac{5}{i-3} - \frac{5}{i-1} \right]$$

Step 4

Expanding the partial sums from the previous step give,

$$\begin{aligned} s_n &= \sum_{i=4}^{n} \left[\frac{5}{i-3} - \frac{5}{i-1} \right] \\ &= \left[\frac{5}{1} - \cancel{\frac{5}{3}} \right] + \left[\frac{5}{2} - \cancel{\frac{5}{4}} \right] + \left[\cancel{\frac{5}{3}} - \cancel{\frac{5}{5}} \right] + \left[\cancel{\frac{5}{4}} - \cancel{\frac{5}{6}} \right] + \left[\cancel{\frac{5}{5}} - \cancel{\frac{5}{7}} \right] + \cdots \\ &\quad + \left[\cancel{\frac{5}{n-7}} - \cancel{\frac{5}{n-5}} \right] + \left[\cancel{\frac{5}{n-6}} - \cancel{\frac{5}{n-4}} \right] + \left[\cancel{\frac{5}{n-5}} - \cancel{\frac{5}{n-3}} \right] + \\ &\quad \left[\cancel{\frac{5}{n-4}} - \frac{5}{n-2} \right] + \left[\cancel{\frac{5}{n-3}} - \frac{5}{n-1} \right] \\ &= 5 + \frac{5}{2} - \frac{5}{n-2} - \frac{5}{n-1} \end{aligned}$$

It is important when doing this expanding to expand out from both the initial and final values of i and to expand out until all the parts of a series term cancel. Once that has been done it is safe to assume that the cancelling will continue until we get near the end of the expansion.

Also, as seen above these can be quite messy to expand out until everything starts to cancel out so don't get too excited about it when it does get messy like this. It just happens sometimes and we have to be careful with all the expansion.

Note that at this point we now know that the series was a telescoping series since we got almost all the "interior" terms to cancel out.

Step 5

At this point all we need to do is look at the limit of the partial sums to get,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[\frac{15}{2} - \frac{5}{n-2} - \frac{5}{n-1} \right] = \frac{15}{2}$$

Step 6

The limit of the partial sums exists and is a finite number (*i.e.* not infinity) and so we can see that the series **converges** and its value is,

$$\sum_{n=4}^{\infty} \frac{10}{n^2 - 4n + 3} = \frac{15}{2}$$

10.6 Integral Test

1. Determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$$

Solution

There really isn't all that much to this problem. We could use the Integral Test on this series or we could just use the p-series Test we discussed in the notes for this section.

We can clearly see that $p = \pi > 1$ and so by the p-series Test this series must **converge.**

---

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{2}{3+5n}$$

Step 1

Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2

The series terms are,

$$a_n = \frac{2}{3+5n}$$

We can clearly see that for the range of n in the series the terms are positive and so that condition is met.

Step 3

In this case because there is only one n in the denominator and because all the terms in the denominator are positive it is (hopefully) clear that,

$$a_n = \frac{2}{3+5n} > \frac{2}{3+5(n+1)} = a_{n+1}$$

and so the series terms are decreasing.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4

Now, let's compute the integral for the test.

$$\int_0^\infty \frac{2}{3+5x}\,dx = \lim_{t\to\infty}\int_0^t \frac{2}{3+5x}\,dx = \lim_{t\to\infty}\left(\frac{2}{5}\ln|3+5x|\right)\Big|_0^t$$
$$= \lim_{t\to\infty}\left(\frac{2}{5}\ln|3+5t| - \frac{2}{5}\ln|3|\right) = \infty$$

Step 5

Okay, the integral from the last step is a divergent integral and so by the Integral Test the series must also be a **divergent** series.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{(2n+7)^3}$$

Step 1

Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2

The series terms are,

$$a_n = \frac{1}{(2n+7)^3}$$

We can clearly see that for the range of n in the series the terms are positive and so that condition is met.

Step 3

In this case because there is only one n in the denominator and because all the terms in the denominator are positive it is (hopefully) clear that,

$$a_n = \frac{1}{(2n+7)^3} > \frac{1}{(2(n+1)+7)^3} = a_{n+1}$$

and so the series terms are decreasing.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4

Now, let's compute the integral for the test.

$$\begin{aligned}\int_2^\infty \frac{1}{(2x+7)^3}\,dx &= \lim_{t\to\infty}\int_2^t \frac{1}{(2x+7)^3}\,dx = \lim_{t\to\infty}\left.\left(-\frac{1}{4}\frac{1}{(2x+7)^2}\right)\right|_2^t \\ &= \lim_{t\to\infty}\left(-\frac{1}{4}\frac{1}{(2t+7)^2}+\frac{1}{4}\frac{1}{(11)^2}\right) = \frac{1}{484}\end{aligned}$$

Step 5

Okay, the integral from the last step is a convergent integral and so by the Integral Test the series must also be a **convergent** series.

4. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}$$

Step 1

Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2

The series terms are,

$$a_n = \frac{n^2}{n^3+1}$$

We can clearly see that for the range of n in the series the terms are positive and so that condition is met.

Step 3

In this case we need to be a little more careful with checking the decreasing condition. We can't just plug in n + 1 into the series term as we've done in the first couple of problems in this section. Doing that would suggest that both the numerator and denominator will increase and so it's not all that clear cut of a case that the terms will be decreasing.

Therefore, we'll need to do a quick Calculus I increasing/decreasing analysis. Here the function for the series terms and its derivative.

$$f(x) = \frac{x^2}{x^3+1} \qquad f'(x) = \frac{2x - x^4}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

With a quick number line or sign chart we can see that the function will increase for $0 < x < \sqrt[3]{2} = 1.2599$ and will decrease for $\sqrt[3]{2} = 1.2599 < x < \infty$. Because the function and series terms are the same we know that the series terms will have the same increasing/decreasing behavior.

So, from this analysis we can see that the series terms are not always decreasing but will be decreasing for $n > \sqrt[3]{2}$ which is sufficient for us to use to say that this condition is also met.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4

Now, let's compute the integral for the test.

$$\begin{aligned}\int_0^\infty \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{x^2}{x^3+1}\,dx &= \lim_{t\to\infty}\left(\frac{1}{3}\ln\left|x^3+1\right|\right)\Big|_0^t \\ &= \lim_{t\to\infty}\left(\frac{1}{3}\ln\left|t^3+1\right| - \ln(1)\right) = \infty\end{aligned}$$

Step 5

Okay, the integral from the last step is a divergent integral and so by the Integral Test the series must also be a **divergent** series.

5. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty}\frac{3}{n^2-3n+2}$$

Step 1

Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2

The series terms are,

$$a_n = \frac{3}{n^2-3n+2}$$

We can clearly see that for $n \geq 3$ (which matches our range of n for the series) we will have,

$$n^2 \geq 3n \qquad \Rightarrow \qquad n^2 - 3n \geq 0 \qquad \Rightarrow \qquad n^2 - 3n + 2 \geq n^2 - 3n \geq 0$$

Therefore, the series terms are positive and so that condition is met.

Note that on occasion we'll need to do more than just state that the series terms are positive by inspection and do a little work to show that the terms really are positive!

Step 3

In this case we need to be a little more careful with checking the decreasing condition. We can't just plug in n + 1 into the series term as we've done in the first couple of problems in this section.

Doing that the first term in the denominator would be getting larger which would suggest the series term is decreasing. However, because the second term in the denominator is subtracted off if we increase n that would suggest the denominator is getting smaller and hence the series term is increasing.

Because we have these "competing" interests we'll need to do a quick Calculus I increasing/decreasing analysis. Here the function for the series terms and its derivative.

$$f(x) = \frac{3}{x^2 - 3x + 2} \qquad\qquad f'(x) = \frac{9 - 6x}{\left(x^2 - 3x + 2\right)^2}$$

With a quick number line or sign chart we can see that the function will increase for $x < \frac{3}{2}$ and will decrease for $x > \frac{3}{2}$. Because the function and series terms are the same we know that the series terms will have the same increasing/decreasing behavior.

So, from this analysis we can see that the series terms are always decreasing for the range n in our series and so this condition is also met.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4

Now, let's compute the integral for the test. The integral we'll need to compute is,

$$\int_3^\infty \frac{3}{x^2 - 3x + 2}\,dx$$

This integral will however require us to do some quick partial fractions in order to do the evaluation. Here is that quick work.

$$\frac{3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \qquad \to \qquad 3 = A(x-2) + B(x-1)$$

$$\begin{array}{lll} x = 1 & 3 = -A & \\ & & \Rightarrow \\ x = 2 & 3 = B & \end{array} \qquad \begin{array}{l} A = -3 \\ B = 3 \end{array}$$

The integral is then,

$$\begin{aligned} \int_3^\infty \frac{3}{x-2} - \frac{3}{x-1}\,dx &= \lim_{t\to\infty} \int_3^t \frac{3}{x-2} - \frac{3}{x-1}\,dx = \lim_{t\to\infty} \left(3\ln|x-2| - 3\ln|x-1|\right)\Big|_3^t \\ &= \lim_{t\to\infty} \left[3\ln|t-2| - 3\ln|t-1| - (3\ln|1| - 3\ln|2|)\right] \\ &= \lim_{t\to\infty} \left[3\ln\left|\frac{t-2}{t-1}\right| + 3\ln|2|\right] = 3\ln\left(\frac{1}{1}\right) + 3\ln(2) = 3\ln(2) \end{aligned}$$

Be careful with the limit of the first two terms! To correctly compute the limit they need to be combined using logarithm properties as shown and we can then do a L'Hospital's Rule on the argument of the log to compute the limit.

Step 5

Okay, the integral from the last step is a convergent integral and so by the Integral Test the series must also be a **convergent** series.

10.7 Comparison Test/Limit Comparison Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + 1\right)^2$$

Step 1

First, the series terms are,

$$a_n = \left(\frac{1}{n^2} + 1\right)^2$$

and it should pretty obvious in this case that they are positive and so we know that we can use the Comparison Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2

For most of the Comparison Test problems we usually guess the convergence and proceed from there. However, in this case it is hopefully clear that for any n,

$$\left(\frac{1}{n^2} + 1\right)^2 > (1)^2 = 1$$

Now, let's take a look at the following series,

$$\sum_{n=1}^{\infty} 1$$

Because $\lim\limits_{n \to \infty} 1 = 1 \neq 0$ we can see from the Divergence Test that this series will be divergent.

So we've found a divergent series with terms that are smaller than the original series terms. Therefore, by the Comparison Test the series in the problem statement must also be **divergent**.

As a final note for this problem notice that we didn't actually need to do a Comparison Test to arrive at this answer. We could have just used the Divergence Test from the beginning since,

$$\lim_{n \to \infty} \left(\frac{1}{n^2} + 1\right)^2 = 1 \neq 0$$

This is something that you should always keep in mind with series convergence problems. The Divergence Test is a quick test that can, on occasion, be used to quickly determine that a series diverges and hence avoid a lot of the hassles of some of the other tests.

---

2. Determine if the following series converges or diverges.

$$\sum_{n=4}^{\infty} \frac{n^2}{n^3 - 3}$$

Step 1

First, the series terms are,

$$a_n = \frac{n^2}{n^3 - 3}$$

and it should pretty obvious that as long as $n > \sqrt[3]{3}$ (which we'll always have for this series) that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint

Can you make a guess as to whether or not the series should converge or diverge?

Step 2

Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "-3" in the denominator won't really affect the size of the denominator for large enough n and so it seems like for large n that the term will probably behave like,

$$b_n = \frac{n^2}{n^3} = \frac{1}{n}$$

We also know that the series,

$$\sum_{n=4}^{\infty} \frac{1}{n}$$

will diverge because it is a harmonic series or by the p-series Test.

Therefore, it makes some sense that we can guess the series in the problem statement will probably diverge as well.

Hint

Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3

So, because we're guessing that the series diverges we'll need to find a series with smaller terms that we know, or can prove, diverges.

Note as well that we'll also need to prove that the new series terms really are smaller than the terms from the series in the problem statement. We can't just "hope" that the will be smaller.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms smaller by either making the numerator smaller or the denominator larger.

In this case it should be pretty clear that,

$$n^3 > n^3 - 3$$

Therefore, we'll have the following relationship.

$$\frac{n^2}{n^3} < \frac{n^2}{n^3 - 3}$$

You do agree with this right? The numerator in each is the same while the denominator in the left term is larger than the denominator in the right term. Therefore, the rational expression on the left must be smaller than the rational expression on the right.

Step 4

Now, the series,

$$\sum_{n=4}^{\infty} \frac{n^2}{n^3} = \sum_{n=4}^{\infty} \frac{1}{n}$$

is a divergent series (as discussed above) and we've also shown that the series terms in this series are smaller than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **diverge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{7}{n(n+1)}$$

Step 1

First, the series terms are,

$$a_n = \frac{7}{n(n+1)}$$

and it should pretty obvious that for the range of n we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint

Can you make a guess as to whether or not the series should converge or diverge?

Step 2

Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "+1" in the denominator won't really affect the size of the denominator for large enough n and so it seems like for large n that the term will probably behave like,

$$b_n = \frac{7}{n(n)} = \frac{7}{n^2}$$

We also know that the series,

$$\sum_{n=2}^{\infty} \frac{7}{n^2}$$

will converge by the p-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint

Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3

So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case it should be pretty clear that,

$$n < n + 1 \qquad \Rightarrow \qquad n(n) < n(n+1)$$

Therefore, we'll have the following relationship.

$$\frac{7}{n(n)} > \frac{7}{n(n+1)}$$

You do agree with this right? The numerator in each is the same while the denominator in the left term is smaller than the denominator in the right term. Therefore, the rational expression on the left must be larger than the rational expression on the right.

Step 4

Now, the series,

$$\sum_{n=2}^{\infty} \frac{7}{n\,(n)} = \sum_{n=2}^{\infty} \frac{7}{n^2}$$

is a convergent series (as discussed above) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

---

4. Determine if the following series converges or diverges.

$$\sum_{n=7}^{\infty} \frac{4}{n^2 - 2n - 3}$$

Step 1

First, the series terms are,

$$a_n = \frac{4}{n^2 - 2n - 3}$$

You can verify that for $n \geq 7$ we have $n^2 > 2n + 3$ and so $n^2 - 2n - 3 = n^2 - (2n + 3) > 0$. Therefore, the series terms are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint

Can you make a guess as to whether or not the series should converge or diverge?

Step 2

Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

For large enough n we know that the n^2 (a quadratic term) in the denominator will increase at a much faster rate than the $-2n-3$ (a linear term) portion of the denominator. Therefore the n^2 portion of the denominator will, in all likelihood, define the behavior of the denominator and so the terms should behave like,

$$b_n = \frac{4}{n^2}$$

We also know that the series,

$$\sum_{n=4}^{\infty} \frac{4}{n^2}$$

will converge by the p-series Test $(p = 2 > 1)$.

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint

Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3

So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

We now have a problem however. The obvious thing to try is to drop the last two terms on the denominator. Doing that however gives the following inequality,

$$n^2 > n^2 - 2n - 3$$

This in turn gives the following relationship.

$$\frac{4}{n^2} < \frac{4}{n^2 - 2n - 3}$$

The denominator on the left is larger and so the rational expression on the left must be smaller. This leads to the problem. While the series,

$$\sum_{n=4}^{\infty} \frac{4}{n^2}$$

will definitely converge (as discussed above) it's terms are smaller than the series terms in the problem statement. Just because a series with smaller terms converges does not, in any way, imply a series with larger terms will also converge!

There are other manipulations we might try but they are all liable to run into similar issues or end up with new terms that we wouldn't be able to quickly prove convergence on.

Hint

So, if the Comparison Test won't easily work what else is there to do?

Step 4

So, the Comparison Test won't easily work in this case. That pretty much leaves the Limit Comparison Test to try. This test only requires positive terms (which we have) and a second series that we're pretty sure behaves like the series we want to know the convergence for. Note as well that, for the Limit Comparison Test, we don't care if the terms for the second series are larger or smaller than problem statement series terms.

If you think about it we already have exactly what we need. In Step 2 we used a second series to guess at the convergence of the problem statement series. The terms in the new series are positive (which we need) and we're pretty sure it behaves in the same manner as the problem statement series.

So, let's compute the limit we need for the Limit Comparison Test.

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left[a_n \frac{1}{b_n} \right] = \lim_{n\to\infty} \left[\frac{4}{n^2 - 2n - 3} \frac{n^2}{4} \right] = \lim_{n\to\infty} \left[\frac{n^2}{n^2 - 2n - 3} \right] = 1$$

Step 5

Okay. We now have $0 < c = 1 < \infty$, *i.e.* c is not zero or infinity and so by the Limit Comparison Test the two series must have the same convergence. We determined in Step 2 that the second series converges and so the series given in the problem statement must also **converge**.

Be careful with the Comparison Test. Too often students just try to "force" larger or smaller by just hoping that the second series terms has the correct relationship (*i.e.* larger or smaller as needed) to the problem series terms. The problem is that this often leads to an incorrect answer. Be careful to always prove the larger/smaller nature of the series terms and if you can't get a series term of the correct larger/smaller nature then you may need to resort to the Limit Comparison Test.

---

5. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{n-1}{\sqrt{n^6+1}}$$

Step 1

First, the series terms are,

$$a_n = \frac{n-1}{\sqrt{n^6+1}}$$

and it should pretty obvious that for the range of n we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint

Can you make a guess as to whether or not the series should converge or diverge?

Step 2

Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The “-1” in the numerator and the “$+1$” in the denominator won’t really affect the size of the numerator and denominator respectively for large enough n and so it seems like for large n that the term will probably behave like,

$$b_n = \frac{n}{\sqrt{n^6}} = \frac{n}{n^3} = \frac{1}{n^2}$$

We also know that the series,

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

will converge by the p-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint

Now that we have our guess, if we’re going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3

So, because we’re guessing that the series converge we’ll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we’ll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can’t just “hope” that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case we can work with both the numerator and the denominator. Let’s start with the numerator. It should be pretty clear that,

$$n > n - 1$$

Using this we can make the numerator larger to get the following relationship,

$$\frac{n-1}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6+1}}$$

Now, in the denominator it again is hopefully clear that,

$$n^6 < n^6 + 1$$

Using this we can make the denominator smaller (and hence make the rational expression larger) to get,

$$\frac{n-1}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6}} = \frac{1}{n^2}$$

Step 4

Now, the series,

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

is a convergent series (as discussed above) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

---

6. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2n^3+7}{n^4 \sin^2(n)}$$

Step 1

First, the series terms are,

$$a_n = \frac{2n^3+7}{n^4 \sin^2(n)}$$

and it should pretty obvious that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint

Can you make a guess as to whether or not the series should converge or diverge?

Step 2

Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "$+7$" in the numerator and the "$\sin^2(n)$" in the denominator won't really affect the size of the numerator and denominator respectively for large enough n and so it seems like for large n that the term will probably behave like,

$$b_n = \frac{2n^3}{n^4} = \frac{2}{n}$$

We also know that the series,

$$\sum_{n=1}^{\infty} \frac{2}{n}$$

will diverge because it is a harmonic series or by the p-series Test.

Therefore, it makes some sense that we can guess the series in the problem statement will probably diverge as well.

Hint

Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3

So, because we're guessing that the series diverges we'll need to find a series with smaller terms that we know, or can prove, diverges.

Note as well that we'll also need to prove that the new series terms really are smaller than the terms from the series in the problem statement. We can't just "hope" that the will be smaller.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms smaller by either making the numerator smaller or the denominator larger.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. It should be pretty clear that,

$$2n^3 < 2n^3 + 7$$

Using this we can make the numerator smaller to get the following relationship,

$$\frac{2n^3 + 7}{n^4 \sin^2(n)} > \frac{2n^3}{n^4 \sin^2(n)}$$

Now, we know that $0 \le \sin^2(n) \le 1$ and so in the denominator we can see that if we replace the $\sin^2(n)$ with its largest possible value we have,

$$n^4 \sin^2(n) < n^4(1) = n^4$$

Using this we can make the denominator larger (and hence make the rational expression smaller) to get,

$$\frac{2n^3 + 7}{n^4 \sin^2(n)} > \frac{2n^3}{n^4 \sin^2(n)} > \frac{2n^3}{n^4} = \frac{2}{n}$$

Step 4

Now, the series,

$$\sum_{n=1}^{\infty} \frac{2}{n}$$

is a divergent series (as discussed above) and we've also shown that the series terms in this series are smaller than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **diverge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

---

7. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{2^n \sin^2(5n)}{4^n + \cos^2(n)}$$

Step 1

First, the series terms are,

$$a_n = \frac{2^n \sin^2(5n)}{4^n + \cos^2(n)}$$

and it should pretty obvious that for the range of n we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint

Can you make a guess as to whether or not the series should converge or diverge?

Step 2

Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The trig functions in the numerator and in the denominator won't really affect the size of the numerator and denominator for large enough n and so it seems like for large n that the term will probably behave like,

$$b_n = \frac{2^n}{4^n} = \left(\frac{2}{4}\right)^n = \left(\frac{1}{2}\right)^n$$

We also know that the series,

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

will converge because it is a geometric series with $r = \frac{1}{2} < 1$.

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint

Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3

So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. We know that $0 \le \sin^2(5n) \le 1$ and so replacing the $\sin^2(5n)$ in the numerator with the largest possible value we get,

$$2^n \sin^2(5n) < 2^n(1) = 2^n$$

Using this we can make the numerator larger to get the following relationship,

$$\frac{2^n \sin^2(5n)}{4^n + \cos^2(n)} < \frac{2^n}{4^n + \cos^2(n)}$$

Now, in the denominator we know that $0 \le \cos^2(n) \le 1$ and so replacing the $\cos^2(n)$ with the smallest possible value we get,

$$4^n + \cos^2(n) > 4^n + 0 = 4^n$$

Using this we can make the denominator smaller (and hence make the rational expression larger) to get,

$$\frac{2^n \sin^2(5n)}{4^n + \cos^2(n)} < \frac{2^n}{4^n + \cos^2(n)} < \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n$$

Step 4

Now, the series,

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$$

is a convergent series (as discussed above) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

---

8. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty}\frac{\mathbf{e}^{-n}}{n^2+2n}$$

Step 1

First, the series terms are,

$$a_n = \frac{\mathbf{e}^{-n}}{n^2+2n}$$

and it should pretty obvious that for the range of n we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint

Can you make a guess as to whether or not the series should converge or diverge?

Step 2

In this case let's first notice the exponential in the numerator will go to zero as n goes to infinity. Let's also notice that the denominator is just a polynomial. In cases like this the exponential is going to go to zero so fast that behavior of the denominator will not matter at all and in all probability the series in the problem statement will probably converge as well.

Hint

Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3

So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. We can use some quick Calculus I to prove that $\mathbf{e}^{-n}$ is a decreasing function and so,

$$\mathbf{e}^{-n} < \mathbf{e}^{-3} < 1$$

Using this we can make the numerator larger to get the following relationship,

$$\frac{\mathbf{e}^{-n}}{n^2 + 2n} < \frac{1}{n^2 + 2n}$$

Now, in the denominator it should be fairly clear that,

$$n^2 + 2n > n^2$$

Using this we can make the denominator smaller (and hence make the rational expression larger) to get,

$$\frac{\mathbf{e}^{-n}}{n^2 + 2n} < \frac{1}{n^2 + 2n} < \frac{1}{n^2}$$

Step 4

Now, the series,

$$\sum_{n=3}^{\infty} \frac{1}{n^2}$$

is a convergent series (p-series Test with $p = 2 > 1$) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

---

9. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{4n^2 - n}{n^3 + 9}$$

Step 1

First, the series terms are,

$$a_n = \frac{4n^2 - n}{n^3 + 9}$$

You can verify that for $n \geq 1$ we have $4n^2 > n$ and so $4n^2 - n > 0$. Therefore, the series terms are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint

Can you make a guess as to whether or not the series should converge or diverge?

Step 2

Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

For large enough n we know that the n^2 (a quadratic term) in the numerator will increase at a much faster rate than the $-n$ (a linear term) portion of the numerator. Therefore the n^2 portion of the numerator will, in all likelihood, define the behavior of the numerator. Likewise, the "+9" in the denominator will not affect the size of the denominator for large n and so the terms should behave like,

$$b_n = \frac{4n^2}{n^3} = \frac{4}{n}$$

We also know that the series,

$$\sum_{n=1}^{\infty} \frac{4}{n}$$

will diverge because it is a harmonic series or by the p-series Test.

Therefore, it makes some sense that we can guess the series in the problem statement will probably diverge as well.

Hint

Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3

So, because we're guessing that the series diverge we'll need to find a series with smaller terms that we know, or can prove, diverge.

Note as well that we'll also need to prove that the new series terms really are smaller than the terms from the series in the problem statement. We can't just "hope" that the will be smaller.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms smaller by either making the numerator smaller or the denominator larger.

We now have a problem however. The obvious thing to try is to drop the last term in both the numerator and the denominator. Doing that however gives the following inequalities,

$$4n^2 - n < 4n^2 \qquad\qquad n^3 + 9 > n^3$$

Using these two in the series terms gives the following relationship,

$$\frac{4n^2 - n}{n^3 + 9} < \frac{4n^2}{n^3 + 9} < \frac{4n^2}{n^3} = \frac{4}{n}$$

Now the series,

$$\sum_{n=0}^{\infty} \frac{4}{n}$$

will definitely diverge (as discussed above) it's terms are larger than the series terms in the problem statement. Just because a series with larger terms diverges does not, in any way, imply a series with smaller terms will also diverge!

There are other manipulations we might try but they are all liable to run into similar issues or end up with new terms that we wouldn't be able to quickly prove convergence on.

Hint

So, if the Comparison Test won't easily work what else is there to do?

Step 4

So, the Comparison Test won't easily work in this case. That pretty much leaves the Limit Comparison Test to try. This test only requires positive terms (which we have) and a second series that we're pretty sure behaves like the series we want to know the convergence for. Note as well that, for the Limit Comparison Test, we don't care if the terms for the second series are larger or smaller than problem statement series terms.

If you think about it we already have exactly what we need. In Step 2 we used a second series to guess at the convergence of the problem statement series. The terms in the new series are positive (which we need) and we're pretty sure it behaves in the same manner as the problem statement series.

So, let's compute the limit we need for the Limit Comparison Test.

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left[a_n \frac{1}{b_n}\right] = \lim_{n\to\infty} \left[\frac{4n^2 - n}{n^3 + 9}\,\frac{n}{4}\right] = \lim_{n\to\infty} \left[\frac{4n^3 - n^2}{4n^3 + 36}\right] = 1$$

Step 5

Okay. We now have $0 < c = 1 < \infty$, *i.e.* c is not zero or infinity and so by the Limit Comparison Test the two series must have the same convergence. We determined in Step 2 that the second series diverges and so the series given in the problem statement must also **diverge**.

Be careful with the Comparison Test. Too often students just try to "force" larger or smaller by just hoping that the second series terms has the correct relationship (*i.e.* larger or smaller as needed) to the problem series terms. The problem is that this often leads to an incorrect answer. Be careful to always prove the larger/smaller nature of the series terms and if you can't get a series term of the correct larger/smaller nature then you may need to resort to the Limit Comparison Test.

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10. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n^2+4n+1}}{n^3+9}$$

Step 1

First, the series terms are,

$$a_n = \frac{\sqrt{2n^2+4n+1}}{n^3+9}$$

and it should pretty obvious that for the range of n we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint

Can you make a guess as to whether or not the series should converge or diverge?

Step 2

Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

For large enough n we know that the $2n^2$ (a quadratic term) in the numerator will increase at a much faster rate than the $4n+1$ (a linear term) portion of the numerator. Therefore the $2n^2$ portion of the numerator will, in all likelihood, define the behavior of the numerator. Likewise, the "+9" in the denominator will not affect the size of the denominator for large n and so the terms should behave like,

$$b_n = \frac{\sqrt{2n^2}}{n^3} = \frac{\sqrt{2}}{n^2}$$

We also know that the series,

$$\sum_{n=1}^{\infty} \frac{\sqrt{2}}{n^2}$$

will converge by the p-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint

Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3

So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

We now have a problem however. The obvious thing to try is to drop the last two terms in the numerator and the last term in the denominator. Doing that however gives the following inequalities,

$$2n^2 < 2n^2 + 4n + 1 \qquad\qquad n^3 + 9 > n^3$$

This leads to a real problem! If we use the inequality for the numerator we're going to get a smaller rational expression and if we use the inequality for the denominator we're going to get a larger rational expression. Because these two can't both be used at the same time it will make it fairly difficult to use the Comparison Test since neither one individually give a series we can quickly deal with.

Hint

So, if the Comparison Test won't easily work what else is there to do?

Step 4

So, the Comparison Test won't easily work in this case. That pretty much leaves the Limit Comparison Test to try. This test only requires positive terms (which we have) and a second series that we're pretty sure behaves like the series we want to know the convergence for. Note as well that, for the Limit Comparison Test, we don't care if the terms for the second series are larger or smaller than problem statement series terms.

If you think about it we already have exactly what we need. In Step 2 we used a second series to guess at the convergence of the problem statement series. The terms in the new series are positive (which we need) and we're pretty sure it behaves in the same manner as the problem statement series.

So, let's compute the limit we need for the Limit Comparison Test.

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left[a_n \frac{1}{b_n} \right] = \lim_{n\to\infty} \left[\frac{\sqrt{2n^2+4n+1}}{n^3+9} \frac{n^2}{\sqrt{2}} \right]$$

$$= \lim_{n\to\infty} \left[\frac{n^2\sqrt{n^2\left(2+\frac{4}{n}+\frac{1}{n^2}\right)}}{\sqrt{2}\,n^3\left(1+\frac{9}{n^3}\right)} \right] = \lim_{n\to\infty} \left[\frac{n^2\left(n\right)\sqrt{2+\frac{4}{n}+\frac{1}{n^2}}}{\sqrt{2}\,n^3\left(1+\frac{9}{n^3}\right)} \right] = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

Step 5

Okay. We now have $0 < c = 1 < \infty$, *i.e.* c is not zero or infinity and so by the Limit Comparison Test the two series must have the same convergence. We determined in Step 2 that the second series converges and so the series given in the problem statement must also **converge**.

Be careful with the Comparison Test. Too often students just try to "force" larger or smaller by just hoping that the second series terms has the correct relationship (*i.e.* larger or smaller as needed) to the problem series terms. The problem is that this often leads to an incorrect answer. Be careful to always prove the larger/smaller nature of the series terms and if you can't get a series term of the correct larger/smaller nature then you may need to resort to the Limit Comparison Test.

10.8 Alternating Series Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$$

Step 1

First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{1}{7+2n}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2

Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{7+2n} = 0$$

So, the limit is zero and so the first condition is met.

Step 3

Now let's take care of the decreasing check. In this case it should be pretty clear that,

$$\frac{1}{7+2n} > \frac{1}{7+2(n+1)}$$

since increasing n will only increase the denominator and hence force the rational expression to be smaller.

Therefore the b_n form a decreasing sequence.

Step 4

So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

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2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+3}}{n^3 + 4n + 1}$$

Step 1

First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{1}{n^3 + 4n + 1}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2

Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^3 + 4n + 1} = 0$$

So, the limit is zero and so the first condition is met.

Step 3

Now let's take care of the decreasing check. In this case it should be pretty clear that,

$$\frac{1}{n^3 + 4n + 1} > \frac{1}{(n+1)^3 + 4(n+1) + 1}$$

since increasing n will only increase the denominator and hence force the rational expression to be smaller.

Therefore the b_n form a decreasing sequence.

Step 4

So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{1}{(-1)^n (2^n + 3^n)}$$

Step 1

Do not get excited about the $(-1)^n$ is in the denominator! This is still an alternating series! All the $(-1)^n$ does is change the sign regardless of whether or not it is in the numerator.

Also note that we could just as easily rewrite the terms as,

$$\frac{1}{(-1)^n (2^n + 3^n)} = \frac{(-1)^n}{(-1)^n} \frac{1}{(-1)^n (2^n + 3^n)} = \frac{(-1)^n}{(-1)^{2n} (2^n + 3^n)} = \frac{(-1)^n}{(2^n + 3^n)}$$

Note that $(-1)^{2n} = 1$ because the exponent is always even!

So, we now know that this is an alternating series with,

$$b_n = \frac{1}{2^n + 3^n}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2

Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2^n + 3^n} = 0$$

So, the limit is zero and so the first condition is met.

Step 3

Now let's take care of the decreasing check. In this case it should be pretty clear that,

$$\frac{1}{2^n + 3^n} > \frac{1}{2^{n+1} + 3^{n+1}}$$

since increasing n will only increase the denominator and hence force the rational expression to be smaller.

Therefore the b_n form a decreasing sequence.

Step 4

So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

---

4. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+6} n}{n^2 + 9}$$

Step 1

First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{n}{n^2 + 9}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2

Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+9} = 0$$

So, the limit is zero and so the first condition is met.

Step 3

Now let's take care of the decreasing check. In this case increasing n will increase both the numerator and denominator and so we can't just say that clearly the terms are decreasing as we did in the first few problems.

We will have no choice but to do a little Calculus I work for this problem. Here is the function and derivative for that work.

$$f(x) = \frac{x}{x^2+9} \qquad f'(x) = \frac{9-x^2}{(x^2+9)^2}$$

It should be pretty clear that the function will be increasing in $0 \le x < 3$ and decreasing in $x > 3$ (the range of x that corresponds to our range of n).

So, the b_n do not actually form a decreasing sequence but they are decreasing for $n > 3$ and so we can say that they are eventually decreasing and as discussed in the notes that will be sufficient for us.

Step 4

So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

---

5. Determine if the following series converges or diverges.

$$\sum_{n=4}^{\infty} \frac{(-1)^{n+2}(1-n)}{3n-n^2}$$

Step 1

First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{1-n}{3n-n^2}$$

and b_n are positive for $n \geq 4$ and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2

Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1-n}{3n-n^2} = 0$$

So, the limit is zero and so the first condition is met.

Step 3

Now let's take care of the decreasing check. In this case increasing n will increase both the numerator and denominator and so we can't just say that clearly the terms are decreasing as we did in the first few problems.

We will have no choice but to do a little Calculus I work for this problem. Here is the function and derivative for that work.

$$f(x) = \frac{1-x}{3x-x^2} \qquad\qquad f'(x) = \frac{-x^2+2x-3}{(3x-x^2)^2}$$

The numerator of the derivative is never zero for any real number (we'll leave that to you to verify) and since it is clearly negative at $x=0$ we know that the function will always be decreasing for $x \geq 4$.

Therefore the b_n form a decreasing sequence.

Step 4

So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

10.9 Absolute Convergence

1. Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^3+1}$$

Step 1

Okay, let's first see if the series converges or diverges if we put absolute value on the series terms.

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^{n+1}}{n^3+1} \right| = \sum_{n=2}^{\infty} \frac{1}{n^3+1}$$

Now, notice that,

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

and we know by the p-series test that

$$\sum_{n=2}^{\infty} \frac{1}{n^3}$$

converges.

Therefore, by the Comparison Test we know that the series from the problem statement,

$$\sum_{n=2}^{\infty} \frac{1}{n^3+1}$$

will also converge.

Step 2

So, because the series with the absolute value converges we know that the series in the problem statement is **absolutely convergent**.

2. Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-3}}{\sqrt{n}}$$

Step 1

Okay, let's first see if the series converges or diverges if we put absolute value on the series terms.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-3}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

Now, by the by the p-series test we can see that this series will diverge.

Step 2

So, at this point we know that the series in the problem statement is not absolutely convergent so all we need to do is check to see if it's conditionally convergent or divergent. To do this all we need to do is check the convergence of the series in the problem statement.

The series in the problem statement is an alternating series with,

$$b_n = \frac{1}{\sqrt{n}}$$

Clearly the b_n are positive so we can use the Alternating Series Test on this series. It is hopefully clear that the b_n are a decreasing sequence and $\lim_{n\to\infty} b_n = 0$.

Therefore, by the Alternating Series Test the series from the problem statement is convergent.

Step 3

So, because the series with the absolute value diverges and the series from the problem statement converges we know that the series in the problem statement is **conditionally convergent**.

3. Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=3}^{\infty} \frac{(-1)^{n+1}(n+1)}{n^3+1}$$

Step 1

Okay, let's first see if the series converges or diverges if we put absolute value on the series terms.

$$\sum_{n=3}^{\infty} \left| \frac{(-1)^{n+1}(n+1)}{n^3+1} \right| = \sum_{n=3}^{\infty} \frac{n+1}{n^3+1}$$

We know by the p-series test that the following series converges.

$$\sum_{n=3}^{\infty} \frac{1}{n^2}$$

If we now compute the following limit,

$$c = \lim_{n\to\infty} \left[\frac{n+1}{n^3+1} \frac{n^2}{1} \right] = \lim_{n\to\infty} \left[\frac{n^3+n^2}{n^3+1} \right] = 1$$

we know by the Limit Comparison Test that the two series in this problem have the same convergence because c is neither zero or infinity and because $\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges we know that the series from the problem statement must also converge.

Step 2

So, because the series with the absolute value converges we know that the series in the problem statement is **absolutely convergent**.

10.10 Ratio Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2+1}$$

Step 1

We'll need to compute L.

$$\begin{aligned} L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{1-2(n+1)}}{(n+1)^2+1}\,\frac{n^2+1}{3^{1-2n}}\right| \\ &= \lim_{n\to\infty} \left|\frac{3^{-1-2n}}{(n+1)^2+1}\,\frac{n^2+1}{3^{1-2n}}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)^2+1}\,\frac{n^2+1}{3^2}\right| \\ &= \lim_{n\to\infty} \left|\frac{n^2+1}{9\left[(n+1)^2+1\right]}\right| = \frac{1}{9} \end{aligned}$$

When computing a_{n+1} be careful to pay attention to any coefficients of n and powers of n. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2

Okay, we can see that $L = \frac{1}{9} < 1$ and so by the Ratio Test the series **converges**.

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(2n)!}{5n+1}$$

Step 1

We'll need to compute L.

$$\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2\,(n+1))!}{5\,(n+1)+1}\,\frac{5n+1}{(2n)!}\right| \\
&= \lim_{n\to\infty} \left|\frac{(2n+2)!}{5n+6}\,\frac{5n+1}{(2n)!}\right| = \lim_{n\to\infty} \left|\frac{(2n+2)\,(2n+1)\,(2n)!}{5n+6}\,\frac{5n+1}{(2n)!}\right| \\
&= \lim_{n\to\infty} \left|\frac{(2n+2)\,(2n+1)\,(5n+1)}{5n+6}\right| = \infty
\end{aligned}$$

When computing a_{n+1} be careful to pay attention to any coefficients of n and powers of n. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2

Okay, we can see that $L = \infty > 1$ and so by the Ratio Test the series **diverges**.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{(-2)^{1+3n}\,(n+1)}{n^2 5^{1+n}}$$

Step 1

We'll need to compute L.

$$\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-2)^{1+3(n+1)}\,(n+1+1)}{(n+1)^2 5^{1+n+1}}\,\frac{n^2 5^{1+n}}{(-2)^{1+3n}\,(n+1)}\right| \\
&= \lim_{n\to\infty} \left|\frac{(-2)^{4+3n}\,(n+2)}{(n+1)^2 5^{2+n}}\,\frac{n^2 5^{1+n}}{(-2)^{1+3n}\,(n+1)}\right| = \lim_{n\to\infty} \left|\frac{(-2)^3\,(n+2)}{(n+1)^2\,(5)}\,\frac{n^2}{(n+1)}\right| \\
&= \lim_{n\to\infty} \left|\frac{-8n^2\,(n+2)}{5(n+1)^2\,(n+1)}\right| = \frac{8}{5}
\end{aligned}$$

When computing a_{n+1} be careful to pay attention to any coefficients of n and powers of n. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2

Okay, we can see that $L = \frac{8}{5} > 1$ and so by the Ratio Test the series **diverges**.

4. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{\mathbf{e}^{4n}}{(n-2)!}$$

Step 1

We'll need to compute L.

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4(n+1)}}{(n+1-2)!}\frac{(n-2)!}{\mathbf{e}^{4n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4n+4}}{(n-1)!}\frac{(n-2)!}{\mathbf{e}^{4n}}\right| = \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4n+4}}{(n-1)\,(n-2)!}\frac{(n-2)!}{\mathbf{e}^{4n}}\right| = \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4}}{n-1}\right| = 0 \end{aligned}$$

When computing a_{n+1} be careful to pay attention to any coefficients of n and powers of n. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2

Okay, we can see that $L = 0 < 1$ and so by the Ratio Test the series **converges**.

5. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{6n+7}$$

Step 1

We'll need to compute L.

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1+1}}{6(n+1)+7}\,\frac{6n+7}{(-1)^{n+1}}\right|$$
$$= \lim_{n\to\infty} \left|\frac{(-1)^{n+2}}{6n+13}\,\frac{6n+7}{(-1)^{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(-1)(6n+7)}{6n+13}\right| = 1$$

When computing a_{n+1} be careful to pay attention to any coefficients of n and powers of n. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2

Okay, we can see that $L = 1$ and so by the Ratio Test tells us nothing about this series.

Step 3

Just because the Ratio Test doesn't tell us anything doesn't mean we can't determine if this series converges or diverges.

In fact, it's actually quite simple to do in this case. This is an Alternating Series with,

$$b_n = \frac{1}{6n+7}$$

The b_n are clearly positive and it should be pretty obvious (hopefully) that they also form a decreasing sequence. Finally, we also can see that $\lim_{n\to\infty} b_n = 0$ and so by the Alternating Series Test this series will **converge**.

Note, that if this series were not in this section doing this as an Alternating Series from the start would probably have been the best way of approaching this problem.

10.11 Root Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\left(\frac{3n+1}{4-2n}\right)^{2n}$$

Step 1

We'll need to compute L.

$$L=\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\left|\left(\frac{3n+1}{4-2n}\right)^{2n}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\left(\frac{3n+1}{4-2n}\right)^{2}\right|=\left(-\frac{3}{2}\right)^{2}=\frac{9}{4}$$

Step 2

Okay, we can see that $L=\frac{9}{4}>1$ and so by the Root Test the series **diverges**.

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty}\frac{n^{1-3n}}{4^{2n}}$$

Step 1

We'll need to compute L.

$$L=\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\left|\frac{n^{1-3n}}{4^{2n}}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\frac{n^{\frac{1}{n}-3}}{4^{2}}\right|=\left|\frac{n^{\frac{1}{n}}\,n^{-3}}{4^{2}}\right|=\frac{(1)\,(0)}{16}=0$$

Step 2

Okay, we can see that $L=0<1$ and so by the Root Test the series **converges**.

3. Determine if the following series converges or diverges.

$$\sum_{n=4}^{\infty} \frac{(-5)^{1+2n}}{2^{5n-3}}$$

Step 1

We'll need to compute L.

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{(-5)^{1+2n}}{2^{5n-3}}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{(-5)^{\frac{1}{n}+2}}{2^{5-\frac{3}{n}}}\right| = \left|\frac{(-5)^2}{2^5}\right| = \frac{25}{32}$$

Step 2

Okay, we can see that $L = \frac{25}{32} < 1$ and so by the Root Test the series **converges**.

10.12 Strategy for Series

Problems have not yet been written for this section.

I was finding it very difficult to come up with a good mix of new problems and decided my time was better spent writing problems for later sections rather than trying to come up with a sufficient number of problems for what is essentially a review section. I intend to come back at a later date when I have more time to devote to this section and add problems then.

10.13 Estimating the Value of a Series

1. Use the Integral Test and $n = 10$ to estimate the value of $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\left(n^2+1\right)^2}$.

Step 1

Since we are being asked to use the Integral Test to estimate the value of the series we should first make sure that the Integral Test can actually be used on this series.

First, the series terms are clearly positive so that condition is met.

Now, let's do a little Calculus I on the following function.

$$f\left(x\right) = \frac{x}{\left(x^2+1\right)^2} \qquad\qquad f'\left(x\right) = \frac{1-3x^2}{\left(x^2+1\right)^3}$$

The derivative of the function will be negative for $x > \frac{1}{\sqrt{3}} = 0.5774$ and so the function will be decreasing in this range. Because the function and the series terms are the same we can also see that the series terms are decreasing for the range of n in our series.

Therefore, the conditions required to use the Integral Test are met! Note that it is really important to test these conditions before proceeding with the problem. It doesn't make any sense to use a test to estimate the value of a series if the test can't be used on the series. We shouldn't just assume that because we are being asked to use a test here that the test can actually be used!

Step 2

Let's start off with the partial sum using $n = 10$. This is,

$$s_{10} = \sum_{n=1}^{10} \frac{n}{\left(n^2+1\right)^2} = 0.392632317$$

Step 3

Now, to increase the accuracy of the partial sum from the previous step we know we can use each of the following two integrals.

$$\begin{aligned}\int_{10}^{\infty}\frac{x}{\left(x^2+1\right)^2}\,dx=\lim_{t\to\infty}\int_{10}^{t}\frac{x}{\left(x^2+1\right)^2}\,dx&=\lim_{t\to\infty}\left[-\frac{1}{2\left(x^2+1\right)}\right]\Bigg|_{10}^{t}\\&=\lim_{t\to\infty}\left[\frac{1}{202}-\frac{1}{2\left(t^2+1\right)}\right]=\frac{1}{202}\end{aligned}$$

$$\begin{aligned}\int_{11}^{\infty}\frac{x}{\left(x^2+1\right)^2}\,dx=\lim_{t\to\infty}\int_{11}^{t}\frac{x}{\left(x^2+1\right)^2}\,dx&=\lim_{t\to\infty}\left[-\frac{1}{2\left(x^2+1\right)}\right]\Bigg|_{11}^{t}\\&=\lim_{t\to\infty}\left[\frac{1}{244}-\frac{1}{2\left(t^2+1\right)}\right]=\frac{1}{244}\end{aligned}$$

Step 4

Okay, we know from the notes in this section that if s represents that actual value of the series that it must be in the following range.

$$\begin{gathered}0.392632317+\frac{1}{244}<s<0.392632317+\frac{1}{202}\\0.396730678<s<0.397582813\end{gathered}$$

Step 5

Finally, if we average the two numbers above we can get a better estimate of,

$$\boxed{s\approx 0.397156745}$$

2. Use the Comparison Test and $n = 20$ to estimate the value of $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^3 \ln(n)}$.

Step 1

Since we are being asked to use the Comparison Test to estimate the value of the series we should first make sure that the Comparison Test can actually be used on this series.

In this case that is easy enough because, for our range of n, the series terms are clearly positive and so we can use the Comparison Test.

Note that it is really important to test these conditions before proceeding with the problem. It doesn't make any sense to use a test to estimate the value of a series if the test can't be used on the series. We shouldn't just assume that because we are being asked to use a test here that the test can actually be used!

Step 2

Let's start off with the partial sum using $n = 20$. This is,

$$s_{20} = \sum_{n=3}^{20} \frac{1}{n^3 \ln(n)} = 0.057315878$$

Step 3

Now, let's see if we can get can get an error estimate on this approximation of the series value. To do that we'll first need to do the Comparison Test on this series.

That is easy enough for this series once we notice that $\ln(n)$ is an increasing function and so $\ln(n) \ge \ln(3)$. Therefore, we get,

$$\frac{1}{n^3 \ln(n)} \le \frac{1}{n^3 \ln(3)} = \frac{1}{\ln(3)} \frac{1}{n^3}$$

Step 4

We now know, from the discussion in the notes, that an upper bound on the value of the remainder (*i.e.* the error between the approximation and exact value) is,

$$\begin{aligned} R_{20} \le T_{20} &= \sum_{n=21}^{\infty} \frac{1}{n^3 \ln(3)} < \int_{20}^{\infty} \frac{1}{x^3 \ln(3)}\,dx \\ &= \lim_{t\to\infty} \int_{20}^{t} \frac{1}{x^3 \ln(3)}\,dx = \lim_{t\to\infty} \left(-\frac{1}{2x^2 \ln(3)} \right)\Bigg|_{20}^{t} \\ &= \lim_{t\to\infty} \left(\frac{1}{800 \ln(3)} - \frac{1}{2t^2 \ln(3)} \right) = \frac{1}{800 \ln(3)} \end{aligned}$$

Step 5

So, we can estimate that the value of the series is,

$$\boxed{s \approx 0.057315878}$$

and the error on this estimate will be no more than $\frac{1}{800\ln(3)} = 0.001137799$.

3. Use the Alternating Series Test and $n = 16$ to estimate the value of $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+1}$.

Step 1

Since we are being asked to use the Alternating Series Test to estimate the value of the series we should first make sure that the Alternating Series Test can actually be used on this series.

First, note that the b_n for this series are,

$$b_n = \frac{n}{n^2+1}$$

and they are positive and with a quick derivative we can see they are decreasing and so the Alternating Series Test can be used here.

Note that it is really important to test these conditions before proceeding with the problem. It doesn't make any sense to use a test to estimate the value of a series if the test can't be used on the series. We shouldn't just assume that because we are being asked to use a test here that the test can actually be used!

Step 2

Let's start off with the partial sum using $n = 16$. This is,

$$s_{16} = \sum_{n=2}^{16} \frac{(-1)^n n}{n^2 + 1} = 0.260554530$$

Step 3

Now, we know, from the discussion in the notes, that an upper bound on the absolute value of the remainder (*i.e.* the error between the approximation and exact value) is nothing more than,

$$b_{17} = \frac{17}{290} = 0.058620690$$

Step 4

So, we can estimate that the value of the series is,

$$\boxed{s \approx 0.260554530}$$

and the error on this estimate will be no more than 0.058620690.

---

4. Use the Ratio Test and $n = 8$ to estimate the value of $\displaystyle\sum_{n=1}^{\infty} \frac{3^{1+n}}{n\,2^{3+2n}}$.

Step 1

First notice that the terms are positive and so we can use the Ratio Test to do the estimate. Remember that this is a requirement only to use the Ratio Test to get an estimate of the series value and is not an actual requirement to use the Ratio Test to determine if the series converges or diverges.

So, let's start off with the partial sum using $n = 8$. This is,

$$s_8 = \sum_{n=1}^{8} \frac{3^{1+n}}{n\,2^{3+2n}} = 0.509881435$$

Step 2

Now, to get an upper bound on the value of the remainder (*i.e.* the error between the approximation and exact value) we need the following ratio,

$$r_n = \frac{a_{n+1}}{a_n} = \frac{3^{2+n}}{(n+1)\ 2^{5+2n}} \frac{n\,2^{3+2n}}{3^{1+n}} = \frac{3n}{4\,(n+1)}$$

We'll also potentially need the limit,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3n}{4\,(n+1)} = \frac{3}{4}$$

Step 3

Next, we need to know if the r_n form an increasing or decreasing sequence. A quick application of Calculus I will answer this.

$$f\,(x) = \frac{3x}{4\,(x+1)} \qquad\qquad f'\,(x) = \frac{3}{4(x+1)^2} > 0$$

As noted above the derivative is always positive and so the function, and hence the r_n are increasing.

Step 4

The upper bound on the remainder is then,

$$R_8 \le \frac{a_9}{1-L} = \frac{\frac{6561}{2{,}097{,}152}}{1-\frac{3}{4}} = 0.012514114$$

Step 5

So, we can estimate that the value of the series is,

$$\boxed{s \approx 0.509881435}$$

and the error on this estimate will be no more than 0.012514114.

10.14 Power Series

1. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}(n^2+1)}(4x-12)^n$$

Step 1

Okay, let's start off with the Ratio Test to get our hands on L.

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{(4x-12)^{n+1}}{(-3)^{3+n}\left((n+1)^2+1\right)}\frac{(-3)^{2+n}(n^2+1)}{(4x-12)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(4x-12)}{(-3)\left((n+1)^2+1\right)}\frac{(n^2+1)}{1}\right| \\ &= |4x-12|\lim_{n\to\infty}\frac{(n^2+1)}{3\left((n+1)^2+1\right)} = \frac{1}{3}|4x-12| \end{aligned}$$

Step 2

So, we know that the series will converge if,

$$\frac{1}{3}|4x-12|<1 \qquad \to \qquad \frac{4}{3}|x-3|<1 \qquad \to \qquad |x-3|<\frac{3}{4}$$

Step 3

So, from the previous step we see that the radius of convergence is $\boxed{R=\frac{3}{4}}$.

Step 4

Now, let's start working on the interval of convergence. Let's break up the inequality we got in Step 2.

$$-\frac{3}{4}<x-3<\frac{3}{4} \qquad \to \qquad \frac{9}{4}<x<\frac{15}{4}$$

Step 5

To finalize the interval of convergence we need to check the end points of the inequality from Step 4.

$$x = \frac{9}{4}: \quad \sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}(n^2+1)}(-3)^n = \sum_{n=0}^{\infty} \frac{1}{(-3)^2(n^2+1)} = \sum_{n=0}^{\infty} \frac{1}{9(n^2+1)}$$

$$x = \frac{15}{4}: \quad \sum_{n=0}^{\infty} \frac{1}{(-1)^{2+n}(3)^{2+n}(n^2+1)}(3)^n = \sum_{n=0}^{\infty} \frac{1}{(-1)^{2+n}(3)^2(n^2+1)} = \sum_{n=0}^{\infty} \frac{(-1)^{2+n}}{9(n^2+1)}$$

Now, we can do a quick Comparison Test on the first series to see that it converges and we can do a quick Alternating Series Test on the second series to see that is also converges.

We'll leave it to you to verify both of these statements.

Step 6

The interval of convergence is below and for summary purposes the radius of convergence is also shown.

$$\boxed{\text{Interval}: \frac{9}{4} \le x \le \frac{15}{4}} \qquad \boxed{R = \frac{3}{4}}$$

2. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{n^{2n+1}}{4^{3n}}(2x+17)^n$$

Step 1

Okay, let's start off with the Root Test to get our hands on L.

$$L = \lim_{n\to\infty} \left| \frac{n^{2n+1}}{4^{3n}}(2x+17)^n \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{n^{2+\frac{1}{n}}}{4^3}(2x+17) \right| = |2x+17| \lim_{n\to\infty} \frac{n^{2+\frac{1}{n}}}{4^3}$$

Okay, we can see that , in this case, L will be infinite provided $x \ne -\frac{17}{2}$ and so the series will diverge for $x \neq -\frac{17}{2}$. We also know that the power series will converge for $x = -\frac{17}{2}$ (this is the value of a for this series!).

Step 2

Therefore, we know that the interval of convergence is $\boxed{x = -\dfrac{17}{2}}$ and the radius of convergence is $\boxed{R = 0}$.

3. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{n+1}{(2n+1)!}(x-2)^n$$

Step 1

Okay, let's start off with the Ratio Test to get our hands on L.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(n+2)(x-2)^{n+1}}{(2n+3)!} \frac{(2n+1)!}{(n+1)(x-2)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+2)(x-2)}{(2n+3)(2n+2)(2n+1)!} \frac{(2n+1)!}{(n+1)} \right| \\ &= |x-2| \lim_{n\to\infty} \frac{n+2}{(2n+3)(2n+2)(n+1)} = 0 \end{aligned}$$

Okay, we can see that , in this case, $L = 0$ for every x.

Step 2

Therefore, we know that the interval of convergence is $\boxed{-\infty < x < \infty}$ and the radius of convergence is $\boxed{R = \infty}$.

4. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{4^{1+2n}}{5^{n+1}}(x+3)^n$$

Step 1

Okay, let's start off with the Ratio Test to get our hands on L.

$$L = \lim_{n\to\infty} \left| \frac{4^{3+2n}(x+3)^{n+1}}{5^{n+2}} \frac{5^{n+1}}{4^{1+2n}(x+3)^n} \right| = \lim_{n\to\infty} \left| \frac{4^2(x+3)}{5} \right|$$

$$= |x+3| \lim_{n\to\infty} \frac{16}{5} = \frac{16}{5}|x+3|$$

Step 2

So, we know that the series will converge if,

$$\frac{16}{5}|x+3| < 1 \qquad \rightarrow \qquad |x+3| < \frac{5}{16}$$

Step 3

So, from the previous step we see that the radius of convergence is $\boxed{R = \frac{5}{16}}$.

Step 4

Now, let's start working on the interval of convergence. Let's break up the inequality we got in Step 2.

$$-\frac{5}{16} < x+3 < \frac{5}{16} \qquad \rightarrow \qquad -\frac{53}{16} < x < -\frac{43}{16}$$

Step 5

To finalize the interval of convergence we need to check the end points of the inequality from Step 4.

$$x = -\frac{53}{16} : \quad \sum_{n=0}^{\infty} \frac{4^1 4^{2n}}{5^n 5^1} \left(-\frac{5}{16}\right)^n = \sum_{n=0}^{\infty} \frac{4\left(16^n\right)}{5^n\left(5\right)} \frac{\left(-1\right)^n 5^n}{16^n} = \sum_{n=0}^{\infty} \frac{4(-1)^n}{5}$$

$$x = -\frac{43}{16} : \quad \sum_{n=0}^{\infty} \frac{4^1 4^{2n}}{5^n 5^1} \left(\frac{5}{16}\right)^n = \sum_{n=0}^{\infty} \frac{4\left(16^n\right)}{5^n\left(5\right)} \frac{5^n}{16^n} = \sum_{n=0}^{\infty} \frac{4}{5}$$

Now,

$$\lim_{n\to\infty} \frac{4(-1)^n}{5} \quad - \quad \text{Does not exist} \qquad\qquad \lim_{n\to\infty} \frac{4}{5} = \frac{4}{5}$$

Therefore, each of these two series diverge by the Divergence Test.

Step 6

The interval of convergence is below and for summary purposes the radius of convergence is also shown.

$$\boxed{\text{Interval} : -\frac{53}{16} < x < -\frac{43}{16}} \qquad \boxed{R = \frac{5}{16}}$$

---

5. For the following power series determine the interval and radius of convergence.

$$\sum_{n=1}^{\infty} \frac{6^n}{n} \left(4x-1\right)^{n-1}$$

Step 1

Okay, let's start off with the Ratio Test to get our hands on L.

$$L = \lim_{n\to\infty} \left| \frac{6^{n+1}(4x-1)^n}{n+1} \frac{n}{6^n (4x-1)^{n-1}} \right| = \lim_{n\to\infty} \left| \frac{6n\left(4x-1\right)}{n+1} \right|$$
$$= |4x-1| \lim_{n\to\infty} \frac{6n}{n+1} = 6\left|4x-1\right|$$

Step 2

So, we know that the series will converge if,

$$6\left|4x-1\right|<1 \qquad \to \qquad 24\left|x-\frac{1}{4}\right|<1 \qquad \to \qquad \left|x-\frac{1}{4}\right|<\frac{1}{24}$$

Step 3

So, from the previous step we see that the radius of convergence is $\boxed{R=\frac{1}{24}}$.

Step 4

Now, let's start working on the interval of convergence. Let's break up the inequality we got in Step 2.

$$-\frac{1}{24}<x-\frac{1}{4}<\frac{1}{24} \qquad \to \qquad \frac{5}{24}<x<\frac{7}{24}$$

Step 5

To finalize the interval of convergence we need to check the end points of the inequality from Step 4.

$$x=\frac{5}{24}: \quad \sum_{n=1}^{\infty}\frac{6^n}{n}\left(-\frac{1}{6}\right)^{n-1}=\sum_{n=1}^{\infty}\frac{6^n}{n}\,\frac{(-1)^{n-1}}{6^{n-1}}=\sum_{n=1}^{\infty}\frac{6(-1)^{n-1}}{n}$$

$$x=\frac{7}{24}: \quad \sum_{n=1}^{\infty}\frac{6^n}{n}\left(\frac{1}{6}\right)^{n-1}=\sum_{n=1}^{\infty}\frac{6^n}{n}\,\frac{1}{6^{n-1}}=\sum_{n=1}^{\infty}\frac{6}{n}$$

Now, the first series is an alternating harmonic series which we know converges (or you could just do a quick Alternating Series Test to verify this) and the second series diverges by the p-series test.

Step 6

The interval of convergence is below and for summary purposes the radius of convergence is also shown.

$$\boxed{\text{Interval}: \frac{5}{24} \le x < \frac{7}{24}} \qquad \boxed{R = \frac{1}{24}}$$

10.15 Power Series and Functions

1. Write the following function as a power series and give the interval of convergence.

$$f(x) = \frac{6}{1+7x^4}$$

Step 1

First, in order to use the formula from this section we know that we need the numerator to be a one. That is easy enough to "fix" up as follows,

$$f(x) = 6\frac{1}{1+7x^4}$$

Step 2

Next, we know we need the denominator to be in the form $1-p$ and again that is easy enough, in this case, to rewrite the denominator to get the following form of the function,

$$f(x) = 6\frac{1}{1-(-7x^4)}$$

Step 3

At this point we can use the formula from the notes to write this as a power series. Doing this gives,

$$f(x) = 6\frac{1}{1-(-7x^4)} = 6\sum_{n=0}^{\infty}\left(-7x^4\right)^n \qquad \text{provided } \left|-7x^4\right| < 1$$

Step 4

Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single x with a single exponent on it.

These are easy enough rules to take care of. All we need to do is move whatever is in front of the series to the inside of the series and use basic exponent rules to take care of the x "rule". Doing all this gives,

$$f\left(x\right) = \sum_{n=0}^{\infty} 6(-7)^n \left(x^4\right)^n = \sum_{n=0}^{\infty} 6(-7)^n x^{4n} \qquad \text{provided } \left|-7x^4\right| < 1$$

Step 5

To get the interval of convergence all we need to do is do a little work on the "provided" portion of the result from the last step to get,

$$\left|-7x^4\right| < 1 \quad \rightarrow \quad 7|x|^4 < 1 \quad \rightarrow \quad |x|^4 < \frac{1}{7} \quad \rightarrow \quad |x| < \frac{1}{7^{\frac{1}{4}}} \quad \rightarrow \quad -\frac{1}{7^{\frac{1}{4}}} < x < \frac{1}{7^{\frac{1}{4}}}$$

Note that we don't need to check the endpoints of this interval since we already know that we only get convergence with the strict inequalities and we will get divergence for everything else.

Step 6

The answers for this problem are then,

$$\boxed{\text{Power Series : } \frac{6}{1+7x^4} = \sum_{n=0}^{\infty} 6(-7)^n x^{4n}} \qquad \boxed{\text{Interval : } -\frac{1}{7^{\frac{1}{4}}} < x < \frac{1}{7^{\frac{1}{4}}}}$$

2. Write the following function as a power series and give the interval of convergence.

$$f\left(x\right) = \frac{x^3}{3 - x^2}$$

Step 1

First, in order to use the formula from this section we know that we need the numerator to be a one. That is easy enough to "fix" up as follows,

$$f\left(x\right) = x^3 \frac{1}{3 - x^2}$$

Step 2

Next, we know we need the denominator to be in the form $1-p$ and again that is easy enough, in this case, to rewrite the denominator by factoring a 3 out of the denominator as follows,

$$f(x) = \frac{x^3}{3}\frac{1}{1-\frac{1}{3}x^2}$$

Step 3

At this point we can use the formula from the notes to write this as a power series. Doing this gives,

$$f(x) = \frac{x^3}{3}\frac{1}{1-\frac{1}{3}x^2} = \frac{x^3}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}x^2\right)^n \qquad \text{provided } \left|\frac{1}{3}x^2\right| < 1$$

Step 4

Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single x with a single exponent on it.

These are easy enough rules to take care of. All we need to do is move whatever is in front of the series to the inside of the series and use basic exponent rules to take care of the x "rule". Doing all this gives,

$$\begin{aligned} f(x) = \frac{x^3}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}x^2\right)^n &= \sum_{n=0}^{\infty}\frac{1}{3}x^3\left(\frac{1}{3}\right)^n\left(x^2\right)^n \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n+1}x^{2n+3} \qquad \text{provided } \left|\frac{1}{3}x^2\right| < 1 \end{aligned}$$

Step 5

To get the interval of convergence all we need to do is do a little work on the "provided" portion of the result from the last step to get,

$$\left|\frac{1}{3}x^2\right| < 1 \quad \to \quad \frac{1}{3}|x|^2 < 1 \quad \to \quad |x|^2 < 3 \quad \to \quad |x| < \sqrt{3} \quad \to \quad -\sqrt{3} < x < \sqrt{3}$$

Note that we don't need to check the endpoints of this interval since we already know that we only get convergence with the strict inequalities and we will get divergence for everything else.

Step 6

The answers for this problem are then,

$$\boxed{\text{Power Series : } \frac{x^3}{3-x^2} = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n+1} x^{2n+3}} \qquad \boxed{\text{Interval : } -\sqrt{3} < x < \sqrt{3}}$$

---

3. Write the following function as a power series and give the interval of convergence.

$$f(x) = \frac{3x^2}{5 - 2\sqrt[3]{x}}$$

Step 1

First, in order to use the formula from this section we know that we need the numerator to be a one. That is easy enough to "fix" up as follows,

$$f(x) = 3x^2 \frac{1}{5 - 2\sqrt[3]{x}}$$

Step 2

Next, we know we need the denominator to be in the form $1 - p$ and again that is easy enough, in this case, to rewrite the denominator by factoring a 5 out of the denominator as follows,

$$f(x) = \frac{3x^2}{5}\frac{1}{1 - \frac{2}{5}\sqrt[3]{x}}$$

Step 3

At this point we can use the formula from the notes to write this as a power series. Doing this gives,

$$f(x) = \frac{3x^2}{5}\frac{1}{1-\frac{2}{5}\sqrt[3]{x}} = \frac{3x^2}{5}\sum_{n=0}^{\infty}\left(\frac{2}{5}\sqrt[3]{x}\right)^n \qquad \text{provided } \left|\frac{2}{5}\sqrt[3]{x}\right| < 1$$

Step 4

Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single x with a single exponent on it.

These are easy enough rules to take care of. All we need to do is move whatever is in front of the series to the inside of the series and use basic exponent rules to take care of the x "rule". Doing all this gives,

$$\begin{aligned} f(x) = \frac{3x^2}{5}\sum_{n=0}^{\infty}\left(\frac{2}{5}\sqrt[3]{x}\right)^n &= \sum_{n=0}^{\infty}\frac{3}{5}x^2\left(\frac{2}{5}\right)^n\left(x^{\frac{1}{3}}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{3}{5}\left(\frac{2}{5}\right)^n x^{\frac{1}{3}n+2} \qquad \text{provided } \left|\frac{2}{5}\sqrt[3]{x}\right| < 1 \end{aligned}$$

Step 5

To get the interval of convergence all we need to do is do a little work on the "provided" portion of the result from the last step to get,

$$\left|\frac{2}{5}\sqrt[3]{x}\right| < 1 \quad\to\quad \frac{2}{5}|x|^{\frac{1}{3}} < 1 \quad\to\quad |x|^{\frac{1}{3}} < \frac{5}{2} \quad\to\quad |x| < \frac{125}{8} \quad\to\quad -\frac{125}{8} < x < \frac{125}{8}$$

Note that we don't need to check the endpoints of this interval since we already know that we only get convergence with the strict inequalities and we will get divergence for everything else.

Step 6

The answers for this problem are then,

$$\boxed{\text{Power Series : } \frac{3x^2}{5-2\sqrt[3]{x}} = \sum_{n=0}^{\infty} \frac{3}{5}\left(\frac{2}{5}\right)^n x^{\frac{1}{3}n+2}} \qquad \boxed{\text{Interval : } -\frac{125}{8} < x < \frac{125}{8}}$$

---

4. Give a power series representation for the derivative of the following function.

$$g(x) = \frac{5x}{1-3x^5}$$

Hint

While we could differentiate the function and then attempt to find a power series representation that seems like a lot of work. It's a good think that we know how to differentiate power series.

Step 1

First let's notice that we can quickly find a power series representation for this function. Here is that work.

$$g(x) = 5x\frac{1}{1-3x^5} = 5x\sum_{n=0}^{\infty}\left(3x^5\right)^n = \sum_{n=0}^{\infty} 5x\left(3^n\right)x^{5n} = \sum_{n=0}^{\infty} 5\left(3^n\right)x^{5n+1}$$

Step 2

Now, we know how to differentiate power series and we know that the derivative of the power series representation of a function is the power series representation of the derivative of the function.

Therefore,

$$g'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} 5\left(3^n\right)x^{5n+1}\right] = \boxed{\sum_{n=0}^{\infty} 5(5n+1)\left(3^n\right)x^{5n}}$$

Remember that to differentiate a power series all we need to do is differentiate the term of the power series with respect to x.

5. Give a power series representation for the integral of the following function.

$$h(x) = \frac{x^4}{9 + x^2}$$

Hint

Integrating this function seems like (potentially) a lot of work, not to mention determining a power series representation of the result. It's a good think that we know how to integrate power series.

Step 1

First let's notice that we can quickly find a power series representation for this function. Here is that work.

$$\begin{aligned} h(x) = \frac{x^4}{9} \frac{1}{1 - \left(-\frac{1}{9}x^2\right)} = \frac{x^4}{9} \sum_{n=0}^{\infty} \left(-\frac{1}{9}x^2\right)^n &= \sum_{n=0}^{\infty} \frac{1}{9} x^4 (-1)^n \left(\frac{1}{9}\right)^n x^{2n} \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{9}\right)^{n+1} x^{2n+4} \end{aligned}$$

Step 2

Now, we know how to integrate power series and we know that the integral of the power series representation of a function is the power series representation of the integral of the function.

Therefore,

$$\int h(x)\, dx = \int \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{9}\right)^{n+1} x^{2n+4}\, dx = \boxed{c + \sum_{n=0}^{\infty} \frac{1}{2n+5} (-1)^n \left(\frac{1}{9}\right)^{n+1} x^{2n+5}}$$

Remember that to integrate a power series all we need to do is integrate the term of the power series and we can't forget to add on the "$+c$" since we're doing an indefinite integral.

10.16 Taylor Series

1. Use one of the Taylor Series derived in the notes to determine the Taylor Series for $f(x) = \cos(4x)$ about $x = 0$.

Step 1

There really isn't all that much to do here for this problem. We are working with cosine and want the Taylor series about $x = 0$ and so we can use the Taylor series for cosine derived in the notes to get,

$$\cos(4x) = \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n}}{(2n)!}$$

Step 2

Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single x with a single exponent on it.

In this case we don't have anything out in front of the series to worry about so all we need to do is use the basic exponent rules on the $4x$ term to get,

$$\cos(4x) = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 16^n x^{2n}}{(2n)!}}$$

---

2. Use one of the Taylor Series derived in the notes to determine the Taylor Series for $f(x) = x^6 \mathbf{e}^{2x^3}$ about $x = 0$.

Step 1

There really isn't all that much to do here for this problem. We are working with the exponential function and want the Taylor series about $x = 0$ and so we can use the Taylor series for the exponential function derived in the notes to get,

$$x^6 \mathbf{e}^{2x^3} = x^6 \sum_{n=0}^{\infty} \frac{(2x^3)^n}{n!}$$

Note that we only convert the exponential using the Taylor series derived in the notes and, at this point, we just leave the x^6 alone in front of the series.

Step 2

Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single x with a single exponent on it.

These are easy enough rules to take care of. All we need to do is move whatever is in front of the series to the inside of the series and use basic exponent rules to take care of the x "rule". Doing all this gives,

$$x^6\mathbf{e}^{2x^3} = x^6\sum_{n=0}^{\infty}\frac{\left(2x^3\right)^n}{n!} = \sum_{n=0}^{\infty}x^6\frac{2^n\left(x^3\right)^n}{n!} = \boxed{\sum_{n=0}^{\infty}\frac{2^n x^{3n+6}}{n!}}$$

3. Find the Taylor Series for $f\left(x\right) = \mathbf{e}^{-6x}$ about $x = -4$.

Step 1

Because we are working about $x = -4$ in this problem we are not able to just use the formula derived in class for the exponential function because that requires us to be working about $x = 0$.

Step 2

So, we'll need to start over from the beginning and start taking some derivatives of the function.

$$\begin{aligned}
n &= 0: & f\left(x\right) &= \mathbf{e}^{-6x}\\
n &= 1: & f'\left(x\right) &= -6\mathbf{e}^{-6x}\\
n &= 2: & f''\left(x\right) &= \left(-6\right)^2\mathbf{e}^{-6x}\\
n &= 3: & f^{(3)}\left(x\right) &= \left(-6\right)^3\mathbf{e}^{-6x}\\
n &= 4: & f^{(4)}\left(x\right) &= \left(-6\right)^4\mathbf{e}^{-6x}
\end{aligned}$$

Remember that, in general, we're going to need to go out to at least $n = 4$ for most of these problems to make sure that we can get the formula for the general term in the Taylor Series.

Also, remember to NOT multiply things out when taking derivatives! Doing that will make your life much harder when it comes time to find the general formula.

Step 3

It is now time to see if we can get a formula for the general term in the Taylor Series.

In this case, it is (hopefully) pretty simple to catch the pattern in the derivatives above. The general term is given by,

$$f^{(n)}(x) = (-6)^n \mathbf{e}^{-6x} \qquad n = 0, 1, 2, 3, \ldots$$

As noted this formula works all the way back to $n = 0$. It is important to make sure that you check this formula to determine just how far back it will work. We will, on occasion, get formulas that will not work for the first couple of n's and we need to know that before we start writing down the Taylor Series.

Step 4

Now, recall that we don't really want the general term at any x. We want the general term at $x = -4$. This is,

$$f^{(n)}(-4) = (-6)^n \mathbf{e}^{24} \qquad n = 0, 1, 2, 3, \ldots$$

Step 5

Okay, at this point we can formally write down the Taylor Series for this problem.

$$\mathbf{e}^{-6x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-4)}{n!}(x+4)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-6)^n \mathbf{e}^{24}}{n!}(x+4)^n}$$

4. Find the Taylor Series for $f(x) = \ln(3+4x)$ about $x = 0$.

Step 1

Okay, we'll need to start off this problem by taking a few derivatives of the function.

$$\begin{aligned}
n=0: &\quad f(x) = \ln(3+4x) \\
n=1: &\quad f'(x) = \frac{4}{3+4x} = 4(3+4x)^{-1} \\
n=2: &\quad f''(x) = -4^2(3+4x)^{-2} \\
n=3: &\quad f^{(3)}(x) = 4^3(2)(3+4x)^{-3} \\
n=4: &\quad f^{(4)}(x) = -4^4(2)(3)(3+4x)^{-4} \\
n=5: &\quad f^{(5)}(x) = 4^5(2)(3)(4)(3+4x)^{-5}
\end{aligned}$$

Remember that, in general, we're going to need to go out to at least $n = 4$ for most of these problems to make sure that we can get the formula for the general term in the Taylor Series.

Also, remember to NOT multiply things out when taking derivatives! Doing that will make your life much harder when it comes time to find the general formula. In this case we "merged" all the 4's that came from the chain rule into a single term but left it as an exponent rather than get an actual value. This is not uncommon with these kinds of problems. The exponents we dropped down for the derivatives we left alone with the exception of dealing with the signs.

Step 2

It is now time to see if we can get a formula for the general term in the Taylor Series.

Hopefully you can see the pattern in the derivatives above. The general term is given by,

$$\begin{aligned}
f^{(0)}(x) &= \ln(3+4x) && n=0 \\
f^{(n)}(x) &= (-1)^{n+1}4^n(n-1)!(3+4x)^{-n} && n=1,2,3,\ldots
\end{aligned}$$

As noted this formula works all the way back to $n = 1$ but clearly does not work for $n = 0$. It is problems like this one that make it clear why we always need to check our proposed formula for the general solution to see just how far back it works.

Step 3

Now, recall that we don't really want the general term at any x. We want the general term at $x = 0$. This is,

$$\begin{aligned} f^{(0)}(0) &= \ln(3) \qquad n = 0 \\ f^{(n)}(0) &= (-1)^{n+1} 4^n (n-1)!(3)^{-n} \\ &= (-1)^{n+1} 4^n (n-1)! \frac{1}{3^n} \\ &= (-1)^{n+1} \left(\frac{4}{3}\right)^n (n-1)! \qquad n = 1, 2, 3, \ldots \end{aligned}$$

We did a little simplification for the second one just to make it a little simpler.

Step 4

Okay, at this point we can formally write down the Taylor Series for this problem. However, before we actually do that recall that our general term formula did not work for $n = 0$ and so we'll need to first strip that out of the series before we put the general formula in.

$$\begin{aligned} \ln(3+4x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \ln(3) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \left(\frac{4}{3}\right)^n (n-1)!}{n!} x^n \\ &= \boxed{\ln(3) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \left(\frac{4}{3}\right)^n}{n} x^n} \end{aligned}$$

Don't forget to simplify/cancel where we can in the final answer. In this case we could do some simplifying with the factorials.

5. Find the Taylor Series for $f(x) = \dfrac{7}{x^4}$ about $x = -3$.

Step 1

Okay, we'll need to start off this problem by taking a few derivatives of the function.

$$\begin{aligned}
n = 0: &\qquad f(x) = \frac{7}{x^4} = 7x^{-4}\\
n = 1: &\qquad f'(x) = -7(4)x^{-5}\\
n = 2: &\qquad f''(x) = 7(4)(5)x^{-6}\\
n = 3: &\qquad f^{(3)}(x) = -7(4)(5)(6)x^{-7}\\
n = 4: &\qquad f^{(4)}(x) = 7(4)(5)(6)(7)x^{-8}
\end{aligned}$$

Remember that, in general, we're going to need to go out to at least $n = 4$ for most of these problems to make sure that we can get the formula for the general term in the Taylor Series.

Also, remember to NOT multiply things out when taking derivatives! Doing that will make your life much harder when it comes time to find the general formula. In this case the only "simplification" we did was to multiply out the minus signs that came from the exponents upon doing the derivatives. That is a fairly common thing to do with these kinds of problems.

Step 2

It is now time to see if we can get a formula for the general term in the Taylor Series.

Hopefully you can see the pattern in the derivatives above. The general term is given by,

$$\begin{aligned}
f^{(n)}(x) &= 7(-1)^n \frac{(2)(3)}{(2)(3)}(4)(5)(6)\cdots(n+3)\,x^{-(n+4)}\\
&= 7(-1)^n \frac{(2)(3)(4)(5)(6)\cdots(n+3)}{6}x^{-(n+4)}\\
&= \frac{7}{6}(-1)^n(n+3)!x^{-(n+4)} \qquad\qquad n = 0,1,2,3,\ldots
\end{aligned}$$

This formula may have been a little trickier to get. We almost had a factorial in the derivatives but each one was missing the $(2)(3)$ part that would be needed to get the factorial to show up. Because that was all that was missing and it was missing in each of the derivatives we multiplied each derivative by $\frac{(2)(3)}{(2)(3)}$ (*i.e.* a really fancy way of writing one...). We could then use the numerator of this to complete the factorial and the denominator was just left alone.

Also, as noted this formula works all the way back to $n = 0$. It is important to make sure that you check this formula to determine just how far back it will work. We will, on occasion, get formulas that will not work for the first couple of n's and we need to know that before we start writing down the Taylor Series.

Step 3

Now, recall that we don't really want the general term at any x. We want the general term at $x = -3$. This is,

$$\begin{aligned} f^{(n)}(-3) &= \frac{7}{6}(-1)^n (n+3)!(-3)^{-(n+4)} \\ &= \frac{7(-1)^n (n+3)!}{6(-3)^{n+4}} \\ &= \frac{7(-1)^n (n+3)!}{6(-1)^{n+4}(3)^{n+4}} \\ &= \frac{7(n+3)!}{6(-1)^4(3)^{n+4}} \\ &= \frac{7(n+3)!}{6(3)^{n+4}} \qquad\qquad n = 1,2,3,\ldots \end{aligned}$$

We did a little simplification here so we could cancel out all the alternating signs that were present in the term.

Step 4

Okay, at this point we can formally write down the Taylor Series for this problem.

$$\begin{aligned} \frac{7}{x^4} = \sum_{n=0}^{\infty}\frac{f^{(n)}(-3)}{n!}(x+3)^n &= \sum_{n=0}^{\infty}\frac{7(n+3)!}{6(3)^{n+4}n!}(x+3)^n \\ &= \boxed{\sum_{n=0}^{\infty}\frac{7(n+3)(n+2)(n+1)}{6(3)^{n+4}}(x+3)^n} \end{aligned}$$

Don't forget to simplify/cancel where we can in the final answer. In this case we could do some simplifying with the factorials.

6. Find the Taylor Series for $f(x) = 7x^2 - 6x + 1$ about $x = 2$.

Step 1

First, let's not get too excited about the fact that we have a polynomial here for this problem. It works exactly the same way with a few small differences.

We'll start off by taking a few derivatives of the function and evaluating them at $x = 2$

$$\begin{array}{lll} n = 0: & f(x) = 7x^2 - 6x + 1 & f(2) = 17 \\ n = 1: & f'(x) = 14x - 6 & f'(2) = 22 \\ n = 2: & f''(x) = 14 & f''(2) = 14 \\ n \geq 3: & f^{(n)}(x) = 0 & f^{(n)}(2) = 0 \end{array}$$

Okay, this is where one of the differences between a polynomial and the other types of functions we typically see with Taylor Series problems. After some point all the derivatives will be zero. That is not something to get excited about. In fact, it actually makes the problem a little easier!

Because all the derivatives are zero after some point we don't need a formula for the general term. All we need are the values of the non-zero derivative terms.

Step 2

Once we have the values from the previous step all we need to do is write down the Taylor Series. To do that all we need to do is strip all the non-zero terms from the series and then acknowledge that the remainder will just be zero (all the remaining terms are zero after all!).

Doing this gives,

$$\begin{aligned} 7x^2 - 6x + 1 &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \sum_{n=3}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \boxed{17 + 22(x-2) + 7(x-2)^2} \end{aligned}$$

It looks a little strange but there it is. Do not multiply/simplify this out. This really is the answer we are looking for.

Also, don't think that this is a problem that is just done to make you work another problem. There are applications of series (beyond the scope of this course however...) that really do require this kind of thing to be done as strange as that might sound!

10.17 Applications of Series

1. Determine a Taylor Series about $x = 0$ for the following integral.

$$\int \frac{\mathbf{e}^x - 1}{x}\,dx$$

Step 1

This problem isn't quite as hard as it might first appear. We know how to integrate a series so all we really need to do here is find a Taylor series for the integrand and then integrate that.

Step 2

Okay, let's start out by noting that we are working about $x = 0$ and that means we can use the formula for the Taylor Series of the exponential function. For reference purposes this is,

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Next, let's strip out the $n = 0$ term from this and then subtract one. Doing this gives,

$$\mathbf{e}^x - 1 = \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}\right] - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

Of course, in doing the above step all we really managed to do was eliminate the $n = 0$ term from the series. In fact, that was not a bad thing to have happened as well see shortly.

Finally, let's divide the whole thing by x. This gives,

$$\frac{\mathbf{e}^x - 1}{x} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

We moved the x that was outside the series into the series. This is required in order to do the integral of the series. We only want a single x in the problem and we now have that.

Also note that while the function on the left has a division by zero issue the series on the right does not have this problem. All of the x's in the series have positive or zero exponents! This is a really good thing.

Of course, the other good thing that we have at this point is that we've managed to find a series representation for the integrand!

Step 3

All we need to do now is compute the integral of the series to get a series representation of the integral.

$$\int \frac{\mathbf{e}^{x}-1}{x}\,dx = \int \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}\,dx = \boxed{c + \sum_{n=1}^{\infty} \frac{x^{n}}{(n)\,(n!)}}$$

---

2. Write down $T_2(x)$, $T_3(x)$ and $T_4(x)$ for the Taylor Series of $f(x) = \mathbf{e}^{-6x}$ about $x = -4$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $[-8,-2]$.

Step 1

The first thing we need to do here is get the Taylor Series for $f(x) = \mathbf{e}^{-6x}$ about $x = -4$. Luckily enough for us we did that in Problem 3 of the previous section. Here is the Taylor Series we derived in that problem.

$$\mathbf{e}^{-6x} = \sum_{n=0}^{\infty} \frac{(-6)^{n}\mathbf{e}^{24}}{n!}(x+4)^{n}$$

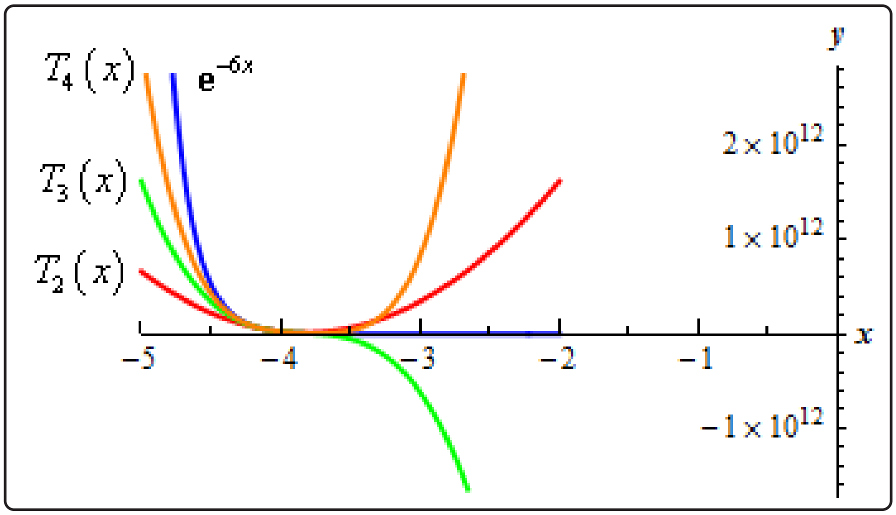
Step 2

Here are the three Taylor polynomials needed for this problem.

$$T_2(x) = \mathbf{e}^{24} - 6\mathbf{e}^{24}(x+4) + 18\mathbf{e}^{24}(x+4)^2$$
$$T_3(x) = \mathbf{e}^{24} - 6\mathbf{e}^{24}(x+4) + 18\mathbf{e}^{24}(x+4)^2 - 36\mathbf{e}^{24}(x+4)^3$$
$$T_4(x) = \mathbf{e}^{24} - 6\mathbf{e}^{24}(x+4) + 18\mathbf{e}^{24}(x+4)^2 - 36\mathbf{e}^{24}(x+4)^3 + 54\mathbf{e}^{24}(x+4)^4$$

Step 3

Here is the graph for this problem.



We can see that as long as we stay "near" $x = -4$ the graphs of the polynomial are pretty close to the graph of the exponential function. However, if we get too far away the graphs really do start to diverge from the graph of the exponential function.

---

3. Write down $T_3(x)$, $T_4(x)$ and $T_5(x)$ for the Taylor Series of $f(x) = \ln(3+4x)$ about $x = 0$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $\left[-\frac{1}{2}, 2\right]$.

Step 1

The first thing we need to do here is get the Taylor Series for $f(x) = \ln(3+4x)$ about $x = 0$. Luckily enough for us we did that in Problem 4 of the previous section. Here is the Taylor Series we derived in that problem.

$$ln(3+4x) = \ln(3) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\frac{4}{3}\right)^n}{n} x^n$$

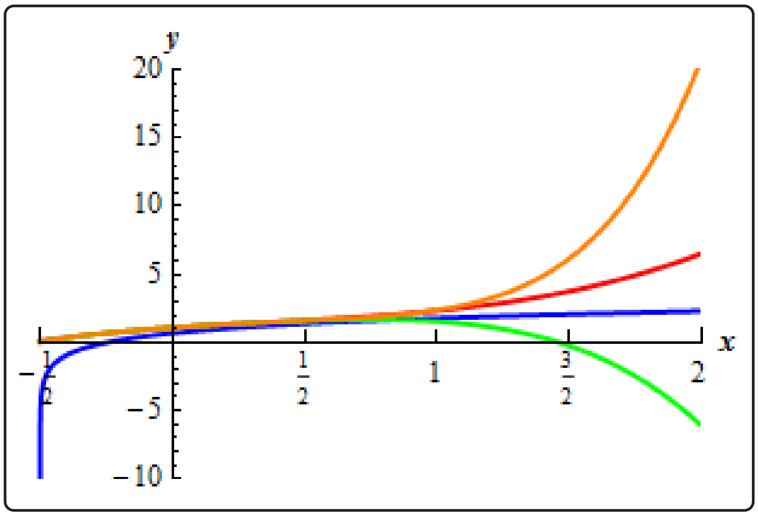
Step 2

Here are the three Taylor polynomials needed for this problem.

$$T_3(x) = \ln(3) + \frac{4}{3}x - \frac{8}{9}x^2 + \frac{64}{81}x^3$$
$$T_4(x) = \ln(3) + \frac{4}{3}x - \frac{8}{9}x^2 + \frac{64}{81}x^3 - \frac{64}{81}x^4$$
$$T_5(x) = \ln(3) + \frac{4}{3}x - \frac{8}{9}x^2 + \frac{64}{81}x^3 - \frac{64}{81}x^4 + \frac{1024}{1215}x^5$$

Step 3

Here is the graph for this problem.



We can see that as long as we stay "near" $x = 0$ the graphs of the polynomial are pretty close to the graph of the exponential function. However, if we get too far away the graphs really do start to diverge from the graph of the exponential function.

10.18 Binomial Series

1. Use the Binomial Theorem to expand $(4+3x)^5$.

Solution

Not really a lot to do with this problem. All we need to do is use the formula from the Binomial Theorem to do the expansion. Here is that work.

$$\begin{aligned}(4+3x)^5 &= \sum_{i=0}^{5}\binom{5}{i}4^{5-i}(3x)^i \\ &= \binom{5}{0}\left(4^5\right)+\binom{5}{1}\left(4^4\right)(3x)^1+\binom{5}{2}\left(4^3\right)(3x)^2+\binom{5}{3}\left(4^2\right)(3x)^3 \\ &\qquad +\binom{5}{4}\left(4^1\right)(3x)^4+\binom{5}{5}(3x)^5 \\ &= 4^5+(5)\left(4^4\right)(3x)+\frac{5\,(4)}{2!}\left(4^3\right)(3x)^2+\frac{5\,(4)\,(3)}{3!}\left(4^2\right)(3x)^3 \\ &\qquad +(5)\,(4)\,(3x)^4+(3x)^5 \\ &= \boxed{1024+3840x+5760x^2+4320x^3+1620x^4+243x^5}\end{aligned}$$

2. Use the Binomial Theorem to expand $(9-x)^4$.

Solution

Not really a lot to do with this problem. All we need to do is use the formula from the Binomial Theorem to do the expansion. Here is that work.

$$\begin{aligned}(9-x)^4 &= \sum_{i=0}^{4}\binom{4}{i}9^{4-i}(-x)^i \\ &= \binom{4}{0}\left(9^4\right)+\binom{4}{1}\left(9^3\right)(-x)^1+\binom{4}{2}\left(9^2\right)(-x)^2+\binom{4}{3}\left(9^1\right)(-x)^3+\binom{4}{4}(-x)^4 \\ &= 9^4+(4)\left(9^3\right)(-x)+\frac{4\,(3)}{2!}\left(9^2\right)(-x)^2+(4)\left(9^1\right)(-x)^3+(-x)^4 \\ &= \boxed{6561-2916x+486x^2-36x^3+x^4}\end{aligned}$$

3. Write down the first four terms in the binomial series for $(1+3x)^{-6}$.

Step 1

First, we need to make sure it is in the proper form to use the Binomial Series from the notes which in this case we are already in the proper form with $k=-6$.

Step 2

Now all we need to do is plug into the formula from the notes and write down the first four terms.

$$\begin{aligned}(1+3x)^{-6} &= \sum_{i=0}^{\infty}\binom{-6}{i}(3x)^i \\ &= 1+(-6)(3x)^1+\frac{(-6)(-7)}{2!}(3x)^2+\frac{(-6)(-7)(-8)}{3!}(3x)^3+\cdots \\ &= \boxed{1-18x+189x^2-1512x^3+\cdots}\end{aligned}$$

---

4. Write down the first four terms in the binomial series for $\sqrt[3]{8-2x}$.

Step 1

First, we need to make sure it is in the proper form to use the Binomial Series. Here is the proper form for this function,

$$\sqrt[3]{8-2x}=\left(8\left(1-\frac{1}{4}x\right)\right)^{\frac{1}{3}}=(8)^{\frac{1}{3}}\left(1-\frac{1}{4}x\right)^{\frac{1}{3}}=2\left(1+\left(-\frac{1}{4}x\right)\right)^{\frac{1}{3}}$$

Recall that for proper from we need it to be in the form “1+” and so we needed to factor the 8 out of the root and “move” the minus sign into the second term. Also, as we can see we will have $k=\frac{1}{3}$

Step 2

Now all we need to do is plug into the formula from the notes and write down the first four terms.

$$\begin{aligned}
\sqrt[3]{8-2x} &= 2\left(1+\left(-\frac{1}{4}x\right)\right)^{\frac{1}{3}} \\
&= 2\sum_{i=0}^{\infty}\binom{\frac{1}{3}}{i}\left(-\frac{1}{4}x\right)^{i} \\
&= 2\left[1+\left(\frac{1}{3}\right)\left(-\frac{1}{4}x\right)^{1}+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(-\frac{1}{4}x\right)^{2}+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(-\frac{1}{4}x\right)^{3}+\dots\right] \\
&= \boxed{2-\frac{1}{6}x-\frac{1}{72}x^{2}-\frac{5}{2592}x^{3}+\cdots}
\end{aligned}$$

11 Vectors

Once again we are completely changing topics from the last chapter. We are going to do a (very) brief introduction to vectors. We'll look at basic notation and concepts involving vectors as well as arithmetic involving vectors. We'll also look at the dot product and cross product of vectors as well as a couple of quick applications of the dot and cross product.

Once we get into the multi-variable Calculus (*i.e.* the topics usually taught in Calculus III) we'll run into vectors on a semi regular basis and so we'll need to be familiar with them and the common notation, concepts and arithmetic involving vectors.

The following sections are the practice problems, with solutions, for this material.

11.1 Vectors - The Basics

1. Give the vector for the line segment from $(-9, 2)$ to $(4, -1)$. Find its magnitude and determine if the vector is a unit vector.

Step 1

Writing down a vector for a line segment is really simple. Just recall that the components of the vector are always the coordinates of the ending point minus the coordinates of the starting point. Always keep in mind that the starting and ending points are important!

Here is the vector for this line segment.

$$\vec{v} = \langle 4 - (-9), -1 - 2 \rangle = \boxed{\langle 13, -3 \rangle}$$

Step 2

To compute the magnitude just recall the formula we gave in the notes. The magnitude of this vector is then,

$$\|\vec{v}\| = \sqrt{(13)^2 + (-3)^2} = \boxed{\sqrt{178}}$$

Step 3

Because we can see that $\|\vec{v}\| = \sqrt{178} \neq 1$ we know that this vector is **not** a unit vector.

2. Give the vector for the line segment from $(4, 5, 6)$ to $(4, 6, 6)$. Find its magnitude and determine if the vector is a unit vector.

Step 1

Writing down a vector for a line segment is really simple. Just recall that the components of the vector are always the coordinates of the ending point minus the coordinates of the starting point. Always keep in mind that the starting and ending points are important!

Here is the vector for this line segment.

$$\vec{v} = \langle 4-4, 6-5, 6-6 \rangle = \boxed{\langle 0, 1, 0 \rangle}$$

Step 2

To compute the magnitude just recall the formula we gave in the notes. The magnitude of this vector is then,

$$\|\vec{v}\| = \sqrt{(0)^2 + (1)^2 + (0)^2} = \boxed{1}$$

Step 3

Because we can see that $\|\vec{v}\| = 1$ we know that this vector **is** a unit vector.

---

3. Give the vector for the position vector for $(-3, 2, 10)$. Find its magnitude and determine if the vector is a unit vector.

Step 1

Writing down a vector for a line segment is really simple. Just recall that the components of the vector are always the coordinates of the ending point minus the coordinates of the starting point.

Just recall that the starting point for any position vector is the origin and the ending point is the point we're working with. In other words, the components of the position vector are simply the coordinates of the point.

Here is the position vector for this point.

$$\boxed{\vec{v} = \langle -3, 2, 10 \rangle}$$

Step 2

To compute the magnitude just recall the formula we gave in the notes. The magnitude of this vector is then,

$$\|\vec{v}\| = \sqrt{(-3)^2 + (2)^2 + (10)^2} = \boxed{\sqrt{113}}$$

Step 3

Because we can see that $\|\vec{v}\| = \sqrt{113} \neq 1$ we know that this vector is **not** a unit vector.

---

4. Give the vector for the position vector for $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Find its magnitude and determine if the vector is a unit vector.

Step 1

Writing down a vector for a line segment is really simple. Just recall that the components of the vector are always the coordinates of the ending point minus the coordinates of the starting point.

Just recall that the starting point for any position vector is the origin and the ending point is the point we're working with. In other words, the components of the position vector are simply the coordinates of the point.

Here is the position vector for this point.

$$\boxed{\vec{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle}$$

Step 2

To compute the magnitude just recall the formula we gave in the notes. The magnitude of this vector is then,

$$\|\vec{v}\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{4}{4}} = \boxed{1}$$

Step 3

Because we can see that $\|\vec{v}\| = 1$ we know that this vector **is** a unit vector.

5. The vector $\vec{v} = \langle 6, -4, 0 \rangle$ starts at the point $P = (-2, 5, -1)$. At what point does the vector end?

Step 1

To answer this problem we just need to recall that the components of the vector are always the coordinates of the ending point minus the coordinates of the starting point.

So, if the ending point of the vector is given by $Q = (x_2, y_2, z_2)$ then we know that the vector $\vec{v}$ can be written as,

$$\vec{v} = \overrightarrow{PQ} = \langle x_2 + 2, y_2 - 5, z_2 + 1 \rangle$$

Step 2

But we also know just what the components of $\vec{v}$ are so we can set the vector from Step 1 above equal to what we know $\vec{v}$ is. Doing this gives,

$$\langle x_2 + 2, y_2 - 5, z_2 + 1 \rangle = \langle 6, -4, 0 \rangle$$

Step 3

Now, if two vectors are equal the corresponding components must be equal. Or,

$$\begin{aligned} x_2 + 2 &= 6 & \Rightarrow \quad & x_2 = 4 \\ y_2 - 5 &= -4 & \Rightarrow \quad & y_2 = 1 \\ z_2 + 1 &= 0 & \Rightarrow \quad & z_2 = -1 \end{aligned}$$

As noted above this results in three very simple equations that we can solve to determine the coordinates of the ending point.

The endpoint of the vector is then,

$$\boxed{Q = (4, 1, -1)}$$

11.2 Vector Arithmetic

1. Given $\vec{a} = \langle 8, 5\rangle$ and $\vec{b} = \langle -3, 6\rangle$ compute each of the following.

   (a) $6\vec{a}$

   (b) $7\vec{b} - 2\vec{a}$

   (c) $\left\| 10\vec{a} + 3\vec{b} \right\|$

Solutions

(a) $6\vec{a}$

Solution

This is just a scalar multiplication problem. Just remember to multiply each component by the scalar, 6 in this case.

$$6\vec{a} = 6\langle 8, 5\rangle = \boxed{\langle 48, 30\rangle}$$

(b) $7\vec{b} - 2\vec{a}$

Solution

Here we'll just do each of the scalar multiplications and then do the subtraction. With the subtraction just remember to subtract corresponding components from each vector and recall that order is important here since we are doing subtraction!

$$7\vec{b} - 2\vec{a} = 7\langle -3, 6\rangle - 2\langle 8, 5\rangle = \langle -21, 42\rangle - \langle 16, 10\rangle = \boxed{\langle -37, 32\rangle}$$

(c) $\left\| 10\vec{a} + 3\vec{b} \right\|$

Solution

So, first we compute the vector inside the magnitude bars and the compute the magnitude.

$$10\vec{a} + 3\vec{b} = 10\langle 8, 5\rangle + 3\langle -3, 6\rangle = \langle 80, 50\rangle + \langle -9, 18\rangle = \langle 71, 68\rangle$$

The magnitude is then,

$$\left\|10\vec{a}+3\vec{b}\right\| = \sqrt{(71)^2+(68)^2} = \boxed{\sqrt{9665}}$$

---

2. Given $\vec{u} = 8\vec{i} - \vec{j} + 3\vec{k}$ and $\vec{v} = 7\vec{j} - 4\vec{k}$ compute each of the following.

(a) $-3\vec{v}$

(b) $12\vec{u}+\vec{v}$

(c) $\|-9\vec{v}-2\vec{u}\|$

Solutions

(a) $-3\vec{v}$

Solution

This is just a scalar multiplication problem. Just remember to multiply each component by the scalar, -3 in this case.

$$-3\vec{v} = -3\left(7\vec{j}-4\vec{k}\right) = \boxed{-21\vec{j}+12\vec{k}}$$

(b) $12\vec{u}+\vec{v}$

Solution

Here we'll just do each of the scalar multiplications and then do the subtraction. With the addition just remember to add corresponding components from each vector.

$$\begin{aligned} 12\vec{u}+\vec{v} = 12\left(8\vec{i}-\vec{j}+3\vec{k}\right)+\left(7\vec{j}-4\vec{k}\right) &= \left(96\vec{i}-12\vec{j}+36\vec{k}\right)+\left(7\vec{j}-4\vec{k}\right) \\ &= \boxed{96\vec{i}-5\vec{j}+32\vec{k}} \end{aligned}$$

(c) $\|-9\vec{v}-2\vec{u}\|$

Solution

So, first we compute the vector inside the magnitude bars and the compute the magnitude.

$$\begin{aligned}-9\vec{v}-2\vec{u} &= -9\left(7\vec{j}-4\vec{k}\right)-2\left(8\vec{i}-\vec{j}+3\vec{k}\right)\\ &= \left(-63\vec{j}+36\vec{k}\right)-\left(16\vec{i}-2\vec{j}+6\vec{k}\right)=-16\vec{i}-61\vec{j}+30\vec{k}\end{aligned}$$

Be careful with the lack of an $\vec{i}$ component in the first vector here. Just recall that means the coefficient of $\vec{i}$ in the first vector is just zero!

The magnitude is then,

$$\|-9\vec{v}-2\vec{u}\|=\sqrt{(-16)^2+(-61)^2+(30)^2}=\boxed{\sqrt{4877}}$$

---

3. Find a unit vector that points in the same direction as $\vec{q}=\vec{i}+3\vec{j}+9\vec{k}$.

Step 1

Of course, the first step here really should be to check and see if we are lucky enough to actually have a unit vector already. It's unlikely we do have a unit vector but you never know until you check!

$$\|\vec{q}\|=\sqrt{(1)^2+(3)^2+(9)^2}=\sqrt{91}$$

Okay, as we pretty much had already guessed, this isn't a unit vector (its magnitude isn't one!) but we can use this to help find the answer.

Step 2

Recall that all we need to do to turn any vector into a unit vector is divide the vector by its magnitude. Doing that for this vector gives,

$$\vec{u}=\frac{\vec{q}}{\|\vec{q}\|}=\frac{1}{\sqrt{91}}\left(\vec{i}+3\vec{j}+9\vec{k}\right)=\boxed{\frac{1}{\sqrt{91}}\vec{i}+\frac{3}{\sqrt{91}}\vec{j}+\frac{9}{\sqrt{91}}\vec{k}}$$

As a quick check, not really required of course, we can compute a quick magnitude to verify that we do in fact have a unit vector.

$$\|\vec{u}\| = \sqrt{\left(\frac{1}{\sqrt{91}}\right)^2 + \left(\frac{3}{\sqrt{91}}\right)^2 + \left(\frac{9}{\sqrt{91}}\right)^2} = \sqrt{\frac{91}{91}} = 1$$

So, we do have a unit vector!

4. Find a vector that points in the same direction as $\vec{c} = \langle -1, 4\rangle$ with a magnitude of 10.

Step 1

At first glance this doesn't appear to be all that similar to the previous problem. However, it's actually very similar.

First, let's check to see what the magnitude of this vector is.

$$\|\vec{c}\| = \sqrt{(-1)^2 + (4)^2} = \sqrt{17}$$

Step 2

Okay, oddly enough let's determine a unit vector that points in the same direction. This doesn't seem all that useful but it's actually a very good thing to do in this case.

The unit vector is,

$$\vec{u} = \frac{\vec{c}}{\|\vec{c}\|} = \frac{1}{\sqrt{17}}\langle -1, 4\rangle = \boxed{\left\langle -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right\rangle}$$

Now, let's think about what we did here. We took the original vector and multiplied it by a number, $\frac{1}{\sqrt{17}}$ in this case, to change its magnitude. The result is a new vector, pointing in the same direction as the original vector, and has a new magnitude of one.

So, how can we use this new vector (and the process by which we found it) to get an answer for this problem?

Step 3

We know that scalar multiplication can change the magnitude of a vector. We've got a vector with magnitude of one that points in the correct direction. To convert this into a vector with magnitude of 10 all we need to do is multiply this new unit vector by 10 to get,

$$\vec{v} = 10\vec{u} = 10\left\langle -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle = \left\langle -\frac{10}{\sqrt{17}}, \frac{40}{\sqrt{17}} \right\rangle$$

Now, let's verify that this does what we want it to do with a quick magnitude computation.

$$\|\vec{v}\| = \sqrt{\left(-\frac{10}{\sqrt{17}}\right)^2 + \left(\frac{40}{\sqrt{17}}\right)^2} = \sqrt{\frac{1700}{17}} = \sqrt{100} = 10$$

So, we do have a vector with magnitude 10 as predicted!

---

5. Determine if $\vec{a} = \langle 3, -5, 1 \rangle$ and $\vec{b} = \langle 6, -2, 2 \rangle$ are parallel vectors.

Step 1

Recall that two vectors are parallel if they are scalar multiples of each other. In other words, these two vectors will be scalar multiples if we can find a number k such that,

$$\vec{a} = k\,\vec{b}$$

Step 2

Let's just take a look at the first component from each vector. It is obvious that $6 = 2\,(3)$. So, to convert the first components we'd need to multiply $\vec{a}$ by 2. However, if we did that we'd get,

$$2\vec{a} = \langle 6, -10, 2 \rangle \neq \vec{b}$$

This is clearly not $\vec{b}$. The first component is correct and the third component is correct but the second isn't correct. Therefore, there is no single number, k, that we can use to convert $\vec{a}$ into $\vec{b}$ through scalar multiplication.

This in turn means that $\vec{a}$ and $\vec{b}$ **cannot possibly be parallel**.

---

6. Determine if $\vec{v} = 9\vec{i} - 6\vec{j} - 24\vec{k}$ and $\vec{w} = \langle -15, 10, 40 \rangle$ are parallel vectors.

Step 1

Recall that two vectors are parallel if they are scalar multiples of each other. In other words, these two vectors will be scalar multiples if we can find a number k such that,

$$\vec{v} = k\,\vec{w}$$

Step 2

Let's just take a look at the first component from each vector. It is should be clear that $-15 = \left(-\frac{5}{3}\right)(9)$. So, to convert the first components we'd need to multiply $\vec{v}$ by $-\frac{5}{3}$.

if we did that we'd get,

$$-\frac{5}{3}\vec{v} = \langle -15, 10, 40 \rangle = \vec{w}$$

So, we were able to find a number k that we could use to convert $\vec{v}$ into $\vec{w}$ through scalar multiplication and so the **two vectors are parallel**.

---

7. Prove the property : $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.

Step 1

These types of proofs always seem mysterious to students the first time they run across them. The main reason for the mystery is probably that it just seems obvious that it is true. That tends to make is difficult to prove.

We know that this property is true for numbers. However, we can't assume that just because it's true for numbers that it will be true for all other types of objects, vectors in this case!

So, let's start off with two general vectors.

$$\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle \qquad\qquad \vec{w} = \langle w_1, w_2, \ldots w_n \rangle$$

To do this type of proof all we need to do is start with the left side perform the indicated operation, addition in this case, and then use properties about numbers that we already know to be true to try and manipulate it to look like the right side.

Step 2

So, let's start off with the vector addition on the left side. All we want to do here is use the definition of vector addition to write the sum of the two vectors. This is,

$$\vec{v} + \vec{w} = \langle v_1, v_2, \ldots, v_n \rangle + \langle w_1, w_2, \ldots w_n \rangle = \langle v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n \rangle$$

Step 3

Okay, as we noted above we know that $2 + 3 = 3 + 2$. In other words, we know that the order we do addition of numbers doesn't matter.

Why bring this up again?

Well, note that each of the components of the "new" vector on the right side is just a sum of two numbers. Therefore, we can use this property to flip the order of the addition in each of the components.

Doing this gives,

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n \rangle = \langle w_1 + v_1, w_2 + v_2, \ldots, w_n + v_n \rangle$$

Step 4

Now, recall that according to the definition of vector arithmetic the first number in the sum in each component of the vector on the right is the component of the first vector while the second number in the sum is the component of the second vector.

So, all we need to do now is "undo" the sum that gave the vector on the right to get,

$$\vec{v} + \vec{w} = \langle w_1 + v_1, w_2 + v_2, \ldots, w_n + v_n \rangle = \vec{w} + \vec{v}$$

This is exactly what we were asked to prove and so we are done!

11.3 Dot Product

1. Determine the dot product, $\vec{a} \cdot \vec{b}$ if $\vec{a} = \langle 9, 5, -4, 2 \rangle$ and $\vec{b} = \langle -3, -2, 7, -1 \rangle$.

Solution

Not really a whole lot to do here. We just need to run through the definition of the dot product.

$$\vec{a} \cdot \vec{b} = (9)(-3) + (5)(-2) + (-4)(7) + (2)(-1) = \boxed{-67}$$

---

2. Determine the dot product, $\vec{a} \cdot \vec{b}$ if $\vec{a} = \langle 0, 4, -2 \rangle$ and $\vec{b} = 2\vec{i} - \vec{j} + 7\vec{k}$.

Solution

Not really a whole lot to do here. We just need to run through the definition of the dot product and do not get excited about the “mixed” notation here. We know that they are equivalent notations!

$$\vec{a} \cdot \vec{b} = (0)(2) + (4)(-1) + (-2)(7) = \boxed{-18}$$

---

3. Determine the dot product, $\vec{a} \cdot \vec{b}$ if $\|\vec{a}\| = 5$, $\left\|\vec{b}\right\| = \dfrac{3}{7}$ and the angle between the two vectors is $\theta = \dfrac{\pi}{12}$.

Solution

Not really a whole lot to do here. We just need to run through the formula from the geometric interpretation of the dot product.

$$\vec{a} \cdot \vec{b} = (5)\left(\frac{3}{7}\right)\cos\left(\frac{\pi}{12}\right) = \boxed{2.0698}$$

---

4. Determine the angle between $\vec{v} = \langle 1, 2, 3, 4 \rangle$ and $\vec{w} = \langle 0, -1, 4, -2 \rangle$.

Solution

Not really a whole lot to do here. All we really need to do is rewrite the formula from the geometric interpretation of the dot product as,

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \; \|\vec{w}\|}$$

This will allow us to quickly determine the angle between the two vectors.

We'll first need the following quantities (we'll leave it to you to verify the arithmetic involved in these computations....).

$$\vec{v} \cdot \vec{w} = 2 \qquad \|\vec{v}\| = \sqrt{30} \qquad \|\vec{w}\| = \sqrt{21}$$

The angle between the vectors is then,

$$\cos(\theta) = \frac{2}{\sqrt{30}\,\sqrt{21}} = 0.07968 \qquad \Rightarrow \qquad \theta = \cos^{-1}(0.07968) = \boxed{1.49103 \text{ radians}}$$

5. Determine the angle between $\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{b} = \langle -9, 1, -5 \rangle$.

Solution

Not really a whole lot to do here. All we really need to do is rewrite the formula from the geometric interpretation of the dot product as,

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \; \left\|\vec{b}\right\|}$$

This will allow us to quickly determine the angle between the two vectors.

We'll first need the following quantities (we'll leave it to you to verify the arithmetic involved in these computations....).

$$\vec{a} \cdot \vec{b} = 4 \qquad \|\vec{a}\| = \sqrt{14} \qquad \left\|\vec{b}\right\| = \sqrt{107}$$

The angle between the vectors is then,

$$\cos(\theta) = \frac{4}{\sqrt{14}\,\sqrt{107}} = 0.1033 \qquad \Rightarrow \qquad \theta = \cos^{-1}(0.1034) = \boxed{1.4673 \text{ radians}}$$

6. Determine if $\vec{q} = \langle 4, -2, 7 \rangle$ and $\vec{p} = -3\vec{i} + \vec{j} + 2\vec{k}$ are parallel, orthogonal or neither.

Step 1

Based on a quick inspection of the components we can see that the first and second components of the two vectors have opposite signs and the third doesn't. This means there is no possible way for these two vectors to be scalar multiples since there is no number that will change the sign on the first two components and leave the sign of the third component unchanged.

Therefore, we can quickly see that the two vectors are **not parallel**.

Step 2

Let's do a quick dot product on the two vectors next.

$$\vec{q} \cdot \vec{p} = 0$$

Okay, the dot product is zero and we know from the notes that this in turn means that the two vectors must be **orthogonal**.

On a side note an alternate method for working this problem is to find the angle between the two vectors and using that to determine the answer.

Depending on which method you find easiest either will get you the correct answer.

7. Determine if $\vec{a} = \langle 3, 10 \rangle$ and $\vec{b} = \langle 4, -1 \rangle$ are parallel, orthogonal or neither.

Step 1

Based on a quick inspection of the components we can see that the first components of the vectors have the same sign and the second have opposite signs. This means there is no possible way for these two vectors to be scalar multiples since there is no number that will change the sign on the second components and leave the sign of the first component unchanged.

Therefore, we can quickly see that the two vectors are **not parallel**.

Step 2

Let's do a quick dot product on the two vectors next.

$$\vec{a} \cdot \vec{b} = 2$$

Okay, the dot product is not zero and we know from the notes that this in turn means that the two vectors are **not orthogonal**.

The answer to the problem is therefore the two vectors are **neither parallel or orthogonal**.

On a side note an alternate method for working this problem is to find the angle between the two vectors and using that to determine the answer.

Depending on which method you find easiest either will get you the correct answer.

---

8. Determine if $\vec{w} = \vec{i} + 4\vec{j} - 2\vec{k}$ and $\vec{v} = -3\vec{i} - 12\vec{j} + 6\vec{k}$ are parallel, orthogonal or neither.

Solution

Based on a quick inspection is seems (hopefully) fairly clear that we have,

$$\vec{v} = -3\vec{w}$$

Therefore, the two vectors are **parallel**.

On a side note an alternate method for working this problem is to find the angle between the two vectors and using that to determine the answer.

Depending on which method you find easiest either will get you the correct answer.

---

9. Given $\vec{a} = \langle -8, 2 \rangle$ and $\vec{b} = \langle -1, -7 \rangle$ compute $\text{proj}_{\vec{a}}\,\vec{b}$.

Solution

All we really need to do here is use the formula from the notes. That will need the following quantities.

$$\vec{a} \cdot \vec{b} = -6 \qquad \|\vec{a}\|^2 = 68$$

The projection is then,

$$\text{proj}_{\vec{a}}\,\vec{b} = \frac{-6}{68}\langle -8, 2 \rangle = \boxed{\left\langle \frac{12}{17}, -\frac{3}{17} \right\rangle}$$

10. Given $\vec{u} = 7\vec{i} - \vec{j} + \vec{k}$ and $\vec{w} = -2\vec{i} + 5\vec{j} - 6\vec{k}$ compute $\text{proj}_{\vec{w}}\,\vec{u}$.

Solution

All we really need to do here is use the formula from the notes. That will need the following quantities.

$$\vec{u}\cdot\vec{w} = -25 \qquad \|\vec{w}\|^2 = 65$$

The projection is then,

$$\text{proj}_{\vec{w}}\,\vec{u} = \frac{-25}{65}\left(-2\vec{i} + 5\vec{j} - 6\vec{k}\right) = \boxed{\frac{10}{13}\vec{i} - \frac{25}{13}\vec{j} + \frac{30}{13}\vec{k}}$$

11. Determine the direction cosines and direction angles for $\vec{r} = \left\langle -3, -\frac{1}{4}, 1\right\rangle$.

Solution

All we really need to do here is use the formulas from the notes. That will need the following quantity.

$$\|\vec{r}\| = \sqrt{\frac{161}{16}} = \frac{\sqrt{161}}{4}$$

The direction cosines and angles are then,

$$\cos\alpha = \frac{-3}{\sqrt{161}/4} = -\frac{12}{\sqrt{161}} \qquad \Rightarrow \qquad \alpha = \cos^{-1}\left(-\frac{12}{\sqrt{161}}\right) = 2.8106 \text{ radians}$$

$$\cos\beta = \frac{-1/4}{\sqrt{161}/4} = -\frac{1}{\sqrt{161}} \qquad \Rightarrow \qquad \beta = \cos^{-1}\left(-\frac{1}{\sqrt{161}}\right) = 1.6497 \text{ radians}$$

$$\cos\gamma = \frac{1}{\sqrt{161}/4} = \frac{4}{\sqrt{161}} \qquad \Rightarrow \qquad \gamma = \cos^{-1}\left(\frac{4}{\sqrt{161}}\right) = 1.2501 \text{ radians}$$

11.4 Cross Product

1. If $\vec{w} = \langle 3, -1, 5\rangle$ and $\vec{v} = \langle 0, 4, -2\rangle$ compute $\vec{v} \times \vec{w}$.

Solution

Not really a whole lot to do here. We just need to run through one of the various methods for computing the cross product. We'll be using the "trick" we used in the notes.

$$\vec{v} \times \vec{w} = \left|\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 0 & 4 & -2 \\ 3 & -1 & 5 \end{array}\right| \begin{array}{cc} \vec{i} & \vec{j} \\ 0 & 4 \\ 3 & -1 \end{array}$$

$$= 20\vec{i} - 6\vec{j} + 0\vec{k} - 0\vec{j} - 2\vec{i} - 12\vec{k} = \boxed{18\vec{i} - 6\vec{j} - 12\vec{k}}$$

2. If $\vec{w} = \langle 1, 6, -8\rangle$ and $\vec{v} = \langle 4, -2, -1\rangle$ compute $\vec{w} \times \vec{v}$.

Solution

Not really a whole lot to do here. We just need to run through one of the various methods for computing the cross product. We'll be using the "trick" we used in the notes.

$$\vec{w} \times \vec{v} = \left|\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 1 & 6 & -8 \\ 4 & -2 & -1 \end{array}\right| \begin{array}{cc} \vec{i} & \vec{j} \\ 1 & 6 \\ 4 & -2 \end{array}$$

$$= -6\vec{i} - 32\vec{j} - 2\vec{k} - \left(-\vec{j}\right) - 16\vec{i} - 24\vec{k} = \boxed{-22\vec{i} - 31\vec{j} - 26\vec{k}}$$

3. Find a vector that is orthogonal to the plane containing the points $P = (3, 0, 1)$, $Q = (4, -2, 1)$ and $R = (5, 3, -1)$.

Step 1

We first need two vectors that are both parallel to the plane. Using the points that we are given (all in the plane) we can quickly get quite a few vectors that are parallel to the plane. We'll use the following two vectors.

$$\overrightarrow{PQ} = \langle 1, -2, 0\rangle \qquad \overrightarrow{PR} = \langle 2, 3, -2\rangle$$

Step 2

Now we know that the cross product of any two vectors will be orthogonal to the two original vectors. Since the two vectors from Step 1 are parallel to the plane (they actually lie in the plane in this case!) we know that the cross product must then also be orthogonal, or normal, to the plane.

So, using the "trick" we used in the notes the cross product is,

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 0 \\ 2 & 3 & -2 \end{array} \right| \begin{array}{cc} \vec{i} & \vec{j} \\ 1 & -2 \\ 2 & 3 \end{array}$$

$$= 4\vec{i} + 0\vec{j} + 3\vec{k} - \left(-2\vec{j}\right) - 0\vec{i} - \left(-4\vec{k}\right) = \boxed{4\vec{i} + 2\vec{j} + 7\vec{k}}$$

---

4. Are the vectors $\vec{u} = \langle 1, 2, -4 \rangle$, $\vec{v} = \langle -5, 3, -7 \rangle$ and $\vec{w} = \langle -1, 4, 2 \rangle$ are in the same plane?

Solution

As discussed in the notes to answer this question all we need to do is compute the following quantity,

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \left| \begin{array}{ccc} 1 & 2 & -4 \\ -5 & 3 & -7 \\ -1 & 4 & 2 \end{array} \right| \begin{array}{cc} 1 & 2 \\ -5 & 3 \\ -1 & 4 \end{array}$$

$$= 6 + 14 + 80 - (-20) - (-28) - 12 = 136$$

Okay, since this is not zero we know that they **are not in the same plane**.

---

12 Three Dimensional Space

In this chapter we will start looking at three dimensional space (3-D space or $\mathbb{R}^3$).As with the last chapter this is preparation for multi-variable Calculus (which we'll be starting in the next chapter) as the vast majority of the multi-variable Calculus material assumes we are in three dimensional (or higher dimensional) space.

In this chapter we will discuss the equations of lines and planes in three dimensional space as well as the equations of many of the standard quadric surfaces (*i.e* equations with at least one quadratic term in it).

We will define a vector function and discuss how to perform basic Calculus operations on vector functions. We will also discuss how to get tangent vectors (a vector tangent to a curve), normal vectors (a vector orthogonal/perpendicular) and the curvature of a curve from the vector function that defines the curve. We'll also have a quick discussion of how to get the velocity and acceleration of an object as it travels along a curve defined by a vector function.

We will close out the chapter with a discussion a couple of alternative coordinates systems for three dimensional space, namely, cylindrical coordinates (a 3D extension of polar coordinates) and spherical coordinates.

The following sections are the practice problems, with solutions, for this material.

12.1 The 3-D Coordinate System

1. Give the projection of $P = (3, -4, 6)$ onto the three coordinate planes.

Solution

There really isn't a lot to do with this problem. We know that the xy-plane is given by the equation $z = 0$ and so the projection into the xy-plane for any point is simply found by setting the z coordinate to zero. We can find the projections for the other two coordinate planes in a similar fashion.

So, the projects are then,

$$xy \text{ - plane : } (3, -4, 0)$$
$$xz \text{ - plane : } (3, 0, 6)$$
$$yz \text{ - plane : } (0, -4, 6)$$

2. Which of the points $P = (4, -2, 6)$ and $Q = (-6, -3, 2)$ is closest to the yz-plane?

Step 1

The shortest distance between any point and any of the coordinate planes will be the distance between that point and its projection onto that plane.

Let's call the projections of P and Q onto the yz-plane $\overline{P}$ and $\overline{Q}$ respectively. They are,

$$\overline{P} = (0, -2, 6) \qquad \overline{Q} = (0, -3, 2)$$

Step 2

To determine which of these is closest to the yz-plane we just need to compute the distance between the point and its projection onto the yz-plane.

Note as well that because only the x-coordinate of the two points are different the distance between the two points will just be the absolute value of the difference between two x coordinates.

Therefore,

$$d\left(P, \overline{P}\right) = 4 \qquad d\left(Q, \overline{Q}\right) = 6$$

Based on this is should be pretty clear that $P = (4, -2, 6)$ is closest to the yz-plane.

3. Which of the points $P = (-1, 4, -7)$ and $Q = (6, -1, 5)$ is closest to the z-axis?

Step 1

First, let's note that the coordinates of any point on the z-axis will be $(0, 0, z)$.

Also, the shortest distance from any point not on the z-axis to the z-axis will occur if we draw a line straight from the point to the z-axis in such a way that it forms a right angle with the z-axis.

So, if we start with any point not on the z-axis, say (x_1, y_1, z_1), the point on the z-axis that will be closest to this point is $(0, 0, z_1)$.

Let's call the point closest to P and Q on the z-axis closest to be $\overline{P}$ and $\overline{Q}$ respectively. They are,

$$\overline{P} = (0, 0, -7) \qquad\qquad \overline{Q} = (0, 0, 5)$$

Step 2

To determine which of these is closest to the z-axis we just need to compute the distance between the point and its projection onto the z-axis.

The distances are,

$$d\left(P, \overline{P}\right) = \sqrt{(-1-0)^2 + (4-0)^2 + (-7-(-7))^2} = \sqrt{17}$$
$$d\left(Q, \overline{Q}\right) = \sqrt{(6-0)^2 + (-1-0)^2 + (5-5)^2} = \sqrt{37}$$

Based on this is should be pretty clear that $P = (-1, 4, -7)$ is closest to the z-axis.

4. List all of the coordinates systems ($\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$) that the following equation will have a graph in. Do not sketch the graph.

$$7x^2 - 9y^3 = 3x + 1$$

Solution

First notice that because there are two variables in this equation it cannot have a graph in $\mathbb{R}$ since equations in that coordinate system can only have a single variable.

There are two variables in the equation so we know that it will have a graph in $\mathbb{R}^2$.

Likewise, the fact that the equation has two variables means that it will also have a graph in $\mathbb{R}^3$. Although in this case the third variable, z, can have any value.

5. List all of the coordinates systems ($\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$) that the following equation will have a graph in. Do not sketch the graph.

$$x^3 + \sqrt{y^2+1} - 6z = 2$$

Solution

This equation has three variables and so it will have a graph in $\mathbb{R}^3$.

On other hand because the equation has three variables in it there will be no graph in $\mathbb{R}^2$ (can have at most two variables) and it will not have a graph in $\mathbb{R}$ (can only have a single variable).

12.2 Equations of Lines

1. Give the equation of the line through the points $(2, -4, 1)$ and $(0, 4, -10)$ in vector form, parametric form and symmetric form.

Step 1

Okay, regardless of the form of the equation we know that we need a point on the line and a vector that is parallel to the line.

We have two points that are on the line. We can use either point and depending on your choice of points you may have different answers that we get here. We will use the first point listed above for our point for no other reason that it is the first one listed.

The parallel vector is really simple to get as well since we can always form the vector from the first point to the second point and this vector will be on the line and so will also be parallel to the line. The vector is,

$$\vec{v} = \langle -2, 8, -11 \rangle$$

Step 2

The vector form of the line is,

$$\boxed{\vec{r}(t) = \langle 2, -4, 1 \rangle + t \langle -2, 8, -11 \rangle = \langle 2 - 2t, -4 + 8t, 1 - 11t \rangle}$$

Step 3

The parametric form of the line is,

$$\boxed{x = 2 - 2t \qquad y = -4 + 8t \qquad z = 1 - 11t}$$

Step 4

To get the symmetric form all we need to do is solve each of the parametric equations for t and then set them all equal to each other. Doing this gives,

$$\boxed{\frac{2-x}{2} = \frac{4+y}{8} = \frac{1-z}{11}}$$

---

2. Give the equation of the line through the point $(-7,2,4)$ and parallel to the line given by $x = 5-8t$, $y = 6+t$, $z = -12t$ in vector form, parametric form and symmetric form.

Step 1

Okay, regardless of the form of the equation we know that we need a point on the line and a vector that is parallel to the line.

We were given a point on the line so no need to worry about that for this problem.

The parallel vector is really simple to get as well since we were told that the new line must be parallel to the given line. We also know that the coefficients of the t's in the equation of the line forms a vector parallel to the line.

So,

$$\vec{v} = \langle -8, 1, -12 \rangle$$

is a vector that is parallel to the given line.

Now, if $\vec{v}$ is parallel to the given line and the new line must be parallel to the given line then $\vec{v}$ must also be parallel to the new line.

Step 2

The vector form of the line is,

$$\boxed{\vec{r}(t) = \langle -7, 2, 4 \rangle + t\langle -8, 1, -12 \rangle = \langle -7-8t, 2+t, 4-12t \rangle}$$

Step 3

The parametric form of the line is,

$$\boxed{x = -7-8t \qquad y = 2+t \qquad z = 4-12t}$$

Step 4

To get the symmetric form all we need to do is solve each of the parametric equations for t and then set them all equal to each other. Doing this gives,

$$\boxed{\frac{-7-x}{8} = y-2 = \frac{4-z}{12}}$$

3. Is the line through the points $(2,0,9)$ and $(-4,1,-5)$ parallel, orthogonal or neither to the line given by $\vec{r}(t) = \langle 5, 1-9t, -8-4t \rangle$?

Step 1

Let's start this off simply by getting vectors parallel to each of the lines.

For the line through the points $(2,0,9)$ and $(-4,1,-5)$ we know that the vector between these two points will lie on the line and hence be parallel to the line. This vector is,

$$\vec{v}_1 = \langle 6, -1, 14 \rangle$$

For the second line the coefficients of the t's are the components of the parallel vector so this vector is,

$$\vec{v}_2 = \langle 0, -9, -4 \rangle$$

Step 2

Now, from the first components of these vectors it is hopefully clear that they are not scalar multiples. There is no number we can multiply to zero to get 6.

Likewise, we can only multiply 6 by zero to get 0. However, if we multiply the first vector by zero all the components would be zero and that is clearly not the case.

Therefore, they are not scalar multiples and so these two vectors are not parallel. This also in turn means that **the two lines can't possibly be parallel** either (since each vector is parallel to its respective line).

Step 3

Next,

$$\vec{v}_1 \cdot \vec{v}_2 = -47$$

The dot product is not zero and so these vectors aren't orthogonal. Because the two vectors are parallel to their respective lines this also means that **the two lines are not orthogonal**.

---

4. Determine the intersection point of the line given by $x = 8 + t$, $y = 5 + 6t$, $z = 4 - 2t$ and the line given by $\vec{r}(t) = \langle -7 + 12t, 3 - t, 14 + 8t \rangle$ or show that they do not intersect.

Step 1

If the two lines do intersect then they must share a point in common. In other words there must be some value, say $t = t_1$, and some (probably) different value, say $t = t_2$, so that if we plug t_1 into the equation of the first line and if we plug t_2 into the equation of the second line we will get the same x, y and z coordinates.

Step 2

This means that we can set up the following system of equations.

$$\begin{aligned} 8 + t_1 &= -7 + 12t_2 \\ 5 + 6t_1 &= 3 - t_2 \\ 4 - 2t_1 &= 14 + 8t_2 \end{aligned}$$

If this system of equations has a solution then the lines will intersect and if it doesn't have a solution then the lines will not intersect.

Step 3

Solving a system of equations with more equations than unknowns is probably not something that you've run into all that often to this point. The basic process is pretty much the same however with a couple of minor (but very important) differences.

Start off by picking any two of the equations (so we now have two equations and two unknowns) and solve that system. For this problem let's just take the first two equations.

We'll worry about the third equation eventually.

Solving a system of two equations and two unknowns is something everyone should be familiar with at this point so we'll not put in any real explanation to the solution work below.

$$\begin{aligned} t_1 = -15 + 12t_2 \quad \rightarrow \quad 5 + 6\left(-15 + 12t_2\right) &= 3 - t_2 \\ -85 + 72t_2 &= 3 - t_2 \\ 73t_2 &= 88 \quad \rightarrow \quad t_2 = \frac{88}{73} \\ & \qquad t_1 = -15 + 12\left(\frac{88}{73}\right) = -\frac{39}{73} \end{aligned}$$

Step 4

Okay, this is a somewhat "messy" solution, but they will often be that way so we shouldn't get too excited about it!

Now, recall that to get this solution we used the first two equations. What this means is that if we use this value of t_1 and t_2 in the equations of the first and second lines respectively then the x and y coordinates will be the same (remember we used the x and y equations to find this solution....).

At this point we need to recall that we did have a third equation that also needs to be satisfied at these values of t. In other words, we need to plug $t_1 = -\frac{39}{73}$ and $t_2 = \frac{88}{73}$ into the third equation and see what we get. Doing this gives,

$$\frac{370}{73} = 4 - 2\left(-\frac{39}{73}\right) \neq 14 + 8\left(\frac{88}{73}\right) = \frac{1726}{73}$$

Okay, the two sides are not the same. Just what does this mean? In terms of systems of equations it means that $t_1 = -\frac{39}{73}$ and $t_2 = \frac{88}{73}$ are NOT a solution to the system of equations in Step 2. Had they been a solution then we would have gotten the same number from both sides.

In terms of whether or not the lines intersect we need to only recall that the third equation corresponds to the z coordinates of the lines. We know that at $t_1 = -\frac{39}{73}$ and $t_2 = \frac{88}{73}$ the two lines will have the same x and y coordinates (since they came from solving the first two equations). However, we've just shown that they will not give the same z coordinate.

In other words, there are no values of t_1 and t_2 for which the two lines will have the same x, y and z coordinates. Hence, we can now say that the **two lines do not intersect**.

Before leaving this problem let's note that it doesn't matter which two equations we solve in Step 3. Different sets of equations will lead to different values of t_1 and t_2 but they will still not satisfy the remaining equation for this problem and we will get the same result of the lines not intersecting.

---

5. Determine the intersection point of the line through the points $(1, -2, 13)$ and $(2, 0, -5)$ and the line given by $\vec{r}(t) = \langle 2 + 4t, -1 - t, 3 \rangle$ or show that they do not intersect.

Step 1

Because we don't have the equation for the first line that will be the first thing we'll need to do. The vector between the two points (and hence parallel to the line) is,

$$\vec{v} = \langle 1, 2, -18 \rangle$$

Using the first point listed the equation of the first line is then,

$$\vec{r}(t) = \langle 1, -2, 13 \rangle + t \langle 1, 2, -18 \rangle = \langle 1 + t, -2 + 2t, 13 - 18t \rangle$$

Step 2

If the two lines do intersect then they must share a point in common. In other words there must be some value, say $t = t_1$, and some (probably) different value, say $t = t_2$, so that if we plug t_1 into the equation of the first line and if we plug t_2 into the equation of the second line we will get the same x, y and z coordinates.

Step 3

This means that we can set up the following system of equations.

$$\begin{aligned} 1 + t_1 &= 2 + 4t_2 \\ -2 + 2t_1 &= -1 - t_2 \\ 13 - 18t_1 &= 3 \end{aligned}$$

If this system of equations has a solution then the lines will intersect and if it doesn't have a solution then the lines will not intersect.

Step 4

Solving a system of equations with more equations than unknowns is probably not something that you've run into all that often to this point. The basic process is pretty much the same however with a couple of minor (but very important) differences.

Start off by picking any two of the equations (so we now have two equations and two unknowns) and solve that system. For this problem let's just take the first and third equation. We'll worry about the second equation eventually.

Note that for this system the third equation should definitely be used here since we can quickly just solve that for t_1.

Solving a system of two equations and two unknowns is something everyone should be familiar with at this point so we'll not put in any real explanation to the solution work below.

$$t_1 = \frac{5}{9} \qquad \to \qquad 1 + \frac{5}{9} = 2 + 4t_2 \qquad \to \qquad t_2 = -\frac{1}{9}$$

Step 5

Now, recall that to get this solution we used the first and third equations. What this means is that if we use this value of t_1 and t_2 in the equations of the first and second lines respectively then the x and z coordinates will be the same (remember we used the x and z equations to find this solution....).

At this point we need to recall that we did have another equation that also needs to be satisfied at these values of t. In other words, we need to plug $t_1 = \frac{5}{9}$ and $t_2 = -\frac{1}{9}$ into the second equation and see what we get. Doing this gives,

$$-2 + 2\left(\frac{5}{9}\right) = -\frac{8}{9} = -1 - \left(-\frac{1}{9}\right)$$

Okay, the two sides are the same. Just what does this mean? In terms of systems of equations it means that $t_1 = \frac{5}{9}$ and $t_2 = -\frac{1}{9}$ is a solution to the system of equations in Step 3.

In terms of whether or not the lines intersect we now know that at $t_1 = \frac{5}{9}$ and $t_2 = -\frac{1}{9}$ the two lines will have the same x, y and z coordinates (since they satisfy all three equations).

In other words, these **two lines do intersect**.

Before leaving this problem let's note that it doesn't matter which two equations we solve in Step 4. Different sets of equations will lead to the same values of t_1 and t_2 leading to the two lines intersecting.

6. Does the line given by $x = 9 + 21t$, $y = -7$, $z = 12 - 11t$ intersect the xy-plane? If so, give the point.

Step 1

If the line intersects the xy-plane there will be a point on the line that is also in the xy-plane. Recall as well that any point in the xy-plane will have a z coordinate of $z = 0$.

Step 2

So, to determine if the line intersects the xy-plane all we need to do is set the equation for the z coordinate equal to zero and solve it for t, if that's possible.

Doing this gives,

$$12 - 11t = 0 \qquad \to \qquad t = \frac{12}{11}$$

Step 3

So, we were able to solve for t in this case and so we can now say that **the line does intersect the xy-plane**.

Step 4

All we need to do to finish this off this problem is find the full point of intersection. We can find this simply by plugging $t = \frac{12}{11}$ into the x and y portions of the equation of the line.

Doing this gives,

$$x = 9 + 21\left(\frac{12}{11}\right) = \frac{351}{11} \qquad y = -7$$

The point of intersection is then : $\boxed{\left(\frac{351}{11}, -7, 0\right)}$.

7. Does the line given by $x = 9 + 21t$, $y = -7$, $z = 12 - 11t$ intersect the xz-plane? If so, give the point.

Step 1

If the line intersects the xz-plane there will be a point on the line that is also in the xz-plane. Recall as well that any point in the xz-plane will have a y coordinate of $y = 0$.

Step 2

So, to determine if the line intersects the xz-plane all we need to do is set the equation for the y coordinate equal to zero and solve it for t, if that's possible.

However, in this case we can see that is clearly not possible since the y equation is simply $y = -7$ and this can clearly never be zero.

Step 3

Therefore, **the line does not intersect the xz-plane**.

12.3 Equations of Planes

1. Write down the equation of the plane containing the points $(4,-3,1)$, $(-3,-1,1)$ and $(4,-2,8)$.

Step 1

To make the work on this problem a little easier let's "name" the points as,

$$P=(4,-3,1) \qquad Q=(-3,-1,1) \qquad R=(4,-2,8)$$

Now, we know that in order to write down the equation of a plane we'll need a point (we have three so that's not a problem!) and a vector that is normal to the plane.

Step 2

We'll need to do a little work to get a normal vector.

First, we'll need two vectors that lie in the plane and we can get those from the three points we're given. Note that there are lots of possible vectors that we could use here. Here are the two that we'll use for this problem.

$$\overrightarrow{PQ}=\langle -7,2,0\rangle \qquad \overrightarrow{PR}=\langle 0,1,7\rangle$$

Step 3

Now, these two vectors lie in the plane and we know that the cross product of any two vectors will be orthogonal to both of the vectors. Therefore, the cross product of these two vectors will also be orthogonal (and so normal!) to the plane.

So, let's get the cross product.

$$\vec{n}=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -7 & 2 & 0\\ 0 & 1 & 7\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ -7 & 2\\ 0 & 1\end{matrix}=14\vec{i}-7\vec{k}-\left(-49\vec{j}\right)=14\vec{i}+49\vec{j}-7\vec{k}$$

Note that we used the "trick" discussed in the notes to compute the cross product here.

Step 4

Now all we need to do is write down the equation.

We have three points to choose form here. We'll use the first point simply because it is the first point listed. Any of the others could also be used.

The equation of the plane is,

$$\boxed{14(x-4)+49(y+3)-7(z-1)=0 \qquad \to \qquad 14x+49y-7z=-98}$$

Note that depending on your choice of vectors in Step 2, the order you chose to use them in the cross product computation in Step 3 and the point chosen here will all affect your answer. However, regardless of your choices the equation you get will be an acceptable answer provided you did all the work correctly.

---

2. Write down the equation of the plane containing the point $(3,0,-4)$ and orthogonal to the line given by $\vec{r}(t)=\langle 12-t,1+8t,4+6t\rangle$.

Step 1

We know that we need a point on the plane and a vector that is normal to the plane. We've were given a point that is in the plane so we're okay there.

Step 2

The normal vector for the plane is actually quite simple to get.

We are told that the plane is orthogonal to the line given in the problem statement. This means that the plane will also be orthogonal to any vector that just happens to be parallel to the line.

From the equation of the line we know that the coefficients of the t's are the components of a vector that is parallel to the line. So, a vector parallel to the line is then,

$$\vec{v}=\langle -1,8,6\rangle$$

Now, as mentioned above because this vector is parallel to the line then it will also need to be orthogonal to the plane and hence be normal to the plane. So, a normal vector for the plane is,

$$\vec{n}=\langle -1,8,6\rangle$$

Step 3

Now all we need to do is write down the equation. The equation of the plane is,

$$\boxed{-(x-3)+8(y-0)+6(z+4)=0 \qquad \to \qquad -x+8y+6z=-27}$$

3. Write down the equation of the plane containing the point $(-8,3,7)$ and parallel to the plane given by $4x+8y-2z=45$.

Step 1

We know that we need a point on the plane and a vector that is normal to the plane. We've were given a point that is in the plane so we're okay there.

Step 2

The normal vector for the plane is actually quite simple to get.

We are told that the plane is parallel to the plane given in the problem statement. This means that any vector normal to one plane will be normal to both planes.

From the equation of the plane we were given we know that the coefficients of the x, y and z are the components of a vector that is normal to the plane. So, a vector normal to the given plane is then,

$$\vec{n}=\langle 4,8,-2\rangle$$

Now, as mentioned above because this vector is normal to the given plane then it will also need to be normal to the plane we want to find the equation for.

Step 3

Now all we need to do is write down the equation. The equation of the plane is,

$$\boxed{4(x+8)+8(y-3)-2(z-7)=0 \qquad \to \qquad 4x+8y-2z=-22}$$

4. Determine if the plane given by $4x - 9y - z = 2$ and the plane given by $x + 2y - 14z = -6$ are parallel, orthogonal or neither.

Step 1

Let's start off this problem by noticing that the vector $\vec{n}_1 = \langle 4, -9, -1 \rangle$ will be normal to the first plane and the vector $\vec{n}_2 = \langle 1, 2, -14 \rangle$ will be normal to the second plane.

Now try to visualize the two planes and these normal vectors. What would the two planes look like if the two normal vectors where parallel to each other? What would the two planes look like if the two normal vectors were orthogonal to each other?

Step 2

If the two normal vectors are parallel to each other the two planes would also need to be parallel.

So, let's take a quick look at the normal vectors. We can see that the first component of each vector have the same sign and the same can be said for the third component. However, the second component of each vector has opposite signs.

Therefore, there is no number that we can multiply to $\vec{n}_1$ that will keep the sign on the first and third component the same and simultaneously changing the sign on the second component. This in turn means the two vectors can't possibly be scalar multiples and this further means they cannot be parallel.

If the two normal vectors can't be parallel then **the two planes are not parallel**.

Step 3

Now, if the two normal vectors are orthogonal the two planes will also be orthogonal.

So, a quick dot product of the two normal vectors gives,

$$\vec{n}_1 \cdot \vec{n}_2 = 0$$

The dot product is zero and so the two normal vectors are orthogonal. Therefore, **the two planes are orthogonal**.

5. Determine if the plane given by $-3x+2y+7z=9$ and the plane containing the points $(-2,6,1)$, $(-2,5,0)$ and $(-1,4,-3)$ are parallel, orthogonal or neither.

Step 1

Let's start off this problem by noticing that the vector $\vec{n}_1 = \langle -3,2,7\rangle$ will be normal to the first plane and it would be nice to have a normal vector for the second plane.

We know (Problem 1 from this section!) how to determine a normal vector given three points in the plane. Here is that work.

$$P=(-2,6,1) \qquad Q=(-2,5,0) \qquad R=(-1,4,-3)$$
$$\overrightarrow{QP}=\langle 0,1,1\rangle \qquad \overrightarrow{RQ}=\langle -1,1,3\rangle$$

$$\vec{n}_2=\overrightarrow{QP}\times\overrightarrow{RQ}=\left|\begin{array}{ccc}\vec{i} & \vec{j} & \vec{k}\\ 0 & 1 & 1\\ -1 & 1 & 3\end{array}\right|\begin{array}{cc}\vec{i} & \vec{j}\\ 0 & 1\\ -1 & 1\end{array}=3\vec{i}-\vec{j}-\vec{i}-\left(-\vec{k}\right)=2\vec{i}-\vec{j}+\vec{k}$$

Note that we used the "trick" discussed in the notes to compute the cross product here.

Now try to visualize the two planes and these normal vectors. What would the two planes look like if the two normal vectors where parallel to each other? What would the two planes look like if the two normal vectors were orthogonal to each other?

Step 2

If the two normal vectors are parallel to each other the two planes would also need to be parallel.

So, let's take a quick look at the normal vectors. We can see that the third component of each vector have the same sign while the first and second components each have opposite signs.

Therefore, there is no number that we can multiply to $\vec{n}_1$ that will keep the sign on the third component the same and simultaneously changing the sign on the first and second components. This in turn means the two vectors can't possibly be scalar multiples and this further means they cannot be parallel.

If the two normal vectors can't be parallel then **the two planes are not parallel**.

Step 3

Now, if the two normal vectors are orthogonal the two planes will also be orthogonal.

So, a quick dot product of the two normal vectors gives,

$$\vec{n}_1 \cdot \vec{n}_2 = -1$$

The dot product is not zero and so the two normal vectors are not orthogonal. Therefore, **the two planes are not orthogonal**.

---

6. Determine if the line given by $\vec{r}(t) = \langle -2t, 2+7t, -1-4t \rangle$ intersects the plane given by $4x + 9y - 2z = -8$ or show that they do not intersect.

Step 1

If the line and the plane do intersect then there must be a value of t such that if we plug that t into the equation of the line we'd get a point that lies on the plane. We also know that if a point (x, y, z) is on the plane the then the coordinates will satisfy the equation of the plane.

Step 2

If you think about it the coordinates of all the points on the line can be written as,

$$(-2t, 2+7t, -1-4t)$$

for all values of t.

Step 3

So, let's plug the "coordinates" of the points on the line into the equation of the plane to get,

$$4(-2t) + 9(2+7t) - 2(-1-4t) = -8$$

Step 4

Let's solve this for t as follows,

$$63t + 20 = -8 \qquad \to \qquad t = -\frac{4}{9}$$

Step 5

We were able to find a t from this equation. What that means is that this is the value of t that, once we plug into the equation of the line, gives the point of intersection of the line and plane.

So, **the line and plane do intersect** and they will intersect at the point $\left(\frac{8}{9}, -\frac{10}{9}, \frac{7}{9}\right)$.

Note that all we did to get the point is plug $t = -\frac{4}{9}$ into the general form for points on the line we wrote down in Step 2.

7. Determine if the line given by $\vec{r}(t) = \langle 4 + t, -1 + 8t, 3 + 2t \rangle$ intersects the plane given by $2x - y + 3z = 15$ or show that they do not intersect.

Step 1

If the line and the plane do intersect then there must be a value of t such that if we plug that t into the equation of the line we'd get a point that lies on the plane. We also know that if a point (x, y, z) is on the plane the then the coordinates will satisfy the equation of the plane.

Step 2

If you think about it the coordinates of all the points on the line can be written as,

$$(4 + t, -1 + 8t, 3 + 2t)$$

for all values of t.

Step 3

So, let's plug the "coordinates" of the points on the line into the equation of the plane to get,

$$2(4+t)-(-1+8t)+3(3+2t)=15$$

Step 4

Let's solve this for t as follows,

$$18 = 15 \; ??$$

Step 5

Hmmm...

So, either we've just managed to prove that 18 and 15 are in fact the same number or there is something else going on here.

Clearly 18 and 15 are not the same number and so something else must be going on. In fact, all this means is that there is no t that will satisfy the equation we wrote down in Step 3. This in turn means that **the line and plane do not intersect.**

---

8. Find the line of intersection of the plane given by $3x+6y-5z=-3$ and the plane given by $-2x+7y-z=24$.

Step 1

Okay, we know that we need a point and vector parallel to the line in order to write down the equation of the line. In this case neither has been given to us.

First let's note that any point on the line of intersection must also therefore be in both planes and it's actually pretty simple to find such a point. Whatever our line of intersection is it must intersect at least one of the coordinate planes. It doesn't have to intersect all three of the coordinate planes but it will have to intersect at least one.

So, let's see if it intersects the xy-plane. Because the point on the intersection line must also be in both planes let's set $z=0$ (so we'll be in the xy-plane!) in both of the equations of our planes.

Doing this gives,

$$\begin{aligned} 3x + 6y &= -3 \\ -2x + 7y &= 24 \end{aligned}$$

Step 2

This is a simple system to solve so we'll leave it to you to verify that the solution is,

$$x = -5 \qquad\qquad y = 2$$

The fact that we were able to find a solution to the system from Step 1 means that the line of intersection does in fact intersect the xy-plane and it does so at the point $(-5, 2, 0)$. This is also then a point on the line of intersection.

Note that if the system from Step 1 didn't have a solution then the line of intersection would not have intersected the xy-plane and we'd need to try one of the remaining coordinate planes.

Step 3

Okay, now we need a vector that is parallel to the line of intersection. This might be a little hard to visualize, but if you think about it the line of intersection would have to be orthogonal to both of the normal vectors from the two planes. This in turn means that any vector orthogonal to the two normal vectors must then be parallel to the line of intersection.

Nicely enough we know that the cross product of any two vectors will be orthogonal to each of the two vectors. So, here are the two normal vectors for our planes and their cross product.

$$\vec{n}_1 = \langle 3, 6, -5 \rangle \qquad\qquad \vec{n}_2 = \langle -2, 7, -1 \rangle$$

$$\begin{aligned} \vec{n}_1 \times \vec{n}_2 &= \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 3 & 6 & -5 \\ -2 & 7 & -1 \end{array} \right| \begin{array}{cc} \vec{i} & \vec{j} \\ 3 & 6 \\ -2 & 7 \end{array} \\ &= -6\vec{i} + 10\vec{j} + 21\vec{k} - \left(-3\vec{j}\right) - \left(-35\vec{i}\right) - \left(-12\vec{k}\right) = 29\vec{i} + 13\vec{j} + 33\vec{k} \end{aligned}$$

Note that we used the "trick" discussed in the notes to compute the cross product here.

Step 4

So, we now have enough information to write down the equation of the line of intersection of the two planes. The equation is,

$$\boxed{\vec{r}\,(t) = \langle -5, 2, 0\rangle + t\,\langle 29, 13, 33\rangle = \langle -5 + 29t, 2 + 13t, 33t\rangle}$$

9. Determine if the line given by $x = 8 - 15t$, $y = 9t$, $z = 5 + 18t$ and the plane given by $10x - 6y - 12z = 7$ are parallel, orthogonal or neither.

Step 1

Let's start off this problem by noticing that the vector $\vec{v} = \langle -15, 9, 18\rangle$ will be parallel to the line and the vector $\vec{n} = \langle 10, -6, -12\rangle$ will be normal to the plane.

Now try to visualize the line and plane and their corresponding vectors. What would the line and plane look like if the two vectors were orthogonal to each other? What would the line and plane look like if the two vectors were parallel to each other?

Step 2

If the two vectors are orthogonal to each other the line would be parallel to the plane. If you think about this it does make sense. If $\vec{v}$ is orthogonal to $\vec{n}$ then it must be parallel to the plane because $\vec{n}$ is orthogonal to the plane. Then because the line is parallel to $\vec{v}$ it must also be parallel to the plane.

So, let's do a quick dot product here.

$$\vec{v} \cdot \vec{n} = -420$$

The dot product is not zero and so the two vectors aren't orthogonal to each other. Therefore, the **line and plane are not parallel**.

Step 3

If the two vectors are parallel to each other the line would be orthogonal to the plane. If you think about this it does make sense. The line is parallel to $\vec{v}$ which we've just assumed is parallel to $\vec{n}$. We also know that $\vec{n}$ is orthogonal to the plane and so anything that is parallel to $\vec{n}$ (the line for instance) must also be orthogonal to the plane.

In this case it looks like we have the following relationship between the two vectors.

$$\vec{v} = -\frac{3}{2}\vec{n}$$

The two vectors are parallel and so the **line and plane are orthogonal**.

12.4 Quadric Surfaces

1. Sketch the following quadric surface.

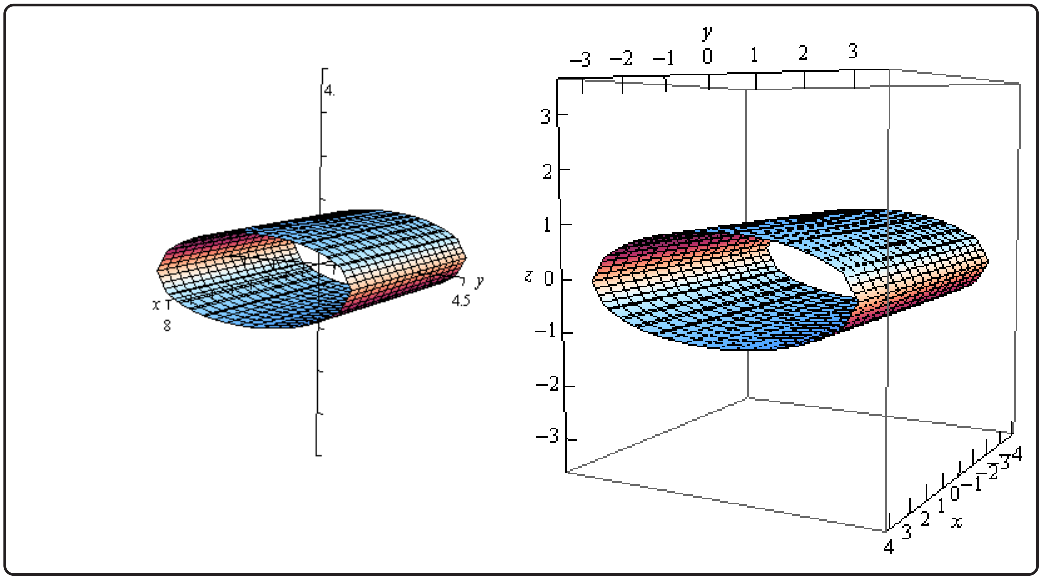
$$\frac{y^2}{9} + z^2 = 1$$

Solution

This is a cylinder that is centered on the x-axis. The cross sections of the cylinder will be ellipses.

Make sure that you can "translate" the equations given in the notes to the other coordinate axes. Once you know what they look like when centered on one of the coordinates axes then a simple and predictable variable change will center them on the other coordinate axes.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.



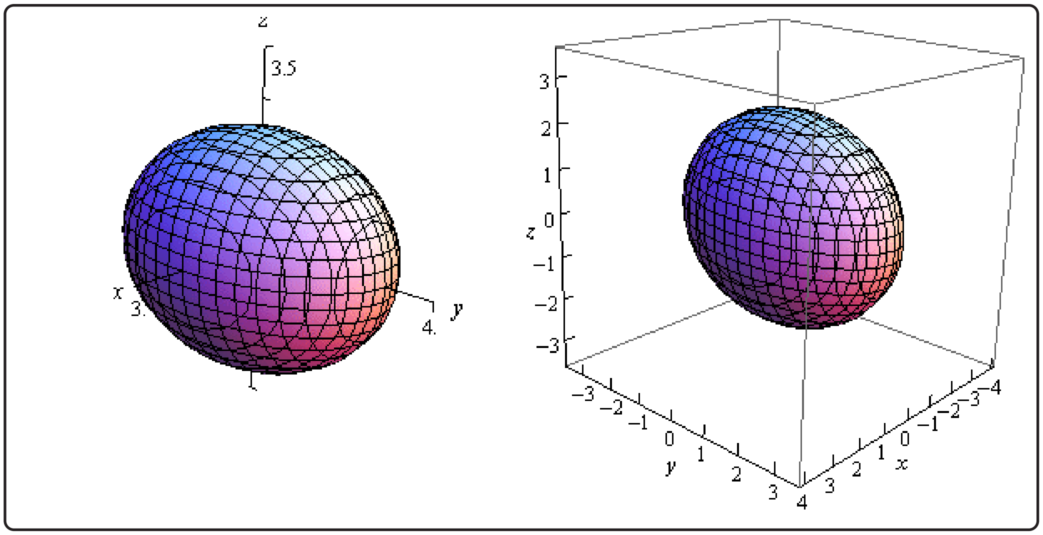
2. Sketch the following quadric surface.

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{6} = 1$$

Solution

This is an ellipsoid and because the numbers in the denominators of each of the terms are not the same we know that it won't be a sphere.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.



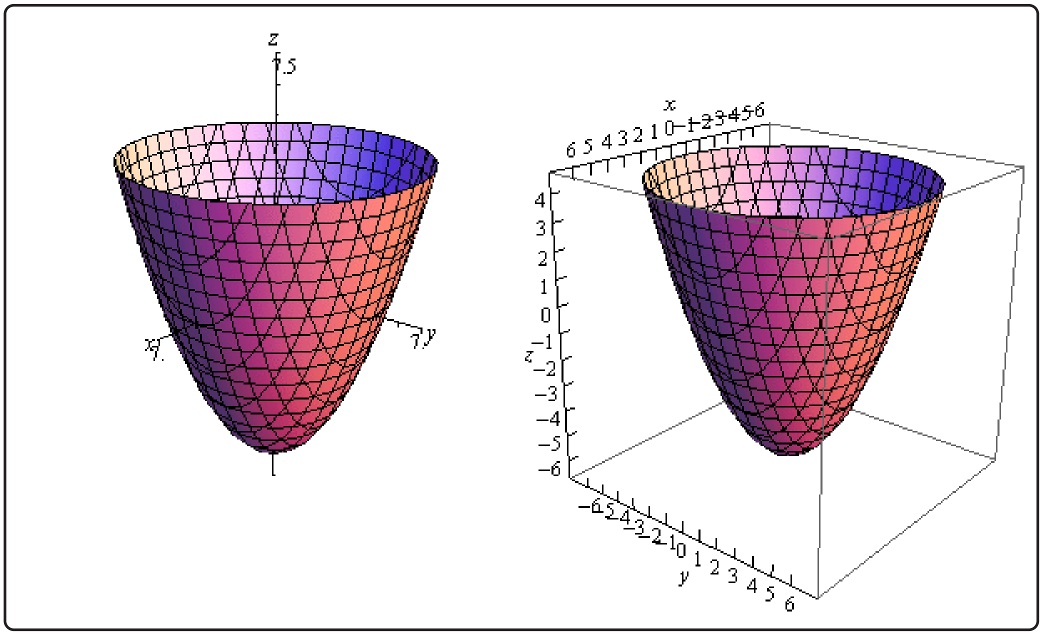
3. Sketch the following quadric surface.

$$z = \frac{x^2}{4} + \frac{y^2}{4} - 6$$

Solution

This is an elliptic paraboloid that is centered on the z-axis. Because the x and y terms are positive we know that it will open upwards. The "-6" tells us that the surface will start at $z = -6$. We can also say that because the coefficients of the x and y terms are identical the cross sections of the surface will be circles.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.



4. Sketch the following quadric surface.

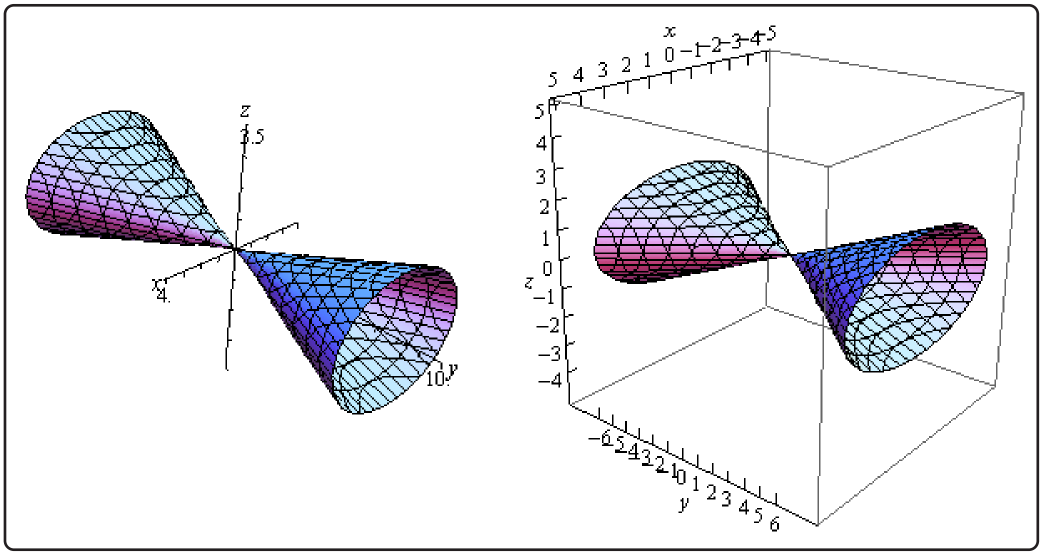
$$y^2 = 4x^2 + 16z^2$$

Solution

This is a cone that is centered on the y-axis and because the coefficients of the x and z terms are different the cross sections of the surface will be ellipses.

Make sure that you can “translate” the equations given in the notes to the other coordinate axes. Once you know what they look like when centered on one of the coordinates axes then a simple and predictable variable change will center them on the other coordinate axes.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as “boxed” axes to help visualize the surface.



5. Sketch the following quadric surface.

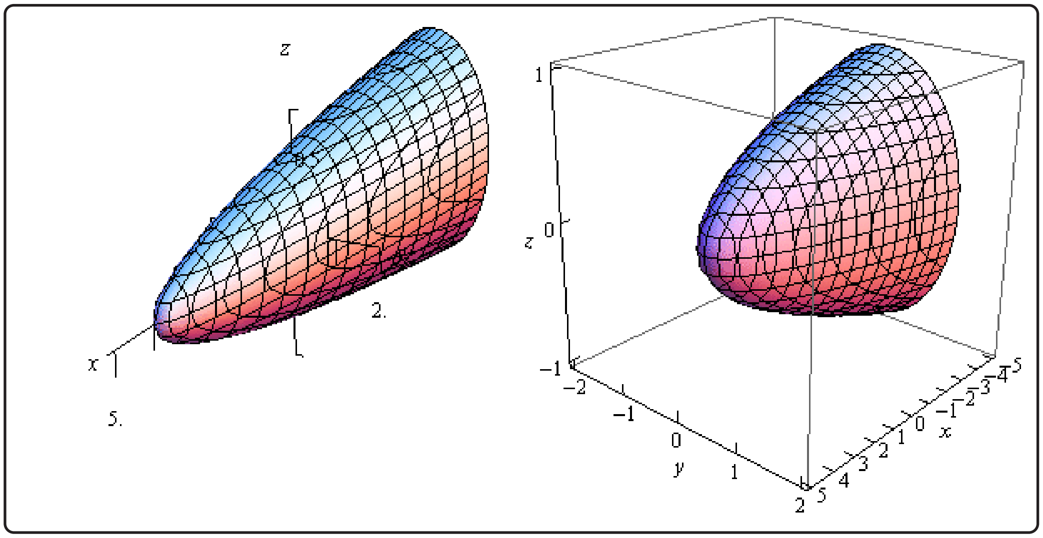
$$x = 4 - 5y^2 - 9z^2$$

Solution

This is an elliptic paraboloid that is centered on the x-axis. Because the y and z terms are negative we know that it will open in the negative x direction. The "4" tells us that the surface will start at $x = 4$. We can also say that because the coefficients of the y and z terms are different the cross sections of the surface will be ellipses.

Make sure that you can "translate" the equations given in the notes to the other coordinate axes. Once you know what they look like when centered on one of the coordinates axes then a simple and predictable variable change will center them on the other coordinate axes.

Here are a couple of sketches of the region. We've given them with the more traditional axes as well as "boxed" axes to help visualize the surface.


z
x
5.
2.
1
0
z
−1
−2
−1
0
y
1
2
5
4
3
2
1
0
−1
−2
−3
−4
−5
x

12.5 Functions of Several Variables

1. Find the domain of the following function.

$$f(x, y) = \sqrt{x^2 - 2y}$$

Solution

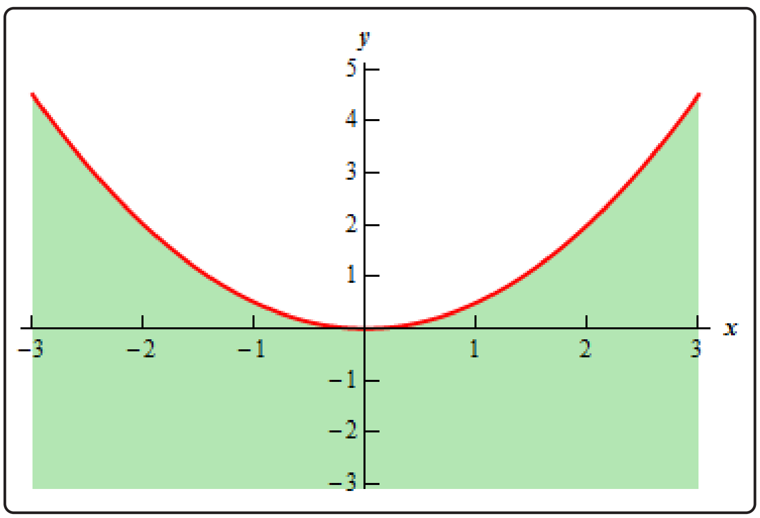
There really isn't all that much to this problem. We know that we can't have negative numbers under the square root and so the we'll need to require that whatever (x, y) is it will need to satisfy,

$$x^2 - 2y \geq 0$$

Let's do a little rewriting on this so we can attempt to sketch the domain.

$$x^2 \geq 2y \qquad \Rightarrow \qquad y \leq \frac{1}{2}x^2$$

So, it looks like we need to be on or below the parabola above. The domain is illustrated by the green area and red line in the sketch below.



2. Find the domain of the following function.

$$f(x,y) = \ln(2x - 3y + 1)$$

Solution

There really isn't all that much to this problem. We know that we can't have negative numbers or zero in a logarithm so we'll need to require that whatever (x,y) is it will need to satisfy,

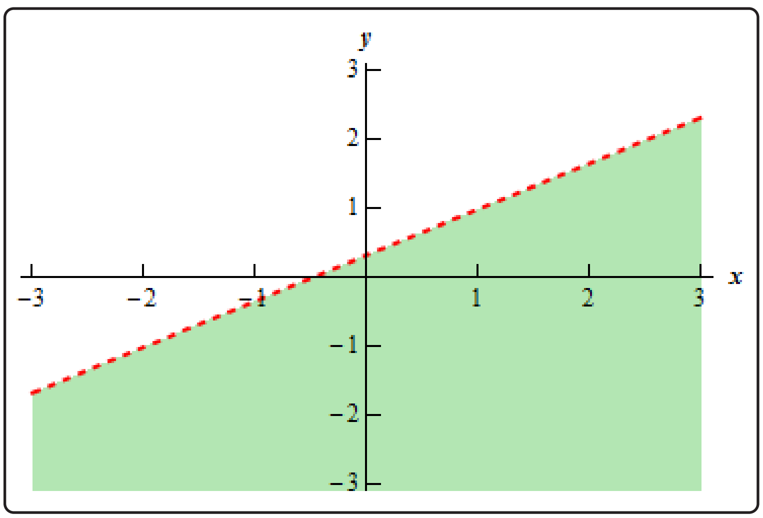
$$2x - 3y + 1 > 0$$

Since this is the only condition we need to meet this is also the domain of the function.

Let's do a little rewriting on this so we can attempt to sketch the domain.

$$2x + 1 > 3y \qquad \Rightarrow \qquad y < \frac{2}{3}x + \frac{1}{3}$$

So, it looks like we need to be below the line above. The domain is illustrated by the green area in the sketch below.



Note that we dashed the graph of the "bounding" line to illustrate that we don't take points from the line itself.

3. Find the domain of the following function.

$$f(x,y,z) = \frac{1}{x^2 + y^2 + 4z}$$

Solution

There really isn't all that much to this problem. We know that we can't have division by zero so we'll need to require that whatever (x,y,z) is it will need to satisfy,

$$x^2 + y^2 + 4z \neq 0$$

Since this is the only condition we need to meet this is also the domain of the function.

Let's do a little rewriting on this so we can attempt to identify the domain a little better.

$$4z \neq -x^2 - y^2 \qquad \Rightarrow \qquad z \neq -\frac{x^2}{4} - \frac{y^2}{4}$$

So, it looks like we need to avoid points, (x,y,z), that are on the elliptic paraboloid given by the equation above.

---

4. Find the domain of the following function.

$$f(x,y) = \frac{1}{x} + \sqrt{y+4} - \sqrt{x+1}$$

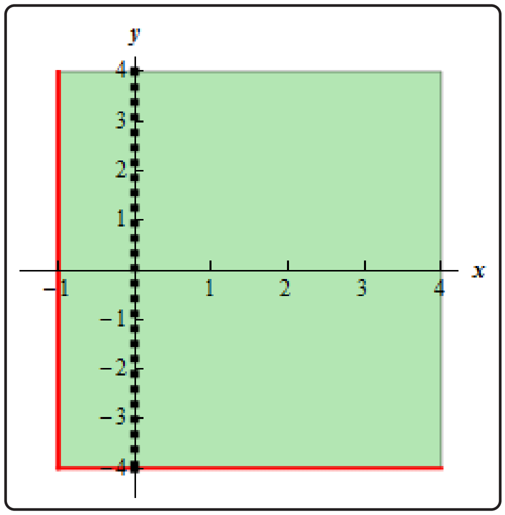
Solution

There really isn't all that much to this problem. We know that we can't have division by zero and we can't take square roots of negative numbers and so we'll need to require that whatever (x,y) is it will need to satisfy the following three conditions.

$$x \geq -1 \qquad x \neq 0 \qquad y \geq -4$$

This is also our domain since these are the only conditions require in order for the function to exist.

A sketch of the domain is shown below. We can take any point in the green area or on the red lines with the exception of the y-axis (*i.e.* $x \neq 0$) as indicated by the black dashes on the y-axis.



5. Identify and sketch the level curves (or contours) for the following function.

$$2x - 3y + z^2 = 1$$

Step 1

We know that level curves or contours are given by setting $z = k$. Doing this in our equation gives,

$$2x - 3y + k^2 = 1$$

Step 2

A quick rewrite of the equation from the previous step gives us,

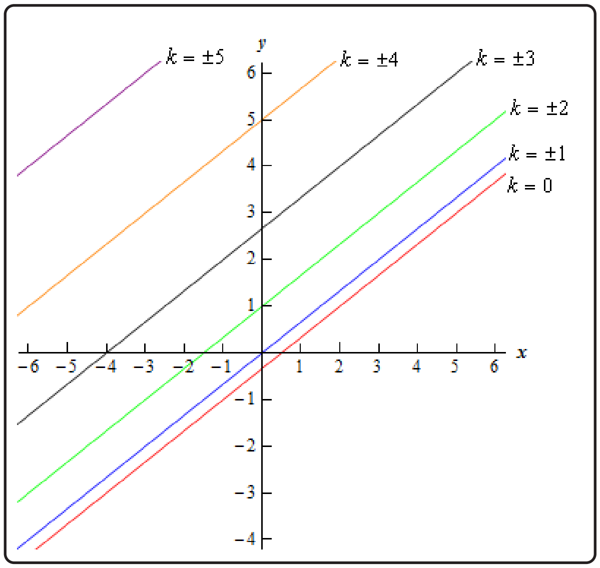
$$y = \frac{2}{3}x + \frac{k^2 - 1}{3}$$

So, the level curves for this function will be lines with slope $\frac{2}{3}$ and a y-intercept of $\left(0, \frac{k^2-1}{3}\right)$.

Note as well that there will be no restrictions on the values of k that we can use, as there sometimes are. Also note that the sign of k will not matter so, with the exception of the level curve for $k = 0$, each level curve will in fact arise from two different values of k.

Step 3

Below is a sketch of some level curves for some values of k for this function.



6. Identify and sketch the level curves (or contours) for the following function.

$$4z + 2y^2 - x = 0$$

Step 1

We know that level curves or contours are given by setting $z = k$. Doing this in our equation gives,

$$4k + 2y^2 - x = 0$$

Step 2

A quick rewrite of the equation from the previous step gives us,

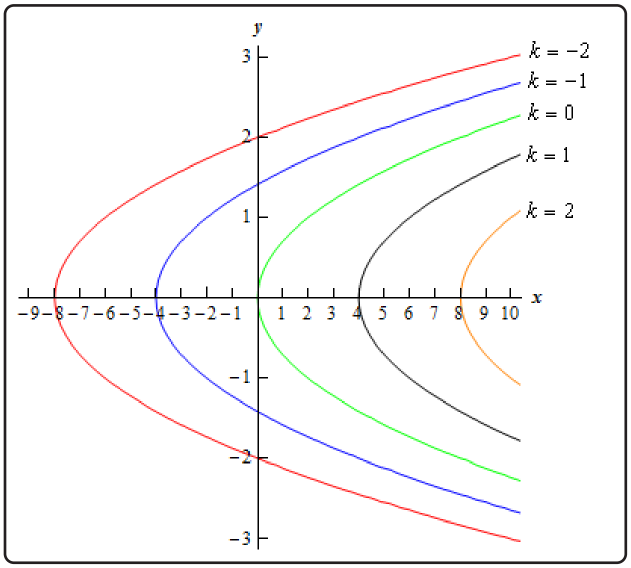
$$x = 2y^2 + 4k$$

So, the level curves for this function will be parabolas opening to the right and starting at 4 k.

Note as well that there will be no restrictions on the values of k that we can use, as there sometimes are.

Step 3

Below is a sketch of some level curves for some values of k for this function.



7. Identify and sketch the level curves (or contours) for the following function.

$$y^2 = 2x^2 + z$$

Step 1

We know that level curves or contours are given by setting $z = k$. Doing this in our equation gives,

$$y^2 = 2x^2 + k$$

Step 2

For this problem the value of k will affect the type of graph of the level curve.

First, if $k = 0$ the equation will be,

$$y^2 = 2x^2 \qquad \Rightarrow \qquad y = \pm\sqrt{2}\, x$$

So, in this case the level curve (actually curves if you think about it) will be two lines through the origin. One is increasing and the other is decreasing.

Next, let's take a look at what we get if $k > 0$. In this case a quick rewrite of the equation from Step 2 gives,

$$\frac{y^2}{k} - \frac{2x^2}{k} = 1$$

Because we know that k is positive we see that we have a hyperbola with the y term the positive term and the x term the negative term. This means that the hyperbola will be symmetric about the y-axis and opens up and down.

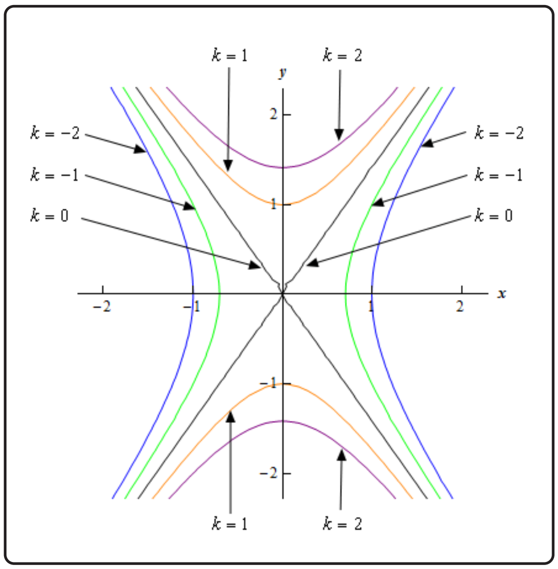
Finally, what do we get if $k < 0$. In this case the equation is,

$$-\frac{2x^2}{k} + \frac{y^2}{k} = 1$$

Now, be careful with this equation. In this case we have negative values of k. This means that the x term is in fact positive (the minus sign will cancel against the minus sign in the k). Likewise, the y term will be negative (it's got a negative k in the denominator). Therefore, we'll have a hyperbola that is symmetric about the x-axis and opens right and left.

Step 3

Below is a sketch of some level curves for some values of k for this function.



8. Identify and sketch the traces for the following function.

$$2x - 3y + z^2 = 1$$

Step 1

We have two traces. One we get by plugging $x = a$ into the equation and the other we get by plugging $y = b$ into the equation. Here is what we get for each of these.

$$x = a : \quad 2a - 3y + z^2 = 1 \quad \rightarrow \quad y = \frac{1}{3}z^2 + \frac{2a-1}{3}$$

$$y = b : \quad 2x - 3b + z^2 = 1 \quad \rightarrow \quad x = -\frac{1}{2}z^2 + \frac{3b+1}{2}$$

Step 2

Okay, we're now into a realm that many students have issues with initially. We no longer have equations in terms of x and y. Instead we have one equation in terms of x and z and another in terms of y and z.

Do not get excited about this! They work the same way that equations in terms of x and y work! The only difference is that we need to make a decision on which variable will be the horizontal axis variable and which variable will be the vertical axis variable.

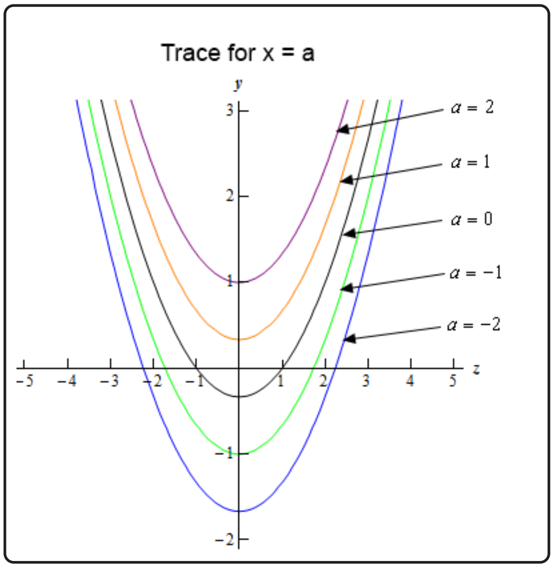
Just because we have an x doesn't mean that it must be the horizontal axis and just because we have a y doesn't mean that it must be the vertical axis! We set up the axis variables in a way that will be convenient for us.

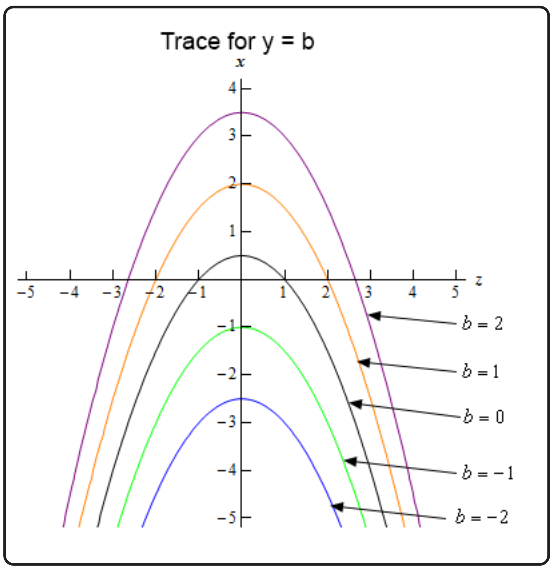
In this case since both equation have a z in them and it is squared we'll let z be the horizontal axis variable for both of the equations.

So, given that convention for the axis variables this means that for the $x = a$ trace we'll have a parabola that opens upwards with vertex at $\left(0, \frac{2a-1}{3}\right)$ and for the $y = b$ trace we'll have a parabola that opens downwards with vertex at $\left(0, \frac{3b+1}{2}\right)$.

Step 3

Below is a sketch for each of the traces.





9. Identify and sketch the traces for the following function.

$$4z + 2y^2 - x = 0$$

Step 1

We have two traces. One we get by plugging $x = a$ into the equation and the other we get by plugging $y = b$ into the equation. Here is what we get for each of these.

$$\begin{aligned} x = a: &\qquad 4z + 2y^2 - a = 0 &\qquad \to &\qquad z = -\frac{1}{2}y^2 + \frac{a}{4} \\ y = b: &\qquad 4z + 2b^2 - x = 0 &\qquad \to &\qquad x = 4z + 2b^2 \end{aligned}$$

Step 2

Okay, we're now into a realm that many students have issues with initially. We no longer have equations in terms of x and y. Instead we have one equation in terms of x and z and another in terms of y and z.

Do not get excited about this! They work the same way that equations in terms of x and y work! The only difference is that we need to make a decision on which variable will be the horizontal axis variable and which variable will be the vertical axis variable.

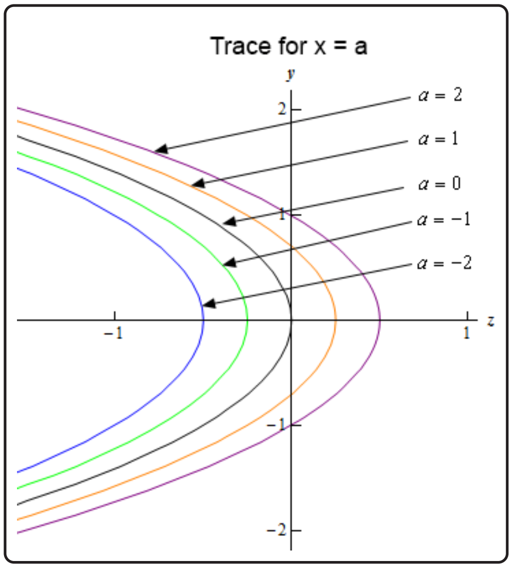
Just because we have an x doesn't mean that it must be the horizontal axis and just because we have a y doesn't mean that it must be the vertical axis! We set up the axis variables in a way that will be convenient for us.

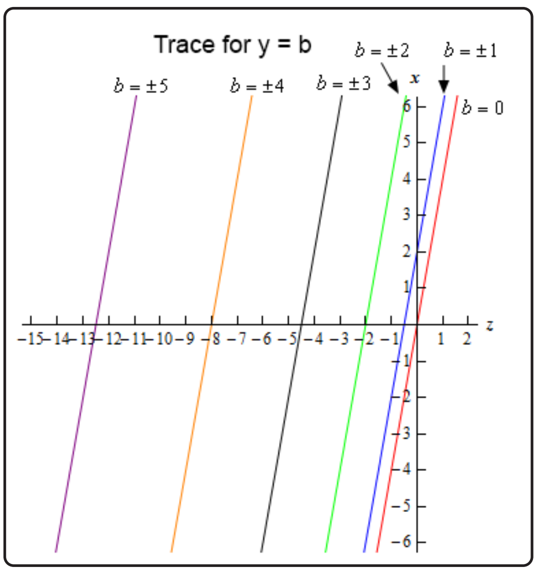
In this case since both equation have a z in them we'll let z be the horizontal axis variable for both of the equations.

So, given that convention for the axis variables this means that for the $x = a$ trace we'll have a parabola that opens to the left with vertex at $\left(\frac{a}{4}, 0\right)$ and for the $y = b$ trace we'll have a line with slope of 4 and an x-intercept at $\left(0, 2b^2\right)$.

Step 3

Below is a sketch for each of the traces.


Trace for x = a
y
z
a = 2
a = 1
a = 0
a = −1
a = −2


Trace for y = b
b = ±5
b = ±4
b = ±3
b = ±2
b = ±1
b = 0
x
z

12.6 Vector Functions

1. Find the domain for the vector function : $\vec{r}(t) = \left\langle t^2 + 1, \frac{1}{t+2}, \sqrt{t+4} \right\rangle$

Step 1

The domain of the vector function is simply the largest possible set of t's that we can use in all the components of the vector function.

The first component will exist for all values of t and so we won't exclude any values of t from that component.

The second component clearly requires us to avoid $t = -2$ so we don't have division by zero in that component.

We'll also need to require that $t \geq -4$ so avoid taking the square root of negative numbers in the third component.

Step 2

Putting all of the information from the first step together we can see that the domain of this function is,

$$\boxed{t \geq -4, \quad t \neq -2}$$

Note that we can't forget to add the $t \neq -2$ onto this since -2 is larger than -4 and would be included otherwise!

2. Find the domain for the vector function : $\vec{r}(t) = \left\langle \ln\left(4 - t^2\right), \sqrt{t+1} \right\rangle$

Step 1

The domain of the vector function is simply the largest possible set of t's that we can use in all the components of the vector function.

We know that we can't take logarithms of negative values or zero and so from the first term we know that we'll need to require that,

$$4 - t^2 > 0 \qquad \to \qquad -2 < t < 2$$

We'll also need to require that $t \geq -1$ so avoid taking the square root of negative numbers in the second component.

Step 2

Putting all of the information from the first step together we can see that the domain of this function is,

$$\boxed{-1 \le t < 2}$$

Remember that we want the largest possible set of t's for which all the components will exist. So we can't take values of $-2 < t < -1$ because even though those are okay in the first component but they aren't in the second component. Likewise, even though we can include $t \ge 2$ in the second component we can't plug them into the first component and so we can't include them in the domain of the function.

3. Sketch the graph of the vector function : $\vec{r}\,(t) = \langle 4t, 10 - 2t \rangle$

Step 1

One way to sketch the graph of vector functions of course is to just compute a bunch of vectors and then recall that we consider them to be position vectors and plot the "points" we get out of them.

This will work provided we pick the "correct" values of t that gives us good points that we can use to actually determine what the graph is.

So, to avoid doing that, recall that because we consider these to be position vectors we can write down a corresponding set of parametric equations that we can use to sketch the graph. The parametric equations for this vector function are,

$$\begin{aligned} x &= 4t \\ y &= 10 - 2t \end{aligned}$$

Step 2

Now, recall from when we looked at parametric equations we eliminated the parameter from the parametric equations to get an equation involving only x and y that will have the same graph as the vector function.

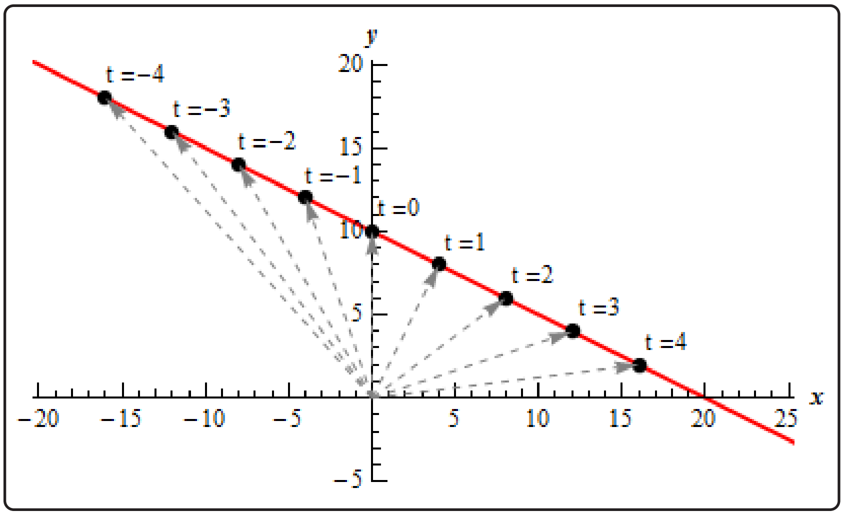
We can do this as follows,

$$x = 4t \qquad \to \qquad t = \frac{1}{4}x \qquad \to \qquad y = 10 - 2\left(\frac{1}{4}x\right) = 10 - \frac{1}{2}x$$

So, it looks like the graph of the vector function will be a line with slope $-\frac{1}{2}$ and y-intercept of $(0, 10)$.

Step 3

A sketch of the graph is below.



For illustration purposes we also put in a set of vectors for variety of t's just to show that with enough them we would have also gotten the graph. Of course, it was easier to eliminate the parameter and just graph the algebraic equations (*i.e.* the equation involving only x and y).

---

4. Sketch the graph of the vector function : $\vec{r}(t) = \left\langle t+1, \frac{1}{4}t^2+3 \right\rangle$

Step 1

One way to sketch the graph of vector functions of course is to just compute a bunch of vectors and then recall that we consider them to be position vectors and plot the "points" we get out of them.

This will work provided we pick the "correct" values of t that gives us good points that we can use to actually determine what the graph is.

So, to avoid doing that, recall that because we consider these to be position vectors we can write down a corresponding set of parametric equations that we can use to sketch the graph. The parametric equations for this vector function are,

$$x = t + 1$$
$$y = \frac{1}{4}t^2 + 3$$

Step 2

Now, recall from when we looked at parametric equations we eliminated the parameter from the parametric equations to get an equation involving only x and y that will have the same graph as the vector function.

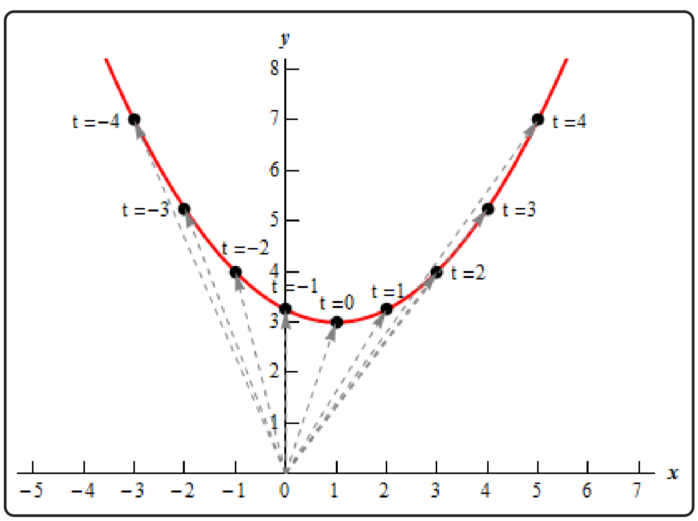
We can do this as follows,

$$x = t + 1 \qquad \to \qquad t = x - 1 \qquad \to \qquad y = \frac{1}{4}(x-1)^2 + 3$$

So, it looks like the graph of the vector function will be a parabola with vertex $(1,3)$ and opening upwards.

Step 3

A sketch of the graph is below.



For illustration purposes we also put in a set of vectors for variety of t's just to show that with enough them we would have also gotten the graph. Of course, it was easier to eliminate the parameter and just graph the algebraic equations (*i.e.* the equation involving only x and y).

5. Sketch the graph of the vector function : $\vec{r}(t) = \langle 4\sin(t), 8\cos(t)\rangle$

Step 1

One way to sketch the graph of vector functions of course is to just compute a bunch of vectors and then recall that we consider them to be position vectors and plot the "points" we get out of them.

This will work provided we pick the "correct" values of t that gives us good points that we can use to actually determine what the graph is.

So, to avoid doing that, recall that because we consider these to be position vectors we can write down a corresponding set of parametric equations that we can use to sketch the graph. The parametric equations for this vector function are,

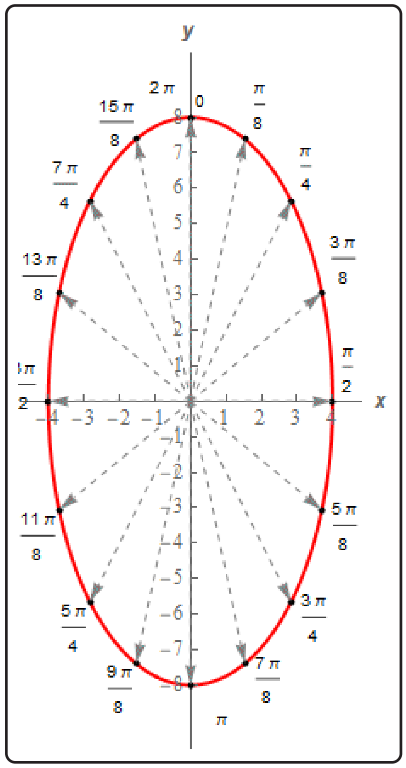
$$x = 4\sin(t)$$
$$y = 8\cos(t)$$

Step 2

Now, recall from our look at parametric equations we now know that this will be the graph of an ellipse centered at the origin with right/left points a distance of 4 from the origin and top/bottom points a distance of 8 from the origin.

Step 3

A sketch of the graph is below.



For illustration purposes we also put in a set of vectors for variety of t's just to show that with enough them we would have also gotten the graph. Of course, it was easier to eliminate the parameter and just graph the algebraic equations (*i.e.* the equation involving only x and y).

---

6. Identify the graph of the vector function without sketching the graph.

$$\vec{r}(t) = \langle 3\cos(6t), -4, \sin(6t)\rangle$$

Step 1

To identify the graph of this vector function (without graphing) let's first write down the set of parametric equations we get from this vector function. They are,

$$x = 3\cos(6t)$$
$$y = -4$$
$$z = \sin(6t)$$

Step 2

Now, from the x and z equations we can see that we have an ellipse in the xz-plane that is given by the following equation.

$$\frac{x^2}{9} + z^2 = 1$$

From the y equation we know that this ellipse will not actually be in the xz-plane but parallel to the xz-plane at $y = -4$.

---

7. Identify the graph of the vector function without sketching the graph.

$$\vec{r}(t) = \langle 2 - t, 4 + 7t, -1 - 3t \rangle$$

Solution

There really isn't a lot to do with this problem. The equation should look very familiar to you. We saw quite a few of these types of equations in the Equations of Lines and Equations of Planes sections.

From those sections we know that the graph of this equation is a line in $\mathbb{R}^3$ that goes through the point $(2, 4, -1)$ and parallel to the vector $\vec{v} = \langle -1, 7, -3 \rangle$.

---

8. Write down the equation of the line segment starting at $(1, 3)$ and ending at $(-4, 6)$.

Solution

There really isn't a lot to do with this problem. All we need to do is use the formula we derived in the notes for this section.

The line segment is,

$$\boxed{\vec{r}(t) = (1 - t)\langle 1, 3 \rangle + t\langle -4, 6 \rangle \quad 0 \le t \le 1}$$

Don't forget the limits on t! Without that you have the full line that goes through those two points instead of the line segment from $(1, 3)$ to $(-4, 6)$.

---

9. Write down the equation of the line segment starting at $(0,2,-1)$ and ending at $(7,-9,2)$.

Solution

There really isn't a lot to do with this problem. All we need to do is use the formula we derived in the notes for this section.

The line segment is,

$$\boxed{\vec{r}(t) = (1-t)\langle 0,2,-1\rangle + t\langle 7,-9,2\rangle \quad 0 \le t \le 1}$$

Don't forget the limits on t! Without that you have the full line that goes through those two points instead of the line segment from $(0,2,-1)$ to $(7,-9,2)$.

12.7 Calculus with Vector Functions

1. Evaluate the following limit.

$$\lim_{t\to 1}\left\langle \mathbf{e}^{t-1}, 4t, \frac{t-1}{t^2-1}\right\rangle$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the limit of all the components of the vector.

$$\begin{aligned}\lim_{t\to 1}\left\langle \mathbf{e}^{t-1}, 4t, \frac{t-1}{t^2-1}\right\rangle &= \left\langle \lim_{t\to 1}\mathbf{e}^{t-1}, \lim_{t\to 1}4t, \lim_{t\to 1}\frac{t-1}{t^2-1}\right\rangle \\ &= \left\langle \lim_{t\to 1}\mathbf{e}^{t-1}, \lim_{t\to 1}4t, \lim_{t\to 1}\frac{1}{2t}\right\rangle = \left\langle \mathbf{e}^{0}, 4, \frac{1}{2}\right\rangle = \boxed{\left\langle 1, 4, \frac{1}{2}\right\rangle}\end{aligned}$$

Don't forget L'Hospital's Rule! We needed that for the third term.

2. Evaluate the following limit.

$$\lim_{t\to -2}\left(\frac{1-\mathbf{e}^{t+2}}{t^2+t-2}\vec{i}+\vec{j}+\left(t^2+6t\right)\vec{k}\right)$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the limit of all the components of the vector.

$$\begin{aligned}\lim_{t\to -2}\left(\frac{1-\mathbf{e}^{t+2}}{t^2+t-2}\vec{i}+\vec{j}+\left(t^2+6t\right)\vec{k}\right) &= \lim_{t\to -2}\frac{1-\mathbf{e}^{t+2}}{t^2+t-2}\vec{i}+\lim_{t\to -2}\vec{j}+\lim_{t\to -2}\left(t^2+6t\right)\vec{k} \\ &= \lim_{t\to -2}\frac{-\mathbf{e}^{t+2}}{2t+1}\vec{i}+\lim_{t\to -2}\vec{j}+\lim_{t\to -2}\left(t^2+6t\right)\vec{k} \\ &= \boxed{\frac{1}{3}\vec{i}+\vec{j}-8\vec{k}}\end{aligned}$$

Don't forget L'Hospital's Rule! We needed that for the first term.

3. Evaluate the following limit.

$$\lim_{t\to\infty}\left\langle \frac{1}{t^2}, \frac{2t^2}{1-t-t^2}, \mathbf{e}^{-t}\right\rangle$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the limit of all the components of the vector.

$$\begin{aligned}\lim_{t\to\infty}\left\langle \frac{1}{t^2}, \frac{2t^2}{1-t-t^2}, \mathbf{e}^{-t}\right\rangle &= \left\langle \lim_{t\to\infty}\frac{1}{t^2}, \lim_{t\to\infty}\frac{2t^2}{1-t-t^2}, \lim_{t\to\infty}\mathbf{e}^{-t}\right\rangle \\ &= \left\langle \lim_{t\to\infty}\frac{1}{t^2}, \lim_{t\to\infty}\frac{2t^2}{t^2\left(\frac{1}{t^2}-\frac{1}{t}-1\right)}, \lim_{t\to\infty}\mathbf{e}^{-t}\right\rangle = \boxed{\langle 0, -2, 0\rangle}\end{aligned}$$

Don't forget your basic limit at infinity processes/facts.

---

4. Compute the derivative of the following limit.

$$\vec{r}(t) = \left(t^3-1\right)\vec{i} + \mathbf{e}^{2t}\,\vec{j} + \cos(t)\,\vec{k}$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the derivative of all the components of the vector.

$$\boxed{\vec{r}\,'(t) = 3t^2\vec{i} + 2\mathbf{e}^{2t}\,\vec{j} - \sin(t)\,\vec{k}}$$

---

5. Compute the derivative of the following limit.

$$\vec{r}(t) = \left\langle \ln\left(t^2+1\right), t\mathbf{e}^{-t}, 4\right\rangle$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the derivative of all the components of the vector.

$$\boxed{\vec{r}\,'(t) = \left\langle \frac{2t}{t^2+1}, \mathbf{e}^{-t} - t\mathbf{e}^{-t}, 0\right\rangle}$$

Make sure you haven't forgotten your basic differentiation formulas such as the chain rule (the first term) and the product rule (the second term).

6. Compute the derivative of the following limit.

$$\vec{r}(t) = \left\langle \frac{t+1}{t-1}, \tan(4t), \sin^2(t) \right\rangle$$

Solution

There really isn't a lot to do here with this problem. All we need to do is take the derivative of all the components of the vector.

$$\begin{aligned} \vec{r}'(t) &= \left\langle \frac{(1)(t-1)-(t+1)(1)}{(t-1)^2}, 4sec^2(4t), 2\sin(t)\cos(t) \right\rangle \\ &= \boxed{\left\langle \frac{-2}{(t-1)^2}, 4\sec^2(4t), 2\sin(t)\cos(t) \right\rangle} \end{aligned}$$

Make sure you haven't forgotten your basic differentiation formulas such as the quotient rule (the first term) and the chain rule (the third term).

7. Evaluate $\int \vec{r}(t)\,dt$, where $\vec{r}(t) = t^3\,\vec{i} - \frac{2t}{t^2+1}\vec{j} + \cos^2(3t)\,\vec{k}$.

Solution

There really isn't a lot to do here with this problem. All we need to do is integrate of all the components of the vector.

$$\begin{aligned} \int \vec{r}(t)\,dt &= \int t^3\,dt\,\vec{i} - \int \frac{2t}{t^2+1}\,dt\,\vec{j} + \int \cos^2(3t)\,dt\,\vec{k} \\ &= \int t^3\,dt\,\vec{i} - \int \frac{2t}{t^2+1}\,dt\,\vec{j} + \int \frac{1}{2}(1+\cos(6t))\,dt\,\vec{k} \\ &= \boxed{\frac{1}{4}t^4\,\vec{i} - \ln\left|t^2+1\right|\,\vec{j} + \frac{1}{2}\left(t + \frac{1}{6}\sin(6t)\right)\vec{k} + \vec{c}} \end{aligned}$$

Don't forget the basic integration rules such as the substitution rule (second term) and some of the basic trig formulas (half angle and double angle formulas) you need to do some of the integrals (third term).

We didn't put a lot of the integration details into the solution on the assumption that you do know your integration skills well enough to follow what is going on. If you are somewhat rusty with your integration skills then you'll need to go back to the integration material from both Calculus I and Calculus II to refresh your integration skills.

8. Evaluate $\displaystyle\int_{-1}^{2} \vec{r}(t)\,dt$ where $\vec{r}(t) = \langle 6, 6t^2 - 4t, t\mathbf{e}^{2t}\rangle$

Solution

There really isn't a lot to do here with this problem. All we need to do is integrate of all the components of the vector.

$$\begin{aligned}\int \vec{r}(t)\,dt &= \left\langle \int 6\,dt, \int 6t^2 - 4t\,dt, \int t\mathbf{e}^{2t}\,dt \right\rangle \\ &= \left\langle \int 6\,dt, \int 6t^2 - 4t\,dt, \frac{1}{2}t\mathbf{e}^{2t} - \frac{1}{2}\int \mathbf{e}^{2t}\,dt \right\rangle = \left\langle 6t, 2t^3 - 2t^2, \frac{1}{2}t\mathbf{e}^{2t} - \frac{1}{4}\mathbf{e}^{2t} \right\rangle\end{aligned}$$

Don't forget the basic integration rules such integration by parts (third term). Also recall that one way to do definite integral is to first do the indefinite integral and then do the evaluation.

The answer for this problem is then,

$$\begin{aligned}\int_{-1}^{2} \vec{r}(t)\,dt &= \left\langle 6t, 2t^3 - 2t^2, \frac{1}{2}t\mathbf{e}^{2t} - \frac{1}{4}\mathbf{e}^{2t} \right\rangle \Bigg|_{-1}^{2} \\ &= \left\langle 12, 8, \frac{3}{4}\mathbf{e}^{4} \right\rangle - \left\langle -6, -4, -\frac{3}{4}\mathbf{e}^{-2} \right\rangle = \boxed{\left\langle 18, 12, \frac{3}{4}\left(\mathbf{e}^{4} + \mathbf{e}^{-2}\right) \right\rangle}\end{aligned}$$

We didn't put a lot of the integration details into the solution on the assumption that you do know your integration skills well enough to follow what is going on. If you are somewhat rusty with your integration skills then you'll need to go back to the integration material from both Calculus I and Calculus II to refresh your integration skills.

9. Evaluate $\int \vec{r}(t)\,dt$, where $\vec{r}(t) = \left\langle (1-t)\cos\left(t^2-2t\right), \cos(t)\sin(t), \sec^2(4t) \right\rangle$.

Solution

There really isn't a lot to do here with this problem. All we need to do is integrate of all the components of the vector.

$$\begin{aligned}\int \vec{r}(t)\,dt &= \left\langle \int (1-t)\cos\left(t^2-2t\right)\,dt, \int \cos(t)\sin(t)\,dt, \int \sec^2(4t)\,dt \right\rangle \\ &= \left\langle \int (1-t)\cos\left(t^2-2t\right)\,dt, \int \frac{1}{2}\sin(2t)\,dt, \int \sec^2(4t)\,dt \right\rangle \\ &= \boxed{\left\langle -\frac{1}{2}\sin\left(t^2-2t\right), -\frac{1}{4}\cos(2t), \frac{1}{4}\tan(4t) \right\rangle + \vec{c}}\end{aligned}$$

Don't forget the basic integration rules such as the substitution rule (all terms) and some of the basic trig formulas (half angle and double angle formulas) you need to do some of the integrals (second term).

We didn't put a lot of the integration details into the solution on the assumption that you do know your integration skills well enough to follow what is going on. If you are somewhat rusty with your integration skills then you'll need to go back to the integration material from both Calculus I and Calculus II to refresh your integration skills.

12.8 Tangent, Normal and Binormal Vectors

1. Find the unit tangent vector for the vector function : $\vec{r}(t) = \left\langle t^2+1, 3-t, t^3 \right\rangle$

Step 1

From the notes in this section we know that to get the unit tangent vector all we need is the derivative of the vector function and its magnitude. Here are those quantities.

$$\vec{r}\,'(t) = \left\langle 2t, -1, 3t^2 \right\rangle$$

$$\left\|\vec{r}\,'(t)\right\| = \sqrt{(2t)^2 + (-1)^2 + (3t^2)^2} = \sqrt{1+4t^2+9t^4}$$

Step 2

The unit tangent vector for this vector function is then,

$$\begin{aligned}\vec{T}(t) &= \frac{1}{\sqrt{1+4t^2+9t^4}}\left\langle 2t, -1, 3t^2 \right\rangle \\ &= \boxed{\left\langle \frac{2t}{\sqrt{1+4t^2+9t^4}}, -\frac{1}{\sqrt{1+4t^2+9t^4}}, \frac{3t^2}{\sqrt{1+4t^2+9t^4}} \right\rangle}\end{aligned}$$

2. Find the unit tangent vector for the vector function : $\vec{r}(t) = t\mathbf{e}^{2t}\,\vec{i} + \left(2-t^2\right)\vec{j} - \mathbf{e}^{2t}\vec{k}$

Step 1

From the notes in this section we know that to get the unit tangent vector all we need is the derivative of the vector function and its magnitude. Here are those quantities.

$$\vec{r}\,'(t) = \left(\mathbf{e}^{2t} + 2t\mathbf{e}^{2t}\right)\vec{i} - 2t\,\vec{j} - 2\mathbf{e}^{2t}\vec{k}$$

$$\left\|\vec{r}\,'(t)\right\| = \sqrt{\left(\mathbf{e}^{2t}+2t\mathbf{e}^{2t}\right)^2 + (-2t)^2 + \left(-2\mathbf{e}^{2t}\right)^2} = \sqrt{\left(\mathbf{e}^{2t}+2t\mathbf{e}^{2t}\right)^2 + 4t^2 + 4\mathbf{e}^{4t}}$$

Step 2

The unit tangent vector for this vector function is then,

$$\boxed{\vec{T}(t) = \frac{1}{\sqrt{\left(\mathbf{e}^{2t} + 2t\mathbf{e}^{2t}\right)^2 + 4t^2 + 4\mathbf{e}^{4t}}}\left(\left(\mathbf{e}^{2t} + 2t\mathbf{e}^{2t}\right)\vec{i} - 2t\,\vec{j} - 2\mathbf{e}^{2t}\vec{k}\right)}$$

Note that because of the "mess" with this one we didn't distribute the magnitude through to each term as we usually do with these. This problem is a good example of just how "messy" these can get.

---

3. Find the tangent line to $\vec{r}(t) = \cos(4t)\,\vec{i} + 3\sin(4t)\,\vec{j} + t^3\vec{k}$ at $t = \pi$.

Step 1

First, we'll need to get the tangent vector to the vector function. The tangent vector is,

$$\vec{r}\,'(t) = -4\sin(4t)\,\vec{i} + 12\cos(4t)\,\vec{j} + 3t^2\vec{k}$$

Note that we could use the unit tangent vector here if we wanted to but given how messy those tend to be we'll just go with this.

Step 2

Now we actually need the tangent vector at the value of t given in the problem statement and not the "full" tangent vector. We'll also need the point on the vector function at the value of t from the problem statement. These are,

$$\vec{r}(\pi) = \cos(4\pi)\,\vec{i} + 3\sin(4\pi)\,\vec{j} + \pi^3\vec{k} = \vec{i} + \pi^3\vec{k}$$
$$\vec{r}\,'(\pi) = -4\sin(4\pi)\,\vec{i} + 12\cos(4\pi)\,\vec{j} + 3\pi^2\vec{k} = 12\vec{j} + 3\pi^2\vec{k}$$

Step 3

To write down the equation of the tangent line we need a point on the line and a vector parallel to the line. Of course, these are just the quantities we found in the previous step.

The tangent line is then,

$$\boxed{\vec r(t) = \vec i + \pi^3 \vec k + t\left(12\vec j + 3\pi^2 \vec k\right) = \vec i + 12t\,\vec j + \left(\pi^3 + 3\pi^2 t\right)\vec k}$$

---

4. Find the tangent line to $\vec r(t) = \left\langle 7\mathbf{e}^{2-t}, \frac{16}{t^3}, 5-t \right\rangle$ at $t = 2$.

Step 1

First, we'll need to get the tangent vector to the vector function. The tangent vector is,

$$\vec r\,'(t) = \left\langle -7\mathbf{e}^{2-t}, -\frac{48}{t^4}, -1 \right\rangle$$

Note that we could use the unit tangent vector here if we wanted to but given how messy those tend to be we'll just go with this.

Step 2

Now we actually need the tangent vector at the value of t given in the problem statement and not the "full" tangent vector. We'll also need the point on the vector function at the value of t from the problem statement. These are,

$$\begin{aligned}\vec r(2) &= \langle 7, 2, 3\rangle \\ \vec r\,'(2) &= \langle -7, -3, -1\rangle\end{aligned}$$

Step 3

To write down the equation of the tangent line we need a point on the line and a vector parallel to the line. Of course, these are just the quantities we found in the previous step.

The tangent line is then,

$$\boxed{\vec r(t) = \langle 7, 2, 3\rangle + t\langle -7, -3, -1\rangle = \langle 7 - 7t, 2 - 3t, 3 - t\rangle}$$

5. Find the unit normal and the binormal vectors for the following vector function.

$$\vec{r}(t) = \langle \cos(2t), \sin(2t), 3 \rangle$$

Step 1

We first need the unit tangent vector so let's get that.

$$\vec{r}\,'(t) = \langle -2\sin(2t), 2cos(2t), 0 \rangle \qquad \|\vec{r}\,'(t)\| = \sqrt{4\sin^2(2t) + 4\cos^2(2t)} = 2$$
$$\vec{T}(t) = \tfrac{1}{2}\langle -2\sin(2t), 2cos(2t), 0 \rangle = \langle -\sin(2t), cos(2t), 0 \rangle$$

Step 2

The unit normal vector is then,

$$\vec{T}\,'(t) = \langle -2\cos(2t), -2\sin(2t), 0 \rangle \qquad \left\|\vec{T}\,'(t)\right\| = \sqrt{4\cos^2(2t) + 4\sin^2(2t)} = 2$$
$$\boxed{\vec{N}(t) = \frac{1}{2}\langle -2\cos(2t), -2\sin(2t), 0 \rangle = \langle -\cos(2t), -\sin(2t), 0 \rangle}$$

Step 3

Finally, the binormal vector is,

$$\begin{aligned} \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(2t) & \cos(2t) & 0 \\ -\cos(2t) & -\sin(2t) & 0 \end{vmatrix} \\ &= \sin^2(2t)\,\vec{k} - \left(-\cos^2(2t)\,\vec{k}\right) \\ &= \left(\sin^2(2t) + \cos^2(2t)\right)\vec{k} \\ &= \boxed{\vec{k} = \langle 0, 0, 1 \rangle = \vec{B}(t)} \end{aligned}$$

12.9 Arc Length with Vector Functions

1. Determine the length of $\vec{r}(t) = (3-4t)\,\vec{i} + 6t\,\vec{j} - (9+2t)\,\vec{k}$ from $-6 \le t \le 8$.

Step 1

We first need the magnitude of the derivative of the vector function. This is,

$$\vec{r}'(t) = -4\vec{i} + 6\,\vec{j} - 2\vec{k}$$

$$\left\|\vec{r}'(t)\right\| = \sqrt{16+36+4} = \sqrt{56} = 2\sqrt{14}$$

Step 2

The length of the curve is then,

$$L = \int_{-6}^{8} 2\sqrt{14}\,dt = 2\sqrt{14}t\Big|_{-6}^{8} = \boxed{28\sqrt{14}}$$

2. Determine the length of $\vec{r}(t) = \left\langle \frac{1}{3}t^3, 4t, \sqrt{2}t^2 \right\rangle$ from $0 \le t \le 2$.

Step 1

We first need the magnitude of the derivative of the vector function. This is,

$$\vec{r}'(t) = \left\langle t^2, 4, 2\sqrt{2}t \right\rangle$$

$$\left\|\vec{r}'(t)\right\| = \sqrt{t^4+16+8t^2} = \sqrt{t^4+8t^2+16} = \sqrt{\left(t^2+4\right)^2} = t^2+4$$

For these kinds of problems make sure to simplify the magnitude as much as you can. It can mean the difference between a really simple problem and an incredibly difficult problem.

Step 2

The length of the curve is then,

$$L = \int_0^2 t^2 + 4\,dt = \left(\frac{1}{3}t^3 + 4t\right)\Big|_0^2 = \boxed{\frac{32}{3}}$$

Note that if we'd not simplified the magnitude this would have been a very difficult integral to compute!

---

3. Find the arc length function for $\vec{r}(t) = \langle t^2, 2t^3, 1 - t^3 \rangle$.

Step 1

We first need the magnitude of the derivative of the vector function. This is,

$$\vec{r}\,'(t) = \langle 2t, 6t^2, -3t^2 \rangle$$

$$\begin{aligned}\left\|\vec{r}\,'(t)\right\| &= \sqrt{4t^2 + 36t^4 + 9t^4} = \sqrt{t^2\left(4 + 45t^2\right)} = \sqrt{t^2}\sqrt{4 + 45t^2} \\ &= |t|\sqrt{4 + 45t^2} = t\sqrt{4 + 45t^2}\end{aligned}$$

For these kinds of problems make sure to simplify the magnitude as much as you can. It can mean the difference between a really simple problem and an incredibly difficult problem.

Note as well that because we are assuming that we are starting at $t = 0$ for this kind of problem it is safe to assume that $t \geq 0$ and so $\sqrt{t^2} = |t| = t$.

Step 2

The arc length function is then,

$$s(t) = \int_0^t u\sqrt{4 + 45u^2}\,du = \frac{1}{135}\left(4 + 45u^2\right)^{\frac{3}{2}}\Big|_0^t = \boxed{\frac{1}{135}\left[\left(4 + 45t^2\right)^{\frac{3}{2}} - 8\right]}$$

---

4. Find the arc length function for $\vec{r}(t) = \left\langle 4t, -2t, \sqrt{5}\, t^2 \right\rangle$.

Step 1

We first need the magnitude of the derivative of the vector function. This is,

$$\vec{r}'(t) = \left\langle 4, -2, 2\sqrt{5}\, t \right\rangle$$

$$\left\| \vec{r}'(t) \right\| = \sqrt{16 + 4 + 20t^2} = \sqrt{20 + 20t^2} = \sqrt{20}\sqrt{1+t^2} = 2\sqrt{5}\sqrt{1+t^2}$$

Step 2

The arc length function is then,

$$s(t) = \int_0^t 2\sqrt{5}\sqrt{1+u^2}\, du$$

Do not always expect these integrals to be "simple" integrals. They will often require techniques more involved than just a standard Calculus I substitution. In this case we need the following trig substitution.

$$u = \tan(\theta) \qquad du = \sec^2(\theta)\, d\theta \qquad \sqrt{1+u^2} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)} = |\sec(\theta)|$$

The limits of the integral become,

$$u = 0 : 0 = \tan(\theta) \quad \to \quad \theta = 0 \qquad u = t > 0 : t = \tan(\theta) \quad \to \quad \theta = \tan^{-1}(t)$$

Now, as noted we know that $t > 0$ and so we can safely assume that from the $u = t$ limit we will get $0 < \theta < \frac{\pi}{2}$. This in turn means that we will always be in the first quadrant and we know that secant is positive in the first quadrant. Therefore, we can remove the absolute values bars on the secant above.

The arc length function is now,

$$\begin{aligned} s(t) &= \int_0^{\tan^{-1}(t)} 2\sqrt{5}\sec^3(\theta)\, d\theta = \sqrt{5}(\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|)\Big|_0^{\tan^{-1}(t)} \\ &= \sqrt{5}\left[\sec\left(\tan^{-1}(t)\right)\tan\left(\tan^{-1}(t)\right) + \ln\left|\sec\left(\tan^{-1}(t)\right) + \tan\left(\tan^{-1}(t)\right)\right|\right] \end{aligned}$$

Now we know that $\tan\left(\tan^{-1}(t)\right) = t$ so that will simplify our answer a little. Let's take a look at the secant term to see if we can simplify that as well. First, from our limit work recall that $\theta = \tan^{-1}(t)$. Or with a little rewrite we have,

$$\tan(\theta) = t = \frac{t}{1} = \frac{\text{opposite}}{\text{adjacent}}$$

Construct a right triangle with opposite side being t and the adjacent side being 1. The hypotenuse is then $\sqrt{t^2+1}$. This in turn means that $\sec(\theta) = \sqrt{t^2+1}$. So,

$$\sec\left(\tan^{-1}(t)\right) = \sec(\theta) = \sqrt{t^2+1}$$

With this simplification our arc length function is then,

$$s(t) = \sqrt{5}\left[t\sqrt{t^2+1} + \ln\left|\sqrt{t^2+1}+t\right|\right]$$

There was some slightly unpleasant simplification here but once we did that we got a much nicer arc length function.

5. Determine where on the curve given by $\vec{r}(t) = \left\langle t^2, 2t^3, 1-t^3\right\rangle$ we are after traveling a distance of 20.

Step 1

From Problem 3 above we know that the arc length function for this vector function is,

$$s(t) = \frac{1}{135}\left[\left(4+45t^2\right)^{\frac{3}{2}} - 8\right]$$

We need to solve this for t. Doing this gives,

$$\begin{aligned}
\left(4+45t^2\right)^{\frac{3}{2}} - 8 &= 135s \\
\left(4+45t^2\right)^{\frac{3}{2}} &= 135s+8 \\
4+45t^2 &= (135s+8)^{\frac{2}{3}} \\
t^2 &= \frac{1}{45}\left[(135s+8)^{\frac{2}{3}} - 4\right] \qquad \to \qquad t = \sqrt{\frac{1}{45}\left[(135s+8)^{\frac{2}{3}} - 4\right]}
\end{aligned}$$

Note that we only used the positive t after taking the root because the implicit assumption from the arc length function is that t is positive.

Step 2

We could use this to reparametrize the vector function however that would lead to a particularly unpleasant function in this case.

The key here is to simply realize that what we are being asked to compute is the value of the reparametrized vector function, $\vec{r}\,(t\,(s))$ when $s = 20$. Or, in other words, we want to compute $\vec{r}\,(t\,(20))$.

So, first,

$$t\,(20) = \sqrt{\frac{1}{45}\left[(135\,(20) + 8)^{\frac{2}{3}} - 4\right]} = 2.05633$$

Our position after traveling a distance of 20 is then,

$$\vec{r}\,(t\,(20)) = \vec{r}\,(2.05633) = \boxed{\langle 4.22849, 17.39035, -7.69518\rangle}$$

12.10 Curvature

1. Find the curvature of $\vec r\,(t) = \langle \cos(2t), -\sin(2t), 4t\rangle$.

Step 1

We have two formulas we can use here to compute the curvature. One requires us to take the derivative of the unit tangent vector and the other requires a cross product.

Either will give the same result. The real question is which will be easier to use. Cross products can be a pain to compute but then some of the unit tangent vectors can be quite messy to take the derivative of. So, basically, the one we use will be the one that will probably be the easiest to use.

In this case it looks like the unit tangent vector won't be that bad to work with so let's go with that formula. Here's the unit tangent vector work.

$$\begin{aligned}
\vec r\,'(t) &= \langle -2\sin(2t), -2\cos(2t), 4\rangle \\
\left\|\vec r\,'(t)\right\| &= \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 16} = \sqrt{20} = 2\sqrt{5} \\
\vec T(t) &= \frac{1}{2\sqrt{5}}\langle -2\sin(2t), -2\cos(2t), 4\rangle = \left\langle -\frac{\sin(2t)}{\sqrt{5}}, -\frac{\cos(2t)}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle
\end{aligned}$$

Step 2

Now, what we really need is the magnitude of the derivative of the unit tangent vector so here is that work,

$$\begin{aligned}
\vec T'(t) &= \left\langle -\frac{2}{\sqrt{5}}\cos(2t), \frac{2}{\sqrt{5}}\sin(2t), 0\right\rangle \\
\left\|\vec T'(t)\right\| &= \sqrt{\frac{4}{5}\cos^2(2t) + \frac{4}{5}\sin^2(2t)} = \frac{2}{\sqrt{5}}
\end{aligned}$$

Step 3

The curvature is then,

$$\kappa = \frac{\left\|\vec T'(t)\right\|}{\left\|\vec r\,'(t)\right\|} = \frac{{}^{2}/_{\sqrt{5}}}{2\sqrt{5}} = \boxed{\frac{1}{5}}$$

So, in this case, the curvature will be independent of t. That won't always be the case so don't expect this to happen every time.

2. Find the curvature of $\vec{r}(t) = \left\langle 4t, -t^2, 2t^3 \right\rangle$.

Step 1

We have two formulas we can use here to compute the curvature. One requires us to take the derivative of the unit tangent vector and the other requires a cross product.

Either will give the same result. The real question is which will be easier to use. Cross products can be a pain to compute but then some of the unit tangent vectors can be quite messy to take the derivative of. So, basically, the one we use will be the one that will probably be the easiest to use.

In this case it looks like the unit tangent vector will involve lots of quotients that would probably be unpleasant to take the derivative of. So, let's go with the cross product formula this time.

We'll need the first and second derivative of the vector function. Here are those.

$$\vec{r}'(t) = \left\langle 4, -2t, 6t^2 \right\rangle \qquad \vec{r}''(t) = \langle 0, -2, 12t \rangle$$

Step 2

Next, we need the cross product of these two derivatives. Here is that work.

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2t & 6t^2 \\ 0 & -2 & 12t \end{vmatrix} = -24t^2\,\vec{i} - 8\vec{k} - 48t\,\vec{j} + 12t^2\,\vec{i} = -12t^2\,\vec{i} - 48t\,\vec{j} - 8\vec{k}$$

Step 3

We now need a couple of magnitudes.

$$\left\|\vec{r}'(t) \times \vec{r}''(t)\right\| = \sqrt{144t^4 + 2304t^2 + 64} \qquad \left\|\vec{r}'(t)\right\| = \sqrt{16 + 4t^2 + 36t^4}$$

The curvature is then,

$$\kappa = \frac{\left\|\vec{r}'(t) \times \vec{r}''(t)\right\|}{\left\|\vec{r}'(t)\right\|^3} = \boxed{\frac{\sqrt{144t^4 + 2304t^2 + 64}}{\left(16 + 4t^2 + 36t^4\right)^{\frac{3}{2}}}}$$

A fairly messy formula here, but these will often be that way.

12.11 Velocity and Acceleration

1. An objects acceleration is given by $\vec{a} = 3t\,\vec{i} - 4\mathbf{e}^{-t}\,\vec{j} + 12t^2\vec{k}$. The objects initial velocity is $\vec{v}\,(0) = \vec{j} - 3\vec{k}$ and the objects initial position is $\vec{r}\,(0) = -5\vec{i} + 2\vec{j} - 3\vec{k}$. Determine the objects velocity and position functions.

Step 1

To determine the velocity function all we need to do is integrate the acceleration function.

$$\vec{v}\,(t) = \int 3t\,\vec{i} - 4\mathbf{e}^{-t}\,\vec{j} + 12t^2\vec{k}\,dt = \frac{3}{2}t^2\,\vec{i} + 4\mathbf{e}^{-t}\,\vec{j} + 4t^3\vec{k} + \vec{c}$$

Don't forget the "constant" of integration, which in this case is actually the vector $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$. To determine the constant of integration all we need is to use the value $\vec{v}\,(0)$ that we were given in the problem statement.

$$\vec{j} - 3\vec{k} = \vec{v}\,(0) = 4\,\vec{j} + c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

To determine the values of c_1, c_2, and c_3 all we need to do is set the various components equal.

$$\begin{aligned} \vec{i} &: 0 = c_1 \\ \vec{j} &: 1 = 4 + c_2 \qquad \Rightarrow \qquad c_1 = 0, \quad c_2 = -3, \quad c_3 = -3 \\ \vec{k} &: -3 = c_3 \end{aligned}$$

The velocity is then,

$$\boxed{\vec{v}\,(t) = \frac{3}{2}t^2\,\vec{i} + \left(4\mathbf{e}^{-t} - 3\right)\,\vec{j} + \left(4t^3 - 3\right)\vec{k}}$$

Step 2

The position function is simply the integral of the velocity function we found in the previous step.

$$\vec{r}\,(t) = \int \frac{3}{2}t^2\,\vec{i} + \left(4\mathbf{e}^{-t} - 3\right)\,\vec{j} + \left(4t^3 - 3\right)\vec{k}\,dt = \frac{1}{2}t^3\,\vec{i} + \left(-4\mathbf{e}^{-t} - 3t\right)\,\vec{j} + \left(t^4 - 3t\right)\vec{k} + \vec{c}$$

We'll use the value of $\vec{r}\,(0)$ from the problem statement to determine the value of the constant of integration.

$$-5\vec{i} + 2\vec{j} - 3\vec{k} = \vec{r}\,(0) = -4\,\vec{j} + c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

$$\begin{aligned} \vec{i} &: -5 = c_1 \\ \vec{j} &: 2 = -4 + c_2 \qquad \Rightarrow \qquad c_1 = -5, \quad c_2 = 6, \quad c_3 = -3 \\ \vec{k} &: -3 = c_3 \end{aligned}$$

The position function is then,

$$\boxed{\vec{r}(t) = \left(\frac{1}{2}t^3 - 5\right)\vec{i} + \left(-4\mathbf{e}^{-t} - 3t + 6\right)\vec{j} + \left(t^4 - 3t - 3\right)\vec{k}}$$

---

2. Determine the tangential and normal components of acceleration for the object whose position is given by $\vec{r}(t) = \langle \cos(2t), -\sin(2t), 4t \rangle$.

Step 1

First, we need the first and second derivatives of the position function.

$$\vec{r}'(t) = \langle -2sin(2t), -2\cos(2t), 4 \rangle \qquad \vec{r}''(t) = \langle -4\cos(2t), 4sin(2t), 0 \rangle$$

Step 2

Next, we'll need the following quantities.

$$\left\|\vec{r}'(t)\right\| = \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 16} = \sqrt{20} = 2\sqrt{5}$$

$$\vec{r}'(t) \cdot \vec{r}''(t) = 8\sin(2t)\cos(2t) - 8\sin(2t)\cos(2t) + 0 = 0$$

$$\begin{aligned} \vec{r}'(t) \times \vec{r}''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2sin(2t) & -2\cos(2t) & 4 \\ -4\cos(2t) & 4sin(2t) & 0 \end{vmatrix} \\ &= -16\cos(2t)\vec{j} - 8\sin^2(2t)\vec{k} - 8\cos^2(2t)\vec{k} - 16\sin(2t)\vec{i} \\ &= -16\sin(2t)\vec{i} - 16\cos(2t)\vec{j} - 8\vec{k} \end{aligned}$$

$$\left\|\vec{r}'(t) \times \vec{r}''(t)\right\| = \sqrt{256\sin^2(2t) + 256\cos^2(2t) + 64} = \sqrt{320} = 8\sqrt{5}$$

Step 3

The tangential component of the acceleration is,

$$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\|\vec{r}'(t)\|} = \boxed{0}$$

The normal component of the acceleration is,

$$a_N = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|} = \frac{8\sqrt{5}}{2\sqrt{5}} = \boxed{4}$$

12.12 Cylindrical Coordinates

1. Convert the Cartesian coordinates for $(4, -5, 2)$ into Cylindrical coordinates.

Step 1

From the point we're given we have,

$$x = 4 \qquad y = -5 \qquad z = 2$$

So, we already have the z coordinate for the Cylindrical coordinates.

Step 2

Remember as well that for r and θ we're going to do the same conversion work as we did in converting a Cartesian point into Polar coordinates.

So, getting r is easy.

$$r = \sqrt{(4)^2 + (-5)^2} = \sqrt{41}$$

Step 3

Finally, we need to get θ.

$$\theta_1 = \tan^{-1}\left(\frac{-5}{4}\right) = -0.8961 \qquad \theta_2 = -0.8961 + \pi = 2.2455$$

If we look at the three dimensional coordinate system from above we can see that θ_1 is in the fourth quadrant and θ_2 is in the second quadrant. Likewise, from our x and y coordinates the point is in the fourth quadrant (as we look at the point from above).

This in turn means that we need to use θ_1 for our point.

The Cylindrical coordinates are then,

$$\boxed{\left(\sqrt{41}, -0.8961, 2\right)}$$

2. Convert the Cartesian coordinates for $(-4, -1, 8)$ into Cylindrical coordinates.

Step 1

From the point we're given we have,

$$x = -4 \qquad y = -1 \qquad z = 8$$

So, we already have the z coordinate for the Cylindrical coordinates.

Step 2

Remember as well that for r and θ we're going to do the same conversion work as we did in converting a Cartesian point into Polar coordinates.

So, getting r is easy.

$$r = \sqrt{(-4)^2 + (-1)^2} = \sqrt{17}$$

Step 3

Finally, we need to get θ.

$$\theta_1 = \tan^{-1}\left(\frac{-1}{-4}\right) = 0.2450 \qquad \theta_2 = 0.2450 + \pi = 3.3866$$

If we look at the three dimensional coordinate system from above we can see that θ_1 is in the first quadrant and θ_2 is in the third quadrant. Likewise, from our x and y coordinates the point is in the third quadrant (as we look at the point from above).

This in turn means that we need to use θ_2 for our point.

The Cylindrical coordinates are then,

$$\boxed{\left(\sqrt{17}, 3.3866, 8\right)}$$

3. Convert the following equation written in Cartesian coordinates into an equation in Cylindrical coordinates.

$$x^3 + 2x^2 - 6z = 4 - 2y^2$$

Step 1

There really isn't a whole lot to do here. All we need to do is plug in the following x and y polar conversion formulas into the equation where (and if) possible.

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad r^2 = x^2 + y^2$$

Step 2

However, first we'll do a little rewrite.

$$x^3 + 2x^2 + 2y^2 - 6z = 4 \qquad \to \qquad x^3 + 2\left(x^2 + y^2\right) - 6z = 4$$

Step 3

Now let's use the formulas from Step 1 to convert the equation into Cylindrical coordinates.

$$(r\cos(\theta))^3 + 2\left(r^2\right) - 6z = 4 \qquad \to \qquad \boxed{r^3\cos^3(\theta) + 2r^2 - 6z = 4}$$

4. Convert the following equation written in Cylindrical coordinates into an equation in Cartesian coordinates.

$$z\,r = 2 - r^2$$

Solution

There is not really a lot to do here other than plug in $r = \sqrt{x^2 + y^2}$ into the equation. Doing this is,

$$\boxed{z\sqrt{x^2 + y^2} = 2 - \left(x^2 + y^2\right)}$$

5. Convert the following equation written in Cylindrical coordinates into an equation in Cartesian coordinates.

$$4\sin(\theta) - 2\cos(\theta) = \frac{r}{z}$$

Step 1

There really isn't a whole lot to do here. All we need to do is to use the following x and y polar conversion formulas in the equation where (and if) possible.

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad r^2 = x^2 + y^2$$

Step 2

To make the conversion a little easier let's multiply the equation by r to get,

$$4r\sin(\theta) - 2r\cos(\theta) = \frac{r^2}{z}$$

Step 3

Now let's use the formulas from Step 1 to convert the equation into Cartesian coordinates.

$$\boxed{4y - 2x = \frac{x^2 + y^2}{z}}$$

6. Identify the surface generated by the equation : $r^2 - 4r\cos(\theta) = 14$

Step 1

To identify the surface generated by this equation it's probably best to first convert the equation into Cartesian coordinates. In this case that's a pretty simple thing to do.

Here is the equation in Cartesian coordinates.

$$x^2 + y^2 - 4x = 14$$

Step 2

To identify this equation (and you do know what it is!) let's complete the square on the x part of the equation.

$$\begin{aligned} x^2 - 4x + y^2 &= 14 \\ x^2 - 4x + 4 + y^2 &= 14 + 4 \\ (x-2)^2 + y^2 &= 18 \end{aligned}$$

So, we can see that this is a cylinder whose central axis is a vertical line parallel to the z-axis and goes through the point (2, 0) in the xy-plane and the radius of the cylinder is $\sqrt{18}$.

---

7. Identify the surface generated by the equation : $z = 7 - 4r^2$

Step 1

To identify the surface generated by this equation it's probably best to first convert the equation into Cartesian coordinates. In this case that's a pretty simple thing to do.

Here is the equation in Cartesian coordinates.

$$z = 7 - 4\left(x^2 + y^2\right) = 7 - 4x^2 - 4y^2$$

Step 2

From the Cartesian equation in Step 1 we can see that the surface generated by the equation is an elliptic paraboloid that starts at $z = 7$ and opens down.

12.13 Spherical Coordinates

1. Convert the Cartesian coordinates for $(3, -4, 1)$ into Spherical coordinates.

Step 1

From the point we're given we have,

$$x = 3 \qquad y = -4 \qquad z = 1$$

Step 2

Let's first determine ρ.

$$\rho = \sqrt{(3)^2 + (-4)^2 + (1)^2} = \sqrt{26}$$

Step 3

We can now determine φ.

$$\cos(\varphi) = \frac{z}{\rho} = \frac{1}{\sqrt{26}} \qquad \varphi = \cos^{-1}\left(\frac{1}{\sqrt{26}}\right) = 1.3734$$

Step 4

Let's use the x conversion formula to determine θ.

$$\cos(\theta) = \frac{x}{\rho\sin(\varphi)} = \frac{3}{\sqrt{26}\sin(1.3734)} = 0.6 \qquad \to \qquad \theta_1 = \cos^{-1}(0.6) = 0.9273$$

This angle is in the first quadrant and if we sketch a quick unit circle we see that a second angle in the fourth quadrant is $\theta_2 = 2\pi - 0.9273 = 5.3559$.

If we look at the three dimensional coordinate system from above we can see that from our x and y coordinates the point is in the fourth quadrant. This in turn means that we need to use θ_2 for our point.

The Spherical coordinates are then,

$$\boxed{\left(\sqrt{26}, 5.3559, 1.3734\right)}$$

2. Convert the Cartesian coordinates for $(-2, -1, -7)$ into Spherical coordinates.

Step 1

From the point we're given we have,

$$x = -2 \qquad y = -1 \qquad z = -7$$

Step 2

Let's first determine ρ.

$$\rho = \sqrt{(-2)^2 + (-1)^2 + (-7)^2} = \sqrt{54}$$

Step 3

We can now determine φ.

$$\cos(\varphi) = \frac{z}{\rho} = \frac{-7}{\sqrt{54}} \qquad \varphi = \cos^{-1}\left(\frac{-7}{\sqrt{54}}\right) = 2.8324$$

Step 4

Let's use the y conversion formula to determine θ.

$$\sin(\theta) = \frac{-1}{\rho\sin(\varphi)} = \frac{-1}{\sqrt{54}\sin(2.8324)} = -0.4472 \quad \rightarrow \quad \theta_1 = \sin^{-1}(-0.4472) = -0.4636$$

This angle is in the fourth quadrant and if we sketch a quick unit circle we see that a second angle in the third quadrant is $\theta_2 = \pi + 0.4636 = 3.6052$.

If we look at the three dimensional coordinate system from above we can see that from our x and y coordinates the point is in the third quadrant. This in turn means that we need to use θ_2 for our point.

The Spherical coordinates are then,

$$\boxed{\left(\sqrt{54}, 3.6052, 2.8324\right)}$$

3. Convert the Cylindrical coordinates for $(2, 0.345, -3)$ into Spherical coordinates.

Step 1

From the point we're given we have,

$$r = 2 \qquad \theta = 0.345 \qquad z = -3$$

So, we already have the value of θ for the Spherical coordinates.

Step 2

Next, we can determine ρ.

$$\rho = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$$

Step 3

Finally, we can determine φ.

$$\cos(\varphi) = \frac{z}{\rho} = \frac{-3}{\sqrt{13}} \qquad \varphi = \cos^{-1}\left(\frac{-3}{\sqrt{13}}\right) = 2.5536$$

The Spherical coordinates are then,

$$\boxed{\left(\sqrt{13}, 0.345, 2.5536\right)}$$

4. Convert the following equation written in Cartesian coordinates into an equation in Spherical coordinates.

$$x^2 + y^2 = 4x + z - 2$$

Step 1

All we need to do here is plug in the following conversion formulas into the equation and do a little simplification.

$$x = \rho\sin(\varphi)\cos(\theta) \qquad y = \rho\sin(\varphi)\sin(\theta) \qquad z = \rho\cos(\varphi)$$

Step 2

Plugging the conversion formula in gives,

$$(\rho\sin(\varphi)\cos(\theta))^2 + (\rho\sin(\varphi)\sin(\theta))^2 = 4(\rho\sin(\varphi)\cos(\theta)) + \rho\cos(\varphi) - 2$$

The first two terms can be simplified as follows.

$$\begin{aligned}\rho^2\sin^2(\varphi)\cos^2(\theta) + \rho^2\sin^2(\varphi)\sin^2(\theta) &= 4\rho\sin(\varphi)\cos(\theta) + \rho\cos(\varphi) - 2\\ \rho^2\sin^2(\varphi)\left(\cos^2(\theta) + \sin^2(\theta)\right) &= 4\rho\sin(\varphi)\cos(\theta) + \rho\cos(\varphi) - 2\\ \rho^2\sin^2(\varphi) &= 4\rho\sin(\varphi)\cos(\theta) + \rho\cos(\varphi) - 2\end{aligned}$$

5. Convert the equation written in Spherical coordinates into an equation in Cartesian coordinates.

$$\rho^2 = 3 - \cos(\varphi)$$

Step 1

There really isn't a whole lot to do here. All we need to do is to use the following conversion formulas in the equation where (and if) possible

$$x = \rho\sin(\varphi)\cos(\theta) \qquad y = \rho\sin(\varphi)\sin(\theta) \qquad z = \rho\cos(\varphi)$$
$$\rho^2 = x^2 + y^2 + z^2$$

Step 2

To make this problem a little easier let's first multiply the equation by ρ. Doing this gives,

$$\rho^3 = 3\rho - \rho\cos(\varphi)$$

Doing this makes recognizing the right most term a little easier.

Step 3

Using the appropriate conversion formulas from Step 1 gives,

$$\boxed{\left(x^2+y^2+z^2\right)^{\frac{3}{2}} = 3\sqrt{x^2+y^2+z^2} - z}$$

---

6. Convert the equation written in Spherical coordinates into an equation in Cartesian coordinates.

$$\csc(\varphi) = 2\cos(\theta) + 4\sin(\theta)$$

Step 1

There really isn't a whole lot to do here. All we need to do is to use the following conversion formulas in the equation where (and if) possible

$$x = \rho\sin(\varphi)\cos(\theta) \qquad y = \rho\sin(\varphi)\sin(\theta) \qquad z = \rho\cos(\varphi)$$
$$\rho^2 = x^2+y^2+z^2$$

Step 2

To make this problem a little easier let's first do some rewrite on the equation.

First, let's deal with the cosecant.

$$\frac{1}{\sin(\varphi)} = 2\cos(\theta) + 4\sin(\theta) \qquad \to \qquad 1 = 2\sin(\varphi)\cos(\theta) + 4\sin(\varphi)\sin(\theta)$$

Next, let's multiply everything by ρ to get,

$$\rho = 2\rho\sin(\varphi)\cos(\theta) + 4\rho\sin(\varphi)\sin(\theta)$$

Doing this makes recognizing the terms on the right a little easier.

Step 3

Using the appropriate conversion formulas from Step 1 gives,

$$\boxed{\sqrt{x^2+y^2+z^2} = 2x + 4y}$$

7. Identify the surface generated by the given equation : $\varphi = \dfrac{4\pi}{5}$

Solution

Okay, as we discussed this type of equation in the notes for this section. We know that all points on the surface generated must be of the form $\left(\rho, \theta, \frac{4\pi}{5}\right)$.

So, we can rotate around the z-axis as much as want them to (*i.e.* θ can be anything) and we can move as far as we want from the origin (*i.e.* ρ can be anything). All we need to do is make sure that the point will always make an angle of $\frac{4\pi}{5}$ with the positive z-axis.

In other words, we have a cone. It will open downwards and the "wall" of the cone will form an angle of $\frac{4\pi}{5}$ with the positive z-axis and it will form an angle of $\frac{\pi}{5}$ with the negative z-axis.

8. Identify the surface generated by the given equation : $\rho = -2\sin(\varphi)\cos(\theta)$

Step 1

Let's first multiply each side of the equation by ρ to get,

$$\rho^2 = -2\rho\sin(\varphi)\cos(\theta)$$

Step 2

We can now easily convert this to Cartesian coordinates to get,

$$\begin{aligned} x^2 + y^2 + z^2 &= -2x \\ x^2 + 2x + y^2 + z^2 &= 0 \end{aligned}$$

Let's complete the square on the x portion to get,

$$\begin{aligned} x^2 + 2x + 1 + y^2 + z^2 &= 0 + 1 \\ (x+1)^2 + y^2 + z^2 &= 1 \end{aligned}$$

Step 3

So, it looks like we have a sphere with radius 1 that is centered at $(-1, 0, 0)$.

13 Partial Derivatives

To this point, with the exception of the occasional section in the last chapter, we've been working almost exclusively with functions of a single variable. It is not time to formally start multi-variable Calculus, *i.e.* Calculus involving functions of two or more variables. We will be covering the same basic topics as we do with single variable Calculus. Namely, limits, derivatives and integrals.

In this chapter we will open up with a quick section discussing taking limits of multi-variable functions. We will only be covering limits of multi-variable functions with a single chapter because, as we'll see, many of the concepts from single variable limits still hold, with some natural extensions of course. However, as we'll also see the work will often be significantly longer/harder and so we won't be spending a lot of time discussing limits of multi-variable functions. Luckily enough for us we also won't need to worry all that much about limits of multi-variable functions so the quick discussion of limits in this chapter will suffice.

The rest of the chapter will be discussing how to take derivatives of multi-variable functions. We want to keep the "main" interpretation of derivatives, namely the derivative will still give the rate of change of the function. The issue here is that because we have multiple variables the function can have differing rates of change depending on how we allow the various variables to change.

So, to start out the derivative discussion we will start by defining the partial derivative. These will restrict just how we allow the various variables to change. We will eventually introduce the directional derivative which will allow the variables to change in any arbitrary manner. In the process of introducing the idea of a directional derivative we'll also introduce the concept of a gradient of a function. The gradient will arise in quite a few sections throughout the rest of this multi-variable Calculus material, including integrals.

Finally, as we'll see, if you can take derivatives of single variable functions then you have the majority of the knowledge that you need to take derivatives of multi-variable functions. There are, however, some subtleties that we'll need to remember to deal with. Those subtleties are, generally, the issues that most students run into when taking derivatives of multi-variable functions.

The following sections are the practice problems, with solutions, for this material.

13.1 Limits

1. Evaluate the following limit.

$$\lim_{(x,y,z)\to(-1,0,4)} \frac{x^3 - z\mathbf{e}^{2y}}{6x + 2y - 3z}$$

Solution

In this case there really isn't all that much to do. We can see that the denominator exists and will not be zero at the point in question and the numerator also exists at the point in question. Therefore, all we need to do is plug in the point to evaluate the limit.

$$\lim_{(x,y,z)\to(-1,0,4)} \frac{x^3 - z\mathbf{e}^{2y}}{6x + 2y - 3z} = \boxed{\frac{5}{18}}$$

---

2. Evaluate the following limit.

$$\lim_{(x,y)\to(2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2}$$

Step 1

Okay, with this problem we can see that, if we plug in the point, we get zero in the numerator and the denominator. Unlike most of the examples of this type however, that doesn't just mean that the limit won't exist.

In this case notice that we can factor and simplify the function as follows,

$$\lim_{(x,y)\to(2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2} = \lim_{(x,y)\to(2,1)} \frac{x\left(x - 2y\right)}{\left(x - 2y\right)\left(x + 2y\right)} = \lim_{(x,y)\to(2,1)} \frac{x}{x + 2y}$$

We may not be used to factoring these kinds of polynomials but we can't forget that factoring is still a possibility that we need to address for these limits.

Step 2

Now, that we've factored and simplified the function we can see that we've lost the division by zero issue and so we can now evaluate the limit. Doing this gives,

$$\lim_{(x,y)\to(2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2} = \lim_{(x,y)\to(2,1)} \frac{x}{x + 2y} = \boxed{\frac{1}{2}}$$

3. Evaluate the following limit.

$$\lim_{(x,y)\to(0,0)} \frac{x-4y}{6y+7x}$$

Step 1

Okay, with this problem we can see that, if we plug in the point, we get zero in the numerator and the denominator. Unlike the second problem above however there is no factoring that can be done to make this into a "doable" limit.

Therefore, we will proceed with the problem as if the limit doesn't exist.

Step 2

So, since we've made the assumption that the limit probably doesn't exist that means we need to find two different paths upon which the limit has different values.

There are many different paths to try but let's start this off with the x-axis (*i.e.* $y = 0$).

Along this path we get,

$$\lim_{(x,y)\to(0,0)} \frac{x-4y}{6y+7x} = \lim_{(x,0)\to(0,0)} \frac{x}{7x} = \lim_{(x,0)\to(0,0)} \frac{1}{7} = \frac{1}{7}$$

Step 3

Now let's try the y-axis (*i.e.* $x = 0$) and see what we get.

$$\lim_{(x,y)\to(0,0)} \frac{x-4y}{6y+7x} = \lim_{(0,y)\to(0,0)} \frac{-4y}{6y} = \lim_{(0,y)\to(0,0)} \frac{-2}{3} = -\frac{2}{3}$$

Step 4

So, we have two different paths that give different values of the limit and so we now know that this limit **does not exist**.

4. Evaluate the following limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3}$$

Step 1

Okay, with this problem we can see that, if we plug in the point, we get zero in the numerator and the denominator. Unlike the second problem above however there is no factoring that can be done to make this into a "doable" limit.

Therefore, we will proceed with the problem as if the limit doesn't exist.

Step 2

So, since we've made the assumption that the limit probably doesn't exist that means we need to find two different paths upon which the limit has different values.

In this case note that using the x-axis or y-axis will not work as either one will result in a division by zero issue. So, let's start off using the path $x = y^3$.

Along this path we get,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3} = \lim_{(y^3,y)\to(0,0)} \frac{y^6 - y^6}{y^3y^3} = \lim_{(y^3,y)\to(0,0)} 0 = 0$$

Step 3

Now let's try $x = y$ for the second path.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3} = \lim_{(y,y)\to(0,0)} \frac{y^2 - y^6}{yy^3} = \lim_{(y,y)\to(0,0)} \frac{1 - y^4}{y^2} = \lim_{(y,y)\to(0,0)} \left(\frac{1}{y^2} - y^2\right) = \infty$$

Step 4

So, we have two different paths that give different values of the limit and so we now know that this limit **does not exist**.

13.2 Partial Derivatives

1. Find all the 1^{st} order partial derivatives of the following function.

$$f(x, y, z) = 4x^3y^2 - \mathbf{e}^z y^4 + \frac{z^3}{x^2} + 4y - x^{16}$$

Solution

So, this is clearly a function of x, y and z and so we'll have three 1^{st} order partial derivatives and each of them should be pretty easy to compute.

Just remember that when computing each individual derivative that the other variables are to be treated as constants. So, for instance, when computing the x partial derivative all y's and z's are treated as constants. This in turn means that, for the x partial derivative, the second and fourth terms are considered to be constants (they don't contain any x's) and so differentiate to zero. Dealing with these types of terms properly tends to be one of the biggest mistakes students make initially when taking partial derivatives. Too often students just leave them alone since they don't contain the variable we are differentiating with respect to.

Here are the three 1^{st} order partial derivatives for this problem.

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= f_x = 12x^2y^2 - \frac{2z^3}{x^3} - 16x^{15} \\ \frac{\partial f}{\partial y} &= f_y = 8x^3y - 4\mathbf{e}^z y^3 + 4 \\ \frac{\partial f}{\partial z} &= f_z = -\mathbf{e}^z y^4 + \frac{3z^2}{x^2} \end{aligned}}$$

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

---

2. Find all the 1^{st} order partial derivatives of the following function.

$$w = \cos\left(x^2 + 2y\right) - \mathbf{e}^{4x - z^4 y} + y^3$$

Solution

This function isn't written explicitly with the (x, y, z) part but it is (hopefully) clearly a function of x, y and z and so we'll have three 1^{st} order partial derivatives and each of them should be pretty easy to compute.

Just remember that when computing each individual derivative that the other variables are to be treated as constants. So, for instance, when computing the x partial derivative all y's and z's are treated as constants. This in turn means that, for the x partial derivative, the third term is considered to be a constant (it doesn't contain any x's) and so differentiates to zero. Dealing with these types of terms properly tends to be one of the biggest mistakes students make initially when taking partial derivatives. Too often students just leave them alone since they don't contain the variable we are differentiating with respect to.

Be careful with chain rule. Again, one of the biggest issues with partial derivatives is students forgetting the "rules" of partial derivatives when it comes to differentiating the inside function. Just remember that you're just doing the partial derivative of a function and remember which variable we are differentiating with respect to.

Here are the three 1^{st} order partial derivatives for this problem.

$$\boxed{\begin{aligned} \frac{\partial w}{\partial x} &= w_x = -2x\sin\left(x^2+2y\right) - 4\mathbf{e}^{4x-z^4y} \\ \frac{\partial w}{\partial y} &= w_y = -2\sin\left(x^2+2y\right) + z^4\mathbf{e}^{4x-z^4y} + 3y^2 \\ \frac{\partial w}{\partial z} &= w_z = 4z^3y\mathbf{e}^{4x-z^4y} \end{aligned}}$$

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

---

3. Find all the 1^{st} order partial derivatives of the following function.

$$f(u,v,p,t) = 8u^2t^3p - \sqrt{v}\,p^2t^{-5} + 2u^2t + 3p^4 - v$$

Solution

So, this is clearly a function of u, v, p, and t and so we'll have four 1^{st} order partial derivatives and each of them should be pretty easy to compute.

Just remember that when computing each individual derivative that the other variables are to be treated as constants. So, for instance, when computing the u partial derivative all v's, p's and t's are treated as constants. This in turn means that, for the u partial derivative, the second, fourth and fifth terms are considered to be constants (they don't contain any u's) and so differentiate to zero. Dealing with these types of terms properly tends to be one of the biggest mistakes students make initially when taking partial derivatives. Too often students just leave them alone since they don't contain the variable we are differentiating with respect to.

Here are the four 1^{st} order partial derivatives for this problem.

$$\begin{aligned}\frac{\partial f}{\partial u} &= f_u = 16ut^3p + 4ut\\ \frac{\partial f}{\partial v} &= f_v = -\frac{1}{2}v^{-\frac{1}{2}}p^2t^{-5} - 1\\ \frac{\partial f}{\partial p} &= f_p = 8u^2t^3 - 2\sqrt{v}\,pt^{-5} + 12p^3\\ \frac{\partial f}{\partial t} &= f_t = 24u^2t^2p + 5\sqrt{v}\,p^2t^{-6} + 2u^2\end{aligned}$$

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

---

4. Find all the 1^{st} order partial derivatives of the following function.

$$f(u,v) = u^2\sin\left(u+v^3\right) - \sec(4u)\tan^{-1}(2v)$$

Solution

For this problem it looks like we'll have two 1^{st} order partial derivatives to compute.

Be careful with product rules with partial derivatives. For example, the second term, while definitely a product, will not need the product rule since each "factor" of the product only contains u's or v's. On the other hand, the first term will need a product rule when doing the u partial derivative since there are u's in both of the "factors" of the product. However, just because we had to product rule with first term for the u partial derivative doesn't mean that we'll need to product rule for the v partial derivative as only the second "factor" in the product has a v in it.

Basically, be careful to not "overthink" product rules with partial derivatives. Do them when required but make sure to not do them just because you see a product. When you see a product look at the "factors" of the product. Do both "factors" have the variable you are differentiating with respect to or not and use the product rule only if they both do.

Here are the two 1^{st} order partial derivatives for this problem.

$$\begin{aligned}\frac{\partial f}{\partial u} &= f_u = 2u\sin\left(u+v^3\right) + u^2\cos\left(u+v^3\right) - 4\sec(4u)\,tan(4u)\tan^{-1}(2v)\\ \frac{\partial f}{\partial v} &= f_v = 3v^2u^2cos\left(u+v^3\right) - \frac{2\sec(4u)}{1+4v^2}\end{aligned}$$

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

5. Find all the 1^{st} order partial derivatives of the following function.

$$f\left(x,z\right) = \mathbf{e}^{-x}\sqrt{z^4+x^2} - \frac{2x+3z}{4z-7x}$$

Solution

For this problem it looks like we'll have two 1^{st} order partial derivatives to compute.

Be careful with product rules and quotient rules with partial derivatives. For example, the first term, while clearly a product, will only need the product rule for the x derivative since both "factors" in the product have x's in them. On the other hand, the first "factor" in the first term does not contain a z and so we won't need to do the product rule for the z derivative. In this case the second term will require a quotient rule for both derivatives.

Basically, be careful to not "overthink" product/quotient rules with partial derivatives. Do them when required but make sure to not do them just because you see a product/quotient. When you see a product/quotient look at the "factors" of the product/quotient. Do both "factors" have the variable you are differentiating with respect to or not and use the product/quotient rule only if they both do.

Here are the two 1^{st} order partial derivatives for this problem.

$$\boxed{\begin{aligned}\frac{\partial f}{\partial x} &= f_x = -\mathbf{e}^{-x}\left(z^4+x^2\right)^{\frac{1}{2}} + x\,\mathbf{e}^{-x}\left(z^4+x^2\right)^{-\frac{1}{2}} - \frac{29z}{\left(4z-7x\right)^2}\\ \frac{\partial f}{\partial z} &= f_z = 2z^3\mathbf{e}^{-x}\left(z^4+x^2\right)^{-\frac{1}{2}} + \frac{29x}{\left(4z-7x\right)^2}\end{aligned}}$$

Note that we did a little bit of simplification in the derivative work here and didn't actually show the "first" step of the problem under the assumption that by this point of your mathematical career you can do the product and quotient rule and don't really need us to show that step to you.

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

6. Find all the 1^{st} order partial derivatives of the following function.

$$g(s,t,v) = t^2 \ln(s+2t) - \ln(3v)\left(s^3 + t^2 - 4v\right)$$

Solution

For this problem it looks like we'll have three 1^{st} order partial derivatives to compute.

Be careful with product rules with partial derivatives. The first term will only need a product rule for the t derivative and the second term will only need the product rule for the v derivative. Do not "overthink" product rules with partial derivatives. Do them when required but make sure to not do them just because you see a product. When you see a product look at the "factors" of the product. Do both "factors" have the variable you are differentiating with respect to or not and use the product rule only if they both do.

Here are the three 1^{st} order partial derivatives for this problem.

$$\boxed{\begin{aligned} \frac{\partial g}{\partial s} &= g_s = \frac{t^2}{s+2t} - 3s^2 \ln(3v) \\ \frac{\partial g}{\partial t} &= g_t = 2t\ln(s+2t) + \frac{2t^2}{s+2t} - 2t\ln(3v) \\ \frac{\partial g}{\partial v} &= g_v = 4\ln(3v) - \frac{s^3+t^2-4v}{v} \end{aligned}}$$

Make sure you can differentiate natural logarithms as they will come up fairly often. Recall that, with the chain rule, we have,

$$\frac{d}{dx}\Big[\ln(f(x))\Big] = \frac{f'(x)}{f(x)}$$

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

---

7. Find all the 1^{st} order partial derivatives of the following function.

$$R(x,y) = \frac{x^2}{y^2+1} - \frac{y^2}{x^2+y}$$

Solution

For this problem it looks like we'll have two 1^{st} order partial derivatives to compute.

Be careful with quotient rules with partial derivatives. For example the first term, while clearly a quotient, will not require the quotient rule for the x derivative and will only require the quotient rule for the y derivative if we chose to leave the y^2+1 in the denominator (recall we could just bring it up to the numerator as $\left(y^2+1\right)^{-1}$ if we wanted to). The second term on the other hand clearly has y's in both the numerator and the denominator and so will require a quotient rule for the y derivative.

Here are the two 1^{st} order partial derivatives for this problem.

$$\boxed{\begin{aligned}\frac{\partial R}{\partial x} &= R_x = \frac{2x}{y^2+1} + \frac{2xy^2}{\left(x^2+y\right)^2}\\ \frac{\partial R}{\partial y} &= R_y = -\frac{2yx^2}{\left(y^2+1\right)^2} - \frac{2yx^2+y^2}{\left(x^2+y\right)^2}\end{aligned}}$$

Note that we did a little bit of simplification in the derivative work here and didn't actually show the "first" step of the problem under the assumption that by this point of your mathematical career you can do the quotient rule and don't really need us to show that step to you.

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

8. Find all the 1^{st} order partial derivatives of the following function.

$$z = \frac{p^2\left(r+1\right)}{t^3} + pr\,\mathbf{e}^{2p+3r+4t}$$

Solution

For this problem it looks like we'll have three 1^{st} order partial derivatives to compute. Here they are,

$$\boxed{\begin{aligned}\frac{\partial z}{\partial p} &= z_p = \frac{2p\left(r+1\right)}{t^3} + r\,\mathbf{e}^{2p+3r+4t} + 2pr\,\mathbf{e}^{2p+3r+4t}\\ \frac{\partial z}{\partial r} &= z_r = \frac{p^2}{t^3} + p\,\mathbf{e}^{2p+3r+4t} + 3pr\,\mathbf{e}^{2p+3r+4t}\\ \frac{\partial z}{\partial t} &= z_t = -\frac{3p^2\left(r+1\right)}{t^4} + 4pr\,\mathbf{e}^{2p+3r+4t}\end{aligned}}$$

The notation used for the derivative doesn't matter so we used both here just to make sure we're familiar with both forms.

9. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the following function.

$$x^2 \sin\left(y^3\right) + x\mathbf{e}^{3z} - \cos\left(z^2\right) = 3y - 6z + 8$$

Step 1

Okay, we are basically being asked to do implicit differentiation here and recall that we are assuming that z is in fact $z(x, y)$ when we do our derivative work.

Let's get $\frac{\partial z}{\partial x}$ first and that requires us to differentiate with respect to x. Just recall that any product involving x and z will require the product rule because we're assuming that z is a function of x. Also recall to properly chain rule any derivative of z to pick up the $\frac{\partial z}{\partial x}$ when differentiating the "inside" function.

Differentiating the equation with respect to x gives,

$$2x \sin\left(y^3\right) + \mathbf{e}^{3z} + 3\frac{\partial z}{\partial x}x\mathbf{e}^{3z} + 2z\frac{\partial z}{\partial x}\sin\left(z^2\right) = -6\frac{\partial z}{\partial x}$$

Solving for $\frac{\partial z}{\partial x}$ gives,

$$2x \sin\left(y^3\right) + \mathbf{e}^{3z} = \left(-6 - 3x\mathbf{e}^{3z} - 2z\sin\left(z^2\right)\right)\frac{\partial z}{\partial x}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{2x \sin\left(y^3\right) + \mathbf{e}^{3z}}{-6 - 3x\mathbf{e}^{3z} - 2z\sin\left(z^2\right)}}$$

Step 2

Now we get to do it all over again except this time we'll differentiate with respect to y in order to find $\frac{\partial z}{\partial y}$. So, differentiating gives,

$$3y^2x^2 \cos\left(y^3\right) + 3\frac{\partial z}{\partial y}x\mathbf{e}^{3z} + 2z\frac{\partial z}{\partial y}sin\left(z^2\right) = 3 - 6\frac{\partial z}{\partial y}$$

Solving for $\frac{\partial z}{\partial y}$ gives,

$$3y^2x^2 \cos\left(y^3\right) - 3 = \left(-6 - 3x\mathbf{e}^{3z} - 2zsin\left(z^2\right)\right)\frac{\partial z}{\partial y}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{3y^2x^2 \cos\left(y^3\right) - 3}{-6 - 3x\mathbf{e}^{3z} - 2zsin\left(z^2\right)}}$$

13.3 Interpretations of Partial Derivatives

1. Determine if $f(x,y) = x \ln(4y) + \sqrt{x+y}$ is increasing or decreasing at $(-3,6)$ if

   (a) we allow x to vary and hold y fixed.

   (b) we allow y to vary and hold x fixed.

Solutions

(a) allow x to vary and hold y fixed

Solution

So, we want to determine the increasing/decreasing nature of a function at a point. We know that this means a derivative from our basic Calculus knowledge. Also, from the problem statement we know we want to allow x to vary while y is held fixed. This means that we will need the x partial derivative.

The x partial derivative and its value at the point is,

$$f_x(x,y) = \ln(4y) + \frac{1}{2}(x+y)^{-\frac{1}{2}} \qquad \rightarrow \qquad f_x(-3,6) = \ln(24) + \frac{1}{2\sqrt{3}} = 3.4667$$

So, we can see that $f_x(-3,6) > 0$ and so at $(-3,6)$ if we allow x to vary and hold y fixed the function will be increasing.

(b) allow y to vary and hold x fixed

Solution

This part is pretty much the same as the previous part. The only difference is that here we are allowing y to vary and we'll hold x fixed. This means we'll need the y partial derivative.

The y partial derivative and its value at the point is,

$$f_y(x,y) = \frac{x}{y} + \frac{1}{2}(x+y)^{-\frac{1}{2}} \qquad \rightarrow \qquad f_y(-3,6) = -\frac{1}{2} + \frac{1}{2\sqrt{3}} = -0.2113$$

So, we can see that $f_y(-3,6) < 0$ and so at $(-3,6)$ if we allow y to vary and hold x fixed the function will be decreasing.

Note that, because of the three dimensional nature of the graph of this function we can't expect the increasing/decreasing nature of the function in one direction to be the same as in any other direction!

2. Determine if $f(x,y) = x^2 \sin\left(\frac{\pi}{y}\right)$ is increasing or decreasing at $\left(-2, \frac{3}{4}\right)$ if

(a) we allow x to vary and hold y fixed.

(b) we allow y to vary and hold x fixed.

Solutions

(a) allow x to vary and hold y fixed

Solution

So, we want to determine the increasing/decreasing nature of a function at a point. We know that this means a derivative from our basic Calculus knowledge. Also, from the problem statement we know we want to allow x to vary while y is held fixed. This means that we will need the x partial derivative.

The x partial derivative and its value at the point is,

$$f_x(x,y) = 2x \sin\left(\frac{\pi}{y}\right) \qquad \rightarrow \qquad f_x\left(-2, \frac{3}{4}\right) = 2\sqrt{3}$$

So, we can see that $f_x\left(-2, \frac{3}{4}\right) > 0$ and so at $\left(-2, \frac{3}{4}\right)$ if we allow x to vary and hold y fixed the function will be increasing.

(b) allow y to vary and hold x fixed

Solution

This part is pretty much the same as the previous part. The only difference is that here we are allowing y to vary and we'll hold x fixed. This means we'll need the y partial derivative.

The y partial derivative and its value at the point is,

$$f_y(x,y) = -\frac{\pi x^2}{y^2} \cos\left(\frac{\pi}{y}\right) \qquad \rightarrow \qquad f_y\left(-2, \frac{3}{4}\right) = \frac{32\pi}{9}$$

So, we can see that $f_y\left(-2, \frac{3}{4}\right) > 0$ and so at $\left(-2, \frac{3}{4}\right)$ if we allow y to vary and hold x fixed the function will be increasing.

Note that, because of the three dimensional nature of the graph of this function we can't expect the increasing/decreasing nature of the function in one direction to be the same as in any other direction! In this case it did happen to be the same behavior but there is no reason to expect that in general.

3. Write down the vector equations of the tangent lines to the traces for $f(x,y) = x\,\mathbf{e}^{2x-y^2}$ at $(2,0)$.

Step 1

We know that there are two traces. One for $x = 2$ (*i.e.* x is fixed and y is allowed to vary) and one for $y = 0$ (*i.e.* y is fixed and x is allowed to vary). We also know that $f_y(2,0)$ will be the slope for the first trace (y varies and x is fixed!) and $f_x(2,0)$ will be the slope for the second trace (x varies and y is fixed!).

So, we'll need the value of both of these partial derivatives. Here is that work,

$$f_x(x,y) = \mathbf{e}^{2x-y^2} + 2x\,\mathbf{e}^{2x-y^2} \qquad \to \qquad f_x(2,0) = 5\mathbf{e}^4 = 272.9908$$
$$f_y(x,y) = -2yx\,\mathbf{e}^{2x-y^2} \qquad \to \qquad f_y(2,0) = 0$$

Step 2

Now, we need to write down the vector equation of the line and so we don't (at some level) need the "slopes" as listed in the previous step. What we need are tangent vectors that give these slopes.

Recall from the notes that the tangent vector for the first trace is,

$$\langle 0, 1, f_y(2,0)\rangle = \langle 0,1,0\rangle$$

Likewise, the tangent vector for the second trace is,

$$\langle 1, 0, f_x(2,0)\rangle = \left\langle 1,0,5\mathbf{e}^4\right\rangle$$

Step 3

Next, we'll also need the position vector for the point on the surface that we are looking at. This is,

$$\langle 2, 0, f(2,0)\rangle = \left\langle 2,0,2\mathbf{e}^4\right\rangle$$

Step 4

Finally, the vector equation of the tangent line for the first trace is,

$$\boxed{\vec{r}\,(t) = \left\langle 2, 0, 2\mathbf{e}^{4} \right\rangle + t\,\langle 0, 1, 0\rangle = \left\langle 2, t, 2\mathbf{e}^{4} \right\rangle}$$

and the trace for the second trace is,

$$\boxed{\vec{r}\,(t) = \left\langle 2, 0, 2\mathbf{e}^{4} \right\rangle + t\left\langle 1, 0, 5\mathbf{e}^{4} \right\rangle = \left\langle 2 + t, 0, 2\mathbf{e}^{4} + 5\mathbf{e}^{4}t \right\rangle}$$

13.4 High Order Partial Derivatives

1. Verify Clairaut's Theorem for the following function.

$$f(x,y) = x^3y^2 - \frac{4y^6}{x^3}$$

Step 1

First, we know we'll need the two 1^{st} order partial derivatives. Here they are,

$$f_x = 3x^2y^2 + 12x^{-4}y^6 \qquad f_y = 2x^3y - 24x^{-3}y^5$$

Step 2

Now let's compute each of the mixed second order partial derivatives.

$$f_{x\,y} = (f_x)_y = 6x^2y + 72x^{-4}y^5 \qquad f_{y\,x} = (f_y)_x = 6x^2y + 72x^{-4}y^5$$

Okay, we can see that $f_{x\,y} = f_{y\,x}$ and so Clairaut's theorem has been verified for this function.

---

2. Verify Clairaut's Theorem for the following function.

$$A(x,y) = \cos\left(\frac{x}{y}\right) - x^7y^4 + y^{10}$$

Step 1

First, we know we'll need the two 1^{st} order partial derivatives. Here they are,

$$A_x = -\frac{1}{y}\sin\left(\frac{x}{y}\right) - 7x^6y^4 \qquad A_y = \frac{x}{y^2}\sin\left(\frac{x}{y}\right) - 4x^7y^3 + 10y^9$$

Step 2

Now let's compute each of the mixed second order partial derivatives.

$$A_{x\,y} = (A_x)_y = \frac{1}{y^2}\sin\left(\frac{x}{y}\right) + \frac{x}{y^3}\cos\left(\frac{x}{y}\right) - 28x^6y^3$$
$$A_{y\,x} = (A_y)_x = \frac{1}{y^2}\sin\left(\frac{x}{y}\right) + \frac{x}{y^3}\cos\left(\frac{x}{y}\right) - 28x^6y^3$$

Okay, we can see that $A_{x\,y} = A_{y\,x}$ and so Clairaut's theorem has been verified for this function.

---

3. Find all 2^{nd} order derivatives for the following function.

$$g(u,v) = u^3v^4 - 2u\sqrt{v^3} + u^6 - \sin(3v)$$

Step 1

First, we know we'll need the two 1^{st} order partial derivatives. Here they are,

$$g_u = 3u^2v^4 - 2v^{\frac{3}{2}} + 6u^5 \qquad\qquad g_v = 4u^3v^3 - 3uv^{\frac{1}{2}} - 3\cos(3v)$$

Step 2

Now let's compute each of the second order partial derivatives.

$$g_{u\,u} = (g_u)_u = 6uv^4 + 30u^4$$
$$g_{u\,v} = (g_u)_v = 12u^2v^3 - 3v^{\frac{1}{2}}$$
$$g_{v\,u} = g_{u\,v} = 12u^2v^3 - 3v^{\frac{1}{2}} \qquad \text{by Clairaut's Theorem}$$
$$g_{v\,v} = (g_v)_v = 12u^3v^2 - \frac{3}{2}uv^{-\frac{1}{2}} + 9\sin(3v)$$

---

4. Find all 2^{nd} order derivatives for the following function.

$$f(s,t) = s^2 t + \ln\left(t^2 - s\right)$$

Step 1

First, we know we'll need the two 1^{st} order partial derivatives. Here they are,

$$f_s = 2st - \frac{1}{t^2 - s} \qquad f_t = s^2 + \frac{2t}{t^2 - s}$$

Step 2

Now let's compute each of the second order partial derivatives.

$$f_{s\,s} = (f_s)_s = 2t - \frac{1}{\left(t^2 - s\right)^2}$$
$$f_{s\,t} = (f_s)_t = 2s + \frac{2t}{\left(t^2 - s\right)^2}$$
$$f_{t\,s} = f_{s\,t} = 2s + \frac{2t}{\left(t^2 - s\right)^2} \qquad \text{by Clairaut's Theorem}$$
$$f_{t\,t} = (f_t)_t = \frac{-2t^2 - 2s}{\left(t^2 - s\right)^2}$$

5. Find all 2^{nd} order derivatives for the following function.

$$h(x,y) = \mathbf{e}^{x^4 y^6} - \frac{y^3}{x}$$

Step 1

First, we know we'll need the two 1^{st} order partial derivatives. Here they are,

$$h_x = 4x^3 y^6 \mathbf{e}^{x^4 y^6} + \frac{y^3}{x^2} \qquad h_y = 6y^5 x^4 \mathbf{e}^{x^4 y^6} - \frac{3y^2}{x}$$

Step 2

Now let's compute each of the second order partial derivatives.

$$h_{x\,x} = (h_x)_x = 12x^2y^6\mathbf{e}^{x^4y^6} + 16x^6y^{12}\mathbf{e}^{x^4y^6} - \frac{2y^3}{x^3}$$
$$h_{x\,y} = (h_x)_y = 24x^3y^5\mathbf{e}^{x^4y^6} + 24x^7y^{11}\mathbf{e}^{x^4y^6} + \frac{3y^2}{x^2}$$
$$h_{y\,x} = h_{x\,y} = 24x^3y^5\mathbf{e}^{x^4y^6} + 24x^7y^{11}\mathbf{e}^{x^4y^6} + \frac{3y^2}{x^2} \qquad \text{by Clairaut's Theorem}$$
$$h_{y\,y} = (h_y)_y = 30y^4x^4\mathbf{e}^{x^4y^6} + 36y^{10}x^8\mathbf{e}^{x^4y^6} - \frac{6y}{x}$$

---

6. Find all 2^{nd} order derivatives for the following function.

$$f(x,y,z) = \frac{x^2y^6}{z^3} - 2x^6z + 8y^{-3}x^4 + 4z^2$$

Step 1

First, we know we'll need the three 1^{st} order partial derivatives. Here they are,

$$f_x = 2xy^6z^{-3} - 12x^5z + 32y^{-3}x^3$$
$$f_y = 6x^2y^5z^{-3} - 24y^{-4}x^4$$
$$f_z = -3x^2y^6z^{-4} - 2x^6 + 8z$$

Step 2

Now let's compute each of the second order partial derivatives (and there will be a few of them....).

$$
\begin{aligned}
f_{xx} &= (f_x)_x = 2y^6z^{-3} - 60x^4z + 96y^{-3}x^2 \\
f_{xy} &= (f_x)_y = 12xy^5z^{-3} - 96y^{-4}x^3 \\
f_{xz} &= (f_x)_z = -6xy^6z^{-4} - 12x^5 \\
f_{yx} &= f_{xy} = 12xy^5z^{-3} - 96y^{-4}x^3 && \text{by Clairaut's Theorem} \\
f_{yy} &= (f_y)_y = 30x^2y^4z^{-3} + 96y^{-5}x^4 \\
f_{yz} &= (f_y)_z = -18x^2y^5z^{-4} \\
f_{zx} &= f_{xz} = -6xy^6z^{-4} - 12x^5 && \text{by Clairaut's Theorem} \\
f_{zy} &= f_{yz} = -18x^2y^5z^{-4} && \text{by Clairaut's Theorem} \\
f_{zz} &= (f_z)_z = 12x^2y^6z^{-5} + 8
\end{aligned}
$$

Note that when we used Clairaut's Theorem here we used the natural extension to the Theorem we gave in the notes.

---

7. Given $f(x,y,z) = x^4y^3z^6$ find $\dfrac{\partial^6 f}{\partial y\partial z^2\partial y\partial x^2}$.

Step 1

Through a natural extension of Clairaut's theorem we know we can do these partial derivatives in any order we wish to. However, in this case there doesn't seem to be any reason to do anything other than the order shown in the problem statement.

Here is the first derivative we need to take.

$$\frac{\partial f}{\partial x} = 4x^3y^3z^6$$

Step 2

The second derivative is,

$$\frac{\partial^2 f}{\partial x^2} = 12x^2y^3z^6$$

Step 3

The third derivative is,

$$\frac{\partial^3 f}{\partial y \partial x^2} = 36x^2y^2z^6$$

Step 4

The fourth derivative is,

$$\frac{\partial^4 f}{\partial z \partial y \partial x^2} = 216x^2y^2z^5$$

Step 5

The fifth derivative is,

$$\frac{\partial^5 f}{\partial z^2 \partial y \partial x^2} = 1080x^2y^2z^4$$

Step 6

The sixth and final derivative we need for this problem is,

$$\boxed{\frac{\partial^6 f}{\partial y \partial z^2 \partial y \partial x^2} = 2160x^2yz^4}$$

---

8. Given $w = u^2\mathbf{e}^{-6v} + \cos\left(u^6 - 4u + 1\right)$ find $w_{v\,u\,u\,v\,v}$.

Step 1

Through a natural extension of Clairaut's theorem we know we can do these partial derivatives in any order we wish to. However, in this case there doesn't seem to be any reason to do anything other than the order shown in the problem statement.

Here is the first derivative we need to take.

$$w_v = -6u^2\mathbf{e}^{-6v}$$

Note that if we'd done a couple of u derivatives first the second derivative would have been a product rule. Because we did the v derivative first we won't need to worry about the "messy" u derivatives of the second term as that term differentiates to zero when differentiating with respect to v.

Step 2

The second derivative is,

$$w_{v\,u} = -12u\mathbf{e}^{-6v}$$

Step 3

The third derivative is,

$$w_{v\,u\,u} = -12\mathbf{e}^{-6v}$$

Step 4

The fourth derivative is,

$$w_{v\,u\,u\,v} = 72\mathbf{e}^{-6v}$$

Step 5

The fifth and final derivative we need for this problem is,

$$\boxed{w_{v\,u\,u\,v\,v} = -432\mathbf{e}^{-6v}}$$

---

9. Given $G(x,y) = y^4 \sin(2x) + x^2\left(y^{10} - \cos\left(y^2\right)\right)^7$ find $G_{y\,y\,y\,x\,x\,x\,y}$.

Step 1

Through a natural extension of Clairaut's theorem we know we can do these partial derivatives in any order we wish to.

In this case the y derivatives of the second term will become unpleasant at some point given that we have four of them. However, the second term has an x^2 and there are three x derivatives we'll need to do eventually. Therefore, the second term will differentiate to zero with the third x derivative. So, let's make heavy use of Clairaut's to do the three x derivatives first prior to any of the y derivatives so we won't need to deal with the "messy" y derivatives with the second term.

Here is the first derivative we need to take.

$$G_x = 2y^4 cos\left(2x\right) + 2x\left(y^{10} - \cos\left(y^2\right)\right)^7$$

Note that if we'd done a couple of y derivatives first the second would have been a product rule and because we did the x derivative first we won't need to every work about the "messy" u derivatives of the second term.

Step 2

The second derivative is,

$$G_{x\,x} = -4y^4 sin\left(2x\right) + 2\left(y^{10} - \cos\left(y^2\right)\right)^7$$

Step 3

The third derivative is,

$$G_{x\,x\,x} = -8y^4\cos\left(2x\right)$$

Step 4

The fourth derivative is,

$$G_{x\,x\,x\,y} = -32y^3\cos\left(2x\right)$$

Step 5

The fifth derivative is,

$$G_{x\,x\,x\,y\,y} = -96y^2\cos\left(2x\right)$$

Step 6

The sixth derivative is,

$$G_{xxxyyy} = -192y\cos(2x)$$

Step 7

The seventh and final derivative we need for this problem is,

$$\boxed{G_{yyyxxxy} = G_{xxxyyyy} = -192\cos(2x)}$$

13.5 Differentials

1. Compute the differential of the following function.

$$z = x^2 \sin(6y)$$

Solution

Not much to do here. Just recall that the differential in this case is,

$$dz = z_x\,dx + z_y\,dy$$

The differential is then,

$$\boxed{dz = 2x\sin(6y)\,dx + 6x^2\cos(6y)\,dy}$$

2. Compute the differential of the following function.

$$f(x, y, z) = \ln\left(\frac{xy^2}{z^3}\right)$$

Solution

Not much to do here. Just recall that the differential in this case is,

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$

The differential is then,

$$\boxed{df = \frac{1}{x}dx + \frac{2}{y}dy - \frac{3}{z}dz}$$

13.6 Chain Rule

1. Given the following information use the Chain Rule to determine $\dfrac{dz}{dt}$.

$$z = \cos\left(y\,x^2\right) \qquad x = t^4 - 2t, \quad y = 1 - t^6$$

Solution

Okay, we can just use the "formula" from the notes to determine this derivative. Here is the work for this problem.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= \left[-2xy\sin\left(yx^2\right)\right]\left[4t^3-2\right] + \left[-x^2\sin\left(yx^2\right)\right]\left[-6t^5\right]\\ &= \boxed{\begin{aligned}&-2\left(t^4-2t\right)\left(1-t^6\right)\left(4t^3-2\right)\sin\left(\left(1-t^6\right)\left(t^4-2t\right)^2\right)\\ &\qquad\qquad +6t^5\left(t^4-2t\right)^2\sin\left(\left(1-t^6\right)\left(t^4-2t\right)^2\right)\end{aligned}}\end{aligned}$$

In the second step we added brackets just to make it clear which term came from which derivative in the "formula".

Also, we plugged in for x and y in the third step just to get an equation in t. For some of these, due to the mess of the final formula, it might have been easier to just leave the x's and y's alone and acknowledge their definition in terms of t to keep the answer a little "nicer". You should probably ask your instructor for his/her preference in this regard.

---

2. Given the following information use the Chain Rule to determine $\dfrac{dw}{dt}$.

$$w = \frac{x^2 - z}{y^4} \qquad x = t^3 + 7, \quad y = \cos\left(2t\right), \quad z = 4t$$

Solution

Okay, we can just use a (hopefully) pretty obvious extension of the "formula" from the notes to determine this derivative. Here is the work for this problem.

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= \left[\frac{2x}{y^4}\right]\left[3t^2\right] + \left[\frac{-4\left(x^2-z\right)}{y^5}\right]\left[-2\sin\left(2t\right)\right] + \left[-\frac{1}{y^4}\right]\left[4\right] \\ &= \boxed{\frac{6t^2\left(t^3+7\right)}{\cos^4\left(2t\right)} + \frac{8\sin\left(2t\right)\left(\left(t^3+7\right)^2-4t\right)}{\cos^5\left(2t\right)} - \frac{4}{\cos^4\left(2t\right)}}\end{aligned}$$

In the second step we added brackets just to make it clear which term came from which derivative in the "formula".

Also, we plugged in for x and y in the third step just to get an equation in t. For some of these, due to the mess of the final formula, it might have been easier to just leave the x's and y's alone and acknowledge their definition in terms of t to keep the answer a little "nicer". You should probably ask your instructor for his/her preference in this regard.

---

3. Given the following information use the Chain Rule to determine $\dfrac{dz}{dx}$.

$$z = x^2y^4 - 2y \qquad\qquad y = \sin\left(x^2\right)$$

Solution

Okay, we can just use the "formula" from the notes to determine this derivative. Here is the work for this problem.

$$\begin{aligned}\frac{dz}{dx} &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} \\ &= \left[2xy^4\right] + \left[4x^2y^3 - 2\right]\left[2x\cos\left(x^2\right)\right] \\ &= \boxed{2x\sin^4\left(x^2\right) + 2x\left(4x^2\sin^3\left(x^2\right) - 2\right)\cos\left(x^2\right)}\end{aligned}$$

In the second step we added brackets just to make it clear which term came from which derivative in the "formula".

Also, we plugged in for y in the third step just to get an equation in x. For some of these, due to the mess of the final formula, it might have been easier to just leave the y's alone and acknowledge their definition in terms of x to keep the answer a little "nicer". You should probably ask your instructor for his/her preference in this regard.

4. Given the following information use the Chain Rule to determine $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

$$z = x^{-2}y^6 - 4x \qquad x = u^2 v, \quad y = v - 3u$$

Solution

Okay, we can just use the “formulas” from the notes (with a small change to the letters) to determine this derivative. Here is the work for this problem.

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\\ &= \left[-2x^{-3}y^6 - 4\right]\left[2uv\right] + \left[6x^{-2}y^5\right]\left[-3\right]\\ &= \boxed{2uv\left(-2u^{-6}v^{-3}(v-3u)^6 - 4\right) - 18u^{-4}v^{-2}(v-3u)^5}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\\ &= \left[-2x^{-3}y^6 - 4\right]\left[u^2\right] + \left[6x^{-2}y^5\right]\left[1\right]\\ &= \boxed{u^2\left(-2u^{-6}v^{-3}(v-3u)^6 - 4\right) + 6u^{-4}v^{-2}(v-3u)^5}\end{aligned}$$

In the second step we added brackets just to make it clear which term came from which derivative in the “formula”.

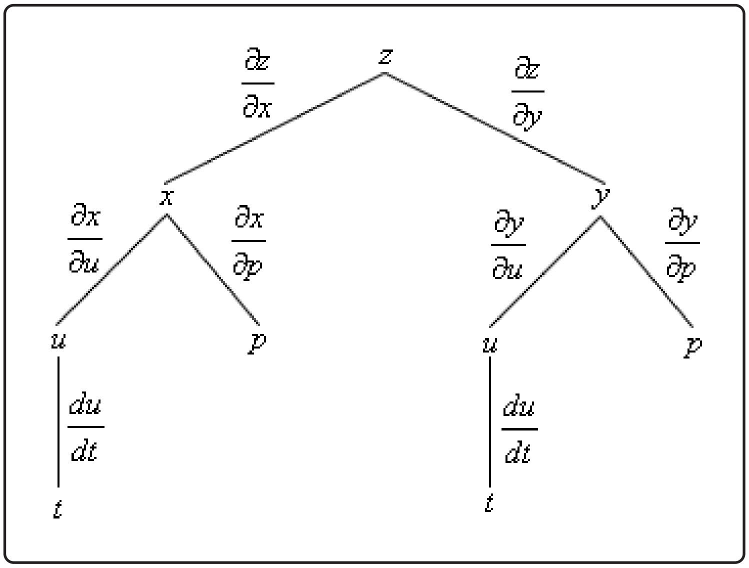
Also, we plugged in for x and y in the third step just to get an equation in u and v. For some of these, due to the mess of the final formula, it might have been easier to just leave the x’s and y’s alone and acknowledge their definition in terms of u and v to keep the answer a little “nicer”. You should probably ask your instructor for his/her preference in this regard.

5. Given the following information use the Chain Rule to determine z_t and z_p .

$$z = 4y\sin(2x) \qquad x = 3u - p, \quad y = p^2 u, \quad u = t^2 + 1$$

Step 1

Okay, we don't have a formula from the notes for this one so we'll need to derive on up first. To do this we'll need the following tree diagram.



Step 2

Here are the formulas for the two derivatives we're being asked to find.

$$z_t = \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} \qquad z_p = \frac{\partial z}{\partial p} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial p}$$

Step 3

Here is the work for this problem.

$$\begin{aligned} z_t &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} \\ &= [8y\cos(2x)][3][2t] + [4\sin(2x)]\left[p^2\right][2t] \\ &= \boxed{48ty\cos(2x) + 8tp^2\sin(2x)} \end{aligned}$$

$$\begin{aligned} z_p &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial p} \\ &= [8y\cos(2x)][-1] + [4\sin(2x)][2pu] \\ &= \boxed{-8y\cos(2x) + 8pu\sin(2x)} \end{aligned}$$

In the second step of each of the derivatives we added brackets just to make it clear which term came from which derivative in the "formula".

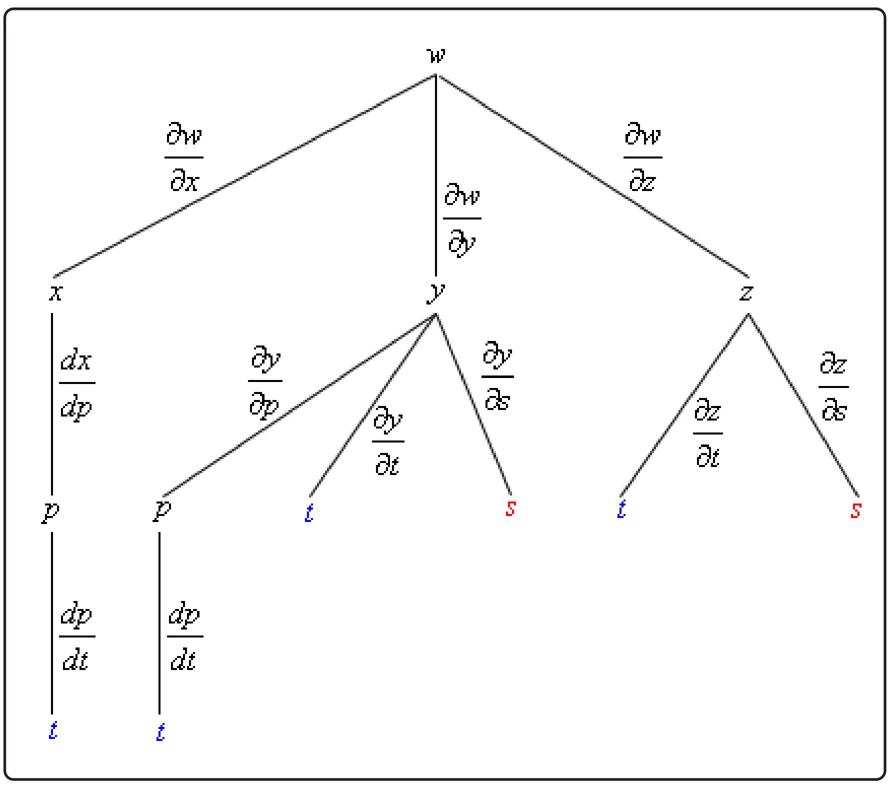
Also, we didn't do any "back substitution" in these derivatives due to the mess that we'd get from each of the derivatives after we got done with all the back substitution.

6. Given the following information use the Chain Rule to determine $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$.

$$w = \sqrt{x^2 + y^2} + \frac{6z}{y} \qquad x = \sin(p), \quad y = p + 3t - 4s, \quad z = \frac{t^3}{s^2}, \quad p = 1 - 2t$$

Step 1

Okay, we don't have a formula from the notes for this one so we'll need to derive on up first. To do this we'll need the following tree diagram.



Some of these tree diagrams can get quite messy. We've colored the variables we're interested in to try and make the branches we need to follow for each derivative a little clearer.

Step 2

Here are the formulas for the two derivatives we're being asked to find.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dp}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Step 3

Here is the work for this problem.

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{dx}{dp}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\
&= \left[\frac{x}{\sqrt{x^2+y^2}}\right]\left[\cos(p)\right]\left[-2\right] + \left[\frac{y}{\sqrt{x^2+y^2}} - \frac{6z}{y^2}\right]\left[1\right]\left[-2\right] + \\
&\qquad + \left[\frac{y}{\sqrt{x^2+y^2}} - \frac{6z}{y^2}\right]\left[3\right] + \left[\frac{6}{y}\right]\left[\frac{3t^2}{s^2}\right] \\
&= \boxed{\frac{-2x\cos(p)}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}} - \frac{6z}{y^2} + \frac{18t^2}{ys^2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= \left[\frac{y}{\sqrt{x^2+y^2}} - \frac{6z}{y^2}\right]\left[-4\right] + \left[\frac{6}{y}\right]\left[-\frac{2t^3}{s^3}\right] \\
&= \boxed{\frac{-4y}{\sqrt{x^2+y^2}} + \frac{24z}{y^2} - \frac{12t^3}{ys^3}}
\end{aligned}$$

In the second step of each of the derivatives we added brackets just to make it clear which term came from which derivative in the "formula" and in general probably aren't needed. We also did a little simplification as needed to get to the third step.

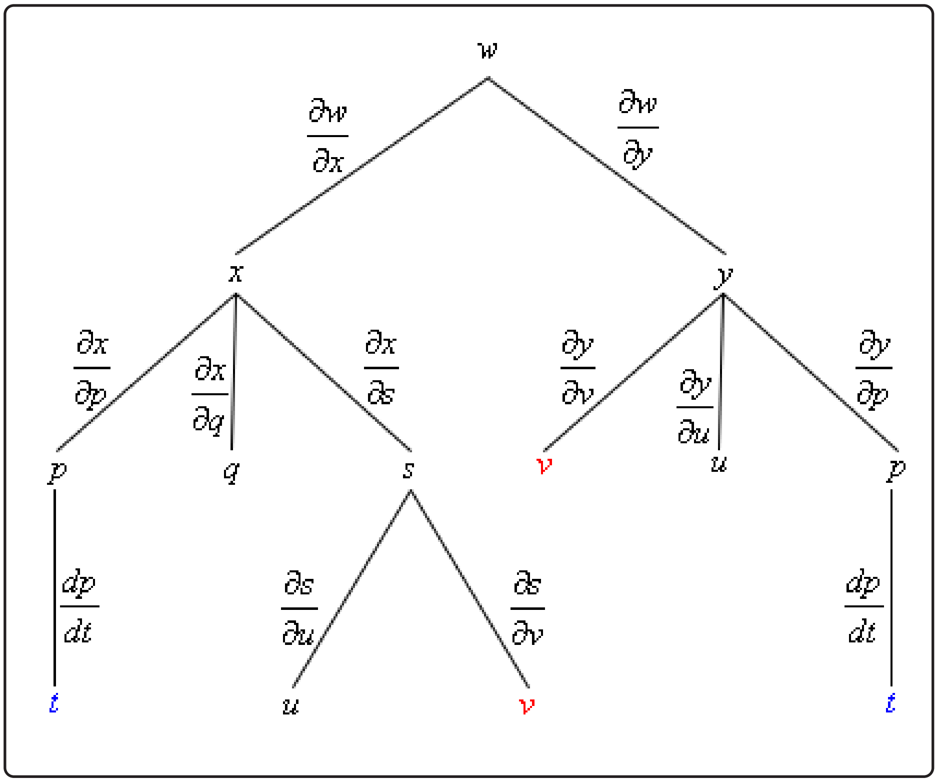
Also, we didn't do any "back substitution" in these derivatives due to the mess that we'd get from each of the derivatives after we got done with all the back substitution.

7. Determine formulas for $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial v}$ for the following situation.

$$w = w(x, y) \qquad x = x(p, q, s), \quad y = y(p, u, v), \quad s = s(u, v), \quad p = p(t)$$

Step 1

To determine the formula for these derivatives we'll need the following tree diagram.



Some of these tree diagrams can get quite messy. We've colored the variables we're interested in to try and make the branches we need to follow for each derivative a little clearer.

Step 2

Here are the formulas we're being asked to find.

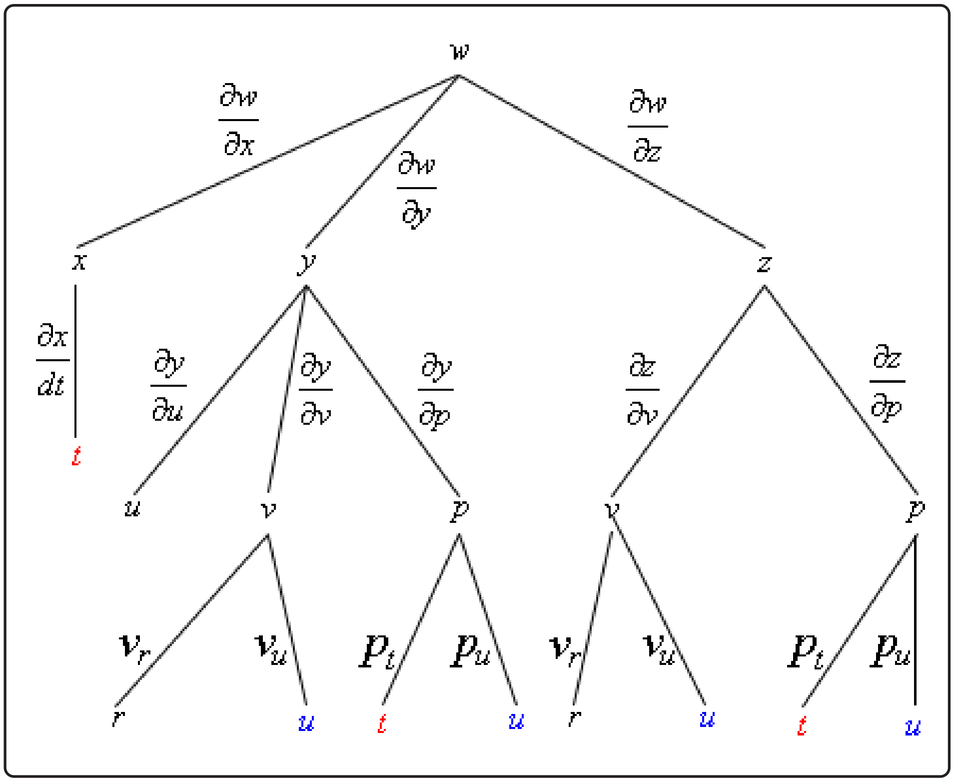
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}\frac{dp}{dt} \qquad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}\frac{\partial s}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$

8. Determine formulas for $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial u}$ for the following situation.

$$w = w(x,y,z) \qquad x = x(t), \quad y = y(u,v,p), \quad z = z(v,p), \quad v = v(r,u), \quad p = p(t,u)$$

Step 1

To determine the formula for these derivatives we'll need the following tree diagram.



Some of these tree diagrams can get quite messy. We've colored the variables we're interested in to try and make the branches we need to follow for each derivative a little clearer.

Also, because the last "row" of branches was getting a little close together we switched to the subscript derivative notation to make it easier to see which derivative was associated with each branch.

Step 2

Here are the formulas we're being asked to find.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial p}\frac{\partial p}{\partial t}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}\frac{\partial v}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}\frac{\partial p}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}\frac{\partial v}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial p}\frac{\partial p}{\partial u}$$

---

9. Compute $\frac{dy}{dx}$ for the following equation.

$$x^2y^4 - 3 = \sin(xy)$$

Step 1

First a quick rewrite of the equation.

$$x^2y^4 - 3 - \sin(xy) = 0$$

Step 2

From the rewrite in the previous step we can see that,

$$F(x, y) = x^2y^4 - 3 - \sin(xy)$$

We can now simply use the formula we derived in the notes to get the derivative.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \boxed{-\frac{2xy^4 - y\cos(xy)}{4x^2y^3 - x\cos(xy)} = \frac{y\cos(xy) - 2xy^4}{4x^2y^3 - x\cos(xy)}}$$

Note that in for the second form of the answer we simply moved the "−" in front of the fraction up to the numerator and multiplied it through. We could just have easily done this with the denominator instead if we'd wanted to.

---

10. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the following equation.

$$\mathbf{e}^{z\,y} + xz^2 = 6xy^4z^3$$

Step 1

First a quick rewrite of the equation.

$$\mathbf{e}^{z\,y} + xz^2 - 6xy^4z^3 = 0$$

Step 2

From the rewrite in the previous step we can see that,

$$F(x, y) = \mathbf{e}^{z\,y} + xz^2 - 6xy^4z^3$$

We can now simply use the formulas we derived in the notes to get the derivatives.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \boxed{-\frac{z^2 - 6y^4z^3}{y\mathbf{e}^{z\,y} + 2xz - 18xy^4z^2} = \frac{6y^4z^3 - z^2}{y\mathbf{e}^{z\,y} + 2xz - 18xy^4z^2}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \boxed{-\frac{z\mathbf{e}^{z\,y} - 24xy^3z^3}{y\mathbf{e}^{z\,y} + 2xz - 18xy^4z^2} = \frac{24xy^3z^3 - z\mathbf{e}^{z\,y}}{y\mathbf{e}^{z\,y} + 2xz - 18xy^4z^2}}$$

Note that in for the second form of each of the answers we simply moved the "−" in front of the fraction up to the numerator and multiplied it through. We could just have easily done this with the denominator instead if we'd wanted to.

11. Determine $f_{u\,u}$ for the following situation.

$$f = f(x, y) \qquad x = u^2 + 3v, \quad y = uv$$

Step 1

These kinds of problems always seem mysterious at first. That is probably because we don't actually know what the function itself is. This isn't really a problem. It simply means that the answers can get a little messy as we'll rarely be able to do much in the way of simplification.

So, the first step here is to get the first derivative and this will require the following chain rule formula.

$$f_u = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

Here is the first derivative,

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}[2u] + \frac{\partial f}{\partial y}[v] = 2u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}$$

Do not get excited about the "unknown" derivatives in our answer here. They will always be present in these kinds of problems.

Step 2

Now, much as we did in the notes, let's do a little rewrite of the answer above as follows,

$$\frac{\partial}{\partial u}(f) = 2u\frac{\partial}{\partial x}(f) + v\frac{\partial}{\partial y}(f)$$

With this rewrite we now have a "formula" for differentiating any function of x and y with respect to u whenever $x = u^2 + 3v$ and $y = uv$. In other words, whenever we have such a function all we need to do is replace the f in the parenthesis with whatever our function is. We'll need this eventually.

Step 3

Now, let's get the second derivative. We know that we find the second derivative as follows,

$$f_{u\,u} = \frac{\partial^2 f}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial u}\left(2u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}\right)$$

Step 4

Now, recall that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are functions of x and y which are in turn defined in terms of u and v as defined in the problem statement. This means that we'll need to do the product rule on the first term since it is a product of two functions that both involve u. We won't need to product rule the second term, in this case, because the first function in that term involves only v's.

Here is that work,

$$f_{u\,u} = 2\frac{\partial f}{\partial x} + 2u\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) + v\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)$$

Because the function is defined only in terms of x and y we cannot "merge" the u and x derivatives in the second term into a "mixed order" second derivative. For the same reason we cannot "merge" the u and y derivatives in the third term.

In each of these cases we are being asked to differentiate functions of x and y with respect to u where x and y are defined in terms of u and v.

Step 5

Now, recall the "formula" from Step 2,

$$\frac{\partial}{\partial u}(f) = 2u\frac{\partial}{\partial x}(f) + v\frac{\partial}{\partial y}(f)$$

Recall that this tells us how to differentiate any function of x and y with respect to u as long as x and y are defined in terms of u and v as they are in this problem.

Well luckily enough for us both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are functions of x and y which in turn are defined in terms of u and v as we need them to be. This means we can use this "formula" for each of the derivatives in the result from Step 4 as follows,

$$\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) &= 2u\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2u\frac{\partial^2 f}{\partial x^2} + v\frac{\partial^2 f}{\partial y\partial x}\\
\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) &= 2u\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2u\frac{\partial^2 f}{\partial x\partial y} + v\frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

Step 6

Okay, all we need to do now is put the results from Step 5 into the result from Step 4 and we'll be done.

$$\begin{aligned}
f_{uu} &= 2\frac{\partial f}{\partial x} + 2u\left[2u\frac{\partial^2 f}{\partial x^2} + v\frac{\partial^2 f}{\partial y\partial x}\right] + v\left[2u\frac{\partial^2 f}{\partial x\partial y} + v\frac{\partial^2 f}{\partial y^2}\right]\\
&= 2\frac{\partial f}{\partial x} + 4u^2\frac{\partial^2 f}{\partial x^2} + 2uv\frac{\partial^2 f}{\partial y\partial x} + 2uv\frac{\partial^2 f}{\partial x\partial y} + v^2\frac{\partial^2 f}{\partial y^2}\\
&= \boxed{2\frac{\partial f}{\partial x} + 4u^2\frac{\partial^2 f}{\partial x^2} + 4uv\frac{\partial^2 f}{\partial x\partial y} + v^2\frac{\partial^2 f}{\partial y^2}}
\end{aligned}$$

Note that we assumed that the two mixed order partial derivative are equal for this problem and so combined those terms. If you can't assume this or don't want to assume this then the second line would be the answer.

13.7 Directional Derivatives

1. Determine the gradient of the following function.

$$f(x,y) = x^2 \sec(3x) - \frac{x^2}{y^3}$$

Solution

Not really a lot to do for this problem. Here is the gradient.

$$\nabla f = \langle f_x, f_y \rangle = \boxed{\left\langle 2x\sec(3x) + 3x^2\sec(3x)\tan(3x) - \frac{2x}{y^3}, \frac{3x^2}{y^4} \right\rangle}$$

---

2. Determine the gradient of the following function.

$$f(x,y,z) = x\cos(xy) + z^2y^4 - 7xz$$

Solution

Not really a lot to do for this problem. Here is the gradient.

$$\nabla f = \langle f_x, f_y, f_z \rangle = \boxed{\langle \cos(xy) - xy\sin(xy) - 7z, -x^2\sin(xy) + 4z^2y^3, 2zy^4 - 7x \rangle}$$

---

3. Determine $D_{\vec{u}}f$ for $f(x,y) = \cos\left(\frac{x}{y}\right)$ in the direction of $\vec{v} = \langle 3, -4 \rangle$.

Step 1

Okay, we know we need the gradient so let's get that first.

$$\nabla f = \left\langle -\frac{1}{y}\sin\left(\frac{x}{y}\right), \frac{x}{y^2} sin\left(\frac{x}{y}\right) \right\rangle$$

Step 2

Also recall that we need to make sure that the direction vector is a unit vector. It is (hopefully) pretty clear that this vector is not a unit vector so let's convert it to a unit vector.

$$\|\vec{v}\| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5 \qquad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{5}\langle 3, -4\rangle = \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$$

Step 3

The directional derivative is then,

$$D_{\vec{u}}f = \left\langle -\frac{1}{y}\sin\left(\frac{x}{y}\right), \frac{x}{y^2}\sin\left(\frac{x}{y}\right)\right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$$
$$= \boxed{-\frac{3}{5y}\sin\left(\frac{x}{y}\right) - \frac{4x}{5y^2}\sin\left(\frac{x}{y}\right) = -\frac{1}{5}\left(\frac{3}{y} + \frac{4x}{y^2}\right) sin\left(\frac{x}{y}\right)}$$

---

4. Determine $D_{\vec{u}}f$ for $f(x,y,z) = x^2y^3 - 4xz$ in the direction of $\vec{v} = \langle -1, 2, 0\rangle$.

Step 1

Okay, we know we need the gradient so let's get that first.

$$\nabla f = \langle 2xy^3 - 4z, 3x^2y^2, -4x\rangle$$

Step 2

Also recall that we need to make sure that the direction vector is a unit vector. It is (hopefully) pretty clear that this vector is not a unit vector so let's convert it to a unit vector.

$$\|\vec{v}\| = \sqrt{(-1)^2 + (2)^2 + (0)^2} = \sqrt{5} \qquad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{5}}\langle -1, 2, 0\rangle = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right\rangle$$

Step 3

The directional derivative is then,

$$D_{\vec{u}}f = \left\langle 2xy^3 - 4z, 3x^2y^2, -4x \right\rangle \cdot \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \right\rangle = \boxed{\frac{1}{\sqrt{5}}\left(4z - 2xy^3 + 6x^2y^2\right)}$$

---

5. Determine $D_{\vec{u}}f\,(3,-1,0)$ for $f\,(x,y,z) = 4x - y^2\mathbf{e}^{3x\,z}$ direction of $\vec{v} = \langle -1, 4, 2 \rangle$.

Step 1

Okay, we know we need the gradient so let's get that first.

$$\nabla f = \left\langle 4 - 3zy^2\mathbf{e}^{3x\,z}, -2y\mathbf{e}^{3x\,z}, -3xy^2\mathbf{e}^{3x\,z} \right\rangle$$

Because we also know that we'll eventually need this evaluated at the point we may as well get that as well.

$$\nabla f\,(3,-1,0) = \langle 4, 2, -9 \rangle$$

Step 2

Also recall that we need to make sure that the direction vector is a unit vector. It is (hopefully) pretty clear that this vector is not a unit vector so let's convert it to a unit vector.

$$\|\vec{v}\| = \sqrt{(-1)^2 + (4)^2 + (2)^2} = \sqrt{21}$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{21}}\langle -1, 4, 2 \rangle = \left\langle -\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right\rangle$$

Step 3

The directional derivative is then,

$$D_{\vec{u}}f\,(3,-1,0) = \langle 4, 2, -9 \rangle \cdot \left\langle -\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right\rangle = \boxed{-\frac{14}{\sqrt{21}}}$$

6. Find the maximum rate of change of $f(x,y) = \sqrt{x^2 + y^4}$ at $(-2,3)$ and the direction in which this maximum rate of change occurs.

Step 1

First, we'll need the gradient and its value at $(-2,3)$.

$$\nabla f = \left\langle \frac{x}{\sqrt{x^2+y^4}}, \frac{2y^3}{\sqrt{x^2+y^4}} \right\rangle \qquad \nabla f(-2,3) = \left\langle \frac{-2}{\sqrt{85}}, \frac{54}{\sqrt{85}} \right\rangle$$

Step 2

Now, by the theorem in class we know that the direction in which the maximum rate of change at the point in question is simply the gradient at $(-2,3)$, which we found in the previous step. So, the direction in which the maximum rate of change of the function occurs is,

$$\nabla f(-2,3) = \boxed{\left\langle \frac{-2}{\sqrt{85}}, \frac{54}{\sqrt{85}} \right\rangle}$$

Step 3

The maximum rate of change is simply the magnitude of the gradient in the previous step. So, the maximum rate of change of the function is,

$$\|\nabla f(-2,3)\| = \sqrt{\frac{4}{85} + \frac{2916}{85}} = \boxed{\sqrt{\frac{584}{17}} = 5.8611}$$

7. Find the maximum rate of change of $f(x,y,z) = \mathbf{e}^{2x}\cos(y-2z)$ at $(4,-2,0)$ and the direction in which this maximum rate of change occurs.

Step 1

First, we'll need the gradient and its value at $(4,-2,0)$.

$$\begin{aligned}\nabla f &= \left\langle 2\mathbf{e}^{2x}\cos(y-2z), -\mathbf{e}^{2x}\sin(y-2z), 2\mathbf{e}^{2x}\sin(y-2z)\right\rangle \\ \nabla f(4,-2,0) &= \left\langle 2\mathbf{e}^{8}\cos(-2), -\mathbf{e}^{8}\sin(-2), 2\mathbf{e}^{8}\sin(-2)\right\rangle \\ &= \langle -2481.03, 2710.58, -5421.15\rangle\end{aligned}$$

Step 2

Now, by the theorem in class we know that the direction in which the maximum rate of change at the point in question is simply the gradient at $(4,-2,0)$, which we found in the previous step. So, the direction in which the maximum rate of change of the function occurs is,

$$\nabla f(4,-2,0) = \boxed{\langle -2481.03, 2710.58, -5421.15\rangle}$$

Step 3

The maximum rate of change is simply the magnitude of the gradient in the previous step. So, the maximum rate of change of the function is,

$$\left\|\nabla f(4,-2,0)\right\| = \sqrt{(-2481.03)^2 + (2710.58)^2 + (-5421.15)^2} = \boxed{6549.17}$$

---

14 Applications of Partial Derivatives

In this chapter we'll take a look at a couple of applications of partial derivatives. The applications here are either very similar to applications we saw for derivatives of single variable functions or extensions of those applications.

For example we will be looking at the tangent plane to a surface rather than tangent lines to curves as we did with single variable functions.

In addition we be finding relative and absolute extrema of multi-variable functions. The difference in this chapter compared to the last time we saw these applications is that they will often involve a lot more work. Because of the increased difficulty of the problems we'll be restricting ourselves to finding the relative and absolute extrema of functions of two variables only.

We will also be looking at Lagrange Multipliers. This is a method that will allow us to optimize a function that is subject to some constraint. That is to say optimizing a function of two or three variables where the variables must also satisfy some constraint (usually in the form on an equation involving the variables).

The following sections are the practice problems, with solutions, for this material.

14.1 Tangent Planes and Linear Approximations

1. Find the equation of the tangent plane to $z = x^2 \cos(\pi y) - \dfrac{6}{xy^2}$ at $(2, -1)$.

Step 1

First, we know we'll need the two 1^{st} order partial derivatives. Here they are,

$$f_x = 2x \cos(\pi y) + \frac{6}{x^2 y^2} \qquad f_y = -\pi x^2 \sin(\pi y) + \frac{12}{xy^3}$$

Step 2

Now we also need the two derivatives from the first step and the function evaluated at $(2, -1)$. Here are those evaluations,

$$f(2,-1) = -7 \qquad f_x(2,-1) = -\frac{5}{2} \qquad f_y(2,-1) = -6$$

Step 3

The tangent plane is then,

$$\boxed{z = -7 - \frac{5}{2}(x-2) - 6(y+1) = -\frac{5}{2}x - 6y - 8}$$

---

2. Find the equation of the tangent plane to $z = x\sqrt{x^2 + y^2} + y^3$ at $(-4, 3)$.

Step 1

First, we know we'll need the two 1^{st} order partial derivatives. Here they are,

$$f_x = \sqrt{x^2 + y^2} + \frac{x^2}{\sqrt{x^2 + y^2}} \qquad f_y = \frac{xy}{\sqrt{x^2 + y^2}} + 3y^2$$

Step 2

Now we also need the two derivatives from the first step and the function evaluated at $(-4,3)$. Here are those evaluations,

$$f(-4,3) = 7 \qquad f_x(-4,3) = \frac{41}{5} \qquad f_y(-4,3) = \frac{123}{5}$$

Step 3

The tangent plane is then,

$$\boxed{z = 7 + \frac{41}{5}(x+4) + \frac{123}{5}(y-3) = \frac{41}{5}x + \frac{123}{5}y - 34}$$

---

3. Find the linear approximation to $z = 4x^2 - y\mathbf{e}^{2x+y}$ at $(-2,4)$.

Step 1

Recall that the linear approximation to a function at a point is really nothing more than the tangent plane to that function at the point.

So, we know that we'll first need the two 1^{st} order partial derivatives. Here they are,

$$f_x = 8x - 2y\mathbf{e}^{2x+y} \qquad f_y = -\mathbf{e}^{2x+y} - y\mathbf{e}^{2x+y}$$

Step 2

Now we also need the two derivatives from the first step and the function evaluated at $(-2,4)$. Here are those evaluations,

$$f(-2,4) = 12 \qquad f_x(-2,4) = -24 \qquad f_y(-2,4) = -5$$

Step 3

The linear approximation is then,

$$\boxed{L(x,y) = 12 - 24(x+2) - 5(y-4) = -24x - 5y - 16}$$

14.2 Gradient Vector, Tangent Planes and Normal Lines

1. Find the tangent plane and normal line to $x^2y = 4z\mathbf{e}^{x+y} - 35$ at $(3, -3, 2)$.

Step 1

First, we need to do a quick rewrite of the equation as,

$$x^2y - 4z\mathbf{e}^{x+y} = -35$$

Step 2

Now we need the gradient of the function on the left side of the equation from Step 1 and its value at $(3, -3, 2)$. Here are those quantities.

$$\nabla f = \left\langle 2xy - 4z\mathbf{e}^{x+y}, x^2 - 4z\mathbf{e}^{x+y}, -4\mathbf{e}^{x+y} \right\rangle \qquad \nabla f\left(3,-3,2\right) = \left\langle -26, 1, -4 \right\rangle$$

Step 3

The tangent plane is then,

$$\boxed{-26\left(x-3\right) + \left(1\right)\left(y+3\right) - 4\left(z-2\right) = 0} \quad \Rightarrow \quad \boxed{-26x + y - 4z = -89}$$

The normal line is,

$$\boxed{\vec r\left(t\right) = \left\langle 3,-3,2\right\rangle + t\left\langle -26,1,-4\right\rangle = \left\langle 3-26t, -3+t, 2-4t\right\rangle}$$

2. Find the tangent plane and normal line to $\ln\left(\frac{x}{2y}\right) = z^2\left(x-2y\right) + 3z + 3$ at $(4, 2, -1)$.

Step 1

First, we need to do a quick rewrite of the equation as,

$$\ln\left(\frac{x}{2y}\right) - z^2\left(x-2y\right) - 3z = 3$$

Step 2

Now we need the gradient of the function on the left side of the equation from Step 1 and its value at $(4, 2, -1)$. Here are those quantities.

$$\nabla f = \left\langle \frac{1}{x} - z^2, -\frac{1}{y} + 2z^2, -2z\left(x - 2y\right) - 3 \right\rangle \qquad \nabla f\left(4, 2, -1\right) = \left\langle -\frac{3}{4}, \frac{3}{2}, -3 \right\rangle$$

Step 3

The tangent plane is then,

$$\boxed{-\frac{3}{4}\left(x - 4\right) + \frac{3}{2}\left(y - 2\right) - 3\left(z + 1\right) = 0} \quad \Rightarrow \quad \boxed{-\frac{3}{4}x + \frac{3}{2}y - 3z = 3}$$

The normal line is,

$$\boxed{\vec r\left(t\right) = \left\langle 4, 2, -1 \right\rangle + t\left\langle -\frac{3}{4}, \frac{3}{2}, -3 \right\rangle = \left\langle 4 - \frac{3}{4}t, 2 + \frac{3}{2}t, -1 - 3t \right\rangle}$$

14.3 Relative Minimums and Maximums

1. Find and classify all the critical points of the following function.

$$f(x,y) = (y-2)x^2 - y^2$$

Step 1

We're going to need a bunch of derivatives for this problem so let's get those taken care of first.

$$f_x = 2(y-2)x \qquad\qquad f_y = x^2 - 2y$$
$$f_{xx} = 2(y-2) \qquad\qquad f_{xy} = 2x \qquad\qquad f_{yy} = -2$$

Step 2

Now, let's find the critical points for this problem. That means solving the following system.

$$f_x = 0 : 2(y-2)x = 0 \qquad \rightarrow \qquad y = 2 \text{ or } x = 0$$
$$f_y = 0 : x^2 - 2y = 0$$

As shown above we have two possible options from the first equation. We can plug each into the second equation to get the critical points for the equation.

$$y = 2 : x^2 - 4 = 0 \quad \rightarrow \quad x = \pm 2 \quad \Rightarrow \quad (2,2) \text{ and } (-2,2)$$

$$x = 0 : -2y = 0 \quad \rightarrow \quad y = 0 \quad \Rightarrow \quad (0,0)$$

Be careful in writing down the solution to this system of equations. One of the biggest mistakes students make here is to just write down all possible combinations of x and y values they get. That is not how these types of systems are solved!

We got $x = \pm 2$ above only because we assumed first that $y = 2$ and so that leads to the two solutions listed in that first line above. Likewise, we only got $y = 0$ because we first assumed that $x = 0$ which leads to the third solution listed above in the second line. The points $(0,2)$, $(-2,0)$ and $(2,0)$ are NOT solutions to this system as can be easily checked by plugging them into the second equation in the system.

So, do not just "mix and match" all possible values of x and y into points and call them all solutions. This will often lead to points that are not solutions to the system of equations. You need to always keep in mind what assumptions you had to make in order to get certain x or y values in the solution process and only match those values up with the assumption you had to make.

So, in summary, this function has three critical points : $(0,0)$, $(-2,2)$, $(2,2)$.

Also, before proceeding with the next step we should note that there are multiple ways to solve this system. The process you used may not be the same as the one we used here. However, regardless of the process used to solve the system, the solutions should always be the same.

Step 3

Next, we'll need the following,

$$D(x,y) = f_{x\,x} f_{y\,y} - [f_{x\,y}]^2 = [2(y-2)][-2] - [2x]^2 = -4(y-2) - 4x^2$$

Step 4

With $D(x,y)$ we can now classify each of the critical points as follows.

$(0,0)$	:	$D(0,0) = 8 > 0$	$f_{x\,x}(0,0) = -4 < 0$	Relative Maximum
$(-2,2)$	:	$D(-2,2) = -16 < 0$		Saddle Point
$(2,2)$	:	$D(2,2) = -16 < 0$		Saddle Point

Don't forget to check the value of $f_{x\,x}$ when D is positive so we can get the correct classification (*i.e.* maximum or minimum) and also recall that for negative D we don't need the second check as we know the critical point will be a saddle point.

---

2. Find and classify all the critical points of the following function.

$$f(x,y) = 7x - 8y + 2xy - x^2 + y^3$$

Step 1

We're going to need a bunch of derivatives for this problem so let's get those taken care of first.

$$f_x = 7 + 2y - 2x \qquad f_y = -8 + 2x + 3y^2$$
$$f_{x\,x} = -2 \qquad f_{x\,y} = 2 \qquad f_{y\,y} = 6y$$

Step 2

Now, let's find the critical points for this problem. That means solving the following system.

$$f_x = 0 : \; 7 + 2y - 2x = 0$$
$$f_y = 0 : -8 + 2x + 3y^2 = 0 \qquad \to \qquad x = 4 - \frac{3}{2}y^2$$

As shown above we solved the second equation for x and we can now plug this into the first equation as follows,

$$0 = 7 + 2y - 2\left(4 - \frac{3}{2}y^2\right) = 3y^2 + 2y - 1 = (3y-1)(y+1) \qquad \to \qquad y = -1, \;\; y = \frac{1}{3}$$

This gives two values of y which we can now plug back into either of our equations to find corresponding x values. Here is that work.

$$y = -1 : x = 4 - \tfrac{3}{2}(-1)^2 = \tfrac{5}{2} \qquad \Rightarrow \qquad \left(\tfrac{5}{2}, -1\right)$$
$$y = \tfrac{1}{3} : x = 4 - \tfrac{3}{2}\left(\tfrac{1}{3}\right)^2 = \tfrac{23}{6} \qquad \Rightarrow \qquad \left(\tfrac{23}{6}, \tfrac{1}{3}\right)$$

Be careful in writing down the solution to this system of equations. One of the biggest mistakes students make here is to just write down all possible combinations of x and y values they get. That is not how these types of systems are solved!

We got $x = \frac{5}{2}$ above only because we assumed first that $y = -1$ and so that leads to the solution listed in first line above. Likewise, we only got $x = \frac{23}{6}$ because we first assumed that $y = \frac{1}{3}$ which leads to the second solution listed in the second line above. The points $\left(\frac{5}{2}, \frac{1}{3}\right)$ and $\left(\frac{23}{6}, -1\right)$ are NOT a solutions to this system as can be easily checked by plugging these points into the either of the equations in the system.

So, do not just "mix and match" all possible values of x and y into points and call them all solutions. This will often lead to points that are not solutions to the system of equations. You need to always keep in mind what assumptions you had to make in order to get certain x or y values in the solution process and only match those values up with the assumption you had to make.

So, in summary, this function has two critical points : $\left(\frac{5}{2}, -1\right)$, $\left(\frac{23}{6}, \frac{1}{3}\right)$.

Before proceeding with the next step we should note that there are multiple ways to solve this system. The process you used may not be the same as the one we used here. However, regardless of the process used to solve the system, the solutions should always be the same.

Step 3

Next, we'll need the following,

$$D(x,y) = f_{x\,x} f_{y\,y} - [f_{x\,y}]^2 = [-2][6y] - [2]^2 = -12y - 4$$

Step 4

With $D(x,y)$ we can now classify each of the critical points as follows.

$$\left(\frac{5}{2},-1\right) \quad : \quad D\left(\frac{5}{2},-1\right) = 8 > 0 \qquad f_{x\,x}\left(\frac{5}{2},-1\right) = -2 < 0 \qquad \text{Relative Maximum}$$
$$\left(\frac{23}{6},\frac{1}{3}\right) \quad : \quad D\left(\frac{23}{6},\frac{1}{3}\right) = -8 < 0 \qquad \text{Saddle Point}$$

Don't forget to check the value of $f_{x\,x}$ when D is positive so we can get the correct classification (*i.e.* maximum or minimum) and also recall that for negative D we don't need the second check as we know the critical point will be a saddle point.

3. Find and classify all the critical points of the following function.

$$f(x,y) = \left(3x + 4x^3\right)\left(y^2 + 2y\right)$$

Step 1

We're going to need a bunch of derivatives for this problem so let's get those taken care of first.

Do not make these derivatives harder than really are! Do not multiply the function out! We just have a function of x's times a function of y's. Take advantage of that when doing the derivatives.

$$f_x = \left(3 + 12x^2\right)\left(y^2 + 2y\right) \qquad f_y = \left(3x + 4x^3\right)(2y + 2)$$
$$f_{x\,x} = 24x\left(y^2 + 2y\right) \qquad f_{x\,y} = \left(3 + 12x^2\right)(2y + 2) \qquad f_{y\,y} = 2\left(3x + 4x^3\right)$$

Step 2

Now, let's find the critical points for this problem. That means solving the following system.

$$f_x = 0: \ \left(3+12x^2\right)\left(y^2+2y\right) = 0$$
$$f_y = 0: \left(3x+4x^3\right)(2y+2) = 0$$

We could start the solution process with either of these equations as both are pretty simple to solve. Let's start with the first equation.

$$\left(3+12x^2\right)\left(y^2+2y\right) = \left(3+12x^2\right)(y)(y+2) = 0 \quad \to \quad y=0, y=-2, \ x = \pm\frac{1}{2}i$$

Okay, we've got something to deal with at this point. We clearly get four different values to work with here. Two of them, however, are complex. One of the unspoken rules here is that we are only going to work with real values and so we will ignore any complex answers and work with only the real values.

So, we now have two possible values of y so let's plug each of them into the second equation as follows,

$$y=0: 2\left(3x+4x^3\right) = 2x\left(3+4x^2\right) = 0 \quad \to \quad x=0, x=\pm\tfrac{\sqrt{3}}{2}i \quad \Rightarrow \quad (0,0)$$
$$y=-2: -2\left(3x+4x^3\right) = 2x\left(3+4x^2\right) = 0 \quad \to \quad x=0, x=\pm\tfrac{\sqrt{3}}{2}i \quad \Rightarrow \quad (0,-2)$$

As with the first part of the solution process we only take the real values and so ignore the complex portions from this part as well.

In the previous two problems we made mention at this point to be careful and not just from up points for all possible combinations of the x and y values we have at this point.

One of the reasons that students often do that is because of problems like this one where it appears that we are doing just that. However, we haven't just randomly formed all combinations here. It just so happened that when we assumed $y=0$ and $y=-2$ that we just happened to get the same value of x, $x=0$. In general, this won't happen and so do not read into this problem that we always just form all possible combinations of the x and y values to get the critical points for a function. We must always pay attention to the assumptions made at the start of each step.

So, in summary, this function has two critical points : $(0,-2)$, $(0,0)$.

Before proceeding with the next step we should note that there are multiple ways to solve this system. The process you used may not be the same as the one we used here. However, regardless of the process used to solve the system, the solutions should always be the same.

Step 3

Next, we'll need the following,

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - [f_{xy}]^2 \\ &= \left[24x\left(y^2+2y\right)\right]\left[2\left(3x+4x^3\right)\right] - \left[\left(3+12x^2\right)(2y+2)\right]^2 \\ &= 48x\left(3x+4x^3\right)\left(y^2+2y\right) - \left(3+12x^2\right)^2(2y+2)^2 \end{aligned}$$

Step 4

With $D(x,y)$ we can now classify each of the critical points as follows.

$(0,-2)$	:	$D(0,-2) = -36 < 0$	Saddle Point
$(0,0)$	:	$D(0,0) = -36 < 0$	Saddle Point

Don't always expect every problem to have at least one relative extrema. As this example has shown it is completely possible to have only saddle points.

---

4. Find and classify all the critical points of the following function.

$$f(x,y) = 3y^3 - x^2y^2 + 8y^2 + 4x^2 - 20y$$

Step 1

We're going to need a bunch of derivatives for this problem so let's get those taken care of first.

$$f_x = -2xy^2 + 8x \qquad\qquad f_y = 9y^2 - 2x^2y + 16y - 20$$

$$f_{xx} = -2y^2 + 8 \qquad\qquad f_{xy} = -4xy \qquad\qquad f_{yy} = 18y - 2x^2 + 16$$

Step 2

Now, let's find the critical points for this problem. That means solving the following system.

$$\begin{aligned} f_x = 0 :&\quad -2xy^2 + 8x = 2x\left(4-y^2\right) = 0 \qquad \to \qquad y = \pm 2 \text{ or } x = 0 \\ f_y = 0 :&\quad 9y^2 - 2x^2y + 16y - 20 = 0 \end{aligned}$$

As shown above we have three possible options from the first equation. We can plug each into the second equation to get the critical points for the equation.

$y = -2 : 4x^2 - 16 = 0 \quad \rightarrow \quad x = \pm 2 \quad \Rightarrow \quad (2,-2)$ and $(-2,-2)$

$y = 2 : -4x^2 + 48 = 0 \quad \rightarrow \quad x = \pm 2\sqrt{3} \quad \Rightarrow \quad \left(2\sqrt{3},2\right)$ and $\left(-2\sqrt{3},2\right)$

$x = 0 : 9y^2 + 16y - 20 = 0 \quad \rightarrow \quad y = \frac{-16 \pm \sqrt{976}}{18} \quad \Rightarrow \quad \left(0, \frac{-16-\sqrt{976}}{18}\right)$ and $\left(0, \frac{-16+\sqrt{976}}{18}\right)$

As we noted in the first two problems in this section be careful to only write down the actual solutions as found in the above work. Do not just write down all possible combinations of x and y from each of the three lines above. If you do that for this problem you will end up with a large number of points that are not critical points.

Also, do not get excited about the "mess" (*i.e.* roots) involved in some of the critical points. They will be a fact of life with these problems on occasion.

So, in summary, this function has the following six critical points.

$$(-2,-2)\,,\ (2,-2)\,,\ \left(2\sqrt{3},2\right),\ \left(-2\sqrt{3},2\right),\ \left(0, \frac{-16-\sqrt{976}}{18}\right),\ \left(0, \frac{-16+\sqrt{976}}{18}\right)$$

.

Step 3

Next, we'll need the following,

$$\begin{aligned} D(x,y) &= f_{x\,x} f_{y\,y} - \left[f_{x\,y}\right]^2 \\ &= \left[-2y^2+8\right]\left[18y-2x^2+16\right] - \left[-4xy\right]^2 \\ &= \left[-2y^2+8\right]\left[18y-2x^2+16\right] - 16x^2y^2 \end{aligned}$$

Step 4

With $D\left(x,y\right)$ we can now classify each of the critical points as follows.

$$\begin{array}{lcll}
(-2,-2) & : & D(-2,-2) = -256 < 0 & \text{Saddle Point} \\
(2,-2) & : & D(2,-2) = -256 < 0 & \text{Saddle Point} \\
\left(-2\sqrt{3},2\right) & : & D\left(-2\sqrt{3},2\right) = -768 < 0 & \text{Saddle Point} \\
\left(2\sqrt{3},2\right) & : & D\left(2\sqrt{3},2\right) = -768 < 0 & \text{Saddle Point} \\
\left(0,\dfrac{-16-\sqrt{976}}{18}\right) & : & D\left(0,\dfrac{-16-\sqrt{976}}{18}\right) = 180.4 > 0 & \\
 & & f_{x\,x}\left(0,\dfrac{-16-\sqrt{976}}{18}\right) = -5.8 < 0 & \text{Relative Maximum} \\
\left(0,\dfrac{-16+\sqrt{976}}{18}\right) & : & D\left(0,\dfrac{-16+\sqrt{976}}{18}\right) = 205.1 > 0 & \\
 & & f_{x\,x}\left(0,\dfrac{-16+\sqrt{976}}{18}\right) = 6.6 > 0 & \text{Relative Minimum}
\end{array}$$

Don't forget to check the value of $f_{x\,x}$ when D is positive so we can get the correct classification (*i.e.* maximum or minimum) and also recall that for negative D we don't need the second check as we know the critical point will be a saddle point.

14.4 Absolute Minimums and Maximums

1. Find the absolute minimum and absolute maximum of $f(x,y) = 192x^3 + y^2 - 4xy^2$ on the triangle with vertices $(0,0)$, $(4,2)$ and $(-2,2)$.

Step 1

We'll need the first order derivatives to start the problem off. Here they are,

$$f_x = 576x^2 - 4y^2 \qquad f_y = 2y - 8xy$$

Step 2

We need to find the critical points for this problem. That means solving the following system.

$$\begin{aligned} f_x = 0 &\quad : \quad 576x^2 - 4y^2 = 0 \\ f_y = 0 &\quad : \quad 2y(1-4x) = 0 \qquad \to \qquad y = 0 \text{ or } x = \frac{1}{4} \end{aligned}$$

So, we have two possible options from the second equation. We can plug each into the first equation to get the critical points for the equation.

$$\begin{aligned} y = 0 &: \quad 576x^2 = 0 \quad \to \quad x = 0 \quad \Rightarrow \quad (0,0) \\ x = \tfrac{1}{4} &: \quad 36 - 4y^2 = 0 \quad \to \quad y = \pm 3 \quad \Rightarrow \quad \left(\tfrac{1}{4}, 3\right) \text{ and } \left(\tfrac{1}{4}, -3\right) \end{aligned}$$

Okay, we have the three critical points listed above. Also recall that we only use critical points that are actually in the region we are working with. In this case, the last two have y values that clearly are out of the region (we've sketched the region in the next step if you aren't sure you believe this!) and so we can ignore them.

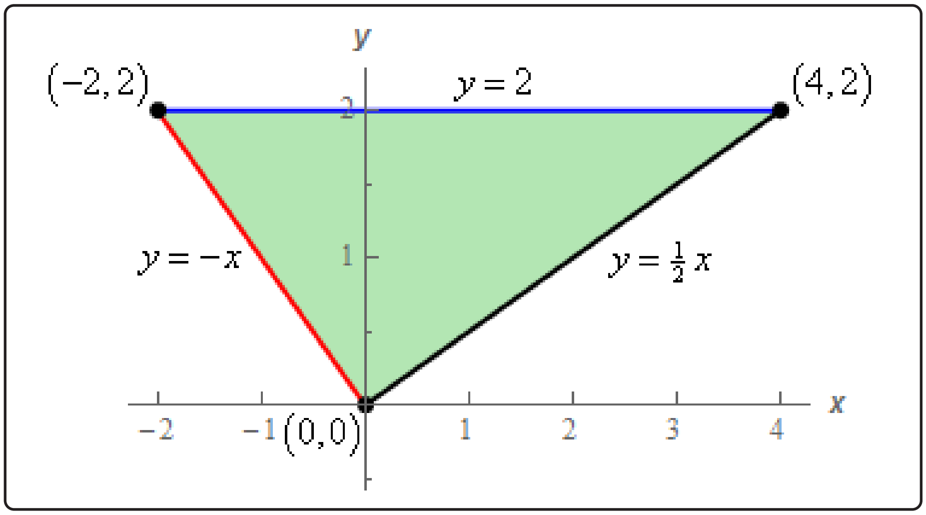
Therefore, the only critical point from this list that we need to use is the first. Note as well that, in this case, this also happens to be one of the points that define the boundary of the region. This will happen on occasion but won't always.

So, we'll need the function value for the only critical point that is actually in our region. Here is that value,

$$f(0,0) = 0$$

Step 3

Now, we know that absolute extrema can occur on the boundary. So, let's start off with a quick sketch of the region we're working on.



Each of the sides of the triangle can then be defined as follows.

Top : $y = 2, \quad -2 \le x \le 4$

Right : $y = \frac{1}{2}x, \quad 0 \le x \le 4$

Left : $y = -x, \quad -2 \le x \le 0$

Now we need to analyze each of these sides to get potential absolute extrema for $f(x, y)$ that might occur on the boundary.

Step 4

Let's first check out the top :$y = 2, \quad -2 \le x \le 4$.

We'll need to identify the points along the top that could be potential absolute extrema for $f(x, y)$. This, in essence, requires us to find the potential absolute extrema of the following equation on the interval $-2 \le x \le 4$.

$$g(x) = f(x, 2) = 192x^3 - 16x + 4$$

This is really nothing more than a Calculus I absolute extrema problem so we'll be doing the work here without a lot of explanation. If you don't recall how to do these kinds of problems you should read through that section in the Calculus I material.

The critical point(s) for $g(x)$ are,

$$g'(x) = 576x^2 - 16 = 0 \qquad \rightarrow \qquad x = \pm\frac{1}{6}$$

So, these two points as well as the x limits for the top give the following four points that are potential absolute extrema for $f(x, y)$.

$$\left(\frac{1}{6}, 2\right) \qquad \left(-\frac{1}{6}, 2\right) \qquad (-2, 2) \qquad (4, 2)$$

Recall that, in this step, we are assuming that $y = 2$! So, the next set of potential absolute extrema for $f(x, y)$ are then,

$$f\left(\frac{1}{6}, 2\right) = \frac{20}{9} \qquad f\left(-\frac{1}{6}, 2\right) = \frac{52}{9} \qquad f(-2, 2) = -1{,}500 \qquad f(4, 2) = 12{,}228$$

Step 5

Next let's check out the right side :$y = \frac{1}{2}x, \;\; 0 \le x \le 4$. For this side we'll need to identify possible absolute extrema of the following function on the interval $0 \le x \le 4$.

$$g(x) = f\left(x, \frac{1}{2}x\right) = \frac{1}{4}x^2 + 191x^3$$

The critical point(s) for the $g(x)$ from this step are,

$$g'(x) = \frac{1}{2}x + 573x^2 = x\left(\frac{1}{2} + 573x\right) = 0 \qquad \rightarrow \qquad x = 0, \quad x = -\frac{1}{1146}$$

Now, recall what we are restricted to the interval $0 \le x \le 4$ for this portion of the problem and so the second critical point above will not be used as it lies outside this interval.

So, the single point from above that is in the interval $0 \le x \le 4$ as well as the x limits for the right give the following two points that are potential absolute extrema for $f(x, y)$.

$$(0, 0) \qquad (4, 2)$$

Recall that, in this step, we are assuming that $y = \frac{1}{2}x$! Also note that, in this case, one of the critical points ended up also being one of the endpoints.

Therefore, the next set of potential absolute extrema for $f(x, y)$ are then,

$$f(0, 0) = 0 \qquad f(4, 2) = 12{,}228$$

Before proceeding to the next step note that both of these have already appeared in previous steps. This will happen on occasion but we can't, in many cases, expect this to happen so we do need to go through and do the work for each boundary.

The main exception to this is usually the endpoints of our intervals as they will always be shared in two of the boundary checks and so, once done, don't really need to be checked again. We just included the endpoints here for completeness.

Step 6

Finally, let's check out the left side :$y = -x, \ \ -2 \le x \le 0$. For this side we'll need to identify possible absolute extrema of the following function on the interval $-2 \le x \le 0$.

$$g(x) = f(x, -x) = x^2 + 188x^3$$

The critical point(s) for the $g(x)$ from this step are,

$$g'(x) = 2x + 564x^2 = 2x(1 + 282x) = 0 \qquad \rightarrow \qquad x = 0, \quad x = -\frac{1}{282}$$

Both of these are in the interval $-2 \le x \le 0$ that we are restricted to for this portion of the problem.

So, the two points from above as well as the x limits for the right give the following three points that are potential absolute extrema for $f(x, y)$.

$$\left(-\frac{1}{282}, \frac{1}{282}\right) \qquad (0,0) \qquad (-2,2)$$

Recall that, in this step we are assuming that $y = -x$! Also note that, in this case, one of the critical points ended up also being one of the endpoints.

Therefore, the next set of potential absolute extrema for $f(x, y)$ are then,

$$f\left(-\frac{1}{282}, \frac{1}{282}\right) = \frac{1}{238,572} \qquad f(0,0) = 0 \qquad f(-2,2) = -1,500$$

As with the previous step we can note that both of the end points above have already occurred previously in the problem and didn't really need to be checked here. They were just included for completeness.

Step 7

Okay, in summary, here are all the potential absolute extrema and their function values for this function on the region we are working on.

$$f\left(\frac{1}{6},2\right)=\frac{20}{9} \qquad f\left(-\frac{1}{6},2\right)=\frac{52}{9} \qquad f\left(-2,2\right)=-1,500$$
$$f\left(0,0\right)=0 \qquad f\left(-\frac{1}{282},\frac{1}{282}\right)=\frac{1}{238,572} \qquad f\left(4,2\right)=12,228$$

From this list we can see that the absolute maximum of the function will be $12,228$ which occurs at $(4,2)$ and the absolute minimum of the function will be $-1,500$ which occurs at $(-2,2)$.

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2. Find the absolute minimum and absolute maximum of $f\left(x,y\right)=\left(9x^2-1\right)\left(1+4y\right)$ on the rectangle given by $-2\leq x\leq 3$, $-1\leq y\leq 4$.

Step 1

We'll need the first order derivatives to start the problem off. Here they are,

$$f_x=18x\left(1+4y\right) \qquad f_y=4\left(9x^2-1\right)$$

Step 2

We need to find the critical points for this problem. That means solving the following system.

$$\begin{aligned} f_x=0 &\quad : \quad 18x\left(1+4y\right)=0 \\ f_y=0 &\quad : \quad 4\left(9x^2-1\right)=0 \qquad \rightarrow \qquad x=\pm\frac{1}{3} \end{aligned}$$

So, we have two possible options from the second equation. We can plug each into the first equation to get the critical points for the equation.

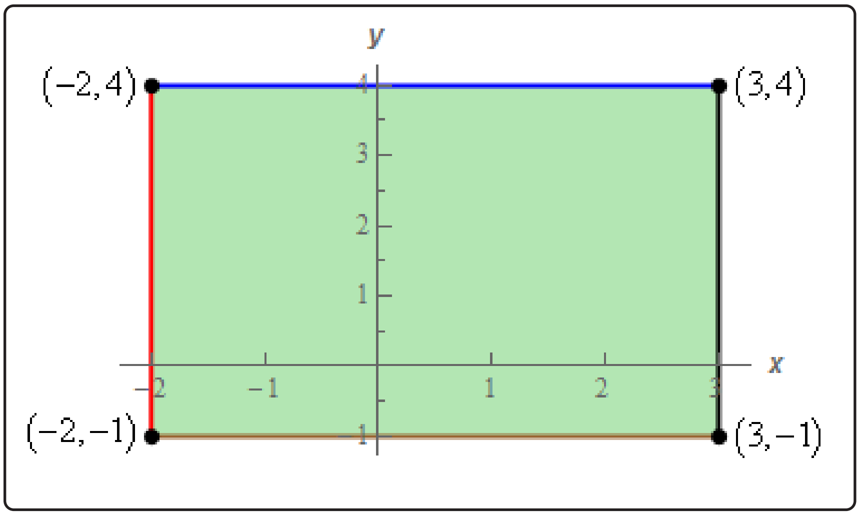
$$x=\frac{1}{3} \;:\; 6\left(1+4y\right)=0 \quad\rightarrow\quad y=-\frac{1}{4} \quad\Rightarrow\quad \left(\frac{1}{3},-\frac{1}{4}\right)$$
$$x=-\frac{1}{3} \;:\; -6\left(1+4y\right)=0 \quad\rightarrow\quad y=-\frac{1}{4} \quad\Rightarrow\quad \left(-\frac{1}{3},-\frac{1}{4}\right)$$

Both of these critical points are in the region we are interested in and so we'll need the function evaluated at both of them. Here are those values,

$$f\left(\frac{1}{3}, -\frac{1}{4}\right) = 0 \qquad f\left(-\frac{1}{3}, -\frac{1}{4}\right) = 0$$

Step 3

Now, we know that absolute extrema can occur on the boundary. So, let's start off with a quick sketch of the region we're working on.



Each of the sides of the rectangle can then be defined as follows.

Top :$y = 4, \quad -2 \le x \le 3$

Bottom : $y = -1, \quad -2 \le x \le 3$

Right : $x = 3, \quad -1 \le y \le 4$

Left : $x = -2, \quad -1 \le y \le 4$

Now we need to analyze each of these sides to get potential absolute extrema for $f(x, y)$ that might occur on the boundary.

Step 4

Let's first check out the top : $y = 4, \quad -2 \le x \le 3$.

We'll need to identify the points along the top that could be potential absolute extrema for $f(x,y)$. This, in essence, requires us to find the potential absolute extrema of the following equation on the interval $-2 \le x \le 3$.

$$g(x) = f(x,4) = 17\left(-1 + 9x^2\right)$$

This is really nothing more than a Calculus I absolute extrema problem so we'll be doing the work here without a lot of explanation. If you don't recall how to do these kinds of problems you should read through that section in the Calculus I material.

The critical point(s) for $g(x)$ are,

$$g'(x) = 306x = 0 \qquad \rightarrow \qquad x = 0$$

This critical point is in the interval we are working on so, this point as well as the x limits for the top give the following three points that are potential absolute extrema for $f(x,y)$.

$$(0,4) \qquad (-2,4) \qquad (3,4)$$

Recall that, in this step, we are assuming that $y = 4$! So, the next set of potential absolute extrema for $f(x,y)$ are then,

$$f(0,4) = -17 \qquad f(-2,4) = 595 \qquad f(3,4) = 1360$$

Step 5

Next, let's check out the bottom : $y = -1, \quad -2 \le x \le 3$. For this side we'll need to identify possible absolute extrema of the following function on the interval $-2 \le x \le 3$.

$$g(x) = f(x,-1) = -3\left(-1 + 9x^2\right)$$

The critical point(s) for the $g(x)$ from this step are,

$$g'(x) = -54x = 0 \qquad \rightarrow \qquad x = 0$$

This critical point is in the interval we are working on so, this point as well as the x limits for the bottom give the following three points that are potential absolute extrema for $f(x,y)$.

$$(0,-1) \qquad (-2,-1) \qquad (3,-1)$$

Recall that, in this step, we are assuming that $y = -1$! So, the next set of potential absolute extrema for $f(x,y)$ are then,

$$f(0,-1) = 3 \qquad f(-2,-1) = -105 \qquad f(3,-1) = -240$$

Step 6

Let's now check out the right side : $x = 3, \quad -1 \le y \le 4$. For this side we'll need to identify possible absolute extrema of the following function on the interval $-1 \le y \le 4$.

$$h(y) = f(3,y) = 80(1+4y)$$

The derivative of the $h(y)$ from this step is,

$$h'(y) = 320$$

In this case there are no critical points of the function along this boundary. So, only the limits for the right side are potential absolute extrema for $f(x,y)$.

$$(3,-1) \qquad (3,4)$$

Recall that, in this step, we are assuming that $x = 3$! Therefore, the next set of potential absolute extrema for $f(x,y)$ are then,

$$f(3,-1) = -240 \qquad f(3,4) = 1360$$

Before proceeding to the next step let's note that both of these points have already been listed in previous steps and so did not really need to be written down here. This will always happen with boundary points (as these are here). Boundary points will always show up in multiple boundary steps.

Step 7

Finally, let's check out the left side : $x = -2, \quad -1 \le y \le 4$. For this side we'll need to identify possible absolute extrema of the following function on the interval $-1 \le y \le 4$.

$$h(y) = f(-2,y) = 35(1+4y)$$

The derivative of the $h(y)$ from this step is,

$$h'(y) = 140$$

In this case there are no critical points of the function along this boundary. So, we only the limits for the right side are potential absolute extrema for $f(x,y)$.

$$(-2,-1) \qquad (-2,4)$$

Recall that, in this step, we are assuming that $x = -2$! Therefore, the next set of potential absolute extrema for $f(x,y)$ are then,

$$f(-2,-1) = -105 \qquad f(-2,4) = 595$$

As with the previous step both of these are boundary points and have appeared in previous steps. They were simply listed here for completeness.

Step 8

Okay, in summary, here are all the potential absolute extrema and their function values for this function on the region we are working on.

$$\begin{array}{ccc} f(0,4) = -17 & f(-2,4) = 595 & f(3,4) = 1360 \\ f(0,-1) = 3 & f(-2,-1) = -105 & f(3,-1) = -240 \end{array}$$

From this list we can see that the absolute maximum of the function will be 1360 which occurs at $(3,4)$ and the absolute minimum of the function will be -240 which occurs at $(3,-1)$.

14.5 Lagrange Multipliers

1. Find the maximum and minimum values of $f(x, y) = 81x^2 + y^2$ subject to the constraint $4x^2 + y^2 = 9$.

Step 1

Before proceeding with the problem let's note because our constraint is the sum of two terms that are squared (and hence positive) the largest possible range of x is $-\frac{3}{2} \le x \le \frac{3}{2}$ (the largest values would occur if $y = 0$). Likewise, the largest possible range of y is $-3 \le y \le 3$ (with the largest values occurring if $x = 0$).

Note that, at this point, we don't know if x and/or y will actually be the largest possible value. At this point we are simply acknowledging what they are. What this allows us to say is that whatever our answers will be they must occur in these bounded ranges and hence by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2

The first actual step in the solution process is then to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} 162x &= 8x\lambda \\ 2y &= 2y\lambda \\ 4x^2 + y^2 &= 9 \end{aligned}$$

Step 3

For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

In this case, simply because the numbers are a little smaller, let's start with the second equation. A little rewrite of the equation gives us the following,

$$2y\lambda - 2y = 2y(\lambda - 1) = 0 \qquad \to \qquad y = 0 \text{ or } \lambda = 1$$

Be careful here to not just divide both sides by y to "simplify" the equation. Remember that you can't divide by anything unless you know for a fact that it won't ever be zero. In this case we can see that y clearly can be zero and if you divide it out to start the solution process you will miss that solution. This is often one of the biggest mistakes that students make when working these kinds of problems.

Step 4

We now have two possibilities from Step 3. Either $y = 0$ or $\lambda = 1$. We'll need to go through both of these possibilities and see what we get.

Let's start by assuming that $y = 0$. In this case we can go directly to the constraint to get,

$$4x^2 = 9 \qquad \to \qquad x = \pm\frac{3}{2}$$

Therefore, from this part we get two points that are potential absolute extrema,

$$\left(-\frac{3}{2}, 0\right) \qquad \left(\frac{3}{2}, 0\right)$$

Step 5

Next, let's assume that $\lambda = 1$. In this case, we can plug this into the first equation to get,

$$162x = 8x \qquad \to \qquad 154x = 0 \qquad \to \qquad x = 0$$

So, under this assumption we must have $x = 0$. We can now plug this into the constraint to get,

$$y^2 = 9 \qquad \to \qquad y = \pm 3$$

So, this part gives us two more points that are potential absolute extrema,

$$(0,-3) \qquad (0,3)$$

Step 6

In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f\left(-\frac{3}{2},0\right)=\frac{729}{4} \qquad f\left(\frac{3}{2},0\right)=\frac{729}{4} \qquad f(0,-3)=9 \qquad f(0,3)=9$$

The absolute maximum is then $\frac{729}{4}=182.25$ which occurs at $\left(-\frac{3}{2},0\right)$ and $\left(\frac{3}{2},0\right)$. The absolute minimum is 9 which occurs at $(0,-3)$ and $(0,3)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

---

2. Find the maximum and minimum values of $f(x,y)=8x^2-2y$ subject to the constraint $x^2+y^2=1$.

Step 1

Before proceeding with the problem let's note because our constraint is the sum of two terms that are squared (and hence positive) the largest possible range of x is $-1 \le x \le 1$ (the largest values would occur if $y=0$). Likewise, the largest possible range of y is $-1 \le y \le 1$ (with the largest values occurring if $x=0$).

Note that, at this point, we don't know if x and/or y will actually be the largest possible value. At this point we are simply acknowledging what they are. What this allows us to say is that whatever our answers will be they must occur in these bounded ranges and hence by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2

The first actual step in the solution process is then to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} 16x &= 2x\lambda \\ -2 &= 2y\lambda \\ x^2 + y^2 &= 1 \end{aligned}$$

Step 3

For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

For this system it looks like maybe the first equation will give us some information to start off with so let's start with that equation. A quick rewrite of the equation gives us the following,

$$16x - 2x\lambda = 2x\left(8 - \lambda\right) = 0 \qquad \rightarrow \qquad x = 0 \ \text{ or } \ \lambda = 8$$

Be careful here to not just divide both sides by x to "simplify" the equation. Remember that you can't divide by anything unless you know for a fact that it won't ever be zero. In this case we can see that x clearly can be zero and if you divide it out to start the solution process you will miss that solution. This is often one of the biggest mistakes that students make when working these kinds of problems.

Step 4

We now have two possibilities from Step 2. Either $x = 0$ or $\lambda = 8$. We'll need to go through both of these possibilities and see what we get.

Let's start by assuming that $x = 0$. In this case we can go directly to the constraint to get,

$$y^2 = 1 \qquad \rightarrow \qquad y = \pm 1$$

Therefore, from this part we get two points that are potential absolute extrema,

$$(0, -1) \qquad\qquad (0, 1)$$

Step 5

Next, let's assume that $\lambda = 8$. In this case, we can plug this into the second equation to get,

$$-2 = 16y \qquad \rightarrow \qquad y = -\frac{1}{8}$$

So, under this assumption we must have $y = -\frac{1}{8}$. We can now plug this into the constraint to get,

$$x^2 + \frac{1}{64} = 1 \qquad \rightarrow \qquad x^2 = \frac{63}{64} \qquad \rightarrow \qquad x = \pm\sqrt{\frac{63}{64}} = \pm\frac{3\sqrt{7}}{8}$$

So, this part gives us two more points that are potential absolute extrema,

$$\left(-\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right) \qquad\qquad \left(\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right)$$

Step 6

In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f\left(-\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right) = \frac{65}{8} \qquad f\left(\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right) = \frac{65}{8} \qquad f(0, -1) = 2 \qquad f(0, 1) = -2$$

The absolute maximum is then $\frac{65}{8} = 8.125$ which occurs at $\left(-\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right)$ and $\left(\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right)$. The absolute minimum is -2 which occurs at $(0, 1)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

3. Find the maximum and minimum values of $f(x, y, z) = y^2 - 10z$ subject to the constraint $x^2 + y^2 + z^2 = 36$.

Step 1

Before proceeding with the problem let's note because our constraint is the sum of three terms that are squared (and hence positive) the largest possible range of x is $-6 \le x \le 6$ (the largest values would occur if $y = 0$ and $z = 0$). Likewise, we'd get the same ranges for both y and z.

Note that, at this point, we don't know if x, y or z will actually be the largest possible value. At this point we are simply acknowledging what they are. What this allows us to say is that whatever our answers will be they must occur in these bounded ranges and hence by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2

The first actual step in the solution process is then to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} 0 &= 2x\lambda \\ 2y &= 2y\lambda \\ -10 &= 2z\lambda \\ x^2 + y^2 + z^2 &= 36 \end{aligned}$$

Step 3

For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

For this system let's start with the third equation and note that because the left side is -10, or more importantly can never by zero, we can see that we must therefore have $z \neq 0$ and $\lambda \neq 0$. The fact that λ can't be zero is really important for this problem.

Step 4

Okay, because we now know that $\lambda \neq 0$ we can see that the only way for the first equation to be true is to have $x = 0$.

Therefore, no matter what else is going on with y and z in this problem we must always have $x = 0$ and we'll need to keep that in mind.

Step 5

Next, let's take a look at the second equation. A quick rewrite of this equation gives,

$$2y - 2y\lambda = 2y(1 - \lambda) = 0 \qquad \to \qquad y = 0 \;\text{ or }\; \lambda = 1$$

Step 6

We now have two possibilities from Step 4. Either $y = 0$ or $\lambda = 1$. We'll need to go through both of these possibilities and see what we get.

Let's start by assuming that $y = 0$ and recall from Step 3 that we also know that $x = 0$. In this case we can plug these values into the constraint to get,

$$z^2 = 36 \qquad \to \qquad z = \pm 6$$

Therefore, from this part we get two points that are potential absolute extrema,

$$(0, 0, -6) \qquad\qquad (0, 0, 6)$$

Step 7

Next, let's assume that $\lambda = 1$. If we head back to the third equation we can see that we now have,

$$-10 = 2z \qquad \rightarrow \qquad z = -5$$

So, under this assumption we must have $z = -5$ and recalling once more from Step 3 that we have $x = 0$ we can now plug these into the constraint to get,

$$y^2 + 25 = 36 \qquad \rightarrow \qquad y^2 = 11 \qquad \rightarrow \qquad y = \pm\sqrt{11}$$

So, this part gives us two more points that are potential absolute extrema,

$$\left(0, -\sqrt{11}, -5\right) \qquad\qquad \left(0, \sqrt{11}, -5\right)$$

Step 8

In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f\left(0, -\sqrt{11}, -5\right) = 61 \qquad f\left(0, \sqrt{11}, -5\right) = 61 \qquad f\left(0, 0, -6\right) = 60 \qquad f\left(0, 0, 6\right) = -60$$

The absolute maximum is then 61 which occurs at $\left(0, -\sqrt{11}, -5\right)$ and $\left(0, \sqrt{11}, -5\right)$. The absolute minimum is -60 which occurs at $(0, 0, 6)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

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4. Find the maximum and minimum values of $f\left(x, y, z\right) = xyz$ subject to the constraint $x + 9y^2 + z^2 = 4$. Assume that $x \geq 0$ for this problem. Why is this assumption needed?

Step 1

Before proceeding with the solution to this problem let's address why the assumption that $x \geq 0$ is needed for this problem.

The answer is simple. Without that assumption this function will not have absolute extrema.

If there are no restrictions on x then we could make x as large and negative as we wanted to and we could still meet the constraint simply by chose a very large y and/or z.

Note as well that because y and z are both squared we could chose them to be either negative or positive.

If we took our choices for x, y and z and plugged them into the function then the function would be similarly large. Also, the larger we chose x the larger we'd need to choose appropriate y and/or z and hence the larger our function would become. Finally, as noted above because we could chose y and z to be either positive or negative we could force the function to be either positive or negative with appropriate choices of signs for y and z.

In other words, if we have no restriction on x, we can make the function arbitrarily large in a positive and negative sense and so this function would not have absolute extrema.

On the other hand, if we put on the restriction on x that we have we now have the sum of three positive terms that must equal four. This in turn leads to the following largest possible values of the three variables in the problem.

$$0 \le x \le 4 \qquad\qquad -\frac{2}{3} \le y \le \frac{2}{3} \qquad\qquad -2 \le z \le 2$$

The largest value of x and the extreme values of y and z would occur when the other two variables are zero and in general there is no way to know ahead of time if any of the variables will in fact take on their largest possible values. However, what we can say now is that because all of our variables are bounded then by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

Note as well that all we really need here is a lower limit for x. It doesn't have to be zero that just makes the above analysis a little bit easier. We could have used the restriction that $x \ge -8$ if we'd wanted to. With this restriction we'd still have a bounded set of ranges for x, y and z and so the function would still have absolute extrema.

This problem shows just why this step is so important for these problems. If this problem did not have a restriction on x and we neglected to do this step we'd get the (very) wrong answer! We could still go through the process below and we'd get values that would appear to be absolute extrema. However, as we've shown above without any restriction on x the function would not have absolute extrema.

The issue here is that the Lagrange multiplier process itself is not set up to detect if absolute extrema exist or not. Before we even start the process we need to first make sure that the values we get out of the process will in fact be absolute extrema (*i.e.* we need to verify that absolute extrema exist).

Step 2

The first step here is to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} yz &= \lambda \\ xz &= 18y\lambda \\ xy &= 2z\lambda \\ x + 9y^2 + z^2 &= 4 \end{aligned}$$

Step 3

For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

With this system let's start out by multiplying the first equation by x, multiplying the second equation by y and multiplying the third equation by z. Doing this gives the following "new" system of equations.

$$\begin{aligned} xyz &= x\lambda \\ xyz &= 18y^2\lambda \\ xyz &= 2z^2\lambda \\ x + 9y^2 + z^2 &= 4 \end{aligned}$$

Let's also note that the constraint won't be true if all three variables are zero simultaneously. One or two of the variables can be zero but we can't have all three be zero.

Step 4

Now, let's set the first and second equations from Step 3 equal. Doing this gives,

$$x\lambda = 18y^2\lambda \quad \rightarrow \quad \left(x - 18y^2\right)\lambda = 0 \quad \rightarrow \quad x = 18y^2 \text{ or } \lambda = 0$$

Let's also set the second and third equation from Step 3 equal. Doing this gives,

$$18y^2\lambda = 2z^2\lambda \quad \rightarrow \quad \left(18y^2 - 2z^2\right)\lambda = 0 \quad \rightarrow \quad z^2 = 9y^2 \text{ or } \lambda = 0$$

Step 5

Okay, from Step 4 we have two possibilities. Either $\lambda = 0$ or we have $x = 18y^2$ and $z^2 = 9y^2$.

Let's take care of the first possibility, $\lambda = 0$. If we go back to the original system this assumption gives us the following system.

$$\begin{aligned} yz &= 0 & &\rightarrow & &y = 0 \text{ or } z = 0 \\ xz &= 0 & &\rightarrow & &x = 0 \text{ or } z = 0 \\ xy &= 0 & &\rightarrow & &x = 0 \text{ or } y = 0 \\ x + 9y^2 + z^2 &= 4 \end{aligned}$$

Step 6

We have all sorts of possibilities from Step 5. From the first equation we have two possibilities. Let's start with $y = 0$. Since the third equation from Step 5 won't really tell us anything (after all it is now $0 = 0$) let's move to the second equation. In this case we get either $x = 0$ or $z = 0$.

Recall that at the end of the third step we noticed that we can't have all three of the variables be zero but we could have two of them be zero. So, this leads to the following two cases that we can plug into the constraint to find the value of the third variable.

$$\begin{aligned} y = 0, x = 0 \quad &: \quad z^2 = 4 \quad \rightarrow \quad z = \pm 2 \\ y = 0, z = 0 \quad &: \quad x = 4 \end{aligned}$$

So, this gives us the following three potential absolute extrema.

$$(0, 0, -2) \qquad (0, 0, 2) \qquad (4, 0, 0)$$

Next, let's take a look at the second possibility from the first equation in Step 5, $z = 0$. In this case the second equation will be $0 = 0$ and so will not be of any use. The third however, has the possibilities of $x = 0$ or $y = 0$. The second of these was already addressed above so all we need to look at is,

$$z = 0, x = 0 \quad : \quad 9y^2 = 4 \quad \rightarrow \quad y = \pm\frac{2}{3}$$

This leads to two more potential absolute extrema.

$$\left(0, -\frac{2}{3}, 0\right) \qquad \left(0, \frac{2}{3}, 0\right)$$

We could now go back and start with the second or third equation but if we did that you'd just end up with the above possibilities (you might want to verify that for yourself...). Therefore, we get a total of five possible absolute extrema from this Step. They are,

$$(0, 0, \pm 2) \qquad \left(0, \pm\frac{2}{3}, 0\right) \qquad (4, 0, 0)$$

We made heavy use of the "$\pm$" notation here to simplify things a little bit. It's not required but will make the rest of the work with these points a little easier as we'll see eventually.

Step 7

Now, way back in Step 5 we had another possibility : $x = 18y^2$ and $z^2 = 9y^2$. We have to now take a look at this case. In this case we can plug each of these directly into the constraint to get the following,

$$18y^2 + 9y^2 + 9y^2 = 36y^2 = 4 \quad \rightarrow \quad y = \pm\frac{1}{3}$$

Now we can go back to the two assumptions we started this step off with to get,

$$x = 18\left(\frac{1}{9}\right) = 2 \qquad z^2 = 9\left(\frac{1}{9}\right) = 1 \quad \rightarrow \quad z = \pm 1$$

Now, in most cases, we can't just "mix and match" all the values of x, y and z to from points. In this case however, we can do exactly that. The $x = 2$ will arise regardless of the sign on y because of the y^2 in the x assumption. Likewise, because of the y^2 in the z assumption each of the z's can arise for either y and so we get all combinations of x, y and z for points in this case.

Therefore, we get the following four possible absolute extrema from this step.

$$\left(2, -\frac{1}{3}, -1\right) \qquad \left(2, -\frac{1}{3}, 1\right) \qquad \left(2, \frac{1}{3}, -1\right) \qquad \left(2, \frac{1}{3}, 1\right)$$

Step 8

In total, it looks like we have nine points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points and with nine points that seems like a lot of work.

However, in this case, it's actually quite simple. Recall that the function we're evaluating is $f(x, y, z) = xyz$. First, this means that if even one of the variables is zero the whole function will be zero. Therefore, the function evaluations for the five points from Step 6 all give,

$$f(0,0,\pm 2) = f\left(0, \pm\frac{2}{3}, 0\right) = f\ (4,0,0) = 0$$

Note the usage of the "$\pm$" notation to "simplify" the work here as well.

Now, the potential points from Step 7 are all the same values, with the exception of signs changing occasionally on the y and z. That means that the function value here will be either $-\frac{2}{3}$ or $\frac{2}{3}$ depending on the number of minus signs in the point. So again, not a lot of effort to compute these function values. Here are the evaluations for the points from Step 7.

$$f\left(2, -\frac{1}{3}, 1\right) = f\left(2, \frac{1}{3}, -1\right) = -\frac{2}{3} \qquad f\left(2, -\frac{1}{3}, -1\right) = f\left(2, \frac{1}{3}, 1\right) = \frac{2}{3}$$

The absolute maximum is then $\frac{2}{3}$ which occurs at $\left(2, -\frac{1}{3}, -1\right)$ and $\left(2, \frac{1}{3}, 1\right)$. The absolute minimum is $-\frac{2}{3}$ which occurs at $\left(2, -\frac{1}{3}, 1\right)$ and $\left(2, \frac{1}{3}, -1\right)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

Before leaving this problem we should note that some of the solution processes for the systems that arise with Lagrange multipliers can be quite involved. It can be easy to get lost in the details of the solution process and forget to go back and take care of one or more possibilities. You need to always be very careful and before finishing a problem go back and make sure that you've dealt with all the possible solution paths in the problem.

5. Find the maximum and minimum values of $f(x, y, z) = 3x^2 + y$ subject to the constraints $4x - 3y = 9$ and $x^2 + z^2 = 9$.

Step 1

Before proceeding with the problem let's note that the second constraint is the sum of two terms that are squared (and hence positive). Therefore, the largest possible range of x is $-3 \le x \le 3$ (the largest values would occur if $z = 0$). We'll get a similar range for z.

Now, the first constraint is not the sum of two (or more) positive numbers. However, we've already established that x is restricted to $-3 \le x \le 3$ and this will give $-7 \le y \le 1$ as the largest possible range of y's. Note that we can easily get this range by acknowledging that the first constraint is just a line and so the extreme values of y will correspond to the extreme values of x.

So, because we now know that our answers must occur in these bounded ranges by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2

The first step here is to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} 6x &= 4\lambda + 2x\mu \\ 1 &= -3\lambda \\ 0 &= 2z\mu \\ 4x - 3y &= 9 \\ x^2 + z^2 &= 9 \end{aligned}$$

Step 3

For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

With this system we get a "freebie" to start off with. Notice that from the second equation we quickly can see that $\lambda = -\frac{1}{3}$ regardless of any of the values of the other variables in the system.

Step 4

Next, from the third equation we can see that we have either $z = 0$ or $\mu = 0$, So, we have 2 possibilities to look at. Let's take a look at $z = 0$ first.

In this case we can go straight to the second constraint to get,

$$x^2 = 9 \qquad \to \qquad x = \pm 3$$

We can in turn plug each of these possibilities into the first constraint to get values for y.

$$\begin{aligned} x = -3 \quad &: \quad -12 - 3y = 9 \quad \to \quad y = -7 \\ x = 3 \quad &: \quad 12 - 3y = 9 \quad \to \quad y = 1 \end{aligned}$$

Okay, from this step we have two possible absolute extrema.

$$(-3, -7, 0) \qquad (3, 1, 0)$$

Step 5

Now let's go back and take a look at what happens if $\mu = 0$. If we plug this into the first equation in our system (and recalling that we also know that $\lambda = -\frac{1}{3}$) we get,

$$6x = -\frac{4}{3} \qquad \to \qquad x = -\frac{2}{9}$$

We can plug this into each of our constraints to get values of y (from the first constraint) and z (form the second constraint). Here is that work,

$$4\left(-\frac{2}{9}\right) - 3y = 9 \qquad \rightarrow \qquad y = -\frac{89}{27}$$
$$\left(-\frac{2}{9}\right)^2 + z^2 = 9 \qquad \rightarrow \qquad z = \pm\frac{5\sqrt{29}}{9}$$

This leads to two more potential absolute extrema.

$$\left(-\frac{2}{9}, -\frac{89}{27}, -\frac{5\sqrt{29}}{9}\right) \qquad \left(-\frac{2}{9}, -\frac{89}{27}, \frac{5\sqrt{29}}{9}\right)$$

Step 6

In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f\left(-3,-7,0\right) = 20 \qquad\qquad f\left(3,1,0\right) = 28$$
$$f\left(-\frac{2}{9}, -\frac{89}{27}, -\frac{5\sqrt{29}}{9}\right) = -\frac{85}{27} \qquad f\left(-\frac{2}{9}, -\frac{89}{27}, \frac{5\sqrt{29}}{9}\right) = -\frac{85}{27}$$

The absolute maximum is then 28 which occurs at $(3,1,0)$. The absolute minimum is $-\frac{85}{27}$ which occurs at $\left(-\frac{2}{9}, -\frac{89}{27}, -\frac{5\sqrt{29}}{9}\right)$ and $\left(-\frac{2}{9}, -\frac{89}{27}, \frac{5\sqrt{29}}{9}\right)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

Before leaving this problem we should note that, in this case, the value of the absolute extrema (as opposed to the location) did not actually depend on the value of z in any way as the function we were optimizing in this problem did not depend on z. This will happen sometimes and we shouldn't get too worried about it when it does.

Note however that we still need the values of z for the location of the absolute extrema. We need the values of z for the location because the points that give the absolute extrema are also required to satisfy the constraint and the second constraint in our problem does involve z's!

15 Multiple Integrals

We now need to start discussing integration of multi-variable functions.

When we looked at definite integrals of single variable functions the values of the independent variable were in some interval $[a, b]$. For functions of multiple variables the values of the independent variables will not just come from intervals anymore. For functions of two variables, for example, the values of the independent variables will come from a two dimensional region. Likewise, for functions of three variables the values of the independent variables will come from a three dimensional region.

We will be discussing Double Integrals (for integrating functions of two variables) and Triple Integrals (for integrating functions of three variables). While most of the integration will be done in terms of Cartesian coordinates we will also discuss converting integrals from Cartesian coordinates into Polar coordinates (for functions of two variables) and Cylindrical or Spherical coordinates (for functions of three variables).

We will also formalize the process for converting an integral from one coordinate system into another. In the process we will derive some of the formulas that were using to convert integrals from Cartesian into Polar, Cylindrical or Spherical coordinates.

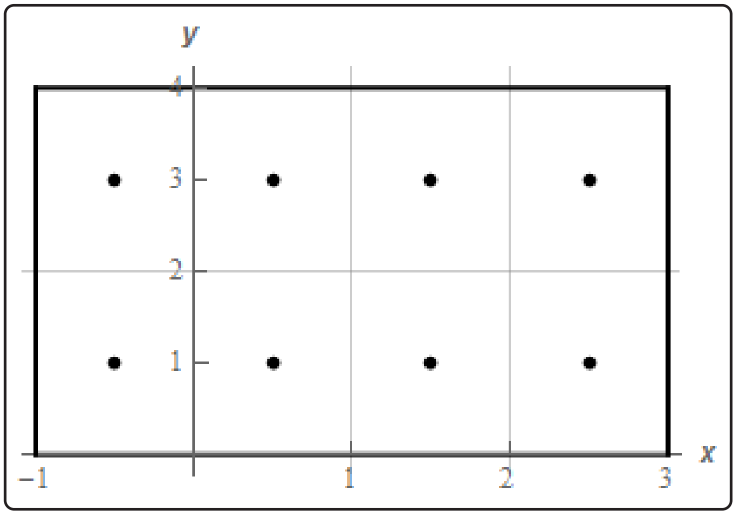
The following sections are the practice problems, with solutions, for this material.

15.1 Double Integrals

1. Use the Midpoint Rule to estimate the volume under $f(x,y) = x^2 + y$ and above the rectangle given by $-1 \le x \le 3$, $0 \le y \le 4$ in the xy-plane. Use 4 subdivisions in the x direction and 2 subdivisions in the y direction.

Step 1

Okay, first let's get a quick sketch of the rectangle we're dealing with here.



The light gray lines show the subdivisions for each direction. The dots in the center of each "block" are the midpoints of each of the blocks.

The coordinates of each of the dots in the lower row are,

$$\left(-\frac{1}{2},1\right) \quad \left(\frac{1}{2},1\right) \quad \left(\frac{3}{2},1\right) \quad \left(\frac{5}{2},1\right)$$

and the coordinates of each of the dots in the upper row are,

$$\left(-\frac{1}{2},3\right) \quad \left(\frac{1}{2},3\right) \quad \left(\frac{3}{2},3\right) \quad \left(\frac{5}{2},3\right)$$

Step 2

We know that the volume we are after is simply,

$$V = \iint\limits_R f(x,y)\, dA$$

and we also know that the Midpoint Rule for this particular case is the following double summation.

$$\iint\limits_R f(x,y)\,dA \approx \sum_{i=1}^{4}\sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A \qquad f(x,y) = x^2 + y$$

Remember that there are four x subdivisions and so the i summation will go to 4. There are two y subdivisions and the j summation will go to 2.

The $\left(\overline{x}_i, \overline{y}_j\right)$ in the formula are simply the midpoints of each of the blocks (*i.e.* the points listed in Step 1 above) and ΔA is the area of each of blocks and so is $\Delta A = (1)(2) = 2$.

Therefore, the volume for this problem is approximately,

$$V \approx \sum_{i=1}^{4}\sum_{j=1}^{2} 2f\left(\overline{x}_i, \overline{y}_j\right) \qquad f(x,y) = x^2 + y$$

Note that we plugged ΔA into the formula.

Step 3

The best way to compute this double summation is probably to first compute the "inner" summation for each value of i.

Each of the inner summations are done for a fixed value of i as i runs from $i = 1$ to $i = 4$. Therefore, the four inner summations are computed using the values of the function at the midpoints for the blocks in each of the columns in the sketch above in Step 1.

This means we need the following summations.

$$i = 1 \; : \; \sum_{j=1}^{2} 2f\left(\overline{x}_1, \overline{y}_j\right) = \sum_{j=1}^{2} 2f\left(-\frac{1}{2}, \overline{y}_j\right) = 2\left[f\left(-\frac{1}{2}, 1\right) + f\left(-\frac{1}{2}, 3\right)\right] = 9$$

$$i = 2 \; : \; \sum_{j=1}^{2} 2f\left(\overline{x}_2, \overline{y}_j\right) = \sum_{j=1}^{2} 2f\left(\frac{1}{2}, \overline{y}_j\right) = 2\left[f\left(\frac{1}{2}, 1\right) + f\left(\frac{1}{2}, 3\right)\right] = 9$$

$$i = 3 \; : \; \sum_{j=1}^{2} 2f\left(\overline{x}_3, \overline{y}_j\right) = \sum_{j=1}^{2} 2f\left(\frac{3}{2}, \overline{y}_j\right) = 2\left[f\left(\frac{3}{2}, 1\right) + f\left(\frac{3}{2}, 3\right)\right] = 17$$

$$i = 4 \; : \; \sum_{j=1}^{2} 2f\left(\overline{x}_4, \overline{y}_j\right) = \sum_{j=1}^{2} 2f\left(\frac{5}{2}, \overline{y}_j\right) = 2\left[f\left(\frac{5}{2}, 1\right) + f\left(\frac{5}{2}, 3\right)\right] = 33$$

Step 4

We can now compute the “outer” summation. This is just the sum of all the inner summations we computed in Step 3.

The volume is then approximately,

$$V \approx \sum_{i=1}^{4} \sum_{j=1}^{2} 2f\left(\overline{x}_i, \overline{y}_j\right) = 9 + 9 + 17 + 33 = \boxed{68}$$

For reference purposes we will eventually be able to verify that the exact volume is $\frac{208}{3} = 69.333\overline{3}$ and so the approximation in this case is actually fairly close.

15.2 Iterated Integrals

1. Compute the following double integral over the indicated rectangle **(a)** by integrating with respect to x first and **(b)** by integrating with respect to y first.

$$\iint\limits_R 12x - 18y\,dA \qquad\qquad R = [-1,4] \times [2,3]$$

(a) Evaluate by integrating with respect to x first.

Step 1

Not too much to do with this problem other than to do the integral in the order given in the problem statement.

Let's first get the integral set up with the proper order of integration.

$$\iint\limits_R 12x - 18y\,dA = \int_2^3 \int_{-1}^4 12x - 18y\,dx\,dy$$

Remember that the first integration is always the "inner" integral and the second integration is always the "outer" integral.

When writing the integral down do not forget the differentials! Many students come out of a Calculus I course with the bad habit of not putting them in. At this point however, that will get you in trouble. You need to be able to recall which variable we are integrating with respect to with each integral and the differentials will tell us that so don't forget about them.

Also, do not forget about the limits and make sure that they get attached to the correct integral. In this case the x integration is first and so the x limits need to go on the inner integral and the y limits need to go on the outer integral. It is easy to get in a hurry and put them on the wrong integral.

Step 2

Okay, let's do the x integration now.

$$\begin{aligned}\iint\limits_R 12x - 18y\,dA &= \int_2^3 \int_{-1}^4 12x - 18y\,dx\,dy \\ &= \int_2^3 \left(6x^2 - 18xy\right)\Big|_{-1}^{4}\,dy = \int_2^3 90 - 90y\,dy\end{aligned}$$

Just remember that because we are integrating with respect to x in this step we treat all y's as if they were a constant and we know how to deal with constants in integrals.

Note that we are assuming that you are capable of doing the evaluation and so did not show the work in this problem and will rarely show it in any of the problems here unless there is a point that needs to be made.

Step 3

Now all we need to take care of in the y integration and that is just a simple Calculus I integral. Here is that work.

$$\iint\limits_R 12x - 18y\,dA = \int_2^3 90 - 90y\,dy = \left(90y - 45y^2\right)\Big|_2^3 = \boxed{-135}$$

(b) Evaluate by integrating with respect to y first.

Step 1

Not too much to do with this problem other than to do the integral in the order given in the problem statement.

Let's first get the integral set up with the proper order of integration.

$$\iint\limits_R 12x - 18y\,dA = \int_{-1}^4 \int_2^3 12x - 18y\,dy\,dx$$

Remember that the first integration is always the "inner" integral and the second integration is always the "outer" integral.

When writing the integral down do not forget the differentials! Many students come out of a Calculus I course with the bad habit of not putting them in. At this point however, that will get you in trouble. You need to be able to recall which variable we are integrating with respect to fist and the differentials will tell us that so don't forget about them.

Also, do not forget about the limits and make sure that they get attached to the correct integral. In this case the y integration is first and so the y limits need to go on the inner integral and the x limits need to go on the outer integral. It is easy to get in a hurry and put them on the wrong integral.

Step 2

Okay, let's do the y integration now.

$$\iint\limits_R 12x - 18y\, dA = \int_{-1}^{4}\int_{2}^{3} 12x - 18y\, dy\, dx$$
$$= \int_{-1}^{4} \left(12xy - 9y^2\right)\Big|_{2}^{3} dx = \int_{-1}^{4} 12x - 45\, dx$$

Just remember that because we are integrating with respect to y in this step we treat all x's as if they were a constant and we know how to deal with constants in integrals.

Note that we are assuming that you are capable of doing the evaluation and so did not show the work in this problem and will rarely show it in any of the problems here unless there is a point that needs to be made.

Step 3

Now all we need to take care of in the x integration and that is just a simple Calculus I integral. Here is that work.

$$\iint\limits_R 12x - 18y\, dA = \int_{-1}^{4} 12x - 45\, dx = \left(6x^2 - 45x\right)\Big|_{-1}^{4} = \boxed{-135}$$

The same answer as from **(a)** which is what we should expect of course. The order of integration will not change the answer. One order may be easier/simpler than the other but the final answer will always be the same regardless of the order.

2. Compute the following double integral over the indicated rectangle.

$$\iint\limits_R 6y\sqrt{x} - 2y^3\, dA \qquad R = [1,4] \times [0,3]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. We know that the order will not affect the final answer so in that sense it doesn't matter which order we decide to use.

Often one of the orders of integration will be easier than the other and so we should keep that in mind when choosing an order. Note as well that often each order will be just as easy/hard as the other order and so order won't really matter all that much in those cases. Of course, there will also be integrals in which one of the orders of integration will be all but impossible, if not impossible, to compute and so you won't really have a choice of orders in cases such as that.

Given all the possibilities discussed above it can seem quite daunting when you need to decide on the order of integration. It generally isn't as bad as it seems however. When choosing an order of integration just take a look at the integral and think about what would need to be done for each order and see if there is one order that seems like it might be easier to take care of first or if maybe the resulting answer will make the second integration somewhat easier.

Sometimes it will not be readily apparent which order will be the easiest until you get into the problem. In these cases the only thing you can really do is just start with one order and see how it goes. If it starts getting too difficult you can always go back and give the other order a try to see if it is any easier.

The better you are at integration the easier/quicker it will be to choose an order of integration. If you are really rusty at integration and/or you didn't learn it all that great back in Calculus I then you should probably head back into the Calculus I material and practice your integration skills a little bit. Multiple integrals will be much easier to deal with if you have good Calculus I skills.

As a final note about choosing an order of integration remember that for the vast majority of the integrals there is not a *correct* choice of order. There are a handful of integrals in which it will be impossible or very difficult to do one order first. In most cases however, either order can be done first and which order is easiest is often a matter of interpretation so don't worry about it if you chose to do a different order here than we do. You will still get the same answer regardless of the order provided you do all the work correctly.

In this case neither orders seem to be particularly difficult so let's integrate with respect to x first for no other reason that it will allow us to get rid of the root right away.

So, here is the integral set up to do the x integration first.

$$\iint\limits_R 6y\sqrt{x} - 2y^3\,dA = \int_0^3 \int_1^4 6yx^{\frac{1}{2}} - 2y^3\,dx\,dy$$

Remember that the first integration is always the "inner" integral and the second integration is always the "outer" integral.

When writing the integral down do not forget the differentials! Many students come out of a Calculus I course with the bad habit of not putting them in. At this point however, that will get you in trouble. You need to be able to recall which variable we are integrating with respect to with each integral and the differentials will tell us that so don't forget about them.

Also, do not forget about the limits and make sure that they get attached to the correct integral. In this case the x integration is first and so the x limits need to go on the inner integral and the y limits need to go on the out integral. It is easy to get in a hurry and put them on the wrong integral.

Step 2

Okay, let's do the x integration now.

$$\begin{aligned}\iint\limits_R 6y\sqrt{x} - 2y^3\, dA &= \int_0^3 \int_1^4 6yx^{\frac{1}{2}} - 2y^3\, dx\, dy \\ &= \int_0^3 \left(4yx^{\frac{3}{2}} - 2xy^3\right)\Big|_1^4 dy \\ &= \int_0^3 28y - 6y^3\, dy\end{aligned}$$

Just remember that because we are integrating with respect to x in this step we treat all y's as if they were a constant and we know how to deal with constants in integrals.

Note that we are assuming that you are capable of doing the evaluation and so did not show the work in this problem and will rarely show it in any of the problems here unless there is a point that needs to be made.

Step 3

Now all we need to take care of is the y integration and that is just a simple Calculus I integral. Here is that work.

$$\iint\limits_R 6y\sqrt{x} - 2y^3\, dA = \int_0^3 28y - 6y^3\, dy = \left(14y^2 - \frac{3}{2}y^4\right)\Big|_0^3 = \boxed{\frac{9}{2}}$$

3. Compute the following double integral over the indicated rectangle.

$$\iint\limits_R \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\, dA \qquad R = [-1, 0] \times [1, 2]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. We know that the order will not affect the final answer so in that sense it doesn't matter which order we decide to use.

Often one of the orders of integration will be easier than the other and so we should keep that in mind when choosing an order. Note as well that often each order will be just as easy/hard as the other order and so order won't really matter all that much in those cases. Of course, there will also be integrals in which one of the orders of integration will be all but impossible, if not impossible, to compute and so you won't really have a choice of orders in cases such as that.

Given all the possibilities discussed above it can seem quite daunting when you need to decide on the order of integration. It generally isn't as bad as it seems however. When choosing an order of integration just take a look at the integral and think about what would need to be done for each order and see if there is one order that seems like it might be easier to take care of first or if maybe the resulting answer will make the second integration somewhat easier.

Sometimes it will not be readily apparent which order will be the easiest until you get into the problem. In these cases the only thing you can really do is just start with one order and see how it goes. If it starts getting too difficult you can always go back and give the other order a try to see if it is any easier.

The better you are at integration the easier/quicker it will be to choose an order of integration. If you are really rusty at integration and/or you didn't learn it all that great back in Calculus I then you should probably head back into the Calculus I material and practice your integration skills a little bit. Multiple integrals will be much easier to deal with if you have good Calculus I skills.

As a final note about choosing an order of integration remember that for the vast majority of the integrals there is not a *correct* choice of order. There are a handful of integrals in which it will be impossible or very difficult to do one order first. In most cases however, either order can be done first and which order is easiest is often a matter of interpretation so don't worry about it if you chose to do a different order here than we do. You will still get the same answer regardless of the order provided you do all the work correctly.

In this case neither orders seem to be particularly difficult so let's integrate with respect to y first for no other reason that it will allow us to get rid of the rational expressions in the integrand after the first integration.

So, here is the integral set up to do the y integration first.

$$\iint\limits_R \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\,dA = \int_{-1}^{0}\int_{1}^{2} \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\,dy\,dx$$

Remember that the first integration is always the "inner" integral and the second integration is always the "outer" integral.

When writing the integral down do not forget the differentials! Many students come out of a Calculus I course with the bad habit of not putting them in. At this point however, that will get you in trouble. You need to be able to recall which variable we are integrating with respect to with each integral and the differentials will tell us that so don't forget about them.

Also, do not forget about the limits and make sure that they get attached to the correct integral. In this case the y integration is first and so the y limits need to go on the inner integral and the x limits need to go on the outer integral. It is easy to get in a hurry and put them on the wrong integral.

Step 2

Okay, let's do the y integration now.

$$\begin{aligned}\iint\limits_R \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\,dA &= \int_{-1}^{0}\int_{1}^{2} \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\,dy\,dx \\ &= \int_{-1}^{0} \left(\frac{1}{2}\ln|y|\,\mathbf{e}^x + \frac{4x-1}{y}\right)\Bigg|_1^2 dx = \int_{-1}^{0} \frac{1}{2}\left(\ln(2)\,\mathbf{e}^x - 4x + 1\right)\,dx\end{aligned}$$

Just remember that because we are integrating with respect to y in this step we treat all x's as if they were a constant and we know how to deal with constants in integrals.

Note that we are assuming that you are capable of doing the evaluation and so did not show the work in this problem and will rarely show it in any of the problems here unless there is a point that needs to be made. Don't forget to do any simplification after the evaluation as that can sometimes greatly simplify the next integration.

Step 3

Now all we need to take care of in the x integration and that is just a simple Calculus I integral. Here is that work.

$$\begin{aligned}\iint\limits_R \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\,dA &= \int_{-1}^{0} \frac{1}{2}\left(\ln(2)\,\mathbf{e}^x - 4x + 1\right)\,dx \\ &= \left(\frac{1}{2}\left(\ln(2)\,\mathbf{e}^x - 2x^2 + x\right)\right)\Bigg|_{-1}^{0} \\ &= \boxed{\frac{1}{2}\left(\ln(2) - \ln(2)\,\mathbf{e}^{-1} + 3\right) = 1.71908}\end{aligned}$$

Do not get excited about "messy" answers. They will happen fairly regularly with these kinds of problems. In those cases it might be easier to reduce everything down to a decimal as we've done here.

---

4. Compute the following double integral over the indicated rectangle.

$$\iint\limits_R \sin(2x) - \frac{1}{1+6y}\,dA \qquad R = \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \times [0,1]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. We know that the order will not affect the final answer so in that sense it doesn't matter which order we decide to use.

Often one of the orders of integration will be easier than the other and so we should keep that in mind when choosing an order. Note as well that often each order will be just as easy/hard as the other order and so order won't really matter all that much in those cases. Of course, there will also be integrals in which one of the orders of integration will be all but impossible, if not impossible, to compute and so you won't really have a choice of orders in cases such as that.

Given all the possibilities discussed above it can seem quite daunting when you need to decide on the order of integration. It generally isn't as bad as it seems however. When choosing an order of integration just take a look at the integral and think about what would need to be done for each order and see if there is one order that seems like it might be easier to take care of first or if maybe the resulting answer will make the second integration somewhat easier.

Sometimes it will not be readily apparent which order will be the easiest until you get into the problem. In these cases the only thing you can really do is just start with one order and see how it goes. If it starts getting too difficult you can always go back and give the other order a try to see if it is any easier.

The better you are at integration the easier/quicker it will be to choose an order of integration. If you are really rusty at integration and/or you didn't learn it all that great back in Calculus I then you should probably head back into the Calculus I material and practice your integration skills a little bit. Multiple integrals will be much easier to deal with if you have good Calculus I skills.

As a final note about choosing an order of integration remember that for the vast majority of the integrals there is not a *correct* choice of order. There are a handful of integrals in which it will be impossible or very difficult to do one order first. In most cases however, either order can be done first and which order is easiest is often a matter of interpretation so don't worry about it if you chose to do a different order here than we do. You will still get the same answer regardless of the order provided you do all the work correctly.

In this case neither orders seem to be particularly difficult so let's integrate with respect to x first for no other reason that the x limits seem a little messier and so this will get rid of them with the first integration.

So, here is the integral set up to do the x integration first.

$$\iint\limits_R \sin(2x) - \frac{1}{1+6y}\, dA = \int_0^1 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2x) - \frac{1}{1+6y}\, dx\, dy$$

Remember that the first integration is always the "inner" integral and the second integration is always the "outer" integral.

When writing the integral down do not forget the differentials! Many students come out of a Calculus I course with the bad habit of not putting them in. At this point however, that will get you in trouble. You need to be able to recall which variable we are integrating with respect to with each integral and the differentials will tell us that so don't forget about them.

Also, do not forget about the limits and make sure that they get attached to the correct integral. In this case the x integration is first and so the x limits need to go on the inner integral and the y limits need to go on the outer integral. It is easy to get in a hurry and put them on the wrong integral.

Step 2

Okay, let's do the x integration now.

$$\iint\limits_R \sin(2x) - \frac{1}{1+6y}\,dA = \int_0^1 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2x) - \frac{1}{1+6y}\,dx\,dy$$
$$= \int_0^1 \left(-\frac{1}{2}\cos(2x) - \frac{x}{1+6y}\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} dy = \int_0^1 \frac{1}{2} - \frac{\frac{\pi}{4}}{1+6y}\,dy$$

Just remember that because we are integrating with respect to x in this step we treat all y's as if they were a constant and we know how to deal with constants in integrals.

Note that we are assuming that you are capable of doing the evaluation and so did not show the work in this problem and will rarely show it in any of the problems here unless there is a point that needs to be made. Don't forget to do any simplification after the evaluation as that can sometimes greatly simplify the next integration.

Step 3

Now all we need to take care of is the y integration and that is just a simple Calculus I integral. Here is that work.

$$\iint\limits_R \sin(2x) - \frac{1}{1+6y}\,dA = \int_0^1 \frac{1}{2} - \frac{\frac{\pi}{4}}{1+6y}\,dy$$
$$= \left(\frac{1}{2}y - \frac{\pi}{24}\ln|1+6y|\right)\Big|_0^1 = \boxed{\frac{1}{2} - \frac{\pi}{24}\ln(7) = 0.24528}$$

5. Compute the following double integral over the indicated rectangle.

$$\iint\limits_R y\mathbf{e}^{y^2-4x}\,dA \qquad\qquad R = [0,2]\times\left[0,\sqrt{8}\right]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. As we discussed in the first few problems of this section this can be daunting task and in those problems the order really did not matter. The order chosen for those problems was mostly a cosmetic choice in the sense that both orders were had pretty much the same level of difficulty.

With this problem the order still really doesn't matter all that much as both will require a substitution. However, we will simplify the integrand a little if we first do the y substitution since that will eliminate the y in front of the exponential. So, in this case, it looks like the integration process might be a little simpler if we integrate with respect to y first so let's do that.

Here is the integral set up to do the y integration first.

$$\iint\limits_R y\mathbf{e}^{y^2-4x}\,dA = \int_0^2 \int_0^{\sqrt{8}} y\mathbf{e}^{y^2-4x}\,dy\,dx$$

Step 2

Okay, let's do the y integration now.

$$\iint\limits_R y\mathbf{e}^{y^2-4x}\,dA = \int_0^2 \int_0^{\sqrt{8}} y\mathbf{e}^{y^2-4x}\,dy\,dx \qquad u = y^2 - 4x \quad \to \quad du = 2y\,dy$$

$$= \int_0^2 \left(\frac{1}{2}\mathbf{e}^{y^2-4x}\right)\Big|_0^{\sqrt{8}}\,dx = \int_0^2 \frac{1}{2}\left(\mathbf{e}^{8-4x} - \mathbf{e}^{-4x}\right)\,dx$$

Just remember that because we are integrating with respect to y in this step we treat all x's as if they were a constant and we know how to deal with constants in integrals.

Be careful with substitutions in the first integration. We showed the substitution we used above as well as the differential. When computing the differential we need to differentiate the right side of the substitution with respect to y (since we are doing a y integration). In other words, we need to do a partial derivative of the right side and so the "$-4x$" will differentiate to zero when differentiating with respect to y!

One of the bigger mistakes that students make here is to leave the "-4 x" in the differential or to "differentiate" it to "-4" which in turn causes the substitution to not work because there is then no way to get rid of the y in front of the exponential.

Mistakes with Calculus I substitutions at this stage is one of the biggest issues that students have when first doing these kinds of integrals so you need to be very careful and always pay attention to which variable you are integrating with respect to and then differentiate your substitution with respect to the same variable.

Step 3

Now all we need to take care of is the x integration and that is just Calculus I integral with a couple of simple substitutions (we'll leave the substitution work to you to verify). Here is that work.

$$\iint\limits_R y\mathbf{e}^{y^2-4x}\,dA = \int_0^2 \frac{1}{2}\left(\mathbf{e}^{8-4x} - \mathbf{e}^{-4x}\right)\,dx$$

$$= \left(\frac{1}{2}\left(-\frac{1}{4}\mathbf{e}^{8-4x} + \frac{1}{4}\mathbf{e}^{-4x}\right)\right)\Bigg|_0^2 = \boxed{\frac{1}{8}\left(\mathbf{e}^{8} + \mathbf{e}^{-8} - 2\right) = 372.3698}$$

Do not get excited about "messy" answers. They will happen fairly regularly with these kinds of problems. In those cases it might be easier to reduce everything down to a decimal as we've done here.

---

6. Compute the following double integral over the indicated rectangle.

$$\iint\limits_R xy^2\sqrt{x^2+y^3}\,dA \qquad\qquad R = [0,3]\times[0,2]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. As we discussed in the first few problems of this section this can be daunting task and in those problems the order really did not matter. The order chosen for those problems was mostly a cosmetic choice in the sense that both orders were had pretty much the same level of difficulty.

With this problem the order still really doesn't matter all that much as both will require a substitution. However, we might get a little more simplification of the integrand if we first do the y substitution since that will eliminate the y^2 in front of the root. Of course, the x substitution would have eliminated the x in front of the root so either way we'll get some simplification, but the y integration seems to offer a little bit more simplification.

So, here is the integral set up to do the y integration first.

$$\iint\limits_R xy^2\sqrt{x^2+y^3}\,dA = \int_0^3\int_0^2 xy^2\sqrt{x^2+y^3}\,dy\,dx$$

Step 2

Okay, let's do the y integration now.

$$\iint\limits_R xy^2\sqrt{x^2+y^3}\,dA = \int_0^3\int_0^2 xy^2\left(x^2+y^3\right)^{\frac{1}{2}}\,dy\,dx \qquad u = x^2+y^3 \quad\rightarrow\quad du = 3y^2\,dy$$

$$= \int_0^3 \left(\frac{2}{9}x\left(x^2+y^3\right)^{\frac{3}{2}}\right)\Bigg|_0^2\,dx = \int_0^3 \frac{2}{9}x\left(x^2+8\right)^{\frac{3}{2}} - \frac{2}{9}x^4\,dx$$

Just remember that because we are integrating with respect to y in this step we treat all x's as if they were a constant and we know how to deal with constants in integrals.

Be careful with substitutions in the first integration. We showed the substitution we used above as well as the differential. When computing the differential we need to differentiate the right side of the substitution with respect to y (since we are doing a y integration). In other words, we need to do a partial derivative of the right side and so the x^2 will differentiate to zero when differentiating with respect to y!

One of the bigger mistakes that students make here is to leave the x^2 in the differential or to "differentiate" it to "$2x$" which in turn causes the substitution to not work because there is then no way to get rid of the y^2 in front of the root.

Mistakes with Calculus I substitutions at this stage is one of the biggest issues that students have when first doing these kinds of integrals so you need to be very careful and always pay attention to which variable you are integrating with respect to and then differentiate your substitution with respect to the same variable.

Step 3

Now all we need to take care of is the x integration and that is just Calculus I integral with a simple substitution (we'll leave the substitution work to you to verify). Here is that work.

$$\iint\limits_R xy^2\sqrt{x^2+y^3}\,dA = \int_0^3 \frac{2}{9}x\left(x^2+8\right)^{\frac{3}{2}} - \frac{2}{9}x^4\,dx$$

$$= \left(\frac{2}{45}\left(x^2+8\right)^{\frac{5}{2}} - \frac{2}{45}x^5\right)\Bigg|_0^3$$

$$= \boxed{\frac{2}{45}\left(17^{\frac{5}{2}} - 243 - 128\sqrt{2}\right) = 34.1137}$$

Do not get excited about "messy" answers. They will happen fairly regularly with these kinds of problems. In those cases it might be easier to reduce everything down to a decimal as we've done here.

7. Compute the following double integral over the indicated rectangle.

$$\iint\limits_R xy\cos\left(y\,x^2\right)\,dA \qquad\qquad R=[-2,3]\times[-1,1]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. As we discussed in the first few problems of this section this can be daunting task and in those problems the order really did not matter. The order chosen for those problems was mostly a cosmetic choice in the sense that both orders were had pretty much the same level of difficulty.

With this problem however, we have a real difference in the orders. If we do the y integration first we will have to do integration by parts. On the other hand, if we do x first we only need to do a Calculus I substitution, which is almost always easier/quicker than integration by parts. On top of all that, if you think about how the substitution will go it looks like we'll also lose the integration by parts for the y after the substitution is done (if you don't see that don't worry you will eventually with enough practice).

So, it looks like integrating with respect to x first is the way to go. Here is the integral set up to do the x integration first.

$$\iint\limits_R xy\cos\left(y\,x^2\right)\,dA=\int_{-1}^{1}\int_{-2}^{3}xy\cos\left(y\,x^2\right)\,dx\,dy$$

Step 2

Okay, let's do the x integration now.

$$\begin{aligned}\iint\limits_R xy\cos\left(y\,x^2\right)\,dA&=\int_{-1}^{1}\int_{-2}^{3}xy\cos\left(y\,x^2\right)\,dx\,dy \qquad u=yx^2 \quad\to\quad du=2yx\,dx\\&=\int_{-1}^{1}\left(\frac{1}{2}\sin\left(y\,x^2\right)\right)\Big|_{-2}^{3}\,dy=\int_{-1}^{1}\frac{1}{2}\left(\sin\left(9y\right)-\sin\left(4y\right)\right)\,dy\end{aligned}$$

So, as noted above, upon doing the substitution we not only lost the x in front of the cosine but we also lost the y and that in turn means we won't have to do integration by parts for the y integral.

So, in this case doing the x integration first completely eliminated the integration by parts from the y integration from the problem. This won't always happen but when it does we'll take advantage of it and when it doesn't we'll be doing integration by parts whether we want to or not.

Step 3

Now all we need to take care of is the y integration and that is just Calculus I integral with a couple of really simple substitutions (we'll leave the substitution work to you to verify). Here is that work.

$$\begin{aligned}\iint\limits_R xy\cos\left(y\,x^2\right)\,dA &= \int_{-1}^{1}\frac{1}{2}\left(\sin\left(9y\right)-\sin\left(4y\right)\right)\,dy\\ &= \frac{1}{2}\left(\frac{1}{4}\cos\left(4y\right)-\frac{1}{9}\cos\left(9y\right)\right)\Big|_{-1}^{1}\\ &= \boxed{\frac{1}{2}\left[\frac{1}{4}\left(\cos\left(4\right)-\cos\left(-4\right)\right)-\frac{1}{9}\left(\cos\left(9\right)-\cos\left(-9\right)\right)\right]=0}\end{aligned}$$

Recall that cosine is an even function and so $\cos\left(-\theta\right)=\cos\left(\theta\right)$!

---

8. Compute the following double integral over the indicated rectangle.

$$\iint\limits_R xy\cos\left(y\right)-x^2\,dA \qquad R=[1,2]\times\left[\frac{\pi}{2},\pi\right]$$

Step 1

The order of integration was not specified in the problem statement so we get to choose the order of integration. As we discussed in the first few problems of this section this can be daunting task and in those problems the order really did not matter. The order chosen for those problems was mostly a cosmetic choice in the sense that both orders were had pretty much the same level of difficulty.

With this problem however, we have a real difference in the orders. If we do the y integration first we will have to do integration by parts. On the other hand, the x integration is a very simple Calculus I integration.

So, it looks like integrating with respect to x first is the way to go. Here is the integral set up to do the x integration first.

$$\iint\limits_R xy\cos(y) - x^2\,dA = \int_{\frac{\pi}{2}}^{\pi}\int_1^2 xy\cos(y) - x^2\,dx\,dy$$

Step 2

Okay, let's do the x integration now.

$$\begin{aligned}\iint\limits_R xy\cos(y) - x^2\,dA &= \int_{\frac{\pi}{2}}^{\pi}\int_1^2 xy\cos(y) - x^2\,dx\,dy \\ &= \int_{\frac{\pi}{2}}^{\pi}\left(\frac{1}{2}x^2y\cos(y) - \frac{1}{3}x^3\right)\Big|_1^2\,dy = \int_{\frac{\pi}{2}}^{\pi}\frac{3}{2}y\cos(y) - \frac{7}{3}\,dy\end{aligned}$$

Note that for this example, unlike the previous one, the integration by parts did not go away after doing the first integration. Be careful to not just expect things like integration by parts to just disappear after doing the first integration. They often won't, and, in fact, it is possible that they might actually show up after doing the first integration!

Step 3

Now all we need to take care of is the y integration. As noted this is integration by parts for the first term and so we should probably split up the integral before doing the integration by parts.

Here is the work for this problem.

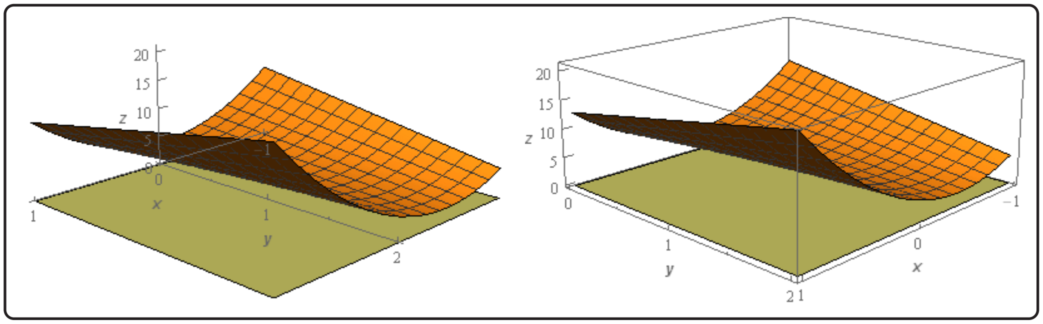
$$\begin{aligned}\iint\limits_R xy\cos(y) - x^2\,dA &= \int_{\frac{\pi}{2}}^{\pi}\frac{3}{2}y\cos(y)\,dy - \int_{\frac{\pi}{2}}^{\pi}\frac{7}{3}\,dy \qquad u = \frac{3}{2}y \qquad dv = \cos(y)\,dy \\ &= \left(\frac{3}{2}y\sin(y) + \frac{3}{2}\cos(y) - \frac{7}{3}y\right)\Big|_{\frac{\pi}{2}}^{\pi} \\ &= \boxed{-\frac{3}{2} - \frac{23}{12}\pi = -7.5214}\end{aligned}$$

Note that we gave the u and dv for the integration by parts work but are leaving the details to you to verify the result.

9. Determine the volume that lies under $f(x,y) = 9x^2 + 4xy + 4$ and above the rectangle given by $[-1,1] \times [0,2]$ in the xy-plane.

Step 1

First, let's start off with a quick sketch of the function.



The greenish rectangle under the surface is R from the problem statement and is given to help visualize the relationship between the surface and R.

Note that the sketch isn't really needed for this problem. It is just here so we can get a feel for what the problem is asking of us. Also, the sketch is presented in two ways to help "see" the surface. Sometimes it is easier to see what is going on with both a "standard" set of axes and a "box frame" set of axes.

Now, we know from the notes that the volume is given by,

$$V = \iint\limits_R 9x^2 + 4xy + 4 \, dA$$

where R is the rectangle given in the problem statement.

Step 2

Now, as with the other problems in this section we need to determine the order of integration. In this case there doesn't seem to be much difference between the two so we'll integrate with respect to x first simply because there are more of them and so maybe we'll "simplify" the integral somewhat after doing that integration.

Here is the x integration work.

$$\begin{aligned} V &= \iint\limits_R 9x^2 + 4xy + 4\,dA \\ &= \int_0^2 \int_{-1}^1 9x^2 + 4xy + 4\,dx\,dy = \int_0^2 \left(3x^3 + 2x^2y + 4x\right)\Big|_{-1}^1\,dy = \int_0^2 14\,dy \end{aligned}$$

In this case the second term involving the y actually canceled out when doing the x limit evaluation. This will happen on occasion and isn't something we should get excited about when it happens.

Step 3

Finally, all we need to do is the very simply y integration. Here is that work.

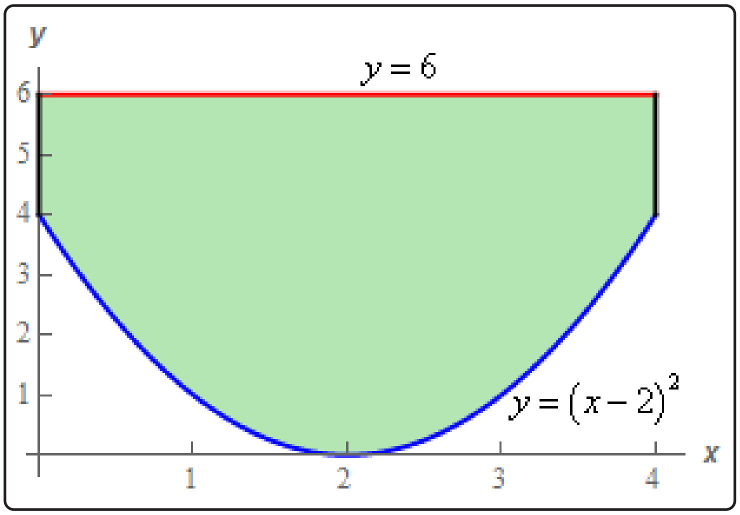
$$V = \int_0^2 14\,dy = \left(14y\right)\Big|_0^2 = \boxed{28}$$

15.3 Double Integrals over General Regions

1. Evaluate $\displaystyle\iint\limits_D 42y^2 - 12x\,dA$ where $D = \left\{(x,y)\,|0 \le x \le 4, (x-2)^2 \le y \le 6\right\}$

Step 1

Below is a quick sketch of the region D.



In general, this sketch is often important to setting the integral up correctly. We'll need to determine the order of integration and often the region will "force" a particular order. Many regions can only be dealt with easily by doing one particular order of integration and sometimes the only way to really see that is to have a sketch of D.

Even if you can do the integral in either order the sketch of D will often help with setting up the limits for the integrals.

Step 2

With this problem we were pretty much given the order of integration by how the region D was specified in the problem statement. Note however, that the sketch shows that this was pretty much the only easy order of integration. The same function is always on the top of the region and the same function is always on the bottom of the region and so it makes sense to integrate y first.

If we wanted to integrate x first we'd have a messier integration to deal with. First the right/left functions change and so we couldn't do the x integration with a single integral.

The x integration would require two integrals in this case. There is also the fact that the lower portion of the region has the same function for both the right and left sides. The equation could be solved for x, as we'd need to in order to x integration first, and often that is either very difficult or would give unpleasant limits. It wouldn't be too difficult in this case but it would put roots into the limits and that often makes for messier integration.

So, let's go with the order of integration specified in the problem statement and because we were given D in the set builder notation we also were given the limits for both x and y which is nice as we usually will need to figure those out on our own.

Here is the integral set up for y integration first.

$$\iint\limits_D 42y^2 - 12x\,dA = \int_0^4 \int_{(x-2)^2}^{6} 42y^2 - 12x\,dy\,dx$$

Step 3

Here is the y integration.

$$\begin{aligned}\iint\limits_D 42y^2 - 12x\,dA &= \int_0^4 \int_{(x-2)^2}^{6} 42y^2 - 12x\,dy\,dx \\ &= \int_0^4 \left(14y^3 - 12xy\right)\Big|_{(x-2)^2}^{6} dx \\ &= \int_0^4 3024 - 72x - 14(x-2)^6 + 12x(x-2)^2\,dx\end{aligned}$$

Step 4

Now, we did not do any real simplification of the integrand in the last step. There was a reason for that.

After doing the first integration students will often just launch into a "simplification" mode and multiply everything out and "simplify" everything. Sometimes that does need to be done and we don't want to give the impression it is never a good thing or never needs to be done.

However, take a look at the third term above. It could be multiplied out if we wanted to but it would take a little bit of time and there is a chance we'd mess up a sign or coefficient somewhere. We are going to be integrating and the third term can be integrated very quickly with a simple Calculus I substitution. In other words, why bother with the messy multiplication with that term when it does not need to be done.

The fourth term, on the other hand, does need to be multiplied out because of the extra x that is in the front of the term.

So, before just launching into "simplification" mode take a quick look at the integrand and see if there are any terms that can be done with a simple substitution as we won't need to mess with those terms. Only multiply out terms that actually need to be multiplied out.

Here is the x integration work. We will leave the Algebra details to you to verify and we'll also be leaving the Calculus I substitution work to you to verify.

$$\begin{aligned}\iint\limits_D 42y^2 - 12x\,dA &= \int_0^4 3024 - 72x - 14(x-2)^6 + 12x(x-2)^2\,dx \\ &= \int_0^4 3024 - 24x - 48x^2 + 12x^3 - 14(x-2)^6\,dx \\ &= \left(3024x - 12x^2 - 16x^3 + 3x^4 - 2(x-2)^7\right)\Big|_0^4 = \boxed{11136}\end{aligned}$$

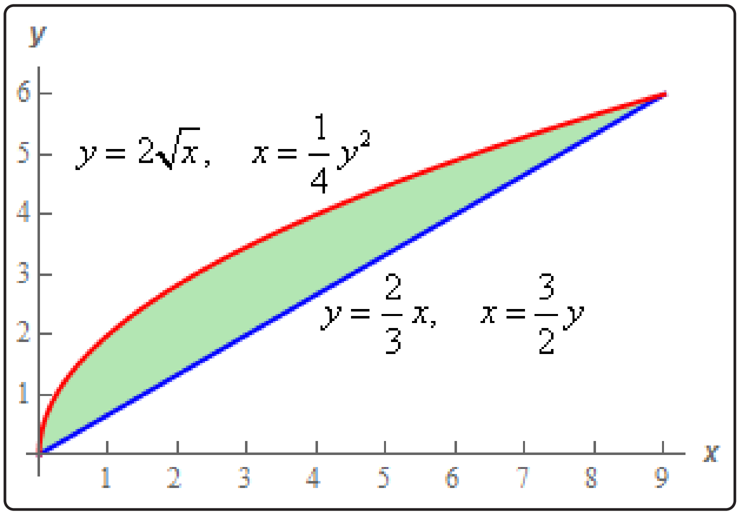
Before leaving this problem let's again note how much easier dealing with the third term was because we did not multiply it out and just used a substitution. Made this problem a lot easier.

Also note that this problem illustrated an important point that needs to be made with many of these integrals. These integrals will often get very messy after the first integration. You need to be ready for that and expect it to happen on occasion. Just because they start off looking "easy" doesn't mean that they will remain easy throughout the whole problem. Just because it becomes a mess doesn't mean you've made a mistake, although that is unfortunately always a possible reason for a messy integral. It may just mean this is one of those integrals that get somewhat messy before they are done.

2. Evaluate $\displaystyle\iint\limits_D 2yx^2 + 9y^3\,dA$ where D is the region bounded by $y = \dfrac{2}{3}x$ and $y = 2\sqrt{x}$.

Step 1

Below is a quick sketch of the region D.



Note that we gave both forms of the equation for each curve to help with the next step.

In general, this sketch is often important to setting the integral up correctly. We'll need to determine the order of integration and often the region will "force" a particular order. Many regions can only be dealt with easily by doing one particular order of integration and sometimes the only way to really see that is to have a sketch of D.

Even if you can do the integral in either order the sketch of D will often help with setting up the limits for the integrals.

Step 2

Now, with this problem, the region will allow either order of integration without any real change of difficulty in the integration.

So, here are the limits for each order of integration that we could use.

$$\begin{array}{c} 0 \le x \le 9 \\ \frac{2}{3}x \le y \le 2\sqrt{x} \end{array} \qquad \text{OR} \qquad \begin{array}{c} 0 \le y \le 6 \\ \frac{1}{4}y^2 \le x \le \frac{3}{2}y \end{array}$$

Step 3

As noted above either order could be done without much real change in difficulty. So, for this problem let's integrate with respect to y first.

Often roots in limits can lead to messier integrands for the second integration. However, in this case notice that both the y terms will integrate to terms with even exponents and that will eliminate the root upon evaluation. This order will also keep the exponents slightly smaller which *may* help a little with the second integration.

Here is the integral set up for y integration first.

$$\iint\limits_D 2yx^2 + 9y^3\,dA = \int_0^9 \int_{\frac{2}{3}x}^{2\sqrt{x}} 2yx^2 + 9y^3\,dy\,dx$$

Step 4

Here is the y integration.

$$\begin{aligned}\iint\limits_D 2yx^2 + 9y^3\,dA &= \int_0^9 \int_{\frac{2}{3}x}^{2\sqrt{x}} 2yx^2 + 9y^3\,dy\,dx \\ &= \int_0^9 \left(y^2x^2 + \frac{9}{4}y^4\right)\Big|_{\frac{2}{3}x}^{2\sqrt{x}}\,dx \\ &= \int_0^9 4x\left(x^2\right) + \frac{9}{4}\left(16x^2\right) - \left[\frac{4}{9}x^2\left(x^2\right) + \frac{9}{4}\left(\frac{16}{81}x^4\right)\right]dx \\ &= \int_0^9 36x^2 + 4x^3 - \frac{8}{9}x^4\,dx\end{aligned}$$

Step 5

In this case there is a small amount of pretty simple simplification that we could do to reduce the complexity of the integrand and so we did that.

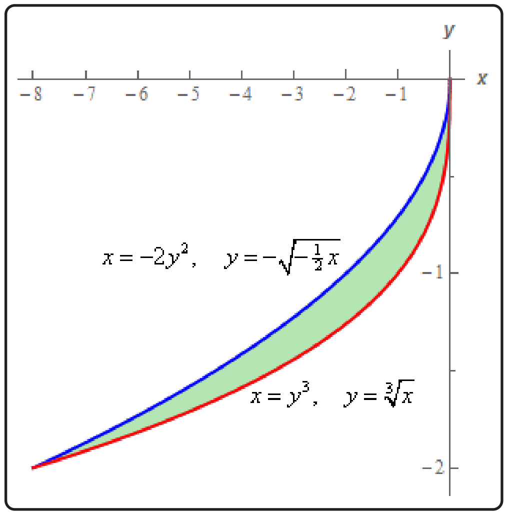
All that is left is to do the x integration. Here is that work.

$$\begin{aligned}\iint\limits_D 2yx^2 + 9y^3\,dA &= \int_0^9 36x^2 + 4x^3 - \frac{8}{9}x^4\,dx \\ &= \left(12x^3 + x^4 - \frac{8}{45}x^5\right)\Big|_0^9 = \boxed{\frac{24057}{5} = 4811.4}\end{aligned}$$

3. Evaluate $\displaystyle\iint\limits_D 10x^2y^3 - 6\,dA$ where D is the region bounded by $x = -2y^2$ and $x = y^3$.

Step 1

Below is a quick sketch of the region D.



Note that we gave both forms of the equation for each curve to help with the next step.

In general, this sketch is often important to setting the integral up correctly. We'll need to determine the order of integration and often the region will "force" a particular order. Many regions can only be dealt with easily by doing one particular order of integration and sometimes the only way to really see that is to have a sketch of D.

Even if you can do the integral in either order the sketch of D will often help with setting up the limits for the integrals.

Step 2

Now, with this problem, the region will allow either order of integration without any real change of difficulty in the integration.

So, here are the limits for each order of integration that we could use.

$$\begin{array}{c} -8 \le x \le 0 \\ \sqrt[3]{x} \le y \le -\sqrt{-\frac{1}{2}x} \end{array} \qquad \text{OR} \qquad \begin{array}{c} -2 \le y \le 0 \\ -2y^2 \le x \le y^3 \end{array}$$

Step 3

As noted above either order could be done without much real change in difficulty. However, for the first set of limits both of the y limits are roots and that might make the second integration a little messier. The upper y limit is also a little messy with all the minus signs.

So, for this problem let's integrate with respect to x first. Here is the integral set up for x integration first.

$$\iint\limits_D 10x^2y^3 - 6\,dA = \int_{-2}^{0}\int_{-2y^2}^{y^3} 10x^2y^3 - 6\,dx\,dy$$

Step 4

Here is the x integration.

$$\begin{aligned}\iint\limits_D 10x^2y^3 - 6\,dA &= \int_{-2}^{0}\int_{-2y^2}^{y^3} 10x^2y^3 - 6\,dx\,dy \\ &= \int_{-2}^{0}\left(\frac{10}{3}x^3y^3 - 6x\right)\bigg|_{-2y^2}^{y^3}\,dy \\ &= \int_{-2}^{0}\frac{10}{3}y^9\left(y^3\right) - 6y^3 - \left[\frac{10}{3}\left(-8y^6\right)\left(y^3\right) - 6\left(-2y^2\right)\right]\,dy \\ &= \int_{-2}^{0}\frac{10}{3}y^{12} + \frac{80}{3}y^9 - 6y^3 - 12y^2\,dy\end{aligned}$$

Step 5

In this case there is a small amount of pretty simple simplification that we could do to reduce the complexity of the integrand and so we did that.

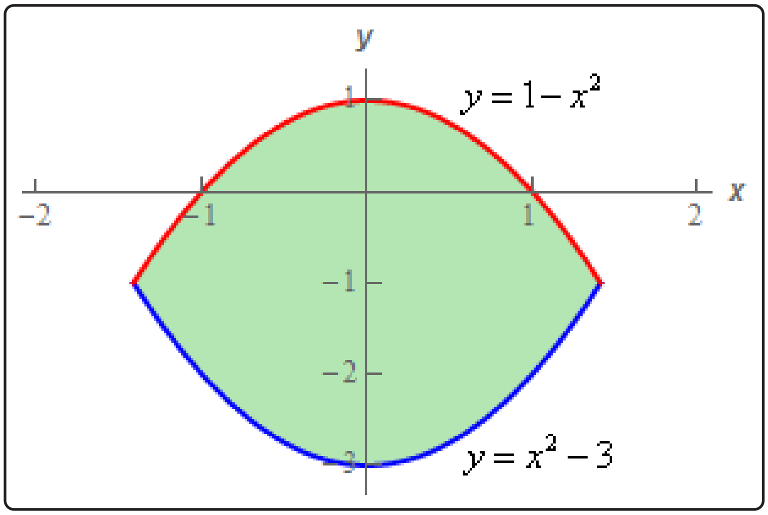
All that is left is to do the y integration. Here is that work.

$$\begin{aligned}\iint\limits_D 10x^2y^3 - 6\,dA &= \int_{-2}^{0}\frac{10}{3}y^{12} + \frac{80}{3}y^9 - 6y^3 - 12y^2\,dy \\ &= \left(\frac{10}{39}y^{13} + \frac{8}{3}y^{10} - \frac{3}{2}y^4 - 4y^3\right)\bigg|_{-2}^{0} = \boxed{-\frac{8296}{13} = -638.1538}\end{aligned}$$

4. Evaluate $\displaystyle\iint\limits_{D} x\left(y-1\right)\,dA$ where D is the region bounded by $y = 1 - x^2$ and $y = x^2 - 3$.

Step 1

Below is a quick sketch of the region D.



In general, this sketch is often important to setting the integral up correctly. We'll need to determine the order of integration and often the region will "force" a particular order. Many regions can only be dealt with easily by doing one particular order of integration and sometimes the only way to really see that is to have a sketch of D.

Even if you can do the integral in either order the sketch of D will often help with setting up the limits for the integrals.

Step 2

With this problem the region is really only set up to integrate y first. Integrating x first would require two integrals (the right/left functions change) and the limits for the x's would be a little messy to deal with.

So, here are the limits for this integral.

$$\begin{gathered} -\sqrt{2} \le x \le \sqrt{2} \\ x^2 - 3 \le y \le 1 - x^2 \end{gathered}$$

The x limits can easily be found by setting the two equations equal and solving for x.

Step 3

Here is the integral set up for y integration first.

$$\iint\limits_D x\,(y-1)\,dA = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{x^2-3}^{1-x^2} x\,(y-1)\,dy\,dx$$

Step 4

Here is the y integration.

$$\begin{aligned}\iint\limits_D x\,(y-1)\,dA &= \int_{-\sqrt{2}}^{\sqrt{2}}\int_{x^2-3}^{1-x^2} x\,(y-1)\,dy\,dx \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \left.\left(x\left(\frac{1}{2}y^2 - y\right)\right)\right|_{x^2-3}^{1-x^2} dx \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{2}x\left(1-x^2\right)^2 - \frac{1}{2}x\left(x^2-3\right)^2 - x\left(1-x^2\right) + x\left(x^2-3\right)\,dx\end{aligned}$$

Step 5

With this problem we have two options in dealing with the simplification of the integrand. The 3^{rd} and 4^{th} terms will need to be simplified. The 1^{st} and 2^{nd} terms however can be simplified, and they aren't that hard to simplify or we could do a fairly quick Calculus I substitutions for each of them.

If you multiply everything out you will get the following integral.

$$\iint\limits_D x\,(y-1)\,dA = \int_{-\sqrt{2}}^{\sqrt{2}} 4x^3 - 8x\,dx = \left.\left(x^4 - 4x^2\right)\right|_{-\sqrt{2}}^{\sqrt{2}} = \boxed{0}$$

There is a lot of cancelation going on with this integrand. It isn't obvious however that there would be that much cancelling at first glance and the multiplication required to do the cancelling is the type where it is easy to miss a minus sign and get the wrong integrand and then a wrong answer.

So, let's also do the substitution path to see the difference. Doing that gives,

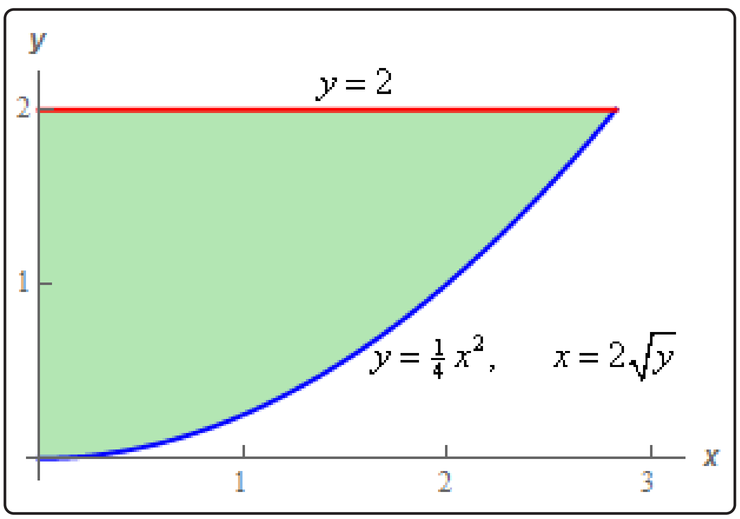
$$\iint\limits_D x(y-1)\,dA = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{2}x\left(1-x^2\right)^2 - \frac{1}{2}x\left(x^2-3\right)^2 + 2x^3 - 4x\,dx$$
$$= \left(-\frac{1}{12}\left(1-x^2\right)^3 - \frac{1}{12}\left(x^2-3\right)^3 + \frac{1}{2}x^4 - 2x^2\right)\Bigg|_{-\sqrt{2}}^{\sqrt{2}} = \boxed{0}$$

So, the same answer which shouldn't be very surprising, but a slightly messier integration and evaluation process. Which path you chose to take and which path you feel is the easier of the two is probably very dependent on the person. As shown however you will get the same answer so you won't need to worry about that.

5. Evaluate $\displaystyle\iint\limits_D 5x^3\cos\left(y^3\right)\,dA$ where D is the region bounded by $y=2$, $y=\frac{1}{4}x^2$ and the y-axis.

Step 1

Below is a quick sketch of the region D.



Note that we gave both forms of the equation for the lower curve to help with the next step.

In general, this sketch is often important to setting the integral up correctly. We'll need to determine the order of integration and often the region will "force" a particular order. Many regions can only be dealt with easily by doing one particular order of integration and sometimes the only way to really see that is to have a sketch of D.

Even if you can do the integral in either order the sketch of D will often help with setting up the limits for the integrals.

Step 2

With this problem the region can be dealt with either order of integration. The integral on the other hand can't. We simply cannot integrate y first as there is no y^2 in front of the cosine we'd need to do the substitution. So, we'll have no choice but to do the x integration first.

So, here are the limits for this integral.

$$\begin{gathered} 0 \le y \le 2 \\ 0 \le x \le 2\sqrt{y} \end{gathered}$$

Step 3

Here is the integral set up for x integration first.

$$\iint\limits_D 5x^3 \cos\left(y^3\right)\, dA = \int_0^2 \int_0^{2\sqrt{y}} 5x^3 \cos\left(y^3\right)\, dx\, dy$$

Step 4

Here is the x integration.

$$\begin{aligned} \iint\limits_D 5x^3 \cos\left(y^3\right)\, dA &= \int_0^2 \int_0^{2\sqrt{y}} 5x^3 \cos\left(y^3\right)\, dx\, dy \\ &= \int_0^2 \left. \left(\frac{5}{4}x^4 \cos\left(y^3\right)\right)\right|_0^{2\sqrt{y}} dy \\ &= \int_0^2 20y^2 \cos\left(y^3\right)\, dy \end{aligned}$$

Step 5

Note that while we couldn't do the y integration first we can do it now. The y^2 we need for the substitution is now there after doing the x integration.

Here is the y integration.

$$\iint\limits_D 5x^3 \cos\left(y^3\right)\, dA = \int_0^2 20y^2 \cos\left(y^3\right)\, dy$$

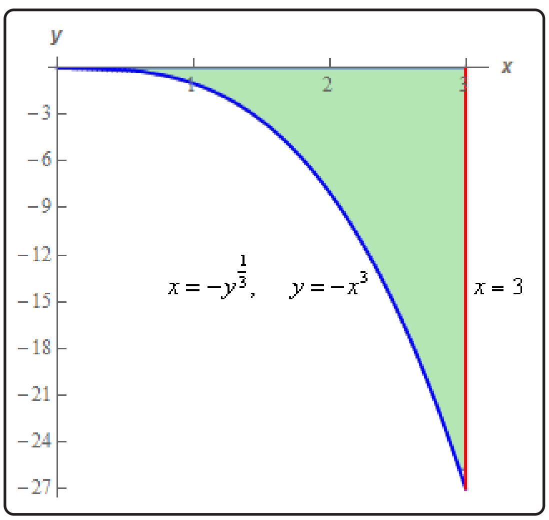
$$= \left(\frac{20}{3}\sin\left(y^3\right)\right)\Bigg|_0^2 = \boxed{\frac{20}{3}\sin\left(8\right) = 6.5957}$$

Remember to have your calculator set to radians when computing to decimals.

6. Evaluate $\displaystyle\iint\limits_D \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\, dA$ where D is the region bounded by $x = -y^{\frac{1}{3}}$, $x = 3$ and the x-axis.

Step 1

Below is a quick sketch of the region D.



Note that we gave both forms of the equation for the left curve to help with the next step.

In general, this sketch is often important to setting the integral up correctly. We'll need to determine the order of integration and often the region will "force" a particular order. Many regions can only be dealt with easily by doing one particular order of integration and sometimes the only way to really see that is to have a sketch of D.

Even if you can do the integral in either order the sketch of D will often help with setting up the limits for the integrals.

Step 2

Now, with this problem, the region will allow either order of integration without any real change of difficulty in the integration.

So, here are the limits for each order of integration that we could use.

$$\begin{array}{c} 0 \le x \le 3 \\ -x^3 \le y \le 0 \end{array} \qquad \text{OR} \qquad \begin{array}{c} -27 \le y \le 0 \\ -y^{\frac{1}{3}} \le x \le 3 \end{array}$$

Step 3

As noted above either order could be done without much real change in difficulty. However, it looks like the limits are a little nicer if we integrate with respect to y first. One of the y limits is zero and the x limits are smaller so it looks like this might be a little easier to deal with.

So, for this problem let's integrate with respect to y first. Here is the integral set up for y integration first.

$$\iint\limits_D \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,dA = \int_0^3 \int_{-x^3}^0 \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,dy\,dx$$

Step 4

Here is the y integration.

$$\begin{aligned} \iint\limits_D \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,dA &= \int_0^3 \int_{-x^3}^0 \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,dy\,dx \\ &= \int_0^3 \left.\left(\frac{\frac{3}{2}y^{\frac{2}{3}}}{\left(x^3+1\right)}\right)\right|_{-x^3}^{0} dx \\ &= \int_0^3 -\frac{\frac{3}{2}\left(-x^3\right)^{\frac{2}{3}}}{\left(x^3+1\right)}\,dx = \int_0^3 -\frac{\frac{3}{2}x^2}{\left(x^3+1\right)}\,dx \end{aligned}$$

Be careful with the minus signs in this integration. The minus sign in front of the integrand comes from the evaluation process (recall we first evaluate at zero!). The minus sign in the numerator will be eliminated when we deal with the exponent (the numerator of the exponent is a two so we need to square things!).

Step 5

Now the x integration is a simple Calculus I substitution. We'll leave the substitution work to you to verify. Here is the x integration.

$$\iint\limits_D \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\,dA = \int_0^3 -\frac{\frac{3}{2}x^2}{\left(x^3+1\right)}\,dx = \left(-\frac{1}{2}\ln\left(x^3+1\right)\right)\Big|_0^3$$

$$= \boxed{-\frac{1}{2}\ln\left(28\right) = -1.6661}$$

Remember that $\ln\left(1\right) = 0$!

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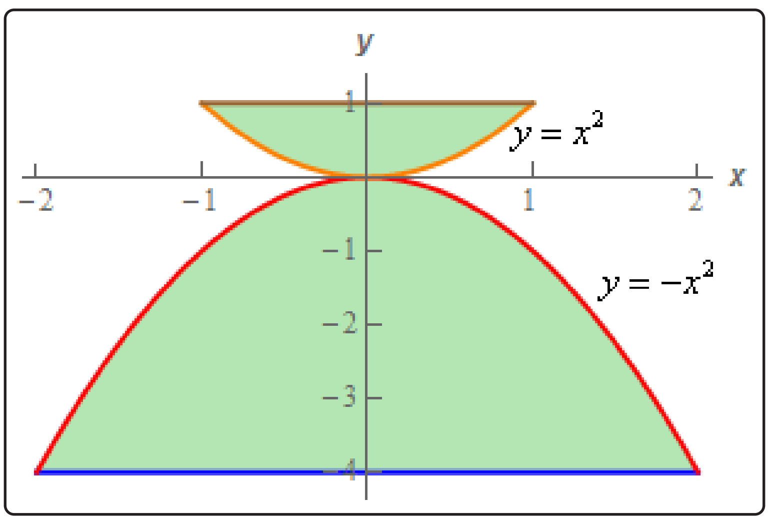
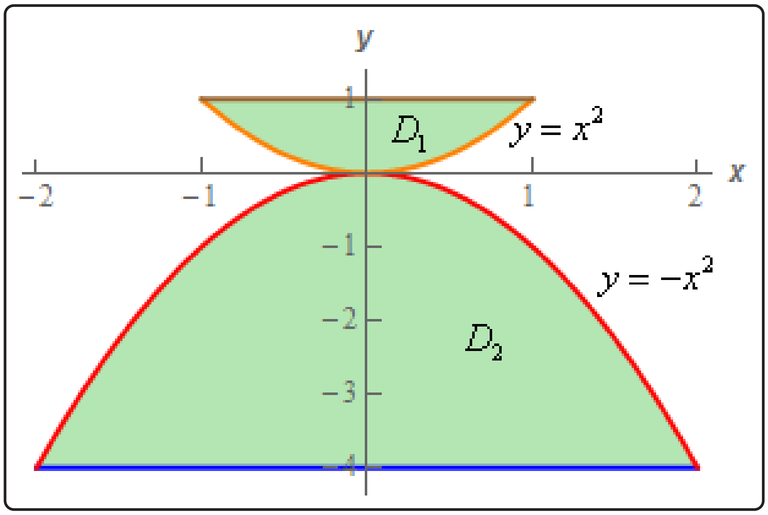
7. Evaluate $\displaystyle\iint\limits_D 3 - 6xy\,dA$ where D is the region shown below.



Step 1

First, let's label the two sub regions in D as shown below.



Step 2

Hopefully it is clear that we cannot get a single set of limits that will completely describe D so we'll need to split the integral as follows.

$$\iint\limits_D 3 - 6xy\,dA = \iint\limits_{D_1} 3 - 6xy\,dA + \iint\limits_{D_2} 3 - 6xy\,dA$$

Step 3

The region should also (hopefully) make it clear that we'll need to integrate y first with both of the integrals. So, here are the limits for each integral.

D_1	D_2
$-1 \le x \le 1$	$-2 \le x \le 2$
$x^2 \le y \le 1$	$-4 \le y \le -x^2$

The integrals are then,

$$\iint\limits_D 3 - 6xy\,dA = \int_{-1}^{1}\int_{x^2}^{1} 3 - 6xy\,dy\,dx + \int_{-2}^{2}\int_{-4}^{-x^2} 3 - 6xy\,dy\,dx$$

Step 4

Not much to do now other than do the integrals. Here is the y integration for both of them.

$$\begin{aligned}\iint\limits_D 3 - 6xy\,dA &= \int_{-1}^{1} \left(3y - 3xy^2\right)\Big|_{x^2}^{1}\,dx + \int_{-2}^{2} \left(3y - 3xy^2\right)\Big|_{-4}^{-x^2}\,dx \\ &= \int_{-1}^{1} 3x^5 - 3x^2 - 3x + 3\,dx + \int_{-2}^{2} 12 + 48x - 3x^2 - 3x^5\,dx\end{aligned}$$

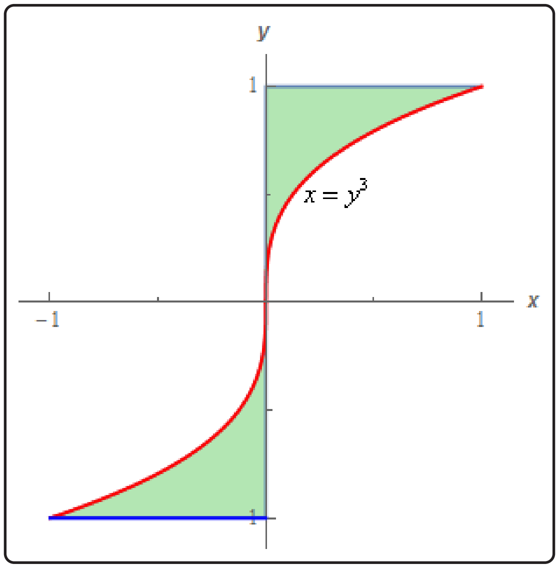
Step 5

Finally, here is the x integration for both of the integrals.

$$\begin{aligned}\iint\limits_D 3 - 6xy\,dA &= \int_{-1}^{1} 3x^5 - 3x^2 - 3x + 3\,dx + \int_{-2}^{2} 12 + 48x - 3x^2 - 3x^5\,dx \\ &= \left(\frac{1}{2}x^6 - x^3 - \frac{3}{2}x^2 + 3x\right)\Big|_{-1}^{1} + \left(12x + 24x^2 - x^3 - \frac{1}{2}x^6\right)\Big|_{-2}^{2} \\ &= \quad 4 \quad + \quad 32 \\ &= \boxed{36}\end{aligned}$$

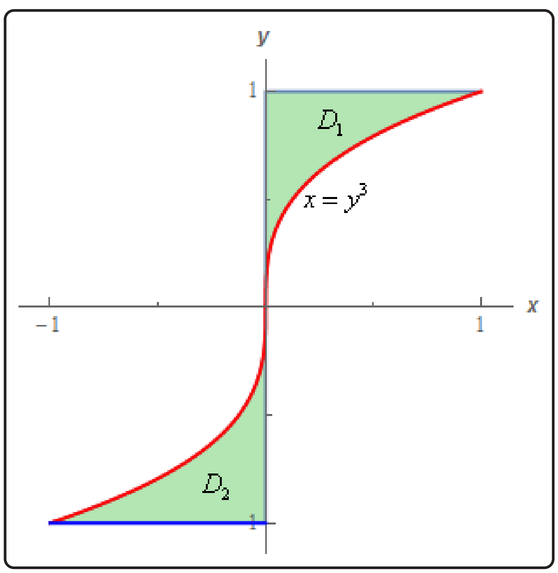
Don't always expect every integral over a region to be done with a single integral. On occasion you will need to split the integral up and do the actual integration over separate sub regions. In this case that was obvious but sometimes it might not be so clear until you get into the problem and realize it would be easier to do over sub regions.

8. Evaluate $\displaystyle\iint\limits_D \mathbf{e}^{y^4}\,dA$ where D is the region shown below.



Step 1

First, let's label the two sub regions in D as shown below.



Step 2

Despite the fact that each of the regions is bounded by the same curve we cannot get a single set of limits that will completely describe D. In the upper region $x = y^3$ is the right boundary and in the lower region $x = y^3$ is the left boundary.

Therefore, each region will need a separate set of limits and so we'll need to split the integral as follows.

$$\iint\limits_D \mathbf{e}^{y^4}\, dA = \iint\limits_{D_1} \mathbf{e}^{y^4}\, dA + \iint\limits_{D_2} \mathbf{e}^{y^4}\, dA$$

Step 3

Hopefully it is clear that we'll need to integrate x first with both of the integrals. So, here are the limits for each integral.

$$\begin{array}{cc} D_1 & D_2 \\ 0 \le y \le 1 & -1 \le y \le 0 \\ 0 \le x \le y^3 & y^3 \le x \le 0 \end{array}$$

The integrals are then,

$$\iint\limits_D \mathbf{e}^{y^4}\, dA = \int_0^1 \int_0^{y^3} \mathbf{e}^{y^4}\, dx\, dy + \int_{-1}^0 \int_{y^3}^0 \mathbf{e}^{y^4}\, dx\, dy$$

Step 4

Not much to do now other than do the integrals. Here is the x integration for both of them.

$$\begin{aligned} \iint\limits_D \mathbf{e}^{y^4}\, dA &= \int_0^1 \left(x\mathbf{e}^{y^4} \right)\Big|_0^{y^3} dy + \int_{-1}^0 \left(x\mathbf{e}^{y^4} \right)\Big|_{y^3}^0 dy \\ &= \int_0^1 y^3\mathbf{e}^{y^4}\, dy + \int_{-1}^0 -y^3\mathbf{e}^{y^4}\, dy \end{aligned}$$

Step 5

Finally, here is the y integration for both of the integrals.

$$\begin{aligned}\iint\limits_D \mathbf{e}^{y^4}\,dA &= \int_0^1 y^3\mathbf{e}^{y^4}\,dy + \int_{-1}^0 -y^3\mathbf{e}^{y^4}\,dy \\ &= \left(\frac{1}{4}\mathbf{e}^{y^4}\right)\Bigg|_0^1 + \left(-\frac{1}{4}\mathbf{e}^{y^4}\right)\Bigg|_{-1}^0 \\ &= \frac{1}{4}(\mathbf{e}-1) + \frac{1}{4}(-1+\mathbf{e}) = \boxed{\frac{1}{2}(\mathbf{e}-1) = 0.8591}\end{aligned}$$

Don't always expect every integral over a region to be done with a single integral. On occasion you will need to split the integral up and do the actual integration over separate sub regions. In this case that was fairly obvious but sometimes it might not be so clear until you get into the problem and realize it would be easier to do over sub regions.

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9. Evaluate $\iint\limits_D 7x^2 + 14y\,dA$ where D is the region bounded by $x = 2y^2$ and $x = 8$ in the order given below.

(a) Integrate with respect to x first and then y.

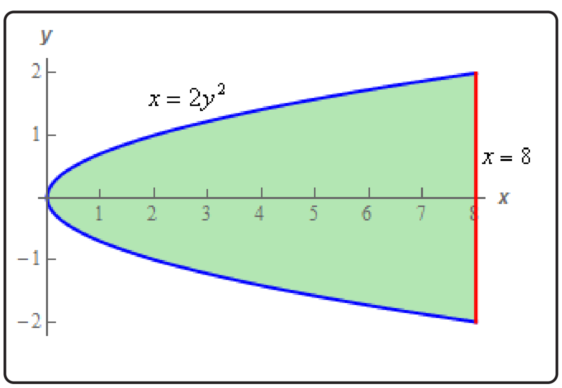
(b) Integrate with respect to y first and then x.

Solutions

(a) Integrate with respect to x first and then y.

Step 1

Here's a quick sketch of the region with the curves labeled for integration with respect to x first.



The limits for the integral for integration with respect to x first are then,

$$\begin{gathered} -2 \le y \le 2 \\ 2y^2 \le x \le 8 \end{gathered}$$

Plugging these limits into the integral is then,

$$\iint\limits_D 7x^2 + 14y\,dA = \int_{-2}^{2}\int_{2y^2}^{8} 7x^2 + 14y\,dx\,dy$$

Step 2

The x integration for this integral is,

$$\begin{aligned} \iint\limits_D 7x^2 + 14y\,dA &= \int_{-2}^{2}\int_{2y^2}^{8} 7x^2 + 14y\,dx\,dy \\ &= \int_{-2}^{2} \left(\frac{7}{3}x^3 + 14xy\right)\bigg|_{2y^2}^{8} dy \\ &= \int_{-2}^{2} \frac{3584}{3} + 112y - 28y^3 - \frac{56}{3}y^6\,dy \end{aligned}$$

Step 3

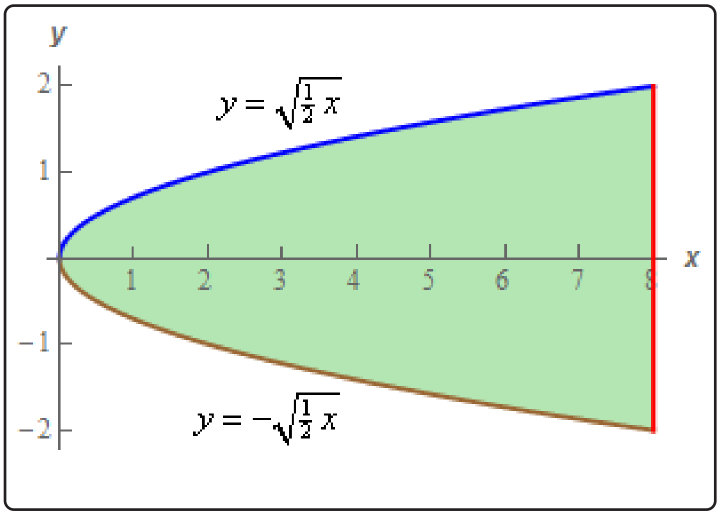
Finally, the y integration is,

$$\begin{aligned} \iint\limits_D 7x^2 + 14y\,dA &= \int_{-2}^{2} \frac{3584}{3} + 112y - 28y^3 - \frac{56}{3}y^6\,dy \\ &= \left(\frac{3584}{3}y + 56y^2 - 7y^4 - \frac{8}{3}y^7\right)\bigg|_{-2}^{2} = \boxed{4096} \end{aligned}$$

(b) Integrate with respect to y first and then x.

Step 1

Here's a quick sketch of the region with the curves labeled for integration with respect to y first.



Note that in order to do y integration first we needed to solve the equation of the parabola for y so the top and bottom curve will have distinct equations in terms of x, which we need to integrate with respect to y first.

The limits for the integral for integration with respect to y first are then,

$$0 \le x \le 8$$
$$-\sqrt{\tfrac{1}{2}x} \le y \le \sqrt{\tfrac{1}{2}x}$$

Plugging these limits into the integral is then,

$$\iint\limits_D 7x^2 + 14y\,dA = \int_0^8 \int_{-\sqrt{\frac{1}{2}x}}^{\sqrt{\frac{1}{2}x}} 7x^2 + 14y\,dy\,dx$$

Step 2

The y integration for this integral is,

$$\begin{aligned}\iint\limits_D 7x^2 + 14y\,dA &= \int_0^8 \int_{-\sqrt{\frac{1}{2}x}}^{\sqrt{\frac{1}{2}x}} 7x^2 + 14y\,dy\,dx \\ &= \int_0^8 \left(7x^2 y + 7y^2\right)\Big|_{-\sqrt{\frac{1}{2}x}}^{\sqrt{\frac{1}{2}x}} dx \\ &= \int_0^8 \frac{14}{\sqrt{2}} x^{\frac{5}{2}}\,dx\end{aligned}$$

Step 3

Finally, the x integration is,

$$\iint\limits_D 7x^2 + 14y\,dA = \int_0^8 \frac{14}{\sqrt{2}} x^{\frac{5}{2}}\,dx = \left(\frac{4}{\sqrt{2}} x^{\frac{7}{2}}\right)\Big|_0^8 = \boxed{4096}$$

We got the same result as the first order of integration as we knew we would.

---

10. Evaluate $\displaystyle\int_0^3 \int_{2x}^6 \sqrt{y^2 + 2}\,dy\,dx$ by first reversing the order of integration.

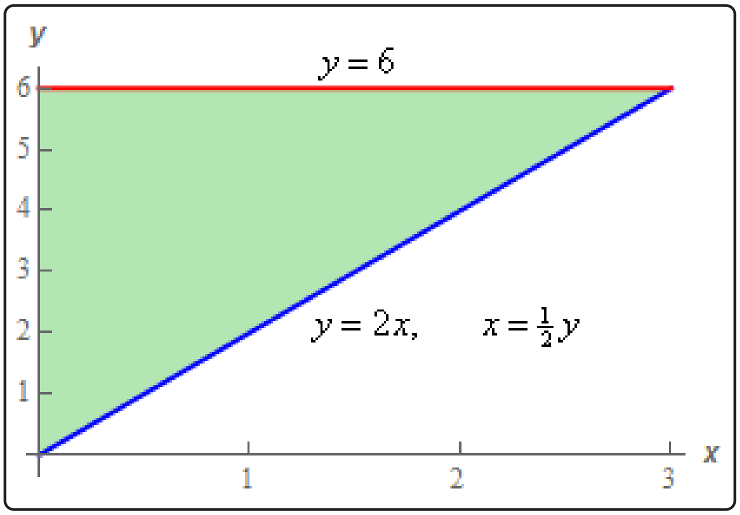
Step 1

Let's start off by noticing that if we were to integrate with respect to y first we'd need to do a trig substitution (which we'll all be thankful if we don't need to do it!) so interchanging the order in this case might well save us a messy integral.

So, here are the limits we get from the integral.

$$\begin{gathered} 0 \le x \le 3 \\ 2x \le y \le 6 \end{gathered}$$

Here is a quick sketch of the region these limits describe.



When reversing the order of integration it is often very helpful to have a sketch of the region to make sure we get the correct limits for the reversed order.

Step 2

Okay, if to reverse the order of integration we need to integrate with respect to x first. The limits for the reversed order are then,

$$0 \le y \le 6$$
$$0 \le x \le \frac{1}{2}y$$

The integral with reversed order is,

$$\int_0^3 \int_{2x}^6 \sqrt{y^2+2}\,dy\,dx = \int_0^6 \int_0^{\frac{1}{2}y} \sqrt{y^2+2}\,dx\,dy$$

Step 3

Now all we need to do is evaluate the integrals. Here is the x integration.

$$\int_0^3 \int_{2x}^6 \sqrt{y^2+2}\,dy\,dx = \int_0^6 \left(x\sqrt{y^2+2} \right) \Big|_0^{\frac{1}{2}y} dy = \int_0^6 \frac{1}{2}y\sqrt{y^2+2}\,dy$$

Step 4

Note that because of the x integration we'll not need to do a trig substitution. All we need is a simple Calculus I integral. Here is the y integration (we'll leave it to you to verify the substitution details).

$$\int_0^3 \int_{2x}^6 \sqrt{y^2+2}\,dy\,dx = \left(\frac{1}{6}\left(y^2+2\right)^{\frac{3}{2}}\right)\Big|_0^6 = \boxed{\frac{1}{6}\left(38^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) = 38.5699}$$

---

11. Evaluate $\displaystyle\int_0^1 \int_{-\sqrt{y}}^{y^2} 6x - y\,dx\,dy$ by first reversing the order of integration.

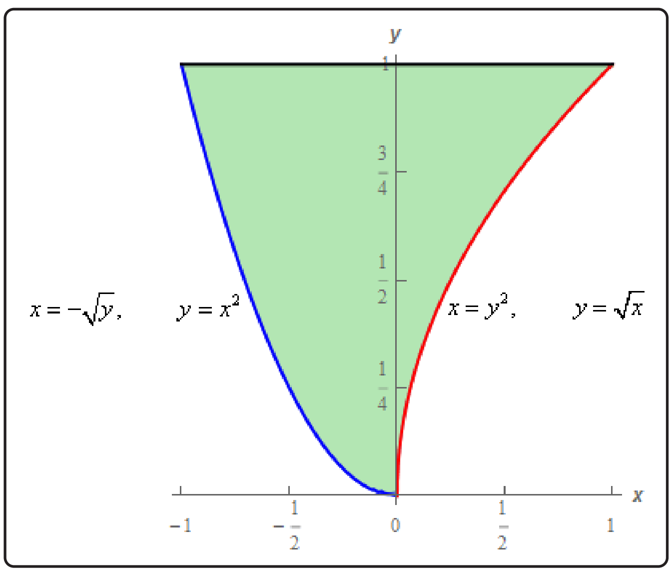
Step 1

Note that with this problem, unlike the previous problem, there is no issues with this order of integration. We could easily do this integral in the given order. We are only reversing the order in the problem because we were told to do so in the problem statement.

Here are the limits we get from the integral.

$$\begin{gathered} 0 \le y \le 1 \\ -\sqrt{y} \le x \le y^2 \end{gathered}$$

Here is a quick sketch of the region these limits describe.



When reversing the order of integration it is often very helpful to have a sketch of the region to make sure we get the correct limits for the reversed order. That is especially the case with this problem.

Step 2

Okay, if we reverse the order of integration we need to integrate with respect to y first. That leads to a small issue however. The lower function changes and so we'll need to split this up into two regions which in turn will mean two integrals when we reverse the order integration.

The limits for each of the regions with the reversed order are then,

$$\begin{array}{ll} -1 \le x \le 0 & \qquad 0 \le x \le 1 \\ x^2 \le y \le 1 & \qquad \sqrt{x} \le y \le 1 \end{array}$$

The integrals with reversed order are then,

$$\int_0^1 \int_{-\sqrt{y}}^{y^2} 6x - y \, dx \, dy = \int_{-1}^0 \int_{x^2}^1 6x - y \, dy \, dx + \int_0^1 \int_{\sqrt{x}}^1 6x - y \, dy \, dx$$

Step 3

Now all we need to do is evaluate the integrals. Here is the y integration for each.

$$\begin{aligned} \int_0^1 \int_{-\sqrt{y}}^{y^2} 6x - y \, dx \, dy &= \int_{-1}^0 \left(6xy - \frac{1}{2}y^2\right)\Big|_{x^2}^1 dx + \int_0^1 \left(6xy - \frac{1}{2}y^2\right)\Big|_{\sqrt{x}}^1 dx \\ &= \int_{-1}^0 \frac{1}{2}x^4 - 6x^3 + 6x - \frac{1}{2} \, dx + \int_0^1 -6x^{\frac{3}{2}} + \frac{13}{2}x - \frac{1}{2} \, dx \end{aligned}$$

Step 4

Finally, the x integration.

$$\begin{aligned}\int_0^1\int_{-\sqrt{y}}^{y^2} 6x - y\,dx\,dy &= \left(\frac{1}{10}x^5 - \frac{3}{2}x^4 + 3x^2 - \frac{1}{2}x\right)\Bigg|_{-1}^{0} + \left(-\frac{12}{5}x^{\frac{5}{2}} + \frac{13}{4}x^2 - \frac{1}{2}x\right)\Bigg|_{0}^{1} \\ &= -\frac{19}{10} + \frac{7}{20} \\ &= \boxed{-\frac{31}{20}}\end{aligned}$$

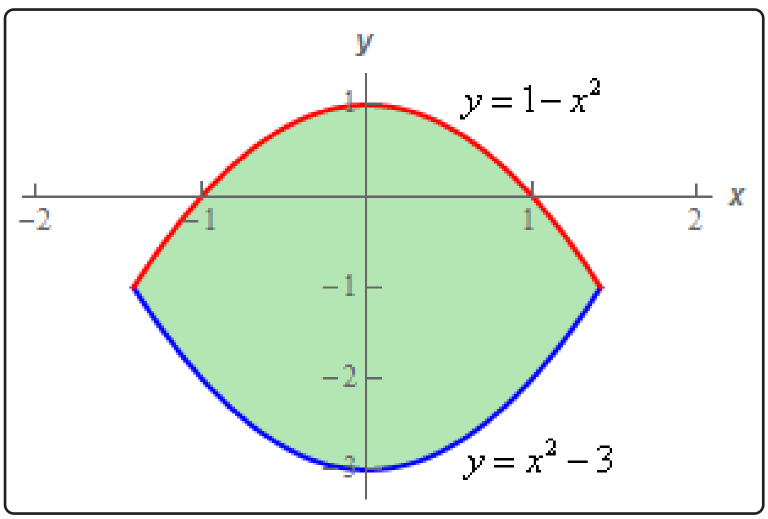
12. Use a double integral to determine the area of the region bounded by $y = 1 - x^2$ and $y = x^2 - 3$.

Step 1

Okay, we know that the area of any region D can be found by evaluating the following double integral.

$$A = \iint\limits_D dA$$

For this problem D is the region sketched below.



Step 2

We've done enough double integrals by this point that it should be pretty obvious the best order of integration is to integrate with respect to y first.

Here are the limits for the integral with this order.

$$-\sqrt{2} \le x \le \sqrt{2}$$
$$x^2 - 3 \le y \le 1 - x^2$$

The x limits can easily be found by setting the two equations equal and solving for x.

Step 3

The integral for the area is then,

$$A = \iint\limits_D dA = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2-3}^{1-x^2} dy\,dx$$

Step 4

Now all we need to do is evaluate the integral. Here is the y integration.

$$A = \iint\limits_D dA = \int_{-\sqrt{2}}^{\sqrt{2}} y\big|_{x^3-3}^{1-x^2}\,dx = \int_{-\sqrt{2}}^{\sqrt{2}} 4 - 2x^2\,dx$$

Step 5

Finally, the x integration and hence the area of D is,

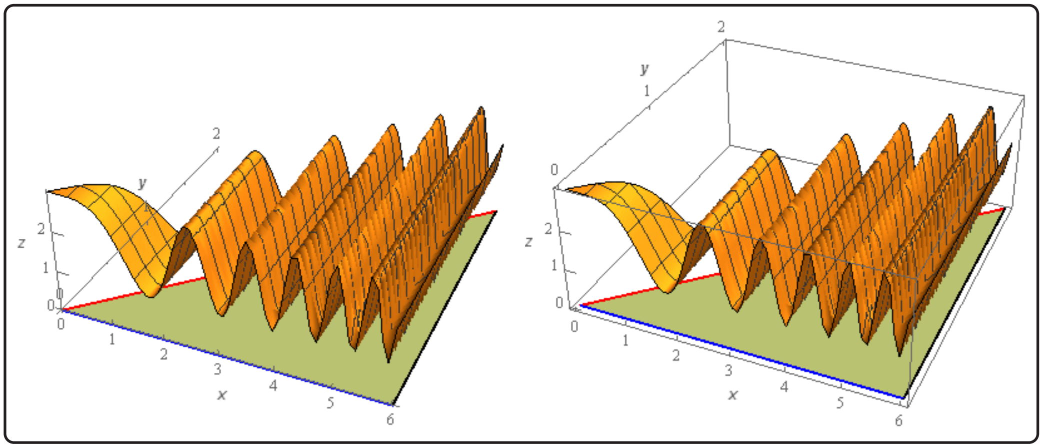
$$A = \left(4x - \frac{2}{3}x^3\right)\Big|_{-\sqrt{2}}^{\sqrt{2}} = \boxed{\frac{16\sqrt{2}}{3}}$$

---

13. Use a double integral to determine the volume of the region that is between the xy-plane and $f(x,y) = 2 + \cos(x^2)$ and is above the triangle with vertices $(0,0)$, $(6,0)$ and $(6,2)$.

Step 1

Let's first get a sketch of the function and the triangle that lies under it.



The surface is sketched with a traditional set of axes and well as a "box frame" set of axes. Sometimes it is easier to see what is going on with the surface when both sketches are present.

The greenish triangle underneath the surface is the triangle referenced in the problem statement.

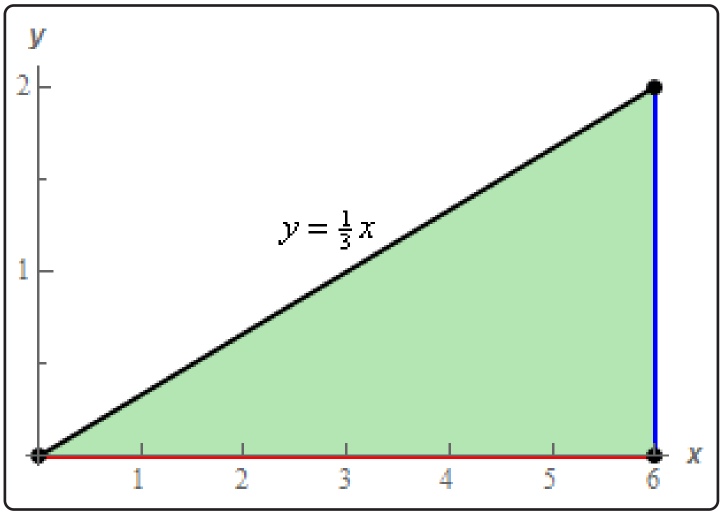
Step 2

Now, the volume we are after is given by the following integral,

$$V = \iint\limits_D 2 + \cos(x^2)\, dA$$

where D is the triangle referenced in the problem statement.

So, in order to evaluate the integral we'll need a sketch of D so we can determine an order of integration as well as limits for the integrals.



Step 3

The region D can easily be described for either order of integration. However, it should be pretty clear that the integral can't be integrated with respect to x first and so we'll need to integrate with respect to y first.

Here are the limits for the integral with this order.

$$0 \le x \le 6$$
$$0 \le y \le \frac{1}{3}x$$

The integral for the volume is then,

$$V = \iint\limits_D 2 + \cos\left(x^2\right)\, dA = \int_0^6 \int_0^{\frac{1}{3}x} 2 + \cos\left(x^2\right)\, dy\, dx$$

Step 4

Now all we need to do is evaluate the integral. Here is the y integration.

$$V = \int_0^6 \left(2y + y\cos\left(x^2\right)\right)\Big|_0^{\frac{1}{3}x} dx = \int_0^6 \frac{2}{3}x + \frac{1}{3}x\cos\left(x^2\right)\, dx$$

Step 5

Finally, the x integration and hence the volume is,

$$V = \left(\frac{1}{3}x^2 + \frac{1}{6}\sin\left(x^2\right)\right)\Big|_0^6 = \boxed{12 + \frac{1}{6}\sin(36) = 11.8347}$$

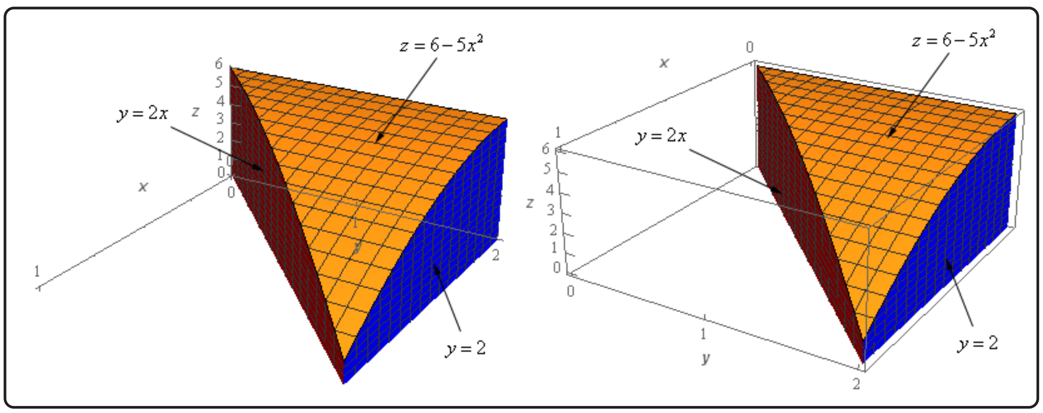
Don't forget to have your calculator set to radians if you are converting to decimals!

---

14. Use a double integral to determine the volume of the region bounded by $z = 6 - 5x^2$ and the planes $y = 2x$, $y = 2$, $x = 0$ and the xy-plane.

Step 1

Let's first get a sketch of the solid that we're working with. If you are not good at visualizing these types of solids in your head these graphs can be invaluable in helping to get the integral set up.



The surface is sketched with a traditional set of axes and well as a "box frame" set of axes. Sometimes it is easier to see what is going on with the surface when both sketches are present.

The upper surface (the orange surface) is the graph of $z = 6 - 5x^2$. The blue plane is the graph of $y = 2$ which is nothing more than the plane parallel to the xz-plane at $y = 2$. The red plane is the graph of $y = 2x$ and this is simply the plane that is perpendicular to the xy-plane and goes through the line $y = 2x$ in the xy-plane. The surface given by $x = 0$ is simply the yz-plane (*i.e* the back of the solid) and is not shown and the xy-plane is the bottom of the surface and again is not shown in the sketch.

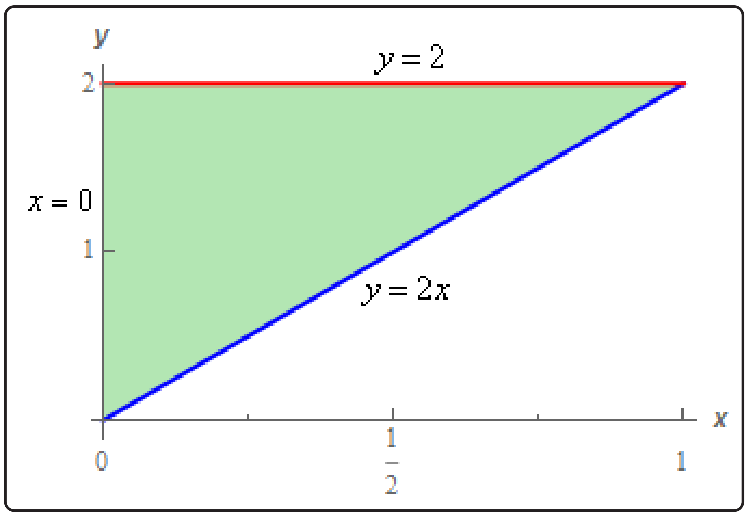
Step 2

In this section the only method that we have for determining the volume of a solid is to find the volume under a surface. In this case it is hopefully clear that we are looking for the surface that is under $z = 6 - 5x^2$ and is above the region D in the xy-plane defined by where the other three planes intersect it. In other words, the region D is the regoin in the xy-plane that is bounded by $y = 2$, $y = 2x$ and $x = 0$.

The integral for the volume is then,

$$V = \iint\limits_D 6 - 5x^2 \, dA$$

where D is sketched below.



Step 3

This integral can be integrated in any order so let's integrate with respect to y first to avoid fractions in the limits (which we'd get with one if we integrated with respect to x first). Here are the limits for our integral.

$$\begin{gathered} 0 \le x \le 1 \\ 2x \le y \le 2 \end{gathered}$$

The integral for the volume is then,

$$V = \iint\limits_D 6 - 5x^2 \, dA = \int_0^1 \int_{2x}^2 6 - 5x^2 \, dy \, dx$$

Step 4

Now all we need to do is evaluate the integral. Here is the y integration.

$$V = \int_0^1 \left(6 - 5x^2\right) y\Big|_{2x}^{2} dx = \int_0^1 \left(6 - 5x^2\right)\left(2 - 2x\right)\, dx = \int_0^1 10x^3 - 10x^2 - 12x + 12\, dx$$

Note that in doing the y integration we acknowledged that the whole integrand contained no y's and so could be considered a constant and so would just be multiplied by y. We could also have done each term individually but sometimes it is just as easy or even easier to do what we've done here. Of course, we then had to multiply out the integrand for the next step but it wasn't too bad.

Step 5

Finally, the x integration and hence the volume is,

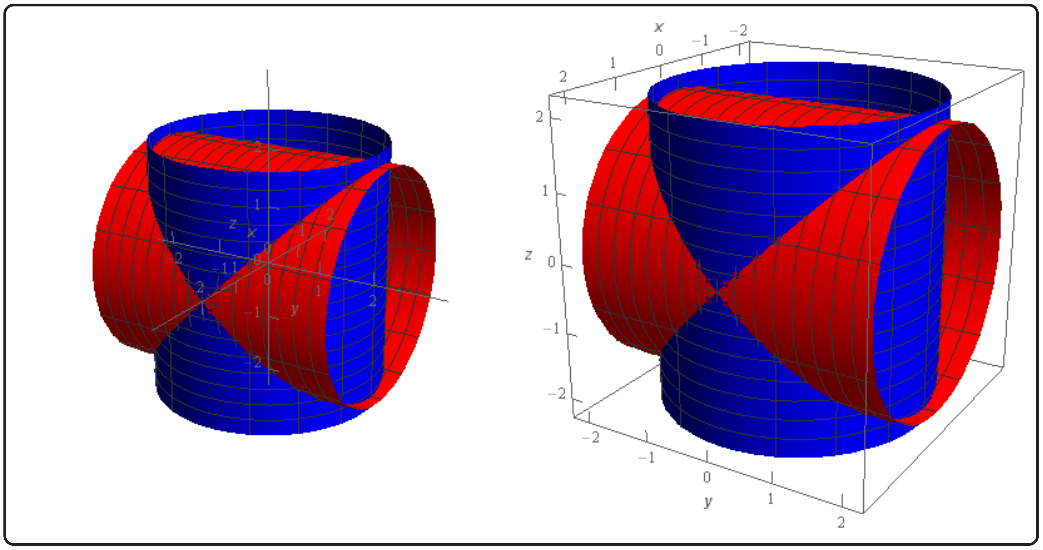
$$V = \left(\frac{5}{2}x^4 - \frac{10}{3}x^3 - 6x^2 + 12x\right)\Big|_0^1 = \boxed{\frac{31}{6}}$$

15. Use a double integral to determine the volume of the region formed by the intersection of the two cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$.

Step 1

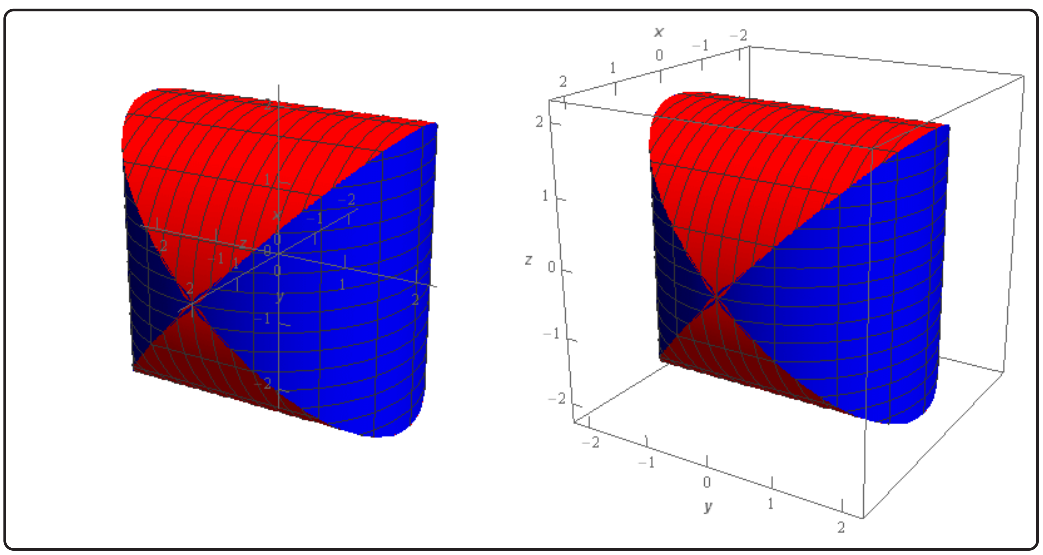
Okay, probably one of the hardest parts of this problem is the visualization of just what this surface looks like so let's start with that.

First, $x^2 + y^2 = 4$ is a cylinder of radius 2 centered on the z-axis and $x^2 + z^2 = 4$ is a cylinder of radius 2 centered on the y-axis and so we will have an intersection of the two around the origin. Here is a sketch of the two cylinders.



As usual we gave two sketches with different axes systems to help visualize the two cylinders.

Now, the solid we want the volume of is the portion of the red cylinder that lies inside the blue cylinder, or equivalently the portion of the blue cylinder that lies inside the red cylinder. Here are a couple of sketches of only the solid that we are after.



The red portion of this solid is the portion of $x^2 + z^2 = 4$ that lies inside $x^2 + y^2 = 4$ and the blue portion of this solid is the portion of $x^2 + y^2 = 4$ that lies inside $x^2 + z^2 = 4$.

Step 2

Now we need to determine an integral that will give the volume of this solid. In this section we really only talked about finding the solid that was under a surface and above a region D in the xy-plane.

This solid also exists below the xy-plane. For this solid, however, this an easy situation to deal with. Recall that the solid is a portion of the cylinder $x^2 + y^2 = 4$ that is inside the cylinder $x^2 + z^2 = 4$, which is centered on the z-axis. This means that the portion of the solid that is below the xy-plane is a simply a mirror image of the portion of the solid that is above the xy-plane. In other words, the volume of the portion of the solid that lies below the xy-plane is the same as the volume of the portion of the solid that lies above the xy-plane.

We can use the ideas from this section to easily find the volume of the solid above the xy-plane and then the volume of the full solid will simply be double this.

Step 3

So, now we need to actually set up the integral for the volume. First, we can solve $x^2 + z^2 = 4$ for z to get,

$$z = \pm\sqrt{4 - x^2}$$

The positive portion of this is the equation for the top half of the cylinder centered on the y-axis and the negative portion is the equation for the bottom half of this cylinder.

This means that the integral for the volume of the top half of the solid is,

$$V = \iint\limits_D \sqrt{4 - x^2}\, dA$$

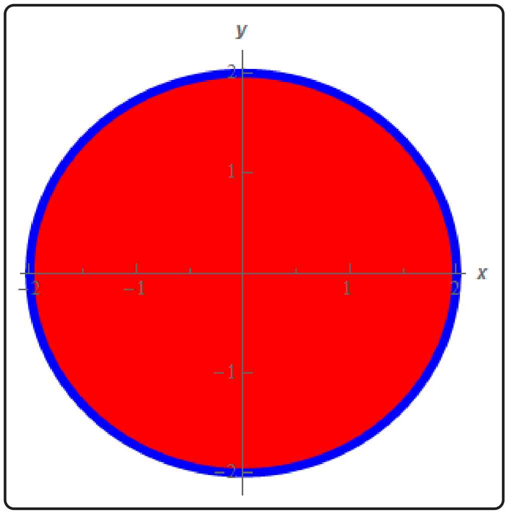
The volume of the whole solid is then,

$$V = 2\iint\limits_D \sqrt{4 - x^2}\, dA$$

Step 4

Now, we need to figure out just what the region D is. We know that, in general, D is the region of the xy-plane that we use to get the graph of the surface we are working with, $z = \sqrt{4 - x^2}$, in this case.

But just what is that for this problem? Look at the solid from directly above it. What you see is the following,



The blue circle is the cylinder $x^2 + y^2 = 4$ that is centered on the z-axis. Since we are looking at the solid from directly above all we see is the walls of the cylinder which of course is just a circle with equation $x^2 + y^2 = 4$. The red portion of this is the walls of the cylinder $x^2 + z^2 = 4$ that lies inside $x^2 + y^2 = 4$.

This is exactly what we need to determine D. Recall once again that D is the region in the xy-plane we use to graph the upper portion of the surface defining the solid. But from our graph above we see that the upper portion of the surface appears to be a circle of radius 2 centered at the origin. This means that is exactly what D is. D is just shape that we see when we look at the region from above and so D is the circle shown in the graph above.

Step 5

Next let's get the limits for our integral. The region D itself doesn't really seem to affect the order of integration and each set of limits are pretty much the same. The only real difference is the x's and y's will be switched around.

Here are the limits for each order of integration.

$$\begin{gathered}-2 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}\end{gathered} \qquad \text{OR} \qquad \begin{gathered}-2 \le y \le 2 \\ -\sqrt{4-y^2} \le x \le \sqrt{4-y^2}\end{gathered}$$

So, as noted above there doesn't appear to be much difference in the limits. However, go back and check the integral we need to compute. We can see that if we integrate x first we'll need to do a trig substitution. So, let's integrate y first.

The integral for the volume of the full solid is then,

$$V = 2\iint\limits_D \sqrt{4-x^2}\,dA = 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2}\,dy\,dx$$

Step 6

Now all we need to do is evaluate the integral. Here is the y integration.

$$\begin{aligned}V = 2\iint\limits_D \sqrt{4-x^2}\,dA &= 2\int_{-2}^{2} \left(y\sqrt{4-x^2}\right)\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\,dx \\ &= 2\int_{-2}^{2} \left(\sqrt{4-x^2} - \left(-\sqrt{4-x^2}\right)\right)\sqrt{4-x^2}\,dx \\ &= 2\int_{-2}^{2} 2\sqrt{4-x^2}\sqrt{4-x^2}\,dx \\ &= 2\int_{-2}^{2} 2\left(4-x^2\right)\,dx = 4\int_{-2}^{2} 4-x^2\,dx\end{aligned}$$

We put a few more steps into the work that absolutely necessary to make the simplifications clear.

Step 7

Finally, the x integration and hence the volume of the full solid is,

$$V = 4\int_{-2}^{2} 4 - x^2 \, dx = 4\left(4x - \frac{1}{3}x^3\right)\Bigg|_{-2}^{2} = \boxed{\frac{128}{3}}$$

So, as we can see the integration portion of the problem was surprisingly simple. The biggest issue here really is just getting this problem set up.

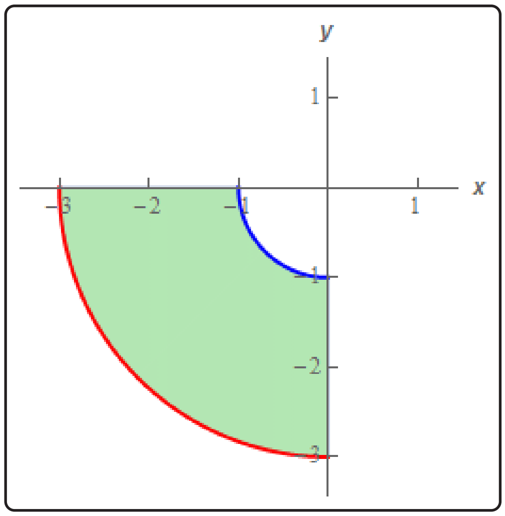
This problem illustrates one of the biggest issues that many students have with some of these problems. You really need to be able to visualize the solids/regions that are being dealt with. Or at the very least have the ability to get the graph of the solid/region. Unfortunately, that is not always something that can be quickly taught. Many folks just seem to naturally be able to visualize these kinds of things but many also are just not able to easily visualize them. If you are in the second set of folks it will make some of these problems a little harder unfortunately but if you persevere you will get through this stuff and hopefully start to be able to do some of the visualization that will be needed on occasion.

15.4 Double Integrals in Polar Coordinates

1. Evaluate $\displaystyle\iint\limits_D y^2 + 3x\,dA$ where D is the region in the 3^{rd} quadrant between $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

Step 1

Below is a quick sketch of the region D.



For double integrals in which polar coordinates are going to be used the sketch of D is often not as useful as for a general region.

However, if nothing else, it does make it clear that polar coordinates will be needed for this problem. Describing this region in terms of Cartesian coordinates is possible but it would take two integrals to do the problem and the most of the limits will involve roots which often (not always, but often) leads to messy integral work.

The sketch shows that the region is at least partially circular and that should always indicate that polar coordinates are not a bad thing to at least think about. In this case, because of the Cartesian limits as discussed above polar coordinates are pretty much the only easy way to do this integral.

Note as well that once we have the sketch determining the polar limits should be pretty simple.

Step 2

Okay, D is just a portion of a ring and so setting up the limits shouldn't be too difficult. Here they are,

$$\pi \le \theta \le \tfrac{3}{2}\pi$$
$$1 \le r \le 3$$

Step 3

The integral in terms of polar coordinates is then,

$$\begin{aligned}\iint\limits_D y^2 + 3x\,dA &= \int_{\pi}^{\frac{3}{2}\pi}\int_1^3 \left((r\sin(\theta))^2 + 3r\cos(\theta)\right) r\,dr\,d\theta\\ &= \int_{\pi}^{\frac{3}{2}\pi}\int_1^3 r^3\sin^2(\theta) + 3r^2\cos(\theta)\,dr\,d\theta\end{aligned}$$

When converting the integral don't forget to convert the x and y into polar coordinates. Also, don't forget that $dA = r\,dr\,d\theta$ and so we'll pickup an extra r in the integrand. Forgetting the extra r is one of the most common mistakes with these kinds of problems.

Step 4

Here is the r integration.

$$\begin{aligned}\iint\limits_D y^2 + 3x\,dA &= \int_{\pi}^{\frac{3}{2}\pi}\int_1^3 r^3\sin^2(\theta) + 3r^2\cos(\theta)\,dr\,d\theta\\ &= \int_{\pi}^{\frac{3}{2}\pi}\left.\left(\frac{1}{4}r^4\sin^2(\theta) + r^3\cos(\theta)\right)\right|_1^3 d\theta\\ &= \int_{\pi}^{\frac{3}{2}\pi} 20\sin^2(\theta) + 26\cos(\theta)\,d\theta\end{aligned}$$

Step 5

Finally, here is the θ integration.

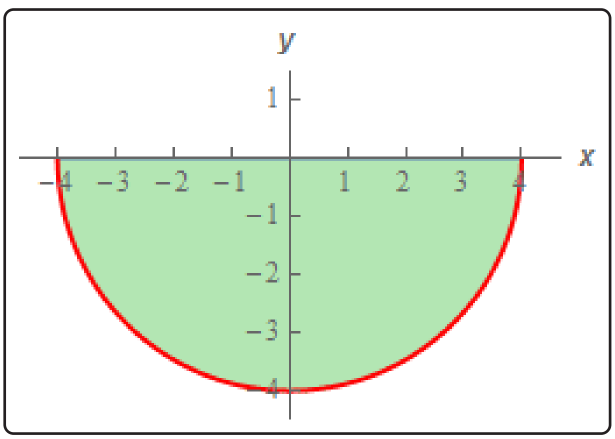
$$\begin{aligned}\iint\limits_D y^2 + 3x\,dA &= \int_{\pi}^{\frac{3}{2}\pi} 20\sin^2(\theta) + 26\cos(\theta)\,d\theta \\ &= \int_{\pi}^{\frac{3}{2}\pi} 10\left(1-\cos(2\theta)\right) + 26\cos(\theta)\,d\theta \\ &= \left(10\theta - 5\sin(2\theta) + 26\sin(\theta)\right)\Big|_{\pi}^{\frac{3}{2}\pi} = \boxed{5\pi - 26}\end{aligned}$$

You'll be seeing a fair amount of $\cos^2(\theta)$, $\sin^2(\theta)$ and $\sin(\theta)\cos(\theta)$ terms in polar integrals so make sure that you know how to integrate these terms! In this case we used a half angle formula to reduce the $\sin^2(\theta)$ into something we could integrate.

2. Evaluate $\displaystyle\iint\limits_D \sqrt{1+4x^2+4y^2}\,dA$ where D is the bottom half of $x^2+y^2=16$.

Step 1

Below is a quick sketch of the region D.



For double integrals in which polar coordinates are going to be used the sketch of D is often not as useful as for a general region.

However, if nothing else, it does make it clear that polar coordinates will be needed for this problem. Describing this region in terms of Cartesian coordinates is possible but one of the limits will involve roots which often (not always, but often) leads to messy integral work.

The sketch shows that the region is at least partially circular and that should always indicate that polar coordinates are not a bad thing to at least think about. In this case, because of the Cartesian limits as discussed above polar coordinates are pretty much the only easy way to do this integral. Also note that this integral would be unpleasant in terms of Cartesian coordinates. Hopefully the polar form will be easier to do.

Note as well that once we have the sketch determining the polar limits should be pretty simple.

Step 2

Okay, D is just a portion of a disk and so setting up the limits shouldn't be too difficult. Here they are,

$$\begin{gathered} \pi \le \theta \le 2\pi \\ 0 \le r \le 4 \end{gathered}$$

Step 3

The integral in terms of polar coordinates is then,

$$\begin{aligned} \iint\limits_D \sqrt{1+4x^2+4y^2}\,dA &= \int_{\pi}^{2\pi}\int_0^4 \left(\sqrt{1+4\left(x^2+y^2\right)}\right) r\,dr\,d\theta \\ &= \int_{\pi}^{2\pi}\int_0^4 r\sqrt{1+4r^2}\,dr\,d\theta \end{aligned}$$

When converting the integral don't forget to convert the x and y into polar coordinates. In this case don't just substitute the polar conversion formulas in for x and y! Recall that $x^2+y^2=r^2$ and the integral will be significantly easier to deal with.

Also, don't forget that $dA = r\,dr\,d\theta$ and so we'll pickup an extra r in the integrand. Forgetting the extra r is one of the most common mistakes with these kinds of problems and in this case without the extra r we'd have a much more unpleasant integral to deal with.

Step 4

Here is the r integration.

$$\begin{aligned}\iint\limits_D \sqrt{1+4x^2+4y^2}\,dA &= \int_{\pi}^{2\pi}\int_0^4 r\sqrt{1+4r^2}\,dr\,d\theta\\ &= \int_{\pi}^{2\pi}\left(\frac{1}{12}\left(1+4r^2\right)^{\frac{3}{2}}\right)\Big|_0^4\,d\theta\\ &= \int_{\pi}^{2\pi}\frac{1}{12}\left(65^{\frac{3}{2}}-1\right)\,d\theta\end{aligned}$$

Step 5

Finally, here is the θ integration.

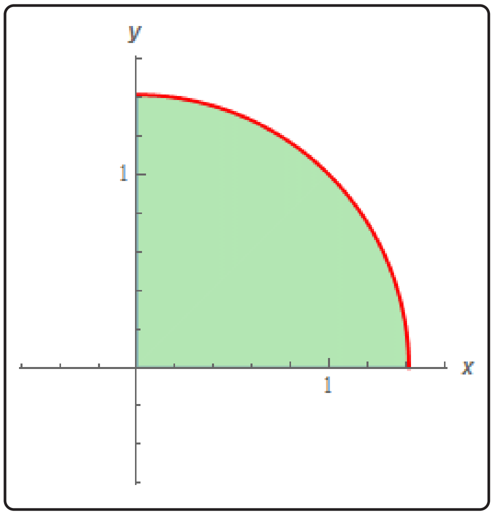
$$\begin{aligned}\iint\limits_D \sqrt{1+4x^2+4y^2}\,dA &= \int_{\pi}^{2\pi}\frac{1}{12}\left(65^{\frac{3}{2}}-1\right)\,d\theta\\ &= \left(\frac{1}{12}\left(65^{\frac{3}{2}}-1\right)\theta\right)\Big|_{\pi}^{2\pi} = \boxed{\frac{1}{12}\pi\left(65^{\frac{3}{2}}-1\right) = 136.9333}\end{aligned}$$

Note that while this was a really simple integral to evaluate you'll be seeing a fair amount of $\cos^2(\theta)$, $\sin^2(\theta)$ and $\sin(\theta)\cos(\theta)$ terms in polar integrals so make sure that you know how to integrate these terms!

3. Evaluate $\displaystyle\iint\limits_D 4xy-7\,dA$ where D is the portion of $x^2+y^2=2$ in the 1^{st} quadrant.

Step 1

Below is a quick sketch of the region D.



For double integrals in which polar coordinates are going to be used the sketch of D is often not as useful as for a general region.

However, if nothing else, it does make it clear that polar coordinates will be needed for this problem. Describing this region in terms of Cartesian coordinates is possible but one of the limits will involve roots which often (not always, but often) leads to messy integral work.

The sketch shows that the region is at least partially circular and that should always indicate that polar coordinates are not a bad thing to at least think about. In this case, because of the Cartesian limits as discussed above polar coordinates are pretty much the only easy way to do this integral.

Note as well that once we have the sketch determining the polar limits should be pretty simple.

Step 2

Okay, D is just a portion of a disk and so setting up the limits shouldn't be too difficult. Here they are,

$$0 \le \theta \le \tfrac{1}{2}\pi$$
$$0 \le r \le \sqrt{2}$$

Step 3

The integral in terms of polar coordinates is then,

$$\iint\limits_D 4xy - 7\,dA = \int_0^{\frac{1}{2}\pi}\int_0^{\sqrt{2}} \left(4\left(r\cos(\theta)\right)\left(r\sin(\theta)\right) - 7\right)\,r\,dr\,d\theta$$
$$= \int_0^{\frac{1}{2}\pi}\int_0^{\sqrt{2}} 4r^3\cos(\theta)\sin(\theta) - 7r\,dr\,d\theta$$

When converting the integral don't forget to convert the x and y into polar coordinates.

Also, don't forget that $dA = r\,dr\,d\theta$ and so we'll pickup an extra r in the integrand. Forgetting the extra r is one of the most common mistakes with these kinds of problems.

Step 4

Here is the r integration.

$$\iint\limits_D 4xy - 7\,dA = \int_0^{\frac{1}{2}\pi}\int_0^{\sqrt{2}} 4r^3\cos(\theta)\sin(\theta) - 7r\,dr\,d\theta$$
$$= \int_0^{\frac{1}{2}\pi} \left.\left(r^4\cos(\theta)\sin(\theta) - \frac{7}{2}r^2\right)\right|_0^{\sqrt{2}}\,d\theta$$
$$= \int_0^{\frac{1}{2}\pi} 4\cos(\theta)\sin(\theta) - 7\,d\theta$$

Step 5

Finally, here is the θ integration.

$$\iint\limits_D 4xy - 7\,dA = \int_0^{\frac{1}{2}\pi} 4\cos(\theta)\sin(\theta) - 7\,d\theta$$
$$= \int_0^{\frac{1}{2}\pi} 2\sin(2\theta) - 7\,d\theta$$
$$= \left.\left(-\cos(2\theta) - 7\theta\right)\right|_0^{\frac{1}{2}\pi} = \boxed{2 - \frac{7}{2}\pi}$$

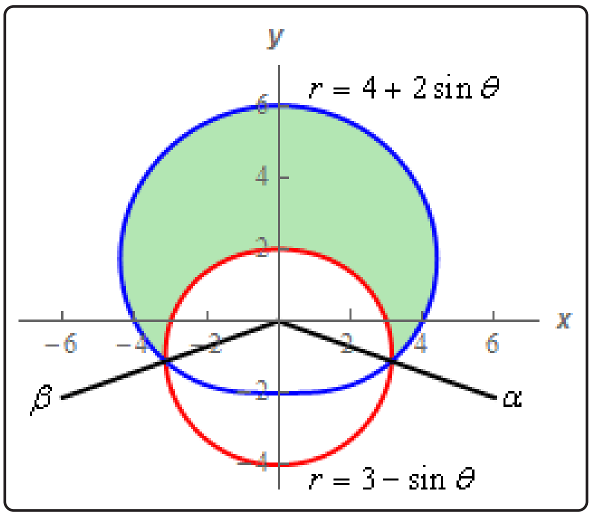
Note that while this was a really simple integral to evaluate you'll be seeing a fair amount of $\cos^2(\theta)$, $\sin^2(\theta)$ and $\sin(\theta)\cos(\theta)$ terms in polar integrals so make sure that you know how to integrate these terms! In this case we used the double angle formula for sine to quickly reduce the $\cos(\theta)\sin(\theta)$ into an easy to integrate term. Of course, we could also have just done a substitution to deal with it but this, in our opinion, is the easiest way to deal with the term.

---

4. Use a double integral to determine the area of the region that is inside $r = 4 + 2\sin(\theta)$ and outside $r = 3 - \sin(\theta)$.

Step 1

Below is a quick sketch of the region D.



For double integrals in which polar coordinates are going to be used the sketch of D is often not as useful as for a general region.

In this case however, the sketch is probably more useful than for most of these problems. First, we can readily identify the correct order for the r limits (*i.e.* which is the "smaller" and which is the "larger" curve"). Secondly, it makes it clear that we are going to need to determine the limits of $\alpha \le \theta \le \beta$ that we'll need to do the integral.

Step 2

So, as noted above the limits of r are (hopefully) pretty clear. To find the limits for θ all we need to do is set the two equations equal. Doing that gives,

$$\begin{aligned} 4 + 2\sin(\theta) &= 3 - \sin(\theta) \\ 3\sin(\theta) &= -1 \\ \sin(\theta) &= -\frac{1}{3} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(-\frac{1}{3}\right) = -0.3398 \end{aligned}$$

This is the angle in the fourth quadrant, *i.e.* $\alpha = -0.3398$. To find the second angle we can note that the angles that the two black lines make with the x axis are the same (ignoring the sign of course). Therefore, the second angle will simply be $\beta = \pi + 0.3398 = 3.4814$.

The limits are then,

$$\begin{gathered} -0.3398 \le \theta \le 3.4814 \\ 3 - \sin(\theta) \le r \le 4 + 2\sin(\theta) \end{gathered}$$

Step 3

We know that the following double integral will give the area of any region D.

$$A = \iint\limits_D dA$$

For our problem the region D is the region given in the problem statement and so the integral in terms of polar coordinates is then,

$$A = \iint\limits_D dA = \int_{-0.3398}^{3.4814} \int_{3-\sin(\theta)}^{4+2\sin(\theta)} r\,dr\,d\theta$$

When converting the integral don't forget that $dA = r\,dr\,d\theta$ and so we'll pickup an extra r in the integrand. Forgetting the extra r is one of the most common mistakes with these kinds of problems and that seems to be even truer for area integrals such as this one.

Step 4

Here is the r integration.

$$\begin{aligned}
A &= \int_{-0.3398}^{3.4814} \int_{3-\sin(\theta)}^{4+2\sin(\theta)} r\, dr\, d\theta \\
&= \int_{-0.3398}^{3.4814} \left. \frac{1}{2} r^2 \right|_{3-\sin(\theta)}^{4+2\sin(\theta)} d\theta \\
&= \int_{-0.3398}^{3.4814} \frac{1}{2} \left[(4+2\sin(\theta))^2 - (3-\sin(\theta))^2 \right] d\theta
\end{aligned}$$

Note that if you remember doing these kinds of area problems back in the Calculus II material that integral should look pretty familiar as it follows the formula we used back in that material.

Step 5

Finally, here is the θ integration.

$$\begin{aligned}
A &= \int_{-0.3398}^{3.4814} \frac{1}{2} \left[(4+2\sin(\theta))^2 - (3-\sin(\theta))^2 \right] d\theta \\
&= \frac{1}{2} \int_{-0.3398}^{3.4814} 16 + 16\sin(\theta) + 4\sin^2(\theta) - \left(9 - 6\sin(\theta) + \sin^2(\theta)\right) d\theta \\
&= \frac{1}{2} \int_{-0.3398}^{3.4814} 7 + 22\sin(\theta) + 3\sin^2(\theta)\, d\theta \\
&= \frac{1}{2} \int_{-0.3398}^{3.4814} 7 + 22\sin(\theta) + \frac{3}{2}\left(1 - \cos(2\theta)\right) d\theta \\
&= \frac{1}{2} \int_{-0.3398}^{3.4814} \frac{17}{2} + 22\sin(\theta) - \frac{3}{2}\cos(2\theta)\, d\theta \\
&= \frac{1}{2} \left. \left(\frac{17}{2}\theta - 22\cos(\theta) - \frac{3}{4}\sin(2\theta) \right) \right|_{-0.3398}^{3.4814} = \boxed{36.5108}
\end{aligned}$$

There was a fair amount of simplification that needed to be done for this problem. That will often be the case. None of it was particularly difficult just tedious and easy to make a mistake if you aren't paying attention. The "hardest" part of the simplification here was using a half angle formula to reduce the $\sin^2(\theta)$ into something we could integrate.

5. Evaluate the following integral by first converting to an integral in polar coordinates.

$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 \mathbf{e}^{x^2+y^2}\, dy\, dx$$

Step 1

The first thing we should do here is to get a feel for the region that we are integrating over. So, here are the x and y limits of this integral.

$$0 \le x \le 3$$
$$-\sqrt{9-x^2} \le y \le 0$$

Now, the lower y limit tells us that the "lower" curve of the region is given by,

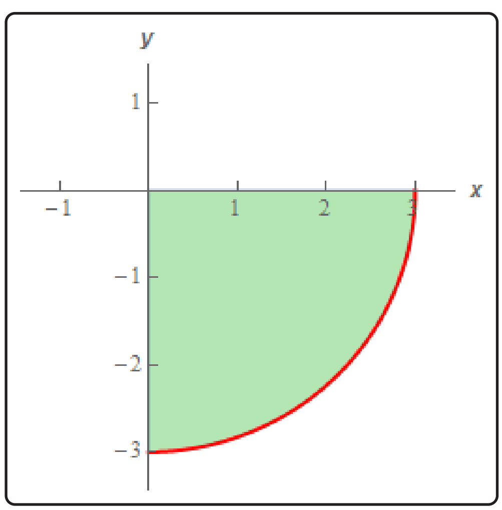
$$y = -\sqrt{9-x^2} \qquad \Rightarrow \qquad x^2 + y^2 = 9$$

As noted above, if we square both sides and do a little rewrite we can see that this is a portion of a circle of radius 3 centered at the origin. In fact, it is the lower portion of the circle (because of the "$-$" in front of the root).

The upper y limit tells us that the region won't go above the x axis and so the from the y limits we can see that the region is at most the lower half of the circle of radius 3 centered at the origin.

From the x limits we can see that x will never be negative and so we now know the region can't contain any portion of the left part of the circle. The upper x limit tells us that x can go all the way out to 3.

Therefore, from the x and y limits we can see that the region we are integrating over is in fact the portion of the circle of radius 3 centered at the origin that is in the 4^{th} quadrant. Here is a quick sketch for the sake of completeness.



Step 2

Now, we will need the polar limits for our integral and they should be pretty easy to determine from the sketch above. Here they are.

$$\frac{3}{2}\pi \le \theta \le 2\pi$$
$$0 \le r \le 3$$

Step 3

Okay, we can now convert the limit to an integral involving polar coordinates. To do this just recall the various relationships between the Cartesian and polar coordinates and make use of them to convert any x's and/or y's into r's and/or θ's.

Also, do not forget that the "*dy dx*" in the integral came from the dA that would have been in the original double integral (which isn't written down but we know that it could be if we wanted to). We also know how that $dA = r\,dr\,d\theta$ and so for this integral we will have $dy\,dx = r\,dr\,d\theta$.

The integral is then,

$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 \mathbf{e}^{x^2+y^2}\,dy\,dx = \int_{\frac{3}{2}\pi}^{2\pi} \int_0^3 r\mathbf{e}^{r^2}\,dr\,d\theta$$

Note that the extra r we picked up from the dA is actually needed here to make this a doable integral! It is important to not forget to properly convert the dA when converting integrals from Cartesian to polar coordinates.

Step 4

Here is the r integration. It needs a simple substitution that we'll leave it to you to verify the results.

$$\begin{aligned}\int_0^3 \int_{-\sqrt{9-x^2}}^0 \mathbf{e}^{x^2+y^2}\,dy\,dx &= \int_{\frac{3}{2}\pi}^{2\pi} \int_0^3 r\mathbf{e}^{r^2}\,dr\,d\theta \\ &= \int_{\frac{3}{2}\pi}^{2\pi} \left.\frac{1}{2}\mathbf{e}^{r^2}\right|_0^3 d\theta \\ &= \int_{\frac{3}{2}\pi}^{2\pi} \frac{1}{2}\left(\mathbf{e}^9 - 1\right)\,d\theta\end{aligned}$$

Step 5

Finally, here is the really simple θ integration.

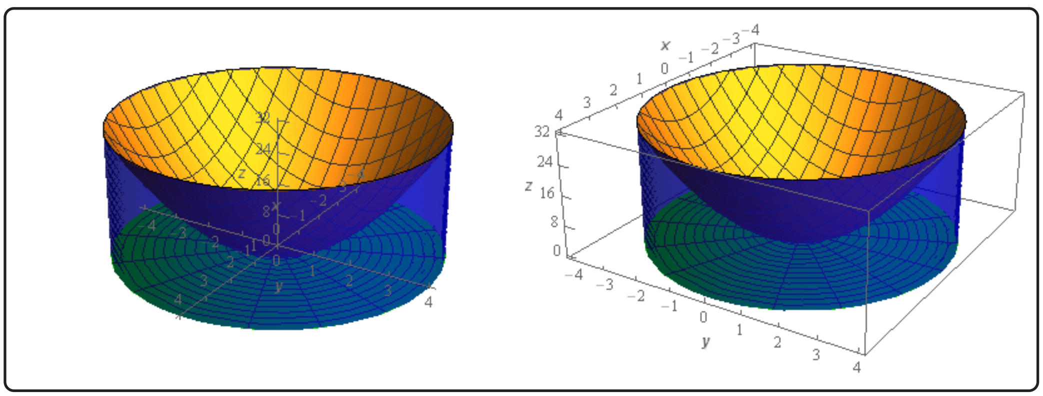
$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 \mathbf{e}^{x^2+y^2}\, dy\, dx = \int_{\frac{3}{2}\pi}^{2\pi} \frac{1}{2}\left(\mathbf{e}^9 - 1\right)\, d\theta$$

$$= \frac{1}{2}\left(\mathbf{e}^9 - 1\right)\theta\Big|_{\frac{3}{2}\pi}^{2\pi} = \boxed{\frac{1}{4}\pi\left(\mathbf{e}^9 - 1\right) = 6363.3618}$$

6. Use a double integral to determine the volume of the solid that is inside the cylinder $x^2+y^2=16$, below $z=2x^2+2y^2$ and above the xy-plane.

Step 1

Let's start off this problem with a quick sketch of the solid we're looking at here in this problem.



The top of the solid is the paraboloid (the gold colored surface in the sketches). The walls of the solid are the cylinder $x^2+y^2=16$ and is shown with the semi translucent surface in the sketch. The bottom of the solid is the xy-plane.

Step 2

So, the volume of the solid that is under the paraboloid and above the region D in the xy-plane is given by,

$$V = \iint\limits_D 2x^2 + 2y^2 \, dA$$

Step 3

The region D is the region from the xy-plane that we use to sketch the surface we are finding the volume under (*i.e.* the paraboloid).

Determining D in this case is pretty simple if you think about it. The solid is defined by the portion of the paraboloid that is inside the cylinder $x^2 + y^2 = 16$. But this is exactly what defines the portion of the xy-plane that we use to graph the surface. We only use the points from the xy-plane that are inside the cylinder and so the region D is then just the disk defined by $x^2 + y^2 \le 16$.

Another way to think of determining D for this case is to look at the solid from directly above it. The 2D region that you see will be the region D. In this case the region we see is the inside of the cylinder or the disk $x^2 + y^2 \le 16$.

Now, the region D is a disk and so this strongly suggests that we use polar coordinates for this problem. The polar limits for this region D is,

$$\begin{gathered} 0 \le \theta \le 2\pi \\ 0 \le r \le 4 \end{gathered}$$

Step 4

Okay, let's step up the integral in terms of polar coordinates.

$$V = \iint\limits_D 2x^2 + 2y^2 \, dA = \int_0^{2\pi} \int_0^4 \left(2r^2\right)(r) \, dr \, d\theta = \int_0^{2\pi} \int_0^4 2r^3 \, dr \, d\theta$$

Don't forget to convert all the x's and y's into r's and θ's and make sure that you simplify the integrand as much as possible. Also, don't forget to add in the r we get from the dA.

Step 5

Here is the simple r integration.

$$V = \int_0^{2\pi}\int_0^4 2r^3\,dr\,d\theta$$
$$= \int_0^{2\pi} \left.\frac{1}{2}r^4\right|_0^4 d\theta = \int_0^{2\pi} 128\,d\theta$$

Step 6

Finally, here is the really simple θ integration.

$$V = \int_0^{2\pi} 128\,d\theta = \boxed{256\pi}$$

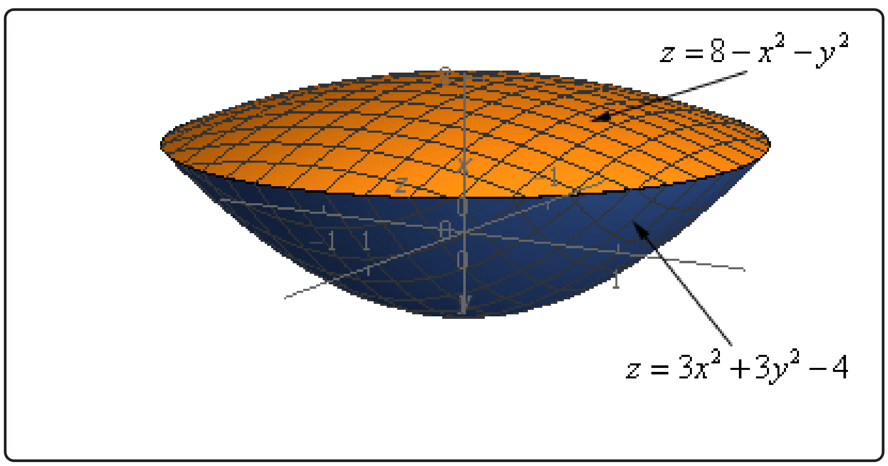
Note that this was a very simple integration and so we didn't actually do any of the work and left it to you to verify the details.

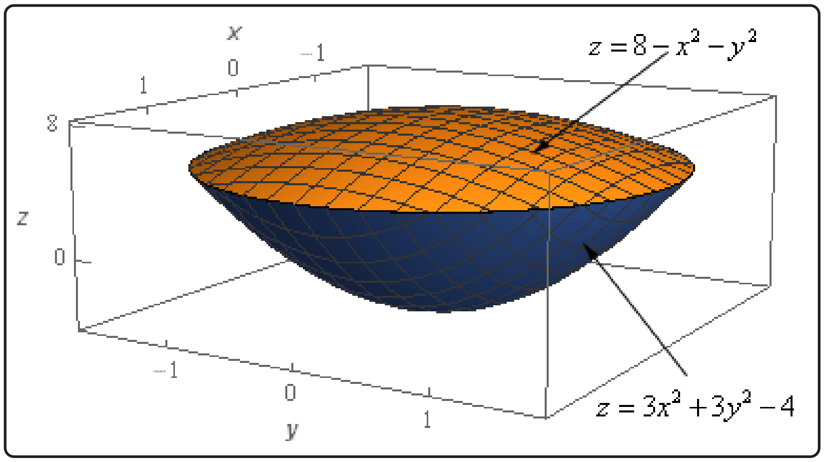
---

7. Use a double integral to determine the volume of the solid that is bounded by $z = 8 - x^2 - y^2$ and $z = 3x^2 + 3y^2 - 4$.

Step 1

Let's start off this problem with a quick sketch of the solid we're looking at here in this problem.





The top of the solid is the paraboloid given by $z = 8 - x^2 - y^2$ (the gold colored surface in the sketches) and the bottom of the solid is the paraboloid given by $z = 3x^2 + 3y^2 - 4$ (the blue colored surface in the sketches).

Step 2

To get the volume of this solid we're going to need to know that value of z where these two surfaces intersect. To let's solve the equation of the upper paraboloid as follow,

$$z = 8 - x^2 - y^2 \qquad \Rightarrow \qquad x^2 + y^2 = 8 - z$$

Now plug this into the equation of the second paraboloid to get,

$$z = 3\left(x^2 + y^2\right) - 4 = 3\left(8 - z\right) - 4 = 20 - 3z \qquad \to \qquad 4z = 20 \qquad \to \qquad z = 5$$

So, we know they will intersect at $z = 5$. Plugging this into the either paraboloid equation and doing a little simplification will give the following equation.

$$x^2 + y^2 = 3$$

Now, what does this tell us? This is the circle where the two paraboloids intersect at $z = 5$. This is also tells us that the region D in the xy-plane that we are going to use in this problem is the disk defined by $x^2 + y^2 \le 3$. This makes sense if you think about it. It is the region in the xy-plane that we'd use to graph each of the paraboloids in the sketches.

Step 3

We don't have a formula to find the volume of this solid at this point so let's see if we can figure out what it is.

Let's start with the volume of just the lower portion of the solid. In other words, what is the volume of the portion of the solid that is below the plane $z = 5$ and above the paraboloid given by $z = 3x^2 + 3y^2 - 4$.

We looked at a solid like this in the notes for this section. Following the same logic in that problem the volume of the lower portion of the solid is given by,

$$V_{\text{lower}} = \iint\limits_D 5 - \left(3x^2 + 3y^2 - 4\right)\, dA = \iint\limits_D 9 - 3x^2 - 3y^2\, dA$$

where D is the disk $x^2 + y^2 \leq 3$ as we discussed above.

Note that even though this paraboloid does slip under the xy-plane the formula is still valid.

Step 4

The volume of the upper portion of the solid, *i.e.* the portion under $z = 8 - x^2 - y^2$ and above the plane $z = 5$ can be found with a similar argument to the one we used for the lower region. The volume of the upper region is then,

$$V_{\text{upper}} = \iint\limits_D 8 - x^2 - y^2 - (5)\, dA = \iint\limits_D 3 - x^2 - y^2\, dA$$

where D is the disk $x^2 + y^2 \leq 3$ as we discussed above.

The volume of the whole solid is then,

$$\begin{aligned} V &= V_{\text{lower}} + V_{\text{upper}} \\ &= \iint\limits_D 9 - 3x^2 - 3y^2\, dA + \iint\limits_D 3 - x^2 - y^2\, dA \\ &= \iint\limits_D 9 - 3x^2 - 3y^2 + \left(3 - x^2 - y^2\right)\, dA \\ &= \iint\limits_D 12 - 4x^2 - 4y^2\, dA \end{aligned}$$

Note that we could combine the two integrals because they were both over the same region D.

Step 5

Okay, as we've already determined D is the disk given by $x^2 + y^2 \leq 3$ and because this is a disk it makes sense to do this integral in polar coordinates. Here are the polar limits for this integral/region.

$$0 \leq \theta \leq 2\pi$$
$$0 \leq r \leq \sqrt{3}$$

Step 6

The volume integral (in terms of polar coordinates) is then,

$$\begin{aligned} V &= \iint\limits_D 12 - 4x^2 - 4y^2 \, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \left(12 - 4r^2\right)(r) \; dr \, d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} 12r - 4r^3 \, dr \, d\theta \end{aligned}$$

Don't forget to convert all the x's and y's into r's and θ's and make sure that you simplify the integrand as much as possible. Also, don't forget to add in the r we get from the dA.

Step 7

Here is the r integration.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\sqrt{3}} 12r - 4r^3 \, dr \, d\theta \\ &= \int_0^{2\pi} \left(6r^2 - r^4\right)\Big|_0^{\sqrt{3}} \, d\theta = \int_0^{2\pi} 9 \, d\theta \end{aligned}$$

Step 8

Finally, here is the really simple θ integration.

$$V = \int_0^{2\pi} 9\,d\theta = \boxed{18\pi}$$

Note that this was a very simple integration and so we didn't actually do any of the work and left it to you to verify the details.

15.5 Triple Integrals

1. Evaluate $\displaystyle\int_2^3 \int_{-1}^4 \int_1^0 4x^2y - z^3\,dz\,dy\,dx$

Step 1

There really isn't all that much to this problem. All we need to do is integrate following the given order and recall that just like with double integrals we start with the "inside" integral and work our way out.

So, here is the z integration.

$$\begin{aligned}\int_2^3 \int_{-1}^4 \int_1^0 4x^2y - z^3\,dz\,dy\,dx &= \int_2^3 \int_{-1}^4 \left(4x^2yz - \frac{1}{4}z^4\right)\Big|_1^0\,dy\,dx\\ &= \int_2^3 \int_{-1}^4 \frac{1}{4} - 4x^2y\,dy\,dx\end{aligned}$$

Remember that triple integration is just like double integration and all the variables other than the one we are integrating with respect to are considered to be constants. So, for the z integration the x's and y's are all considered to be constants.

Step 2

Next, we'll do the y integration.

$$\begin{aligned}\int_2^3 \int_{-1}^4 \int_1^0 4x^2y - z^3\,dz\,dy\,dx &= \int_2^3 \left(\frac{1}{4}y - 2x^2y^2\right)\Big|_{-1}^4\,dx\\ &= \int_2^3 \frac{5}{4} - 30x^2\,dx\end{aligned}$$

Step 3

Finally, we'll do the x integration.

$$\int_2^3 \int_{-1}^4 \int_1^0 4x^2y - z^3\,dz\,dy\,dx = \left(\frac{5}{4}x - 10x^3\right)\Big|_2^3 = \boxed{-\frac{755}{4}}$$

So, not too much to do with this problem since the limits were already set up for us.

2. Evaluate $\displaystyle\int_0^1 \int_0^{z^2} \int_0^3 y \cos\left(z^5\right)\, dx\, dy\, dz$

Step 1

There really isn't all that much to this problem. All we need to do is integrate following the given order and recall that just like with double integrals we start with the "inside" integral and work our way out.

Also note that the fact that one of the limits is not a constant is not a problem. There is nothing that says that triple integrals set up as this is must only have constants as limits!

So, here is the x integration.

$$\begin{aligned}\int_0^1 \int_0^{z^2} \int_0^3 y \cos\left(z^5\right)\, dx\, dy\, dz &= \int_0^1 \int_0^{z^2} \left. \left(y \cos\left(z^5\right) x\right)\right|_0^3 dy\, dz \\ &= \int_0^1 \int_0^{z^2} 3y \cos\left(z^5\right)\, dy\, dz\end{aligned}$$

Remember that triple integration is just like double integration and all the variables other than the one we are integrating with respect to are considered to be constants. So, for the x integration the y's and z's are all considered to be constants.

Step 2

Next, we'll do the y integration.

$$\begin{aligned}\int_0^1 \int_0^{z^2} \int_0^3 y \cos\left(z^5\right)\, dx\, dy\, dz &= \int_0^1 \left. \left(\frac{3}{2} y^2 \cos\left(z^5\right)\right)\right|_0^{z^2} dz \\ &= \int_0^1 \frac{3}{2} z^4 \cos\left(z^5\right)\, dz\end{aligned}$$

Step 3

Finally, we'll do the z integration and note that the only way we are able to do this integration is because of the z^4 that is now in the integrand. Without that present we would not be able to do this integral.

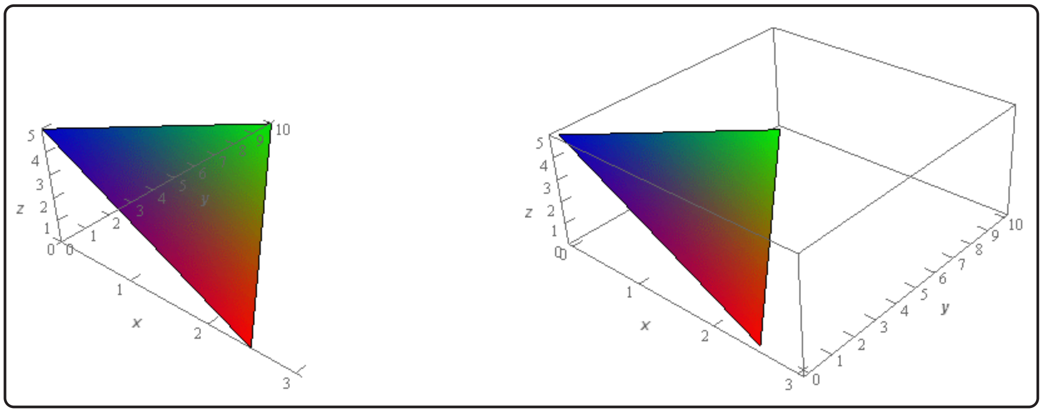
$$\int_0^1 \int_0^{z^2} \int_0^3 y \cos\left(z^5\right)\, dx\, dy\, dz = \left(\frac{3}{10}\sin\left(z^5\right)\right)\Big|_0^1 = \boxed{\frac{3}{10}\sin\left(1\right) = 0.2524}$$

So, not too much to do with this problem since the limits were already set up for us.

3. Evaluate $\displaystyle\iiint\limits_E 6z^2\, dV$ where E is the region below $4x + y + 2z = 10$ in the first octant.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

For this problem the region E is just the region that is under the plane shown above and in the first octant. In other words, the sketch of the plane above is exactly the top of the region E. The bottom of the region is the xy-plane while the sides are simply the yz and xz-planes.

Step 2

So, from the sketch above we know that we'll have the following limits for z.

$$0 \le z \le 5 - 2x - \frac{1}{2}y$$

where we got the upper z limit simply by solving the equation of the plane for z.

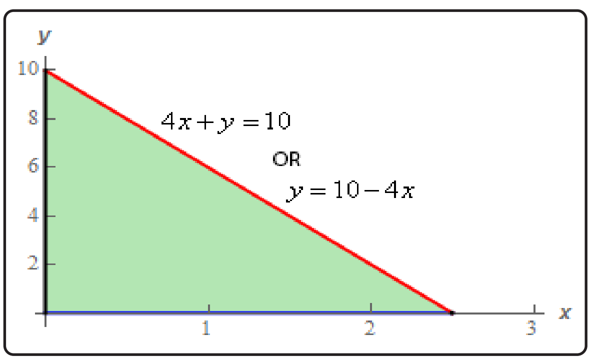
With these limits we can also get the triple integral at least partially set up as follows.

$$\iiint\limits_E 6z^2\,dV = \iint\limits_D \left[\int_0^{5-2x-\frac{1}{2}y} 6z^2\,dz\right] dA$$

Step 3

Next, we'll need limits for D so we can finish setting up the integral. For this problem D is simply the region in the xy-plane (since we are integrating with respect to z first) that we used to graph the plane in Step 1. That also, in this case, makes D the bottom of the region.

Here is a sketch of D.



The hypotenuse of D is simply the intersection of the plane from Step 1 and the xy-plane and so we can quickly get its equation by plugging $z = 0$ into the equation of the plane.

Given the nature of this region as well as the function we'll be integrating it looks like we can use either order of integration for D. So, to keep the limits at least a little nicer we'll integrate y's and then x's.

Here are the limits for the double integral over D.

$$0 \le x \le \frac{5}{2}$$
$$0 \le y \le 10 - 4x$$

The upper x limit was found simply by plugging $y = 0$ into the equation of the hypotenuse and solving for x to determine where the hypotenuse intersected the x-axis.

With these limits plugged into the integral we now have,

$$\iiint\limits_E 6z^2\,dV = \int_0^{\frac{5}{2}} \int_0^{10-4x} \int_0^{5-2x-\frac{1}{2}y} 6z^2\,dz\,dy\,dx$$

Step 4

Okay, now all we need to do is evaluate the integral. Here is the z integration.

$$\begin{aligned}\iiint\limits_E 6z^2\,dV &= \int_0^{\frac{5}{2}} \int_0^{10-4x} \left.\left(2z^3\right)\right|_0^{5-2x-\frac{1}{2}y} dy\,dx \\ &= \int_0^{\frac{5}{2}} \int_0^{10-4x} 2\left(5-2x-\frac{1}{2}y\right)^3 dy\,dx\end{aligned}$$

Do not multiply out the integrand of this integral.

Step 5

As noted in the last step we do not want to multiply out the integrand of this integral. One of the bigger mistakes students make with multiple integrals is to just launch into a simplification mode after the integral and multiply everything out.

Sometimes of course that must be done but, in this case, note that we can easily do the y integration with a simple Calculus I substitution. Here is that work.

$$\begin{aligned}\iiint\limits_E 6z^2\,dV &= \int_0^{\frac{5}{2}} \left.-\left(5-2x-\frac{1}{2}y\right)^4\right|_0^{10-4x} dx \qquad\qquad u = 5-2x-\frac{1}{2}y \\ &= \int_0^{\frac{5}{2}} (5-2x)^4\,dx\end{aligned}$$

We gave the substitution used in this step but are leaving it to you to verify the details of the substitution.

Step 6

Again, notice that we can either do some "simplification" or we can just do another substitution to finish this integral out. Here is the final integration step for this problem. We'll leave it to you to verify the substitution details.

$$\iiint\limits_E 6z^2\,dV = \int_0^{\frac{5}{2}} (5-2x)^4\,dx \qquad\qquad u = 5-2x$$
$$= -\frac{1}{10}(5-2x)^5\Big|_0^{\frac{5}{2}}$$
$$= \boxed{\frac{625}{2}}$$

So, once we got the limits all set up, the integration for this problem wasn't too bad provided we took advantage of the substitutions of course. That will often be the case with these problems. Getting the limits for the integrals set up will often, but not always, be the hardest part of the problem. Once they get set up the integration is often pretty simple.

Also, as noted above do not forget about your basic Calculus I substitutions. Using them will often allow us to avoid some messy algebra that will be easy to make a mistake with.

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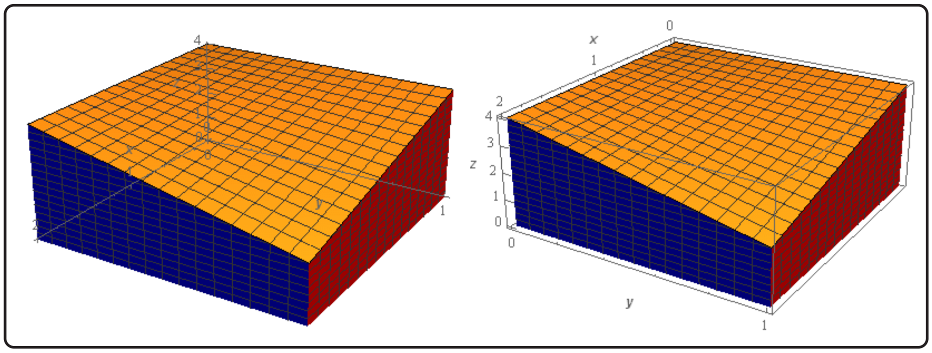
4. Evaluate $\iiint\limits_E 3-4x\,dV$ where E is the region below $z = 4-xy$ and above the region in the xy-plane defined by $0 \le x \le 2$, $0 \le y \le 1$.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

The top portion of the region (the orange colored surface) is the graph of $z = 4 - xy$. The two sides shown (the blue and red surfaces) show the two sides of the region that we can see given the orientation of the region. The bottom of the region is the xy-plane.

Step 2

So, from the sketch above we know that we'll have the following limits for z.

$$0 \le z \le 4 - xy$$

With these limits we can also get the triple integral at least partially set up as follows.

$$\iiint\limits_E 3 - 4x\,dV = \iint\limits_D \left[\int_0^{4-xy} 3 - 4x\,dz\right] dA$$

Step 3

Next, we'll need limits for D so we can finish setting up the integral. In this case that is really simple as we can see from the problem statement that D is just a rectangle in the xy-plane and in fact the limits are given in the problem statement as,

$$\begin{array}{c} 0 \le x \le 2 \\ 0 \le y \le 1 \end{array}$$

There really isn't any advantage to doing one order vs. the other so, in this case, we'll integrate y and then x.

With these limits plugged into the integral we now have,

$$\iiint\limits_E 3 - 4x\,dV = \int_0^2 \int_0^1 \int_0^{4-xy} 3 - 4x\,dz\,dy\,dx$$

Step 4

Okay, now all we need to do is evaluate the integral. Here is the z integration.

$$\iiint\limits_E 3 - 4x\, dV = \int_0^2 \int_0^1 (3 - 4x)\, z\Big|_0^{4-xy} dy\, dx$$
$$= \int_0^2 \int_0^1 (3 - 4x)(4 - xy)\, dy\, dx$$
$$= \int_0^2 \int_0^1 4x^2y - 3xy - 16x + 12\, dy\, dx$$

Note that because the integrand had no z's in it we treated the whole integrand as a constant and just added a single z as shown above. We could just as easily integrated each term individually but it seemed easier to just deal with the integrand as a single term.

Step 5

Now let's do the y integration.

$$\iiint\limits_E 3 - 4x\, dV = \int_0^2 \left(2x^2y^2 - \frac{3}{2}xy^2 - 16xy + 12y\right)\Big|_0^1 dx$$
$$= \int_0^2 12 - \frac{35}{2}x + 2x^2\, dx$$

Step 6

Finally, let's do the x integration.

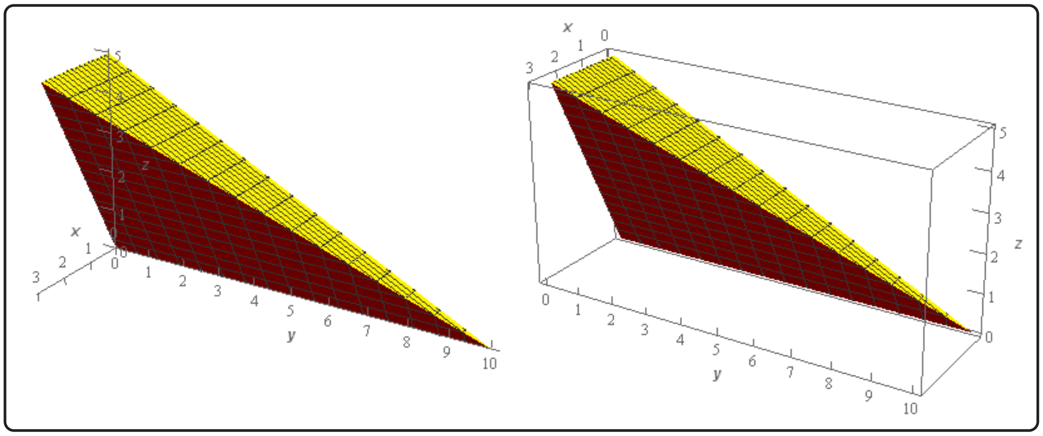
$$\iiint\limits_E 3 - 4x\, dV = \left(12x - \frac{35}{4}x^2 + \frac{2}{3}x^3\right)\Big|_0^2 = \boxed{-\frac{17}{3}}$$

So, once we got the limits all set up, the integration for this problem wasn't too bad. That will often be the case with these problems. Getting the limits for the integrals set up will often, but not always, be the hardest part of the problem. Once they get set up the integration is often pretty simple.

5. Evaluate $\iiint\limits_E 12y - 8x\,dV$ where E is the region behind $y = 10 - 2z$ and in front of the region in the xz-plane bounded by $z = 2x$, $z = 5$ and $x = 0$.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

The top portion of the region (the yellow colored surface) is the graph of the portion $y = 10 - 2z$ that lies in front of the region xz-plane given in the problem statement.

The red surface is the plane defined by $z = 2x$. This is the plane that intersects the xz-plane at the line $z = 2x$ and is bounded above by the surface $y = 10 - 2z$.

Step 2

The region in the xz-plane bounded by $z = 2x$, $z = 5$ and $x = 0$ that is referenced in the problem statement is the region D and so we know that, for this problem, we'll need to be integrating y first (since D is in the xz-plane).

Therefore, we have the following limits for y.

$$0 \le y \le 10 - 2z$$

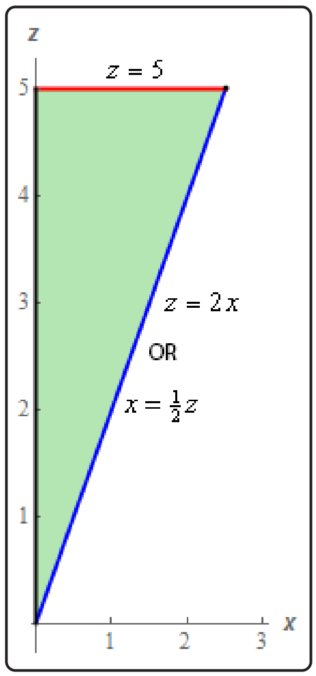
With these limits we can also get the triple integral at least partially set up as follows.

$$\iiint\limits_E 12y - 8x\,dV = \iint\limits_D \left[\int_0^{10-2z} 12y - 8x\,dy\right] dA$$

Step 3

Next, we'll need limits for D so we can finish setting up the integral. For this problem D is the region in the xz-plane from the problem statement as we noted above.

Here is a quick sketch of D.



Another way to think this region is that it is the "back" of the solid sketched in Step 1 and the solid we sketched in Step 1 is all behind this region. In other words the positive y-axis is goes directly back into the page while the negative y-axis comes directly out of the page.

As noted in the sketch of D we can easily defined by either of the following sets of limits.

$$\begin{aligned} 0 \le x \le \frac{5}{2} \\ 2x \le z \le 5 \end{aligned} \qquad \text{OR} \qquad \begin{aligned} 0 \le z \le 5 \\ 0 \le x \le \frac{1}{2}z \end{aligned}$$

The integrand doesn't really suggest one of these would be easier than the other so we'll use the 2^{nd} set of limits for no other reason than the lower limits for both are zero which might make things a little nicer in the integration process.

With these limits plugged into the integral we now have,

$$\iiint\limits_E 12y - 8x\,dV = \int_0^5 \int_0^{\frac{1}{2}z} \int_0^{10-2z} 12y - 8x\,dy\,dx\,dz$$

Step 4

Okay, now all we need to do is evaluate the integral. Here is the y integration.

$$\begin{aligned} \iiint\limits_E 12y - 8x\,dV &= \int_0^5 \int_0^{\frac{1}{2}z} \left.\left(6y^2 - 8xy\right)\right|_0^{10-2z} dx\,dz \\ &= \int_0^5 \int_0^{\frac{1}{2}z} 6(10-2z)^2 - 8x\left(10-2z\right)\,dx\,dz \end{aligned}$$

Note that we did no simplification here because it is not yet clear that we need to do any simplification. The next integration is with respect to x and the first term is a constant as far as that integration is concerned and the second term is multiplied by an x and so it will might well be easier to leave it in that form for the x integration.

Step 5

Now let's do the x integration.

$$\begin{aligned} \iiint\limits_E 12y - 8x\,dV &= \int_0^5 \left.\left[6(10-2z)^2 x - 4x^2\left(10-2z\right)\right]\right|_0^{\frac{1}{2}z} dz \\ &= \int_0^5 3z(10-2z)^2 - z^2\left(10-2z\right)\,dz \\ &= \int_0^5 14z^3 - 130z^2 + 300z\,dz \end{aligned}$$

Step 6

Finally, let's do the z integration.

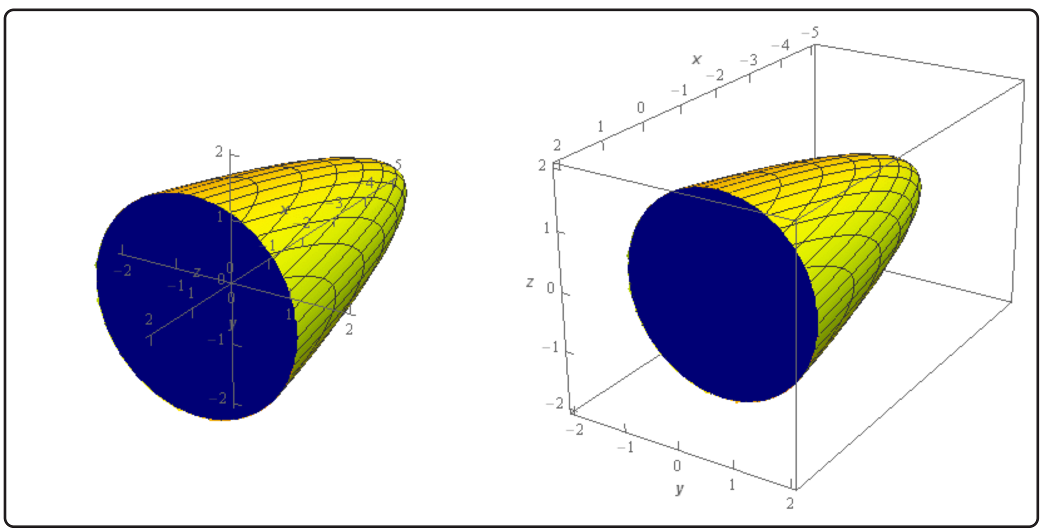
$$\iiint\limits_E 12y - 8x\,dV = \left(\frac{7}{2}z^4 - \frac{130}{3}z^3 + 150z^2\right)\Bigg|_0^5 = \boxed{\frac{3125}{6}}$$

So, once we got the limits all set up, the integration for this problem wasn't too bad. That will often be the case with these problems. Getting the limits for the integrals set up will often, but not always, be the hardest part of the problem. Once they get set up the integration is often pretty simple.

6. Evaluate $\displaystyle\iiint\limits_E yz\,dV$ where E is the region bounded by $x = 2y^2 + 2z^2 - 5$ and the plane $x = 1$.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

We know that $x = 2y^2 + 2z^2 - 5$ is an elliptic paraboloid that is centered on the x-axis and opens in the positive x direction as shown in the sketch above. The blue "cap" on the surface is the portion of the plane $x = 1$ that fits just inside the paraboloid.

Step 2

Okay, for this problem it is hopefully clear that we'll need to integrate with respect to x first and that the region D will be in the y z-plane.

Therefore, we have the following limits for x.

$$2y^2 + 2z^2 - 5 \le x \le 1$$

Remember that when setting these kinds of limits up we go from back to front and so the paraboloid will be the lower limit since it is the surface that is in the "back" of the solid while $x = 1$ is the "front" of the solid and so is the upper limit.

With these limits we can also get the triple integral at least partially set up as follows.

$$\iiint\limits_E yz\,dV = \iint\limits_D \left[\int_{2y^2+2z^2-5}^{1} yz\,dx\right] dA$$

Step 3

Next, we'll need limits for D so we can finish setting up the integral. To determine D we'll need the intersection of the two surfaces. The intersection is,

$$2y^2 + 2z^2 - 5 = 1 \qquad \to \qquad 2y^2 + 2z^2 = 6 \qquad \to \qquad y^2 + z^2 = 3$$

Now, if we looked at the solid from the "front", *i.e.* from along the positive x-axis we'd see the disk $y^2 + z^2 \le 3$ and so this is the region D.

The region D is a disk which clearly suggests polar coordinates, however, it won't be the "standard" xy polar coordinates. Since D is in the yz-plane let's use the following "modified" polar coordinates.

$$y = r\sin(\theta) \qquad z = r\cos(\theta) \qquad y^2 + z^2 = r^2$$

This "definition" of polar coordinates for our problem isn't needed quite yet but will be eventually.

At this point all we need are the polar limits for this circle and those should fairly clearly be given by,

$$\begin{gathered} 0 \le \theta \le 2\pi \\ 0 \le r \le \sqrt{3} \end{gathered}$$

With all the previous problems we'd write the integral down with these limits at this point as well. However, since we are going to have to convert to polar coordinates we'll hold off writing down the integral in polar coordinates until we do the first integration.

Step 4

So, here is the x integration.

$$\begin{aligned}\iiint\limits_E yz\,dV &= \iint\limits_D \left.(xyz)\right|_{2y^2+2z^2-5}^{1} dA\\ &= \iint\limits_D \left[1-\left(2y^2+2z^2-5\right)\right] yz\,dA\\ &= \iint\limits_D \left[6-2\left(y^2+z^2\right)\right] yz\,dA\end{aligned}$$

We did a small amount of simplification here in preparation for the next step.

Step 5

Now we need to convert the integral over to polar coordinates using the modified version we defined in Step 3.

Here is the integral when converted to polar coordinates.

$$\begin{aligned}\iiint\limits_E yz\,dV &= \int_0^{2\pi}\int_0^{\sqrt{3}} \left[6-2r^2\right](r\sin(\theta))(r\cos(\theta))\,r\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^{\sqrt{3}} \left[6r^3-2r^5\right]\sin(\theta)\cos(\theta)\,dr\,d\theta\end{aligned}$$

Step 6

Now let's do the r integration.

$$\iiint\limits_E yz\,dV = \int_0^{2\pi} \left[\frac{3}{2}r^4 - \frac{1}{3}r^6\right] \sin(\theta)\cos(\theta)\Big|_0^{\sqrt{3}}\,d\theta$$

$$= \int_0^{2\pi} \frac{9}{2}\sin(\theta)\cos(\theta)\,d\theta$$

Step 7

Finally, let's do the θ integration and notice that we're going to use the double angle formula for sine to "simplify" the integral slightly prior to the integration.

$$\iiint\limits_E yz\,dV = \int_0^{2\pi} \frac{9}{4}\sin(2\theta)\,d\theta = \left(-\frac{9}{8}\cos(2\theta)\right)\Big|_0^{2\pi} = \boxed{0}$$

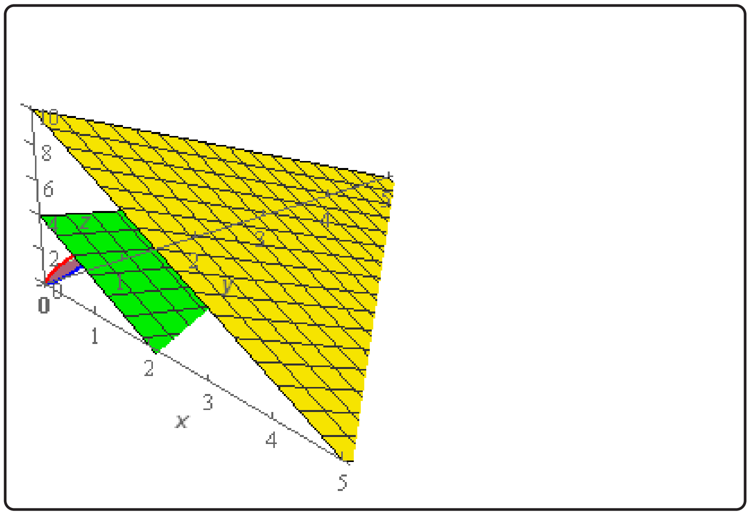
So, once we got the limits all set up, the integration for this problem wasn't too bad. That will often be the case with these problems. Getting the limits for the integrals set up will often, but not always, be the hardest part of the problem. Once they get set up the integration is often pretty simple.

Also, as we saw in this example it is not unusual for polar coordinates to show in the "outer" double integral and there is no reason to expect they will always be the "standard" xy definition of polar coordinates and so you will need to be ready to use them in any of the three orientations (xy, xz or yz) in which they may show up.

7. Evaluate $\iiint\limits_E 15z\,dV$ where E is the region between $2x + y + z = 4$ and $4x + 4y + 2z = 20$ that is in front of the region in the yz-plane bounded by $z = 2y^2$ and $z = \sqrt{4y}$.

Step 1

This region always seems pretty difficult to visualize. So, let's start off with the following sketch.

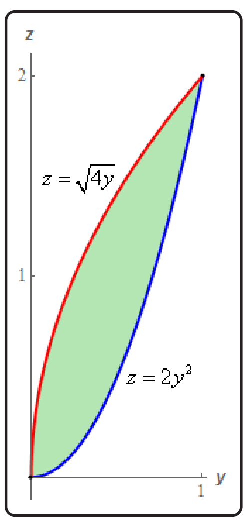


Note that the "orientation" of the x and y axes are different in this sketch form all the other 3D sketches we've done to this point. This is done to help with the visualization. Without the reorientation it would be very difficult to visualize the bounded region.

Do not always expect that the orientation of the axes to always remain fixed and never changing. Sometimes the orientation will need to change so we can visualize a particular surface or region.

Okay, onto the trying to visualize the region we are working with here.

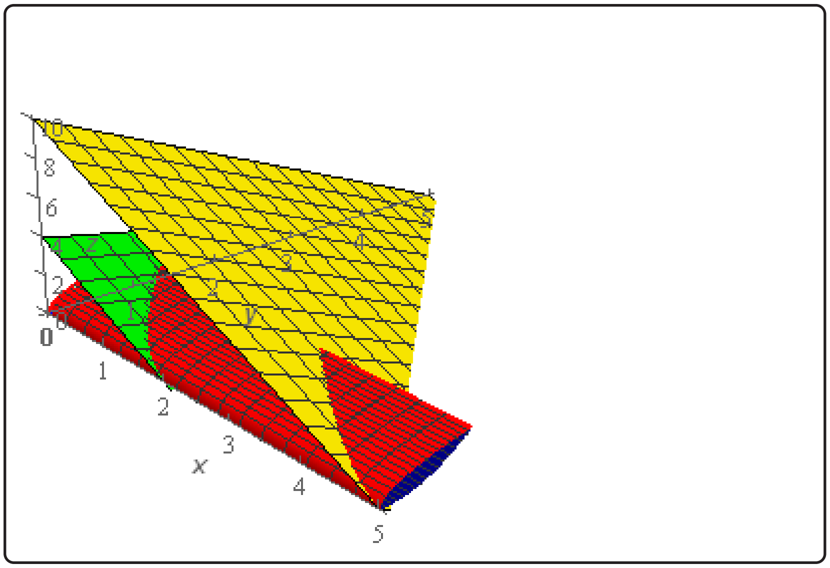
The yellow plane is the graph of $4x + 4y + 2z = 20$ in the 1^{st} octant (the only portion we'll need here) and the green plane is the graph of $2x + y + z = 4$. At the origin (kind of hard to see) is the sketch of the bounded region in the yz-plane referenced in the problem statement. Because this bounded region is hard to see on the graph above here is separate graph of just the bounded region.



Now, to try and visualize the region E, imagine that there is a lump of cookie dough between the two planes in the first sketch above and the bounded region in the yz-plane is a cookie cutter. We move the bounded region away from the yz-plane making sure to always keep it parallel to the yz-plane.

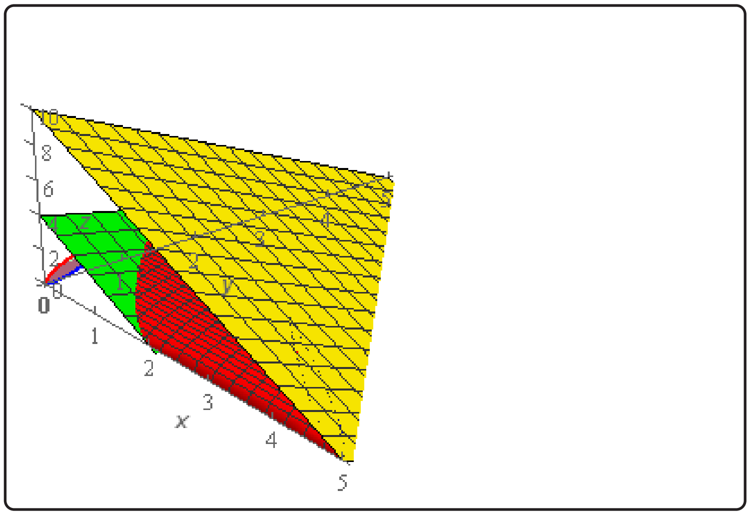
As the bounded region moves out the top of it will first cut into the “cookie dough” when it hits the green plane and will be fully in the cookie dough when the bottom of the bounded region hits the green plane. The bounded region will continue to cut through the cookie dough until it reaches the yellow plane with the top of the bounded region exiting the cookie dough first when it hits the yellow plane. The bounded region will be fully out of the cookie dough when the bottom of the bounded region hits the yellow plane.

Here is a sketch of the path that the bounded region takes as it moves away from the yz-plane.

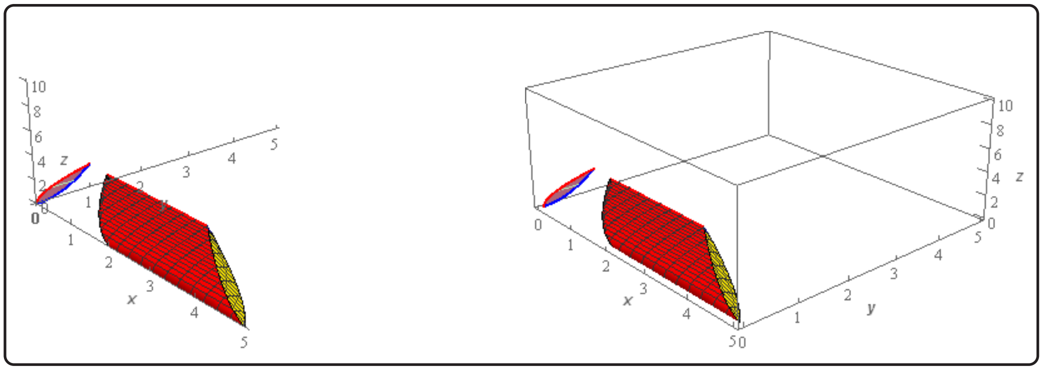


Note that this sketch shows the full path of the bounded region regardless of whether or not it is between the two planes. Or using the analogy from above regardless of whether or not it is cutting out a region from the cookie dough between the two planes. The region E that we are after is just the portion of this path that lies between the two planes.

Here is a sketch of the path of the bounded region that is only between the two planes.



The top of the region E is the red portion shown above. It is kind of hard however to visualize the full region with the two planes still in the sketch. So, here is another sketch with two planes removed from the sketch.



The region E is then the solid shown above. The "front" of E is sloped and follows the yellow plane (and hence is also shaded yellow to make this clearer) and the "rear" of E is sloped and follows the green plane.

With this final sketch of the region E we included both the "traditional" axis system as well as the "boxed" axes system to help visualize the object. We also kept the same x, y, z scale as the previous images to helpful help with the visualization even though there is a lot of "wasted" space in the right side of the axes system.

Step 2

Okay, for this problem the problem statement tells us that the bounded region, *i.e.* D, is in the yz-plane and we we'll need to integrate with respect to x first.

This means we'll need to solve each of the equations of the planes for x and we'll integrate from the back (or green) plane up to the front (or yellow) plane.

Therefore, we have the following limits for x.

$$2 - \frac{1}{2}y - \frac{1}{2}z \le x \le 5 - y - \frac{1}{2}z$$

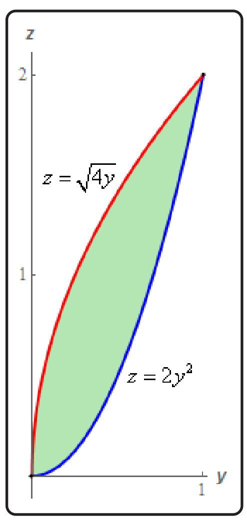
Remember that when setting these kinds of limits up we go from back to front and so the paraboloid will be the lower limit since it is the surface that is in the "back" of the solid while $x = 1$ is the "front" of the solid and so is the upper limit.

With these limits we can also get the triple integral at least partially set up as follows.

$$\iiint\limits_E 15z\,dV = \iint\limits_D \left[\int_{2-\frac{1}{2}y-\frac{1}{2}z}^{5-y-\frac{1}{2}z} 15z\,dx\right] dA$$

Step 3

Next, we'll need limits for D so we can finish setting up the integral. As noted above the region D is just the bounded region given in the problem statement. For reference purposes here is a copy of the sketch of D we gave in the Step 1.



As we can see this region could be integrated in either order but regardless of the order one of the limits will be a quadratic term and one will be a square root. Therefore, we might as well take use the limits as they were given in the problem statement.

So, here are the limits for D.

$$\begin{gathered} 0 \le y \le 1 \\ 2y^2 \le z \le \sqrt{4y} \end{gathered}$$

With these limits plugged into the integral we now have,

$$\iiint\limits_E 15z\,dV = \int_0^1 \int_{2y^2}^{\sqrt{4y}} \int_{2-\frac{1}{2}y-\frac{1}{2}z}^{5-y-\frac{1}{2}z} 15z\,dx\,dz\,dy$$

Step 4

Okay, now all we need to do is evaluate the integral. Here is the x integration.

$$\begin{aligned} \iiint\limits_E 15z\,dV &= \int_0^1 \int_{2y^2}^{\sqrt{4y}} \left.(15zx)\right|_{2-\frac{1}{2}y-\frac{1}{2}z}^{5-y-\frac{1}{2}z} dz\,dy \\ &= \int_0^1 \int_{2y^2}^{\sqrt{4y}} 15z\left(3-\frac{1}{2}y\right) dz\,dy \end{aligned}$$

Step 5

Now let's do the z integration.

$$\begin{aligned} \iiint\limits_E 15z\,dV &= \int_0^1 \left.\frac{15}{2}z^2\left(3-\frac{1}{2}y\right)\right|_{2y^2}^{\sqrt{4y}} dy \\ &= \int_0^1 15y^5 - 90y^4 - 15y^2 + 90y\,dy \end{aligned}$$

Step 6

Finally, let's do the y integration.

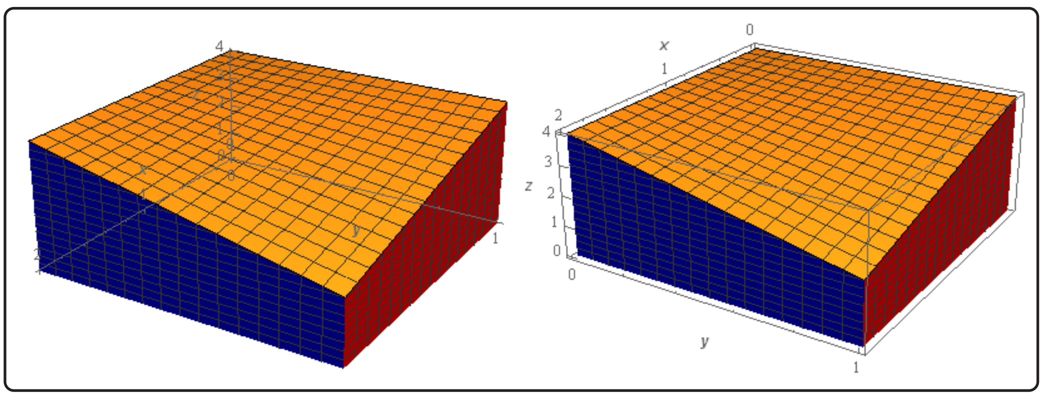
$$\iiint\limits_E 15z\,dV = \left.\left(\frac{5}{2}y^6 - 18y^5 - 5y^3 + 45y^2\right)\right|_0^1 = \boxed{\frac{49}{2}}$$

So, once we got the limits all set up, the integration for this problem wasn't too bad. That will often be the case with these problems. Getting the limits for the integrals set up will often, but not always, be the hardest part of the problem. Once the limits do get set up the integration is often pretty simple and there is no doubt that visualizing the region and getting the limits set up for this problem was probably more difficult that with many of the other problems.

8. Use a triple integral to determine the volume of the region below $z = 4 - xy$ and above the region in the xy-plane defined by $0 \le x \le 2$, $0 \le y \le 1$.

Step 1

Okay, let's start off with a quick sketch of the region we want the volume of so we can get a feel for what we're dealing with. We'll call this region E.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

The top portion of the region (the orange colored surface) is the graph of $z = 4 - xy$. The two sides shown (the blue and red surfaces) show the two sides of the region that we can see given the orientation of the region. The bottom of the region is the xy-plane.

Step 2

The volume of this solid is given by,

$$V = \iiint\limits_E dV$$

Step 3

So, we now need to get the limits set up for the integral. From the sketch above we know that we'll have the following limits for z.

$$0 \le z \le 4 - xy$$

We'll also need limits for D. In this case that is really simple as we can see from the problem statement that D is just a rectangle in the xy-plane and in fact the limits are given in the problem statement as,

$$\begin{aligned} 0 &\le x \le 2 \\ 0 &\le y \le 1 \end{aligned}$$

There really isn't any advantage to doing one order vs. the other so, in this case, we'll integrate y and then x.

Now, plugging all these limits into the integral the volume is,

$$V = \iiint\limits_E dV = \int_0^2 \int_0^1 \int_0^{4-xy} dz\,dy\,dx$$

Step 4

Okay, now all we need to do is evaluate the integral. Here is the z integration.

$$\begin{aligned} V &= \int_0^2 \int_0^1 z\Big|_0^{4-xy} dy\,dx \\ &= \int_0^2 \int_0^1 4 - xy\,dy\,dx \end{aligned}$$

Step 5

Now let's do the y integration.

$$V = \int_0^2 \left(4y - \frac{1}{2}xy^2 \right)\Big|_0^1 \, dx$$
$$= \int_0^2 4 - \frac{1}{2}x \, dx$$

Step 6

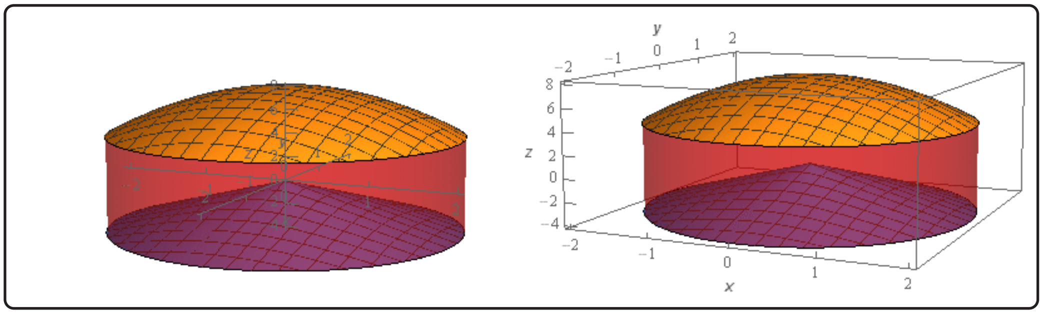
Finally, let's do the x integration to get the volume of the region.

$$V = \iiint\limits_E dV = \left(4x - \frac{1}{4}x^2 \right)\Big|_0^2 = \boxed{7}$$

9. Use a triple integral to determine the volume of the region that is below $z = 8 - x^2 - y^2$ above $z = -\sqrt{4x^2 + 4y^2}$ and inside $x^2 + y^2 = 4$.

Step 1

Okay, let's start off with a quick sketch of the region we want the volume of so we can get a feel for what we're dealing with. We'll call this region E.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

The top of the region (the orange colored surface) is the portion of the graph of the elliptic paraboloid $z = 8 - x^2 - y^2$ that is inside the cylinder $x^2 + y^2 = 4$. The bottom of the region is the portion of the graph of the cone $z = -\sqrt{4x^2 + 4y^2}$ that is inside the cylinder $x^2 + y^2 = 4$. The walls of the region (which are translucent to show the bottom portion) is the cylinder $x^2 + y^2 = 4$.

Step 2

The volume of this solid is given by,

$$V = \iiint\limits_E dV$$

Step 3

So, we now need to get the limits set up for the integral. From the sketch above we can see that we'll need to integrate with respect to z first so here are those limits.

$$-\sqrt{4x^2 + 4y^2} \le z \le 8 - x^2 - y^2$$

We'll also need limits for D. In this case D is just the disk given by $x^2 + y^2 \le 4$ (*i.e.* the portion of the xy-plane that is inside the cylinder. This is the region in the xy-plane that we need to graph the paraboloid and cone and so is D.

Because D is a disk it makes sense to use polar coordinates for integrating over D. Here are the limits for D.

$$0 \le \theta \le 2\pi$$
$$0 \le r \le 2$$

Now, let's set up the volume integral as follows.

$$V = \iiint\limits_E dV = \iint\limits_D \left[\int_{-\sqrt{4x^2+4y^2}}^{8-x^2-y^2} dz \right] dA$$

Because we know that we'll need to do the outer double integral in polar coordinates we'll hold off putting those limits in until we have the z integration done.

Step 4

Okay, let's do the z integration.

$$V = \iint\limits_D z\Big|_{-\sqrt{4x^2+4y^2}}^{8-x^2-y^2} dA$$
$$= \iint\limits_D 8 - x^2 - y^2 + \sqrt{4x^2+4y^2}\, dA$$

Step 5

Now let's convert the integral over to polar coordinates. Don't forget that $x^2 + y^2 = r^2$ and that $dA = r\,dr\,d\theta$.

The volume integral in terms of polar coordinates is then,

$$V = \int_0^{2\pi}\int_0^2 \left[8 - r^2 + 2r\right] r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 8r - r^3 + 2r^2\,dr\,d\theta$$

Step 6

The r integration is then,

$$V = \int_0^{2\pi}\left(4r^2 - \frac{1}{4}r^4 + \frac{2}{3}r^3\right)\Big|_0^2 d\theta = \int_0^{2\pi}\frac{52}{3}\,d\theta$$

Step 7

Finally, we can compute the very simple θ integral to get the volume of the region.

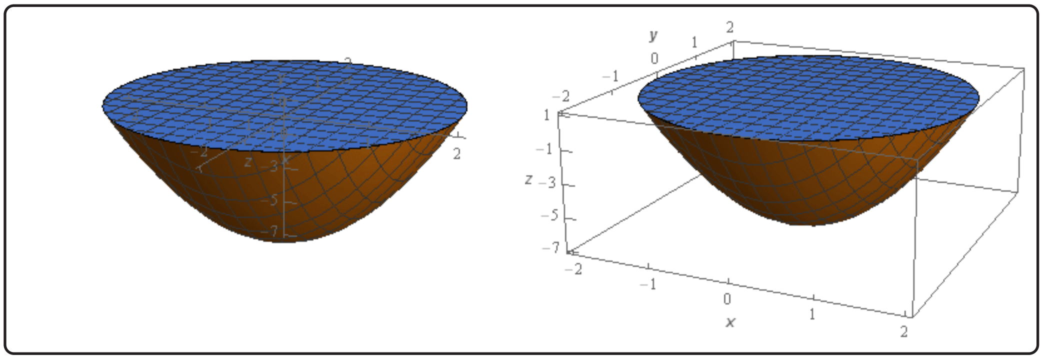
$$V = \int_0^{2\pi}\frac{52}{3}\,d\theta = \boxed{\frac{104}{3}\pi}$$

15.6 Triple Integrals in Cylindrical Coordinates

1. Evaluate $\displaystyle\iiint\limits_E 4xy\,dV$ where E is the region bounded by $z = 2x^2 + 2y^2 - 7$ and $z = 1$.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

Step 2

So, from the sketch above it should be pretty clear that we'll need to integrate z first and so we'll have the following limits for z.

$$2x^2 + 2y^2 - 7 \le z \le 1$$

Step 3

For this problem D is the disk that "caps" the region sketched in Step 1. We can determine the equation of the disk by setting the two equations from the problem statement equal and doing a little rewriting.

$$2x^2 + 2y^2 - 7 = 1 \qquad \to \qquad 2x^2 + 2y^2 = 8 \qquad \to \qquad x^2 + y^2 = 4$$

So, D is the disk $x^2+y^2 \le 4$ and it should be pretty clear that we'll need to use cylindrical coordinates for this integral.

Here are the cylindrical coordinates for this problem.

$$\begin{gathered} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \\ 2r^2 - 7 \le z \le 1 \end{gathered}$$

Don't forget to convert the z limits from Step 2 into cylindrical coordinates as well.

Step 4

Plugging these limits into the integral and converting to cylindrical coordinates gives,

$$\begin{aligned} \iiint\limits_E 4xy\,dV &= \int_0^{2\pi}\int_0^2\int_{2r^2-7}^1 4\left(r\cos(\theta)\right)\left(r\sin(\theta)\right)r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2\int_{2r^2-7}^1 4r^3\cos(\theta)\sin(\theta)\,dz\,dr\,d\theta \end{aligned}$$

Don't forget to convert the x and y's into cylindrical coordinates and also don't forget that $dV = r\,dz\,dr\,d\theta$ and so we pick up another r when converting the dV to cylindrical coordinates.

Step 5

Okay, now all we need to do is evaluate the integral. Here is the z integration.

$$\begin{aligned} \iiint\limits_E 4xy\,dV &= \int_0^{2\pi}\int_0^2 \left.\left(4r^3\cos(\theta)\sin(\theta)z\right)\right|_{2r^2-7}^{1} dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2 4r^3\left(8-2r^2\right)\cos(\theta)\sin(\theta)\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2 \left(32r^3-8r^5\right)\cos(\theta)\sin(\theta)\,dr\,d\theta \end{aligned}$$

Step 6

Next let's do the r integration.

$$\iiint\limits_E 4xy\,dV = \int_0^{2\pi} \left(8r^4 - \frac{4}{3}r^6\right)\cos(\theta)\sin(\theta)\bigg|_0^2 \,d\theta$$
$$= \int_0^{2\pi} \frac{128}{3}\cos(\theta)\sin(\theta)\,d\theta$$

Step 7

Finally, we'll do the θ integration.

$$\iiint\limits_E 4xy\,dV = \int_0^{2\pi} \frac{64}{3}\sin(2\theta)\,d\theta = -\frac{32}{3}\cos(2\theta)\bigg|_0^{2\pi} = \boxed{0}$$

Note that we used the double angle formula for sine to simplify the integrand a little prior to the integration. We could also have done one of two substitutions for this step if we'd wanted to (and we'd get the same answer of course).

---

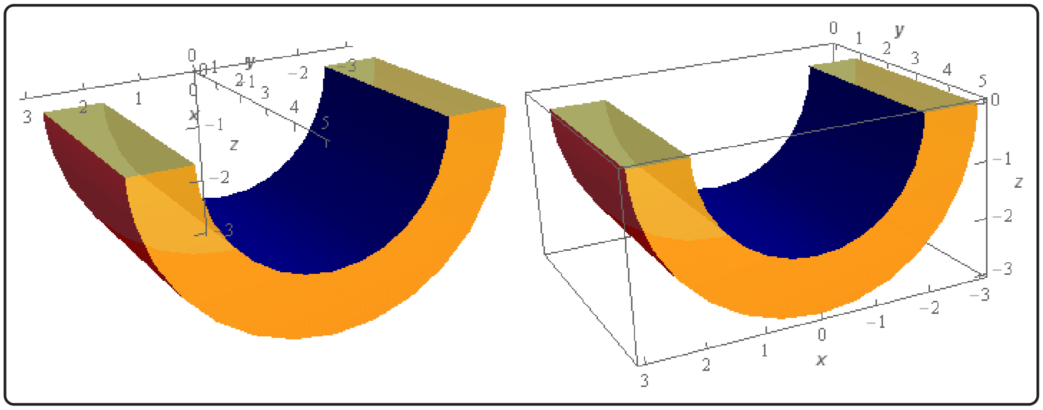
2. Evaluate $\iiint\limits_E \mathbf{e}^{-x^2-z^2}\,dV$ where E is the region between the two cylinders $x^2+z^2=4$ and $x^2+z^2=9$ with $1 \le y \le 5$ and $z \le 0$.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.

We know that $x^2+z^2=4$ and $x^2+z^2=9$ are cylinders of radius 2 and 3 respectively that are centered on the y-axis. The range $1 \le y \le 5$ tells us that we will only have the cylinders in this range of y's. Finally, the $z \le 0$ tells us that we will only have the lower half of each of the cylinders.

Here then is the sketch of E.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

The "front" of E is just the portion of the plane $y = 5$ that "caps" the front and is the orange ring in the sketch. The "back" of E is the portion of the plane $y = 1$ that caps the back of the region and is not shown in the sketch due to the orientation of axis system.

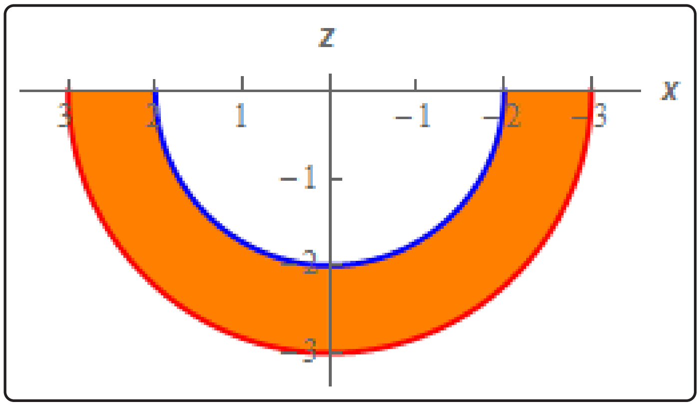
Step 2

So, from the sketch above it looks like the region D will be back in the xz-plane and so we'll need to integrate with respect to y first. In this case this is even easier because both the front and back portions of the surfaces are just the planes $y = 5$ and $y = 1$ respectively. That means that the y limits are,

$$1 \le y \le 5$$

Step 3

Now, let's think about the D for this problem. If we look at the object from along the y axis we see the lower half of the ring with radii 2 and 3 as shown below.



Note that the x-axis orientation is switched from the standard orientation to more accurately match what you'd see if you did look at E from along the y-axis. In other words, the positive x values are on the left side and the negative x values are on the right side. The orientation of the x-axis doesn't change that we still see a portion of a ring that lies in the xz-plane and this is in fact the D for this problem.

Because D is in the xz-plane and it is a portion of a ring that means that we'll need to use the following "modified" version of cylindrical coordinates.

$$x = r\cos(\theta)$$
$$y = y$$
$$z = r\sin(\theta)$$

This also matches up with the fact that we need to integrate y first (as we determined in Step 2) and the first variable of integration with cylindrical coordinates is always the "free" variable (*i.e.* not the one involving the trig functions).

So, we can easily describe the ring in terms of r and θ so here are the cylindrical coordinates for this problem.

$$\pi \le \theta \le 2\pi$$
$$2 \le r \le 3$$
$$1 \le y \le 5$$

Step 4

Plugging these limits into the integral and converting to cylindrical coordinates gives,

$$\iiint\limits_E \mathbf{e}^{-x^2-z^2}\,dV = \int_{\pi}^{2\pi}\int_{2}^{3}\int_{1}^{5} r\,\mathbf{e}^{-r^2}\,dy\,dr\,d\theta$$

Don't forget $x^2 + z^2 = r^2$ under our modified cylindrical coordinates and also don't forget that $dV = r\,dy\,dr\,d\theta$ and so we pick up another r when converting the dV to cylindrical coordinates (that will be very helpful with the r integration).

Step 5

Okay, now all we need to do is evaluate the integral. Here is the y integration.

$$\iiint\limits_E \mathbf{e}^{-x^2-z^2}\,dV = \int_{\pi}^{2\pi}\int_2^3 r\,\mathbf{e}^{-r^2} y\Big|_1^5\,dr\,d\theta$$

$$= \int_{\pi}^{2\pi}\int_2^3 4r\,\mathbf{e}^{-r^2}\,dr\,d\theta$$

Step 6

Next let's do the r integration.

$$\iiint\limits_E \mathbf{e}^{-x^2-z^2}\,dV = \int_{\pi}^{2\pi}\left(-2\,\mathbf{e}^{-r^2}\right)\Big|_2^3\,d\theta$$

$$= \int_{\pi}^{2\pi} 2\left(\mathbf{e}^{-4} - \mathbf{e}^{-9}\right)\,d\theta$$

Step 7

Finally, we'll do the θ integration.

$$\iiint\limits_E \mathbf{e}^{-x^2-z^2}\,dV = 2\left(\mathbf{e}^{-4} - \mathbf{e}^{-9}\right)\theta\Big|_{\pi}^{2\pi} = \boxed{2\pi\left(\mathbf{e}^{-4} - \mathbf{e}^{-9}\right) = 0.1143}$$

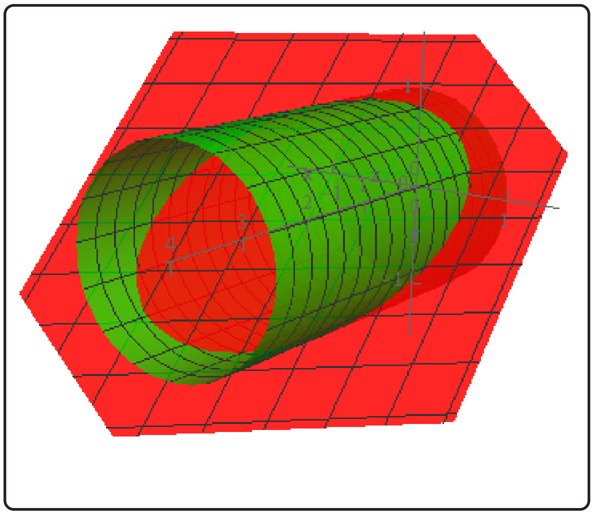
The trickiest part of this one was probably the sketch of E. Once you see that and how to get the D for the integral the rest of the problem was pretty simple for the most part.

3. Evaluate $\displaystyle\iiint\limits_E z\,dV$ where E is the region between the two planes $x+y+z=2$ and $x=0$ and inside the cylinder $y^2+z^2=1$.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.

Let's start off with a quick sketch of the cylinder and the plane $x+y+z=2$.

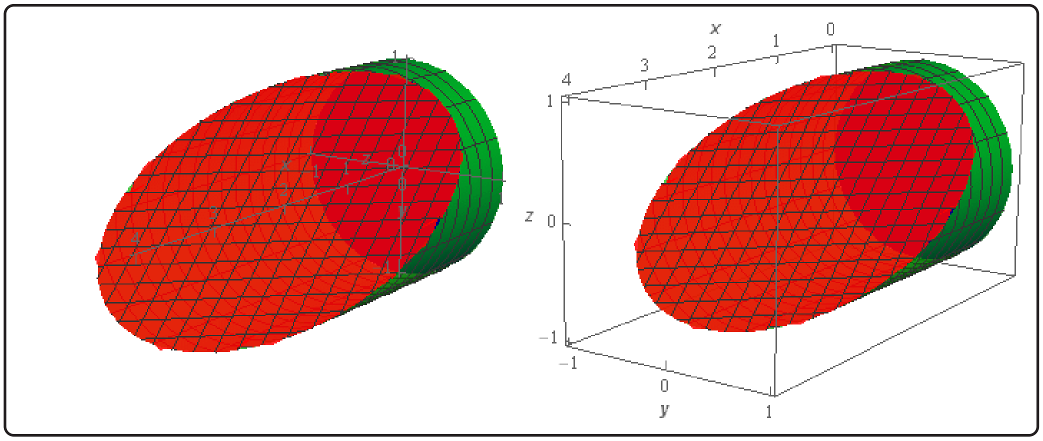


The region E we're working with here is inside the cylinder and between the two planes given in the problem statement. What this basically means is that the two planes will cap the cylinder.

In the graph above we put in more of the cylinder than needed and more of the plane $x+y+z=2$ than needed just to help illustrate the relation between the two surfaces. The plane $x=0$ is just the yz-plane and "caps" the back of the cylinder and so isn't included in the sketch.

Now, let's get rid of the portion of the cylinder that is in front of $x+y+z=2$ since it's not part of the region and let's get rid of the portion of $x+y+z=2$ that is not inside the cylinder. The resulting sketch is the region E.

Here then is the sketch of E.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

Step 2

So, from the sketch it is hopefully clear that the region D will be in the yz-plane and so we'll need to integrate with respect to x first. That means that the x limits are,

$$0 \le x \le 2 - y - z$$

For the upper x limit all we need to do is solve the equation of the plane for x.

Step 3

For this problem D is simply the disk $y^2 + z^2 \le 1$. Because D is in the yz-plane and is a disk we'll need to use the following "modified" version of cylindrical coordinates.

$$\begin{aligned} x &= x \\ y &= r\sin(\theta) \\ z &= r\cos(\theta) \end{aligned}$$

This also matches up with the fact that we need to integrate x first (as we determined in Step 2) and the first variable of integration with cylindrical coordinates is always the "free" variable (*i.e.* not the one involving the trig functions).

So, we can easily describe the disk in terms of r and θ so here are the cylindrical coordinates for this problem.

$$\begin{gathered} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ 0 \le x \le 2 - r\sin(\theta) - r\cos(\theta) \end{gathered}$$

Don't forget to convert the y and z in the x upper limit into cylindrical coordinate as well.

Step 4

Plugging these limits into the integral and converting to cylindrical coordinates gives,

$$\begin{aligned} \iiint\limits_E z\,dV &= \int_0^{2\pi}\int_0^1\int_0^{2-r\sin(\theta)-r\cos(\theta)} \left(r\cos(\theta)\right) r\,dx\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1\int_0^{2-r\sin(\theta)-r\cos(\theta)} r^2\cos(\theta)\,dx\,dr\,d\theta \end{aligned}$$

Don't forget to convert the integrand to our modified cylindrical coordinates and also don't forget that $dV = r\,dx\,dr\,d\theta$ and so we pick up another r when converting the dV to cylindrical coordinates.

Step 5

Okay, now all we need to do is evaluate the integral. Here is the x integration.

$$\begin{aligned} \iiint\limits_E z\,dV &= \int_0^{2\pi}\int_0^1 \left(r^2\cos(\theta)\,x\right)\Big|_0^{2-r\sin(\theta)-r\cos(\theta)} dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 r^2\cos(\theta)\,\left(2 - r\sin(\theta) - r\cos(\theta)\right)\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 2r^2\cos(\theta) - r^3\cos(\theta)\sin(\theta) - r^3\cos^2(\theta)\,dr\,d\theta \end{aligned}$$

Step 6

Next let's do the r integration.

$$\iiint\limits_E z\,dV = \int_0^{2\pi} \left(\frac{2}{3}r^3 \cos(\theta) - \frac{1}{4}r^4 \cos(\theta)\sin(\theta) - \frac{1}{4}r^4 \cos^2(\theta)\right)\Big|_0^1 \, d\theta$$

$$= \int_0^{2\pi} \frac{2}{3}\cos(\theta) - \frac{1}{4}\cos(\theta)\sin(\theta) - \frac{1}{4}\cos^2(\theta)\,d\theta$$

Step 7

Finally, we'll do the θ integration.

$$\iiint\limits_E z\,dV = \int_0^{2\pi} \frac{2}{3}\cos(\theta) - \frac{1}{8}\sin(2\theta) - \frac{1}{8}(1+\cos(2\theta))\,d\theta$$

$$= \left(\frac{2}{3}\sin(\theta) + \frac{1}{16}\cos(2\theta) - \frac{1}{8}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\right)\Big|_0^{2\pi} = \boxed{-\frac{\pi}{4}}$$

Don't forget to simplify the integrand before doing the final integration. In this case we used the sine double angle on the second term and the cosine half angle formula on the third term to simplify the integrand to allow us to quickly do this integration.

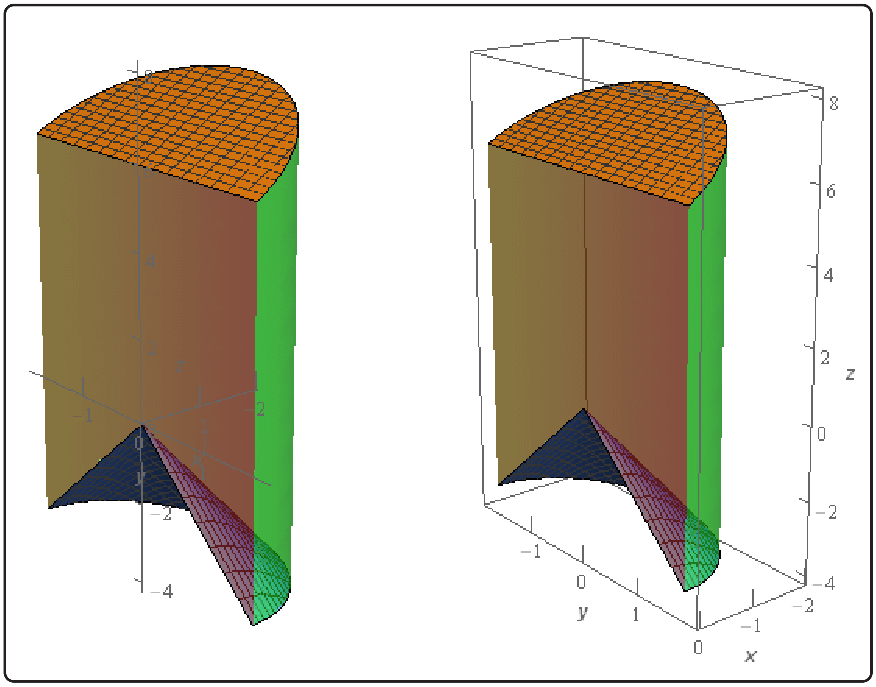
---

4. Use a triple integral to determine the volume of the region below $z = 6 - x$, above $z = -\sqrt{4x^2 + 4y^2}$ inside the cylinder $x^2 + y^2 = 3$ with $x \le 0$.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.

Here then is the sketch of E.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

The plane $z = 6 - x$ is the top "cap" on the cylinder and the cone $z = -\sqrt{4x^2 + 4y^2}$ is the bottom "cap" on the cylinder. We only have half of the cylinder because of the $x \le 0$ portion of the problem statement.

Step 2

The volume of this solid is given by,

$$V = \iiint\limits_E dV$$

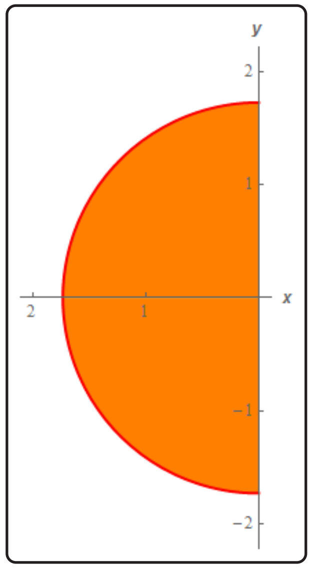
Step 3

So, from the sketch it is hopefully clear that the region D will be in the xy-plane and so we'll need to integrate with respect to z first. That means that the z limits are,

$$-\sqrt{4x^2+4y^2} \le z \le 6-x$$

Step 4

For this problem D is simply the portion of the disk $x^2+y^2 \le \sqrt{3}$ with $x \le 0$. Here is a quick sketch of D to maybe help with the limits.



Since D is clearly a portion of a disk it makes sense that we'll be using cylindrical coordinates. So, here are the cylindrical coordinates for this problem.

$$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$
$$0 \le r \le \sqrt{3}$$
$$-2r \le z \le 6 - r\cos(\theta)$$

Don't forget to convert the z limits into cylindrical coordinates.

Step 5

Plugging these limits into the integral and converting to cylindrical coordinates gives,

$$V = \iiint\limits_E dV = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{\sqrt{3}} \int_{-2r}^{6-r\cos(\theta)} r\,dz\,dr\,d\theta$$

Don't forget that $dV = r\,dx\,dr\,d\theta$ and so we pick up an r when converting the dV to cylindrical coordinates.

Step 6

Okay, now all we need to do is evaluate the integral. Here is the z integration.

$$\begin{aligned} V &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{\sqrt{3}} (rz)\Big|_{-2r}^{6-r\cos(\theta)}\,dr\,d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{\sqrt{3}} 6r - r^2\cos(\theta) + 2r^2\,dr\,d\theta \end{aligned}$$

Step 7

Next let's do the r integration.

$$\begin{aligned} V &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(3r^2 - \frac{1}{3}r^3\cos(\theta) + \frac{2}{3}r^3\right)\Big|_0^{\sqrt{3}}\,d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 9 - \sqrt{3}\cos(\theta) + 2\sqrt{3}\,d\theta \end{aligned}$$

Step 8

Finally, we'll do the θ integration.

$$V = \left(9\theta + 2\sqrt{3}\theta - \sqrt{3}\sin(\theta)\right)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \boxed{2\sqrt{3} + \left(9 + 2\sqrt{3}\right)\pi = 42.6212}$$

5. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_{0}^{\sqrt{5}} \int_{-\sqrt{5-x^2}}^{0} \int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x - 3y \;\; dz\,dy\,dx$$

Step 1

First let's just get the Cartesian limits from the integral.

$$\begin{gathered} 0 \le x \le \sqrt{5} \\ -\sqrt{5-x^2} \le y \le 0 \\ x^2+y^2-11 \le z \le 9-3x^2-3y^2 \end{gathered}$$

Step 2

Now we need to convert the integral into cylindrical coordinates. Let's first deal with the limits.

We are integrating z first in the integral set up to use Cartesian coordinates and so we'll integrate that first in the integral set up to use cylindrical coordinates as well. It is easy to convert the z limits to cylindrical coordinates as follows.

$$r^2 - 11 \le z \le 9 - 3r^2$$

Step 3

Now, the x and y limits. These are the two "outer" integrals in the original integral and so they also define D. So, let's see if we can determine what D is first. Once we have that we should be able to determine the r and θ limits for our integral in cylindrical coordinates.

The lower y limit is $y = -\sqrt{5-x^2}$ and we can see that D will be at most the lower portion of the disk of radius $\sqrt{5}$ centered at the origin.

From the x limits we see that x must be positive and so D is the portion of the disk of radius $\sqrt{5}$ that is in the 4^{th} quadrant.

We now know what D is here so the full set of limits for the integral is,

$$\begin{gathered} \frac{3\pi}{2} \le \theta \le 2\pi \\ 0 \le r \le \sqrt{5} \\ r^2 - 11 \le z \le 9 - 3r^2 \end{gathered}$$

Step 4

Okay, let's convert the integral into cylindrical coordinates.

$$\begin{aligned}\int_0^{\sqrt{5}}\int_{-\sqrt{5-x^2}}^{0}\int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x-3y\;dz\,dy\,dx &= \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\sqrt{5}}\int_{r^2-11}^{9-3r^2} \left(2r\cos(\theta)-3r\sin(\theta)\right) r\,dz\,dr\,d\theta\\ &= \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\sqrt{5}}\int_{r^2-11}^{9-3r^2} r^2\left(2\cos(\theta)-3\sin(\theta)\right)\,dz\,dr\,d\theta\end{aligned}$$

Don't forget that the $dz\,dy\,dz$ in the Cartesian form of the integral comes from the dV in the original triple integral. We also know that, in terms of cylindrical coordinates we have $dV = r\,dz\,dr\,d\theta$ and so we know that $dz\,dy\,dx = r\,dz\,dr\,d\theta$ and we'll pick up an extra r in the integrand.

Step 5

Okay, now all we need to do is evaluate the integral. Here is the z integration.

$$\begin{aligned}\int_0^{\sqrt{5}}\int_{-\sqrt{5-x^2}}^{0}\int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x-3y\;dz\,dy\,dx &= \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\sqrt{5}} \left(r^2\left(2\cos(\theta)-3\sin(\theta)\right)z\right)\Big|_{r^2-11}^{9-3r^2}\,dr\,d\theta\\ &= \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\sqrt{5}} r^2\left(2\cos(\theta)-3\sin(\theta)\right)\left(20-4r^2\right)\,dr\,d\theta\\ &= \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\sqrt{5}} \left(2\cos(\theta)-3\sin(\theta)\right)\left(20r^2-4r^4\right)\,dr\,d\theta\end{aligned}$$

Step 6

Next let's do the r integration.

$$\int_{0}^{\sqrt{5}}\int_{-\sqrt{5-x^2}}^{0}\int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x-3y\;dz\,dy\,dx$$
$$= \int_{\frac{3\pi}{2}}^{2\pi}\left(\left(2\cos(\theta)-3\sin(\theta)\right)\left(\frac{20}{3}r^3-\frac{4}{5}r^5\right)\right)\Bigg|_{0}^{\sqrt{5}}\,d\theta$$
$$= \int_{\frac{3\pi}{2}}^{2\pi}\frac{40}{3}\sqrt{5}\left(2\cos(\theta)-3\sin(\theta)\right)\,d\theta$$

Step 7

Finally, we'll do the θ integration.

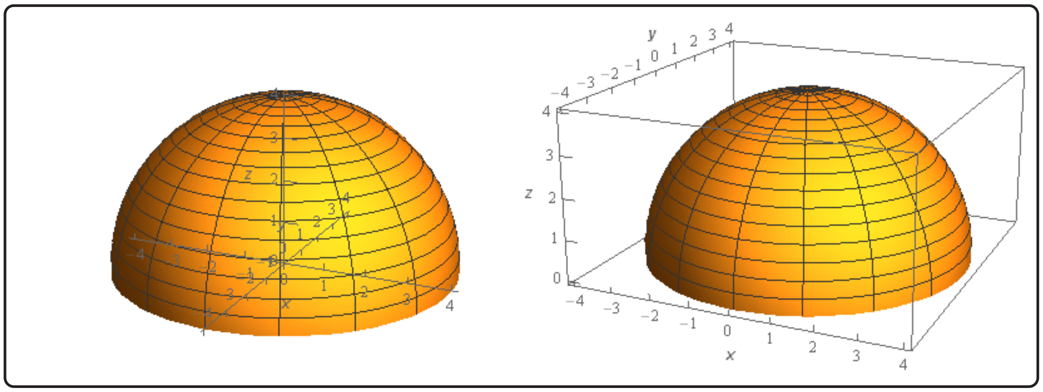
$$\int_{0}^{\sqrt{5}}\int_{-\sqrt{5-x^2}}^{0}\int_{x^2+y^2-11}^{9-3x^2-3y^2} 2x-3y\;dz\,dy\,dx = \left(\frac{40}{3}\sqrt{5}\left(2\sin(\theta)+3\cos(\theta)\right)\right)\Bigg|_{\frac{3\pi}{2}}^{2\pi}$$
$$= \boxed{\frac{200}{3}\sqrt{5}}$$

15.7 Triple Integrals in Spherical Coordinates

1. Evaluate $\displaystyle\iiint\limits_E 10xz + 3\,dV$ where E is the region portion of $x^2 + y^2 + z^2 = 16$ with $z \geq 0$.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

In this case we're dealing with the upper half of a sphere of radius 4.

Step 2

Now, since we are integrating over a portion of a sphere it makes sense to use spherical coordinate for the integral and the limits are,

$$0 \leq \varphi \leq \frac{\pi}{2}$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \rho \leq 4$$

Remember that φ is the angle from the positive z-axis that we rotate through as we cover the region and θ is the angle we rotate around the z-axis as we cover the region.

In this case we have the full upper half of the sphere and so θ will range from 0 to 2π while φ will range from 0 to $\frac{\pi}{2}$.

Step 3

Plugging these limits into the integral and converting to spherical coordinates gives,

$$\iiint\limits_E 10xz + 3\,dV = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^4 \left[10\left(\rho\sin(\varphi)\cos(\theta)\right)\left(\rho\cos(\varphi)\right)+3\right]\left(\rho^2\sin(\varphi)\right)d\rho d\theta d\varphi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^4 10\rho^4\sin^2(\varphi)\cos(\varphi)\cos(\theta)+3\rho^2\sin(\varphi)\,d\rho\,d\theta\,d\varphi$$

Don't forget to convert the x and z into spherical coordinates and also don't forget that $dV = \rho^2\sin(\varphi)\,d\rho\,d\theta\,d\varphi$ and so we'll pick up a couple of extra terms when converting the dV to spherical coordinates.

Step 4

Okay, now all we need to do is evaluate the integral. Here is the ρ integration.

$$\iiint\limits_E 10xz + 3\,dV = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left.\left(2\rho^5\sin^2(\varphi)\cos(\varphi)\cos(\theta)+\rho^3\sin(\varphi)\right)\right|_0^4 d\theta\,d\varphi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} 2048\sin^2(\varphi)\cos(\varphi)\cos(\theta)+64\sin(\varphi)\;d\theta\,d\varphi$$

Step 5

Next let's do the θ integration.

$$\iiint\limits_E 10xz + 3\,dV = \int_0^{\frac{\pi}{2}}\left.\left(2048\sin^2(\varphi)\cos(\varphi)\sin(\theta)+64\theta\sin(\varphi)\right)\right|_0^{2\pi} d\varphi$$

$$= \int_0^{\frac{\pi}{2}} 128\pi\sin(\varphi)\,d\varphi$$

Step 6

Finally, we'll do the φ integration.

$$\iiint\limits_E 10xz + 3\,dV = \left(-128\pi\cos(\varphi)\right)\Big|_0^{\frac{\pi}{2}} = \boxed{128\pi}$$

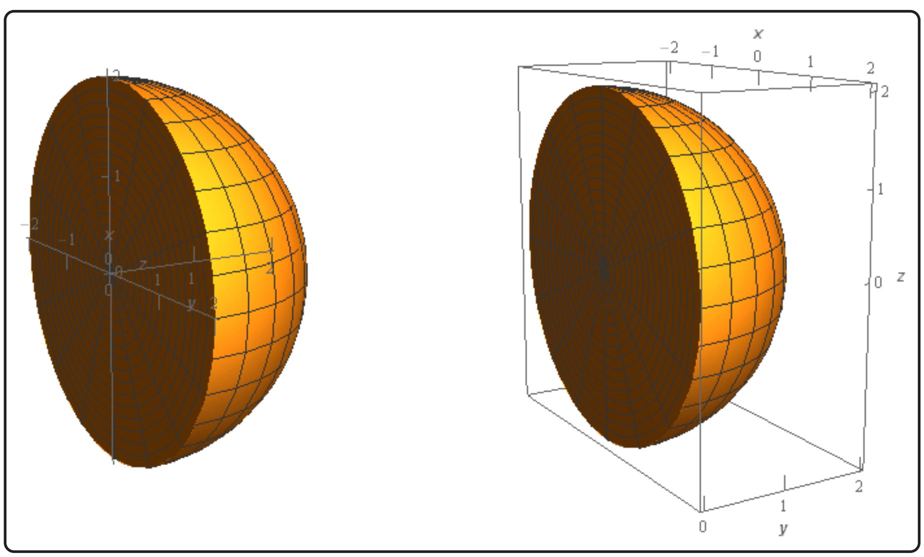
Note that, in this case, because the limits of each of the integrals were all constants we could have done the integration in any order we wanted to. In this case, it might have been "simpler" to do the φ first or second as that would have greatly reduced the integrand for the remaining integral(s).

---

2. Evaluate $\displaystyle\iiint\limits_E x^2 + y^2\,dV$ where E is the region portion of $x^2 + y^2 + z^2 = 4$ with $y \geq 0$.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

In this case we're dealing with the portion of the sphere of radius 2 with $y \geq 0$.

Step 2

Now, since we are integrating over a portion of a sphere it makes sense to use spherical coordinate for the integral and the limits are,

$$\begin{gathered} 0 \le \varphi \le \pi \\ 0 \le \theta \le \pi \\ 0 \le \rho \le 2 \end{gathered}$$

Remember that φ is the angle from the positive z-axis that we rotate through as we cover the region and θ is the angle we rotate around the z-axis as we cover the region.

In this case we have only the portion of the sphere with $y \ge 0$ and so θ will range from 0 to π (remember that we measure θ from the positive x-axis). Because we want the full half of the sphere with $y \ge 0$ we know that φ will range from 0 to π.

Step 3

Plugging these limits into the integral and converting to spherical coordinates gives,

$$\begin{aligned} \iiint\limits_E x^2 + y^2 \, dV &= \int_0^\pi \int_0^\pi \int_0^2 \left[(\rho \sin(\varphi)\cos(\theta))^2 + (\rho \sin(\varphi)\sin(\theta))^2 \right] \left(\rho^2 \sin(\varphi)\right) d\rho d\theta d\varphi \\ &= \int_0^\pi \int_0^\pi \int_0^2 \left[\rho^2 \sin^2(\varphi)\cos^2(\theta) + \rho^2 \sin^2(\varphi)\sin^2(\theta) \right] \left(\rho^2 \sin(\varphi)\right) d\rho d\theta d\varphi \\ &= \int_0^\pi \int_0^\pi \int_0^2 \left[\rho^2 \sin^2(\varphi) \left(\cos^2(\theta) + \sin^2(\theta)\right) \right] \left(\rho^2 \sin(\varphi)\right) d\rho d\theta d\varphi \\ &= \int_0^\pi \int_0^\pi \int_0^2 \rho^4 \sin^3(\varphi) \, d\rho \, d\theta \, d\varphi \end{aligned}$$

Don't forget to convert the x and y into spherical coordinates and also don't forget that $dV = \rho^2 \sin(\varphi) \, d\rho \, d\theta \, d\varphi$ and so we'll pick up a couple of extra terms when converting the dV to spherical coordinates.

In this case we also did a fair amount of simplification that will definitely make the integration easier to deal with. Don't forget to do this kind of simplification when possible!

Step 4

Okay, now all we need to do is evaluate the integral. Here is the ρ integration.

$$\begin{aligned}\iiint\limits_E x^2+y^2\,dV &= \int_0^{\pi}\int_0^{\pi}\left(\frac{1}{5}\rho^5\sin^3(\varphi)\right)\Big|_0^2\,d\theta\,d\varphi\\ &= \int_0^{\pi}\int_0^{\pi}\frac{32}{5}\sin^3(\varphi)\,d\theta\,d\varphi\end{aligned}$$

Step 5

Next let's do the θ integration.

$$\begin{aligned}\iiint\limits_E x^2+y^2\,dV &= \int_0^{\pi}\left(\frac{32}{5}\theta\sin^3(\varphi)\right)\Big|_0^{\pi}\,d\varphi\\ &= \int_0^{\pi}\frac{32}{5}\pi\sin^3(\varphi)\,d\varphi\end{aligned}$$

Step 6

Finally, we'll do the φ integration.

$$\begin{aligned}\iiint\limits_E x^2+y^2\,dV &= \int_0^{\pi}\frac{32}{5}\pi\sin^2(\varphi)\sin(\varphi)\,d\varphi\\ &= \int_0^{\pi}\frac{32}{5}\pi\left(1-\cos^2(\varphi)\right)\sin(\varphi)\,d\varphi\\ &= \left(-\frac{32}{5}\pi\left(\cos(\varphi)-\frac{1}{3}\cos^3(\varphi)\right)\right)\Big|_0^{\pi} = \boxed{\frac{128}{15}\pi}\end{aligned}$$

You do recall how to do the kinds of trig integrals we did in this step don't you? If not you should head back and review some of the Calculus II material as these will be showing up on occasion.

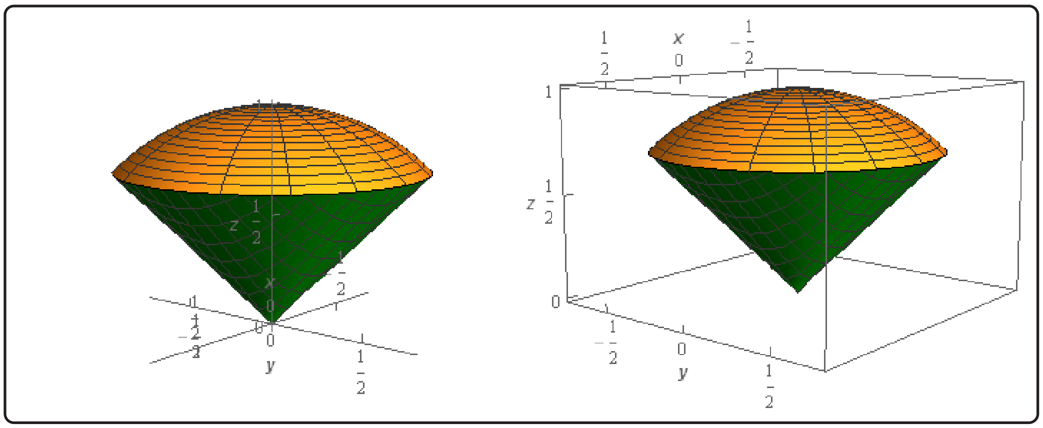
Note that, in this case, because the limits of each of the integrals were all constants we could have done the integration in any order we wanted to.

---

3. Evaluate $\displaystyle\iiint\limits_E 3z\,dV$ where E is the region inside both $x^2 + y^2 + z^2 = 1$ and $z = \sqrt{x^2 + y^2}$.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

In this case we're dealing with the region below the sphere of radius 1 (the orange surface in the sketches) that is inside (*i.e.* above) the cone (the green surface in the sketches).

Step 2

Now, since we are integrating over a portion of a sphere it makes sense to use spherical coordinate for the integral.

The limits for ρ and θ should be pretty clear as those just correspond to the radius of the sphere and to how much of the sphere we get by rotating about the z-axis.

The limits for φ however aren't explicitly given in any way but we can get them from the equation of the cone. First, we know that, in terms of cylindrical coordinates, $\sqrt{x^2 + y^2} = r$ and we know that, in terms of spherical coordinates, $r = \rho\sin(\varphi)$. Therefore, if we convert the equation of the cone into spherical coordinates we get,

$$\rho\cos(\varphi) = \rho\sin(\varphi) \qquad \to \qquad \tan(\varphi) = 1 \qquad \to \qquad \varphi = \frac{\pi}{4}$$

So, the equation of the cone is given by $\varphi = \frac{\pi}{4}$ in terms of spherical coordinates. Because the region we are working on is above the cone we know that φ must therefore range from 0 to $\frac{\pi}{4}$.

The limits are then,

$$0 \le \varphi \le \frac{\pi}{4}$$
$$0 \le \theta \le 2\pi$$
$$0 \le \rho \le 1$$

Step 3

Plugging these limits into the integral and converting to spherical coordinates gives,

$$\begin{aligned}\iiint\limits_E 3z\, dV &= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^1 \left(3\rho\cos(\varphi)\right)\left(\rho^2 \sin(\varphi)\right)\, d\rho\, d\theta\, d\varphi \\ &= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^1 3\rho^3 \cos(\varphi)\sin(\varphi)\, d\rho\, d\theta\, d\varphi\end{aligned}$$

Don't forget to convert the z into spherical coordinates and also don't forget that $dV = \rho^2 \sin(\varphi)\, d\rho\, d\theta\, d\varphi$ and so we'll pick up a couple of extra terms when converting the dV to spherical coordinates.

Step 4

Okay, now all we need to do is evaluate the integral. Here is the ρ integration.

$$\begin{aligned}\iiint\limits_E 3z\, dV &= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \left(\frac{3}{4}\rho^4 \cos(\varphi)\sin(\varphi)\right)\Big|_0^1\, d\theta\, d\varphi \\ &= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \frac{3}{4}\cos(\varphi)\sin(\varphi)\, d\theta\, d\varphi\end{aligned}$$

Step 5

Next let's do the θ integration.

$$\begin{aligned}\iiint\limits_E 3z\,dV &= \int_0^{\frac{\pi}{4}} \left(\frac{3}{4}\theta\cos(\varphi)\sin(\varphi)\right)\Bigg|_0^{2\pi} d\varphi \\ &= \int_0^{\frac{\pi}{4}} \frac{3}{2}\pi\cos(\varphi)\sin(\varphi)\,d\varphi\end{aligned}$$

Step 6

Finally, we'll do the φ integration.

$$\begin{aligned}\iiint\limits_E 3z\,dV &= \int_0^{\frac{\pi}{4}} \frac{3}{4}\pi\sin(2\varphi)\,d\varphi \\ &= \left(-\frac{3}{8}\pi\cos(2\varphi)\right)\Bigg|_0^{\frac{\pi}{4}} = \boxed{\frac{3}{8}\pi}\end{aligned}$$

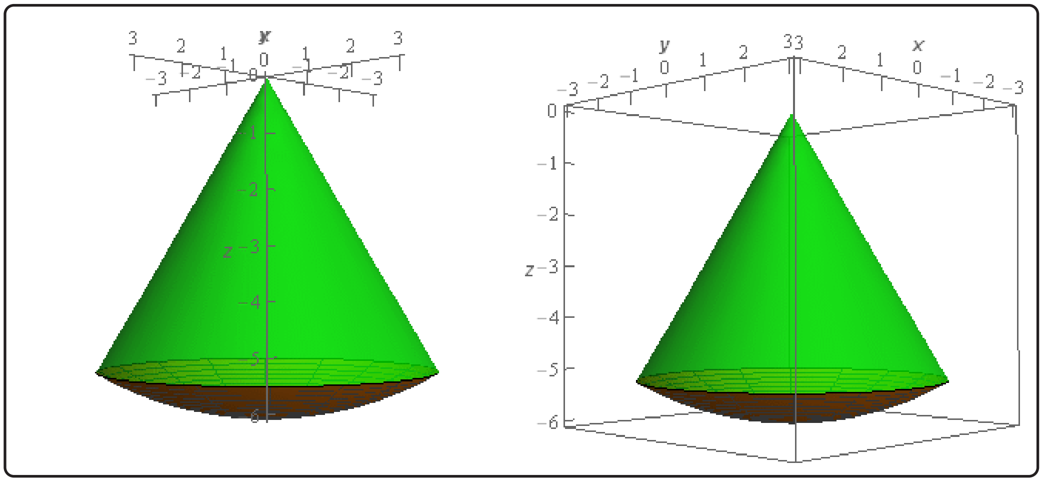
We used the double angle formula for sine to reduce the integral to something that we could quickly do.

Not that, in this case, because the limits of each of the integrals were all constants we could have done the integration in any order we wanted to.

4. Evaluate $\displaystyle\iiint\limits_E x^2\,dV$ where E is the region inside both $x^2+y^2+z^2=36$ and $z=-\sqrt{3x^2+3y^2}$.

Step 1

Okay, let's start off with a quick sketch of the region E so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface and region.

In this case we're dealing with the region above the sphere of radius 6 (the dark orange surface at the bottom in the sketches) that is inside (*i.e.* below) the cone (the green surface in the sketches).

Step 2

Now, since we are integrating over a portion of a sphere it makes sense to use spherical coordinate for the integral.

The limits for ρ and θ should be pretty clear as those just correspond to the radius of the sphere and to how much of the sphere we get by rotating about the z-axis.

The limits for φ however aren't explicitly given in any way but we can get them from the equation of the cone. First, we know that, in terms of cylindrical coordinates, $\sqrt{x^2+y^2}=r$ and we know that, in terms of spherical coordinates, $r=\rho\sin(\varphi)$. Therefore, if we convert the equation of the cone into spherical coordinates we get,

$$\rho\cos(\varphi)=-\sqrt{3}\rho\sin(\varphi) \qquad \to \qquad \tan(\varphi)=-\frac{1}{\sqrt{3}} \qquad \to \qquad \varphi=\frac{5\pi}{6}$$

So, the equation of the cone is given by $\varphi=\frac{5\pi}{6}$ in terms of spherical coordinates. Because the region we are working on is below the cone we know that φ must therefore range from $\frac{5\pi}{6}$ to π.

The limits are then,

$$\frac{5\pi}{6} \le \varphi \le \pi$$
$$0 \le \theta \le 2\pi$$
$$0 \le \rho \le 6$$

Step 3

Plugging these limits into the integral and converting to spherical coordinates gives,

$$\begin{aligned}\iiint\limits_E x^2\,dV &= \int_{\frac{5\pi}{6}}^{\pi}\int_0^{2\pi}\int_0^6 \left(\rho\sin(\varphi)\cos(\theta)\right)^2\left(\rho^2\sin(\varphi)\right)\,d\rho\,d\theta\,d\varphi \\ &= \int_{\frac{5\pi}{6}}^{\pi}\int_0^{2\pi}\int_0^6 \rho^4\sin^3(\varphi)\cos^2(\theta)\,d\rho\,d\theta\,d\varphi\end{aligned}$$

Don't forget to convert the z into spherical coordinates and also don't forget that $dV = \rho^2\sin(\varphi)\,d\rho\,d\theta\,d\varphi$ and so we'll pick up a couple of extra terms when converting the dV to spherical coordinates.

Step 4

Okay, now all we need to do is evaluate the integral. Here is the ρ integration.

$$\begin{aligned}\iiint\limits_E x^2\,dV &= \int_{\frac{5\pi}{6}}^{\pi}\int_0^{2\pi} \left.\left(\frac{1}{5}\rho^5\sin^3(\varphi)\cos^2(\theta)\right)\right|_0^6 d\theta\,d\varphi \\ &= \int_{\frac{5\pi}{6}}^{\pi}\int_0^{2\pi} \frac{7776}{5}\sin^3(\varphi)\cos^2(\theta)\,d\theta\,d\varphi\end{aligned}$$

Step 5

Next let's do the θ integration.

$$\begin{aligned}\iiint\limits_E x^2\,dV &= \int_{\frac{5\pi}{6}}^{\pi}\int_0^{2\pi} \frac{7776}{5}\sin^3(\varphi)\left(\frac{1}{2}\right)(1+\cos(2\theta))\,d\theta\,d\varphi \\ &= \int_{\frac{5\pi}{6}}^{\pi} \left(\frac{3888}{5}\sin^3(\varphi)\left(\theta+\frac{1}{2}\sin(2\theta)\right)\right)\Bigg|_0^{2\pi} d\varphi \\ &= \int_{\frac{5\pi}{6}}^{\pi} \frac{7776}{5}\pi\sin^3(\varphi)\,d\varphi\end{aligned}$$

Step 6

Finally, we'll do the φ integration.

$$\begin{aligned}\iiint\limits_E x^2\,dV &= \int_{\frac{5\pi}{6}}^{\pi} \frac{7776}{5}\pi\sin^2(\varphi)\sin(\varphi)\,d\varphi \\ &= \int_{\frac{5\pi}{6}}^{\pi} \frac{7776}{5}\pi\left(1-\cos^2(\varphi)\right)\sin(\varphi)\,d\varphi \\ &= \left(-\frac{7776}{5}\pi\left(\cos(\varphi)-\frac{1}{3}\cos^3(\varphi)\right)\right)\Bigg|_{\frac{5\pi}{6}}^{\pi} \\ &= \boxed{\frac{7776}{5}\pi\left(\frac{2}{3}-\frac{3\sqrt{3}}{8}\right) = 83.7799}\end{aligned}$$

You do recall how to do the kinds of trig integrals we did in the last two steps don't you? If not you should head back and review some of the Calculus II material as these will be showing up on occasion.

Not that, in this case, because the limits of each of the integrals were all constants we could have done the integration in any order we wanted to.

5. Evaluate the following integral by first converting to an integral in spherical coordinates.

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y\ \,dz\,dy\,dx$$

Step 1

First let's just get the Cartesian limits from the integral.

$$\begin{gathered} -1 \le x \le 0 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \\ \sqrt{6x^2+6y^2} \le z \le \sqrt{7-x^2-y^2} \end{gathered}$$

Step 2

Now we need to convert the integral into spherical coordinates. Let's first take care of the limits first.

From the upper z limit we see that we are under $z = \sqrt{7-x^2-y^2}$ (which is just the equation for the upper portion of a sphere of radius $\sqrt{7}$).

From the lower z limit we see that we are above $z = \sqrt{6x^2+6y^2}$ (which is just the equation of a cone).

So, we appear to be inside an "ice cream cone" shaped region as we usually are when dealing with spherical coordinates.

This leads us to the following ρ limits.

$$0 \le \rho \le \sqrt{7}$$

Step 3

The limits for φ we can get them from the equation of the cone that is the lower z limit referenced in the previous step. First, we know that, in terms of cylindrical coordinates, $\sqrt{x^2+y^2} = r$ and we know that, in terms of spherical coordinates, $r = \rho\sin(\varphi)$. Therefore, if we convert the equation of the cone into spherical coordinates we get,

$$\rho\cos(\varphi) = \sqrt{6}\rho\sin(\varphi) \qquad \to \qquad \tan(\varphi) = \frac{1}{\sqrt{6}} \qquad \to \qquad \varphi = \tan^{-1}\left(\frac{1}{\sqrt{6}}\right) = 0.3876$$

Because the region we are working on is above the cone we know that φ must therefore range from 0 to 0.3876.

Step 4

Finally, let's get the θ limits. For reference purposes here are the x and y limits we found in Step 1.

$$-1 \le x \le 0$$
$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$

All the y limits tell us is that the region D from the original Cartesian coordinates integral is a portion of the circle of radius 1. Note that this should make sense as this is also the intersection of the sphere and cone we get from the z limits (we'll leave it to you to verify this statement).

Now, from the x limits we see that we must have the left side of the circle of radius 1 and so the limits for θ are then,

$$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

The full set of spherical coordinate limits for the integral are then,

$$0 \le \varphi \le \tan^{-1}\left(\frac{1}{\sqrt{6}}\right) = 0.3876$$
$$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$
$$0 \le \rho \le \sqrt{7}$$

Step 5

Okay, let's convert the integral in to spherical coordinates.

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y \,\,dz\,dy\,dx$$
$$= \int_{0}^{0.3876}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{0}^{\sqrt{7}} \left(18\rho\sin(\varphi)\sin(\theta)\right)\left(\rho^2\sin(\varphi)\right)\,d\rho\,d\theta\,d\varphi$$
$$= \int_{0}^{0.3876}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{0}^{\sqrt{7}} 18\rho^3\sin^2(\varphi)\sin(\theta)\,d\rho\,d\theta\,d\varphi$$

Don't forget to convert the y into spherical coordinates. Also, don't forget that the $dz\,dy\,dx$ come from the dV in the original triple integral. We also know that, in terms of spherical coordinates, $dV = \rho^2\sin(\varphi)\,d\rho\,d\theta\,d\varphi$ and so we in turn know that,

$$dz\,dy\,dx = \rho^2\sin(\varphi)\,d\rho\,d\theta\,d\varphi$$

Step 6

Okay, now all we need to do is evaluate the integral. Here is the ρ integration.

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y\ \,dz\,dy\,dx = \int_{0}^{0.3876}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{9}{2}\rho^4\sin^2(\varphi)\sin(\theta)\right)\Big|_{0}^{\sqrt{7}}\,d\theta\,d\varphi$$

$$= \int_{0}^{0.3876}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{441}{2}\sin^2(\varphi)\sin(\theta)\,d\theta\,d\varphi$$

Step 7

Next let's do the θ integration.

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y\ \,dz\,dy\,dx = \int_{0}^{0.3876}\left(-\frac{441}{2}\sin^2(\varphi)\cos(\theta)\right)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\,d\varphi$$

$$= \int_{0}^{0.3876} 0\,d\varphi$$

$$= \boxed{0}$$

So, as noted above once we got the integrand down to zero there was no reason to continue integrating as the answer will continue to be zero for the rest of the problem.

Don't get excited about it when these kinds of things happen. They will on occasion and all it means is that we get to stop integrating a little sooner that we would have otherwise.

15.8 Change of Variables

1. Compute the Jacobian of the following transformation.

$$x = 4u - 3v^2 \qquad y = u^2 - 6v$$

Solution

There really isn't much to do here other than compute the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 4 & -6v \\ 2u & -6 \end{vmatrix} = -24 - (-12uv) = \boxed{-24 + 12uv}$$

---

2. Compute the Jacobian of the following transformation.

$$x = u^2v^3 \qquad y = 4 - 2\sqrt{u}$$

Solution

There really isn't much to do here other than compute the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2uv^3 & 3u^2v^2 \\ -u^{-\frac{1}{2}} & 0 \end{vmatrix} = 0 - \left(-3u^{\frac{3}{2}}v^2\right) = \boxed{3u^{\frac{3}{2}}v^2}$$

---

3. Compute the Jacobian of the following transformation.

$$x = \frac{v}{u} \qquad y = u^2 - 4v^2$$

Solution

There really isn't much to do here other than compute the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ 2u & -8v \end{vmatrix} = \frac{8v^2}{u^2} - (2) = \boxed{\frac{8v^2}{u^2} - 2}$$

---

4. If R is the region inside $\frac{x^2}{4} + \frac{y^2}{36} = 1$ determine the region we would get applying the transformation $x = 2u$, $y = 6v$ to R.

Solution

There really isn't a lot to this problem.

It should be pretty clear that the outer boundary of R is an ellipse. That isn't really important to this problem but this problem will lead to seeing how to set up a nice transformation for elliptical regions.

To determine the transformation of this region all we need to do is plug the transformation boundary equation for R. Doing this gives,

$$\frac{(2u)^2}{4} + \frac{(6v)^2}{36} = 1 \qquad \rightarrow \qquad \frac{4u^2}{4} + \frac{36v^2}{36} = 1 \qquad \rightarrow \qquad u^2 + v^2 = 1$$

So, the boundary equation for R transforms into the equation for the unit circle and so, under this transformation, we can transform an ellipse into a circle (a unit circle in fact...).

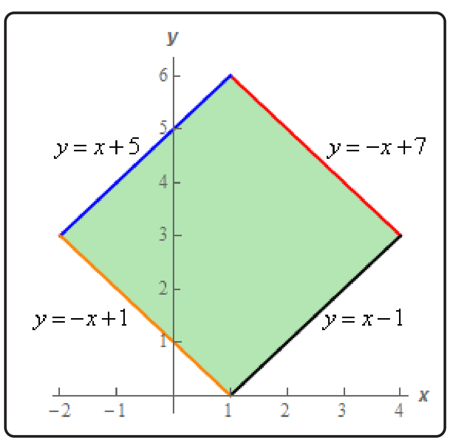
You can see how to determine a transformation that will transform an elliptical region into a circular region can't you? Integrating over an elliptical region would probably be pretty unpleasant but integrating over a unit disk will probably be much nicer so this is a nice transformation to understand how to get!

---

5. If R is the parallelogram with vertices $(1, 0)$, $(4, 3)$, $(1, 6)$ and $(-2, 3)$ determine the region we would get applying the transformation $x = \frac{1}{2}(v - u)$, $y = \frac{1}{2}(v + u)$ to R.

Step 1

Let's start off with a sketch of R.



The equations of each of the boundaries of the region are given in the sketch. Given that we know the coordinates of each of the vertices its simple algebra to determine the equation of a line given two points on the line so we'll leave it to you to verify the equations.

Step 2

Okay, now all we need to do is apply the transformation to each of the boundary equations. To do this we simply plug the transformation into each of the boundary equations.

Let's start with $y = x + 5$. Applying the transformation to this equation gives,

$$\begin{aligned} \frac{1}{2}(v+u) &= \frac{1}{2}(v-u) + 5 \\ v+u &= v-u+10 \\ 2u &= 10 \qquad \Rightarrow \qquad \underline{u=5} \end{aligned}$$

So, this boundary will transform into the equation $u = 5$.

Step 3

Let's now apply the transformation to $y = -x + 7$.

$$\begin{aligned} \frac{1}{2}(v+u) &= -\frac{1}{2}(v-u) + 7 \\ v+u &= -v+u+14 \\ 2v &= 14 \qquad \Rightarrow \qquad \underline{v=7} \end{aligned}$$

This boundary transforms in the equation $v = 7$.

Step 4

Next, let's apply the transformation to $y = x - 1$.

$$\begin{aligned} \frac{1}{2}(v+u) &= \frac{1}{2}(v-u) - 1 \\ v+u &= v-u-2 \\ 2u &= -2 \qquad \Rightarrow \qquad \underline{u=-1} \end{aligned}$$

So, this boundary will transform into the equation $u = -1$.

Step 5

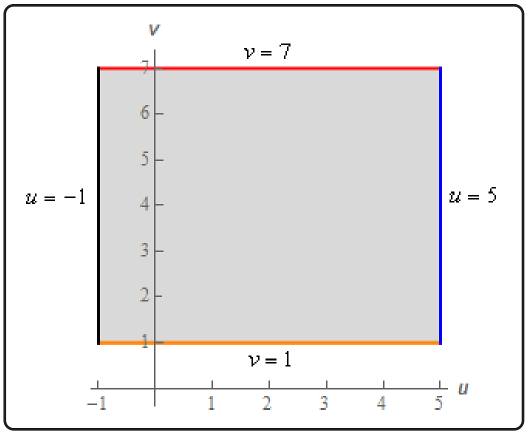
Finally let's apply the transformation to $y = -x + 1$.

$$\begin{aligned} \frac{1}{2}(v+u) &= -\frac{1}{2}(v-u)+1 \\ v+u &= -v+u+2 \\ 2v &= 2 \qquad \Rightarrow \qquad \underline{v=1} \end{aligned}$$

This boundary transforms in the equation $v = 1$.

Step 6

Sketching the transformed equations gives the following region.



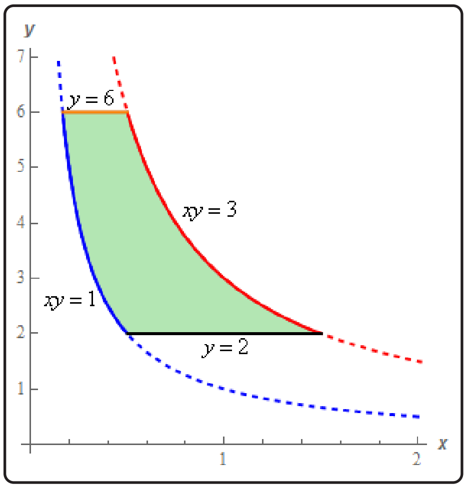
So, we transform the diamond shaped region into a rectangle under the transformation.

Note that we chose u to be the horizontal axis and v to be vertical axis in the transformed region for this problem. There is no real reason for doing that other than it is just what we've always done. Regardless of your choices here make sure to label the axes to make it clear!

6. If R is the region bounded by $xy = 1$, $xy = 3$, $y = 2$ and $y = 6$ determine the region we would get applying the transformation $x = \dfrac{v}{6u}$, $y = 2u$ to R.

Step 1

Let's start off with a sketch of R.



The equations of each of the boundaries of the region are given in the sketch. We also included "extensions" of the two curves (the dotted portions) just to show a fuller sketch of the two curves.

Step 2

Okay, now all we need to do is apply the transformation to each of the boundary equations. To do this we simply plug the transformation into each of the boundary equations.

Let's start with $y = 6$. Applying the transformation to this equation gives,

$$2u = 6 \qquad \Rightarrow \qquad \underline{u = 3}$$

So, this boundary will transform into the equation $u = 3$.

Step 3

Let's now apply the transformation to $xy = 3$.

$$\left(\frac{v}{6u}\right)(2u) = 3$$
$$\frac{v}{3} = 3 \qquad \Rightarrow \qquad \underline{v = 9}$$

This boundary transforms in the equation $v = 9$.

Step 4

Next, let's apply the transformation to $y = 2$.

$$2u = 2 \qquad \Rightarrow \qquad \underline{u = 1}$$

So, this boundary will transform into the equation $u = 1$.

Step 5

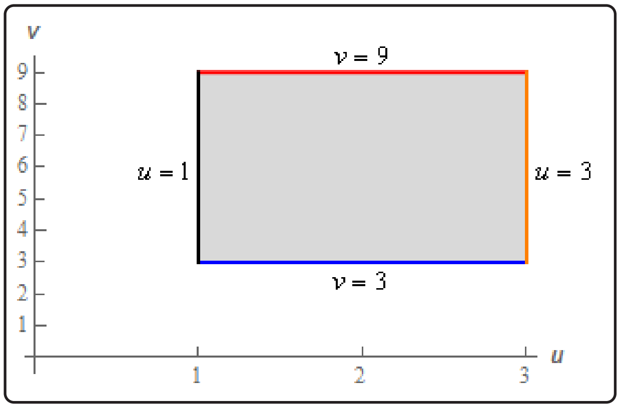
Finally let's apply the transformation to $xy = 1$.

$$\left(\frac{v}{6u}\right)(2u) = 1$$
$$\frac{v}{3} = 1 \qquad \Rightarrow \qquad \underline{v = 3}$$

This boundary transforms in the equation $v = 3$.

Step 6

Sketching the transformed equations gives the following region.



So, we transform the odd shaped region into a rectangle under the transformation.

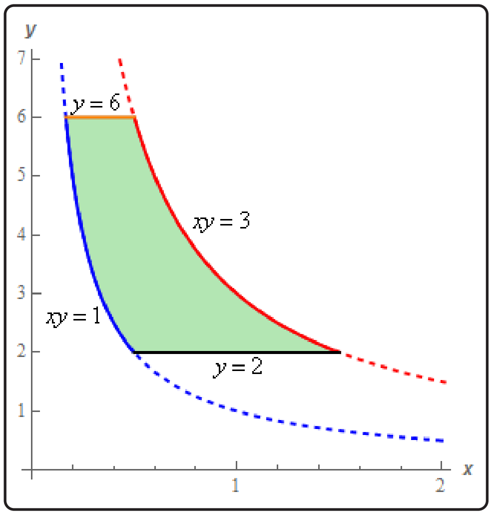
Note that we chose u to be the horizontal axis and v to be vertical axis in the transformed region for this problem. There is no real reason for doing that other than it is just what we've always done. Regardless of your choices here make sure to label the axes to make it clear!

7. Evaluate $\displaystyle\iint\limits_R xy^3\,dA$ where R is the region bounded by $xy = 1$, $xy = 3$, $y = 2$ and $y = 6$ using the transformation $x = \dfrac{v}{6u}$, $y = 2u$.

Step 1

The first thing we need to do is determine the transformation of R. We actually determined the transformation of R in the previous example. However, let's go through the process again (with a few details omitted) just to have it here in this problem.

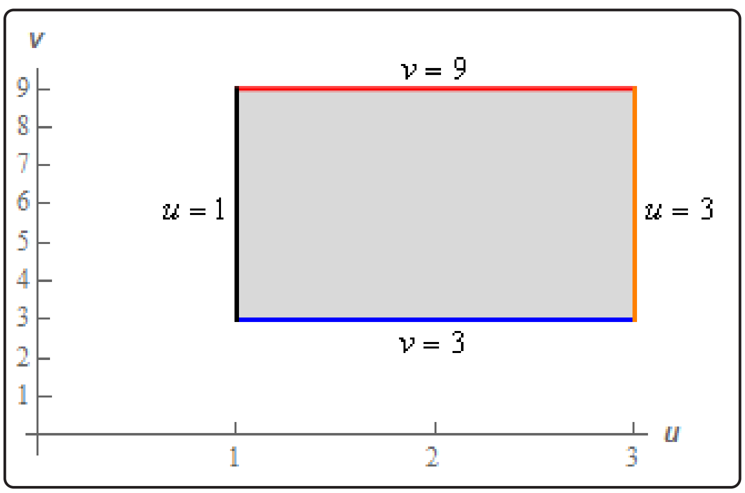
First, a sketch of R.



Now, let's transform each of the boundary curves.

$$y = 6: \qquad 2u = 6 \qquad \to \qquad u = 3$$
$$xy = 3: \qquad \left(\frac{v}{6u}\right)(2u) = 3 \qquad \to \qquad v = 9$$
$$y = 2: \qquad 2u = 2 \qquad \to \qquad u = 1$$
$$xy = 1: \qquad \left(\frac{v}{6u}\right)(2u) = 1 \qquad \to \qquad v = 3$$

Here is a sketch of the transformed region.



So, the limits for the transformed region are,

$$1 \le u \le 3$$
$$3 \le v \le 9$$

Step 2

We'll need the Jacobian of this transformation next.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{v}{6u^2} & \dfrac{1}{6u} \\ 2 & 0 \end{vmatrix} = 0 - \frac{2}{6u} = -\frac{1}{3u}$$

Step 3

We can now write the integral in terms of the "new" uv coordinates system.

$$\begin{aligned}\iint\limits_R xy^3\,dA &= \int_1^3\int_3^9 \left(\frac{v}{6u}\right)(2u)^3\left|-\frac{1}{3u}\right|\,dv\,du \\ &= \int_1^3\int_3^9 \frac{4}{9}v\,u\,dv\,du\end{aligned}$$

Don't forget to add in the Jacobian and don't forget that we need absolute value bars on it. In this case we know that the range of u we're working on (given in Step 1) is positive we know that the quantity in the absolute value bars is negative and so we can drop the absolute value bars by also dropping the minus sign.

Also, the simplified integrand didn't suggest any one order of integration over the other and so we just chose one to work with. The other order would be just as easy to have worked with.

Step 4

Finally, let's evaluate the integral.

$$\begin{aligned}\iint\limits_R xy^3\,dA &= \int_1^3\int_3^9 \frac{4}{9}v\,u\,dv\,du\\ &= \int_1^3 \left(\frac{2}{9}v^2u\right)\Big|_3^9\,du\\ &= \int_1^3 16u\,du\\ &= \left(8u^2\right)\Big|_1^3\\ &= \boxed{64}\end{aligned}$$

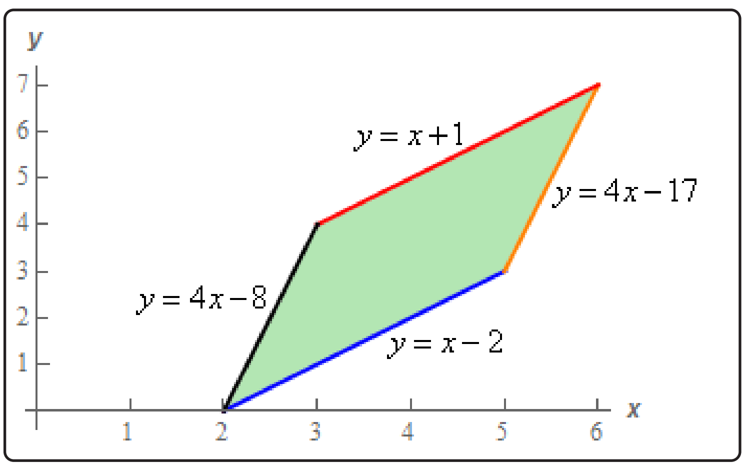
---

8. Evaluate $\displaystyle\iint\limits_R 6x - 3y\,dA$ where R is the parallelogram with vertices $(2,0)$, $(5,3)$, $(6,7)$ and $(3,4)$ using the transformation $x = \frac{1}{3}(v-u)$, $y = \frac{1}{3}(4v-u)$ to R.

Step 1

The first thing we need to do is determine the transformation of R.

First, a sketch of R.



The equations of each of the boundaries of the region are given in the sketch. Given that we know the coordinates of each of the vertices its simple algebra to determine the equation of a line given two points on the line so we'll leave it to you to verify the equations.

Note that integrating over this region would require three integrals regardless of the order. You can see how each order of integration would require three integrals correct?

Now, let's transform each of the boundary curves.

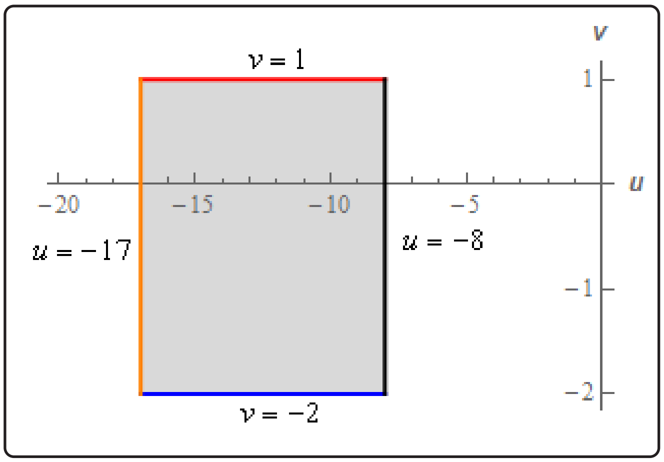
$$y = x + 1: \qquad \frac{1}{3}(4v - u) = \frac{1}{3}(v - u) + 1 \quad \to \quad 3v = 3 \quad \to \quad v = 1$$

$$y = 4x - 17: \qquad \frac{1}{3}(4v - u) = \frac{4}{3}(v - u) - 17 \quad \to \quad 3u = -51 \quad \to \quad u = -17$$

$$y = x - 2: \qquad \frac{1}{3}(4v - u) = \frac{1}{3}(v - u) - 2 \quad \to \quad 3v = -6 \quad \to \quad v = -2$$

$$y = 4x - 8: \qquad \frac{1}{3}(4v - u) = \frac{4}{3}(v - u) - 8 \quad \to \quad 3u = -24 \quad \to \quad u = -8$$

Here is a sketch of the transformed region and note that the transformed region will be much easier to integrate over than the original region.



So, the limits for the transformed region are,

$$\begin{gathered} -17 \le u \le -8 \\ -2 \le v \le 1 \end{gathered}$$

Note as well that this is going to be a much nice region to integrate over than the original region.

Step 2

We'll need the Jacobian of this transformation next.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{1}{3} & \dfrac{1}{3} \\ -\dfrac{1}{3} & \dfrac{4}{3} \end{vmatrix} = -\frac{4}{9} - \left(-\frac{1}{9}\right) = -\frac{1}{3}$$

Step 3

We can now write the integral in terms of the "new" uv coordinates system.

$$\begin{aligned} \iint\limits_R 6x - 3y\,dA &= \int_{-17}^{-8}\int_{-2}^{1}\left[6\left(\frac{1}{3}\right)(v-u) - 3\left(\frac{1}{3}\right)(4v-u)\right]\left|-\frac{1}{3}\right|\,dv\,du \\ &= \int_{-17}^{-8}\int_{-2}^{1} -\frac{1}{3}(2v+u)\,dv\,du \end{aligned}$$

Don't forget to add in the Jacobian and don't forget that we need absolute value bars on it. In this case we can just drop the absolute value bars by also dropping the minus sign since we just have a number in the absolute value.

Also, the simplified integrand didn't suggest any one order of integration over the other and so we just chose one to work with. The other order would be just as easy to have worked with.

Step 4

Finally, let's evaluate the integral.

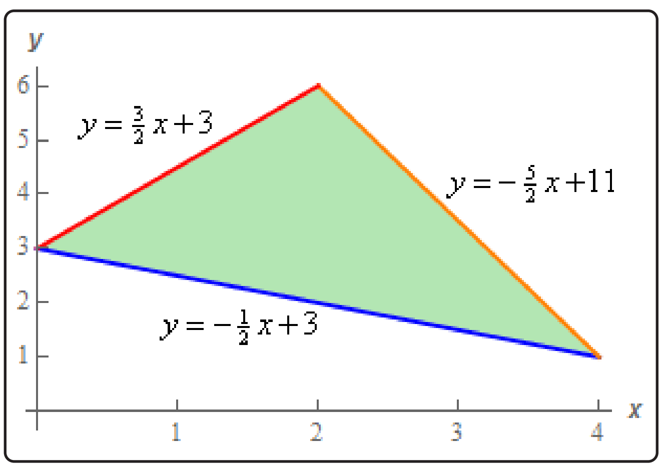
$$\begin{aligned}\iint\limits_R 6x - 3y\,dA &= \int_{-17}^{-8}\int_{-2}^{1} -\frac{1}{3}\left(2v+u\right)\,dv\,du \\ &= \int_{-17}^{-8}\left(-\frac{1}{3}\left(v^2+uv\right)\right)\Big|_{-2}^{1}\,du \\ &= \int_{-17}^{-8} 1-u\,du \\ &= \left(u-\frac{1}{2}u^2\right)\Big|_{-17}^{-8} \\ &= \boxed{\frac{243}{2}}\end{aligned}$$

9. Evaluate $\iint\limits_R x+2y\,dA$ where R is the triangle with vertices $(0,3)$, $(4,1)$ and $(2,6)$ using the transformation $x=\frac{1}{2}(u-v)$, $y=\frac{1}{4}(3u+v+12)$ to R.

Step 1

The first thing we need to do is determine the transformation of R.

First, a sketch of R.



The equations of each of the boundaries of the region are given in the sketch. Given that we know the coordinates of each of the vertices its simple algebra to determine the equation of a line given two points on the line so we'll leave it to you to verify the equations.

Note that integrating over this region would require two integrals regardless of the order. You can see how each order of integration would require two integrals correct?

Now, let's transform each of the boundary curves.

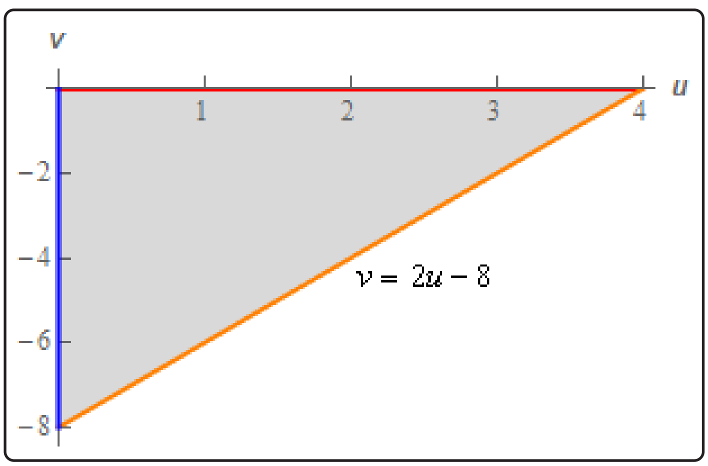
$$y = \frac{3}{2}x + 3: \qquad \frac{1}{4}(3u + v + 12) = \frac{3}{2}\left(\frac{1}{2}\right)(u - v) + 3 \qquad \to \qquad v = 0$$

$$y = -\frac{5}{2}x + 11: \qquad \frac{1}{4}(3u + v + 12) = -\frac{5}{2}\left(\frac{1}{2}\right)(u - v) + 11 \qquad \to \qquad v = 2u - 8$$

$$y = -\frac{1}{2}x + 3: \qquad \frac{1}{4}(3u + v + 12) = -\frac{1}{2}\left(\frac{1}{2}\right)(u - v) + 3 \qquad \to \qquad u = 0$$

Note that, in this case, the first and last boundary equation we looked at above just ended up being transformed into the u-axis and v-axis respectively. That will happen on occasion and might well end up making our life a little easier when it comes to evaluating the integral.

Here is a sketch of the transformed region and note that the transformed region will be much easier to integrate over than the original region.



So, the limits for the transformed region are,

$$0 \le u \le 4$$
$$2u - 8 \le v \le 0$$

Note as well that this is going to be a much nice region to integrate over than the original region.

Step 2

We'll need the Jacobian of this transformation next.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{8} - \left(-\frac{3}{8}\right) = \frac{1}{2}$$

Step 3

We can now write the integral in terms of the "new" uv coordinates system.

$$\begin{aligned}\iint\limits_R x + 2y\,dA &= \int_0^4 \int_{2u-8}^0 \left[\frac{1}{2}(u-v) + 2\left(\frac{1}{4}\right)(3u+v+12)\right]\left|\frac{1}{2}\right| dv\,du \\ &= \int_0^4 \int_{2u-8}^0 u + 3\,dv\,du\end{aligned}$$

Don't forget to add in the Jacobian and don't forget that the absolute value bars on it. In this case we can just drop the absolute value bars since we just have a positive number in the absolute value.

Also, the simplified integrand didn't suggest any one order of integration over the other and so we just chose one to work with. The other order would be just as easy to have worked with.

Step 4

Finally, let's evaluate the integral.

$$\begin{aligned}\iint\limits_R x + 2y\,dA &= \int_0^4 \int_{2u-8}^0 u + 3\,dv\,du\\ &= \int_0^4 (u+3)\,v\Big|_{2u-8}^{0}\,du\\ &= \int_0^4 (u+3)(-2u+8)\,du\\ &= \int_0^4 24 + 2u - 2u^2\,du\\ &= \left(24u + u^2 - \frac{2}{3}u^3\right)\Big|_0^4\\ &= \boxed{\frac{208}{3}}\end{aligned}$$

---

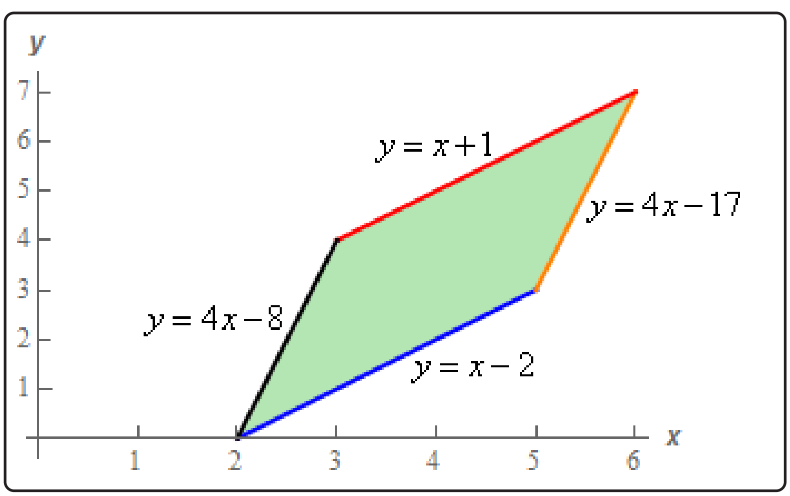
10. Derive the transformation used in problem 8.

Step 1

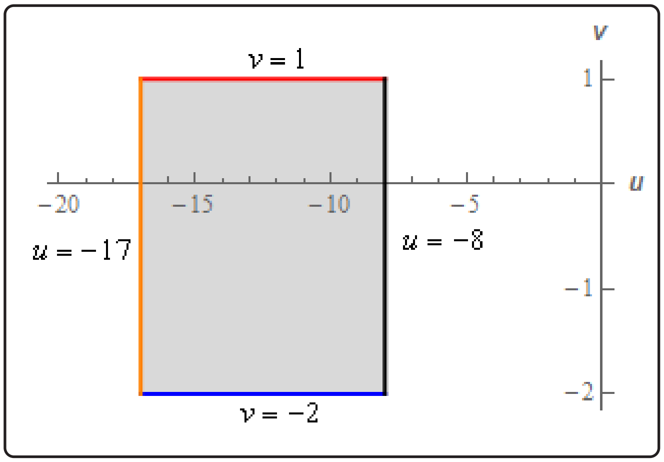
Okay, for reference purposes we want to derive the following transformation,

$$x = \frac{1}{3}(v-u) \qquad\qquad y = \frac{1}{3}(4v-u)$$

We used this transformation on the following region.



The result of the applying the transformation to the region above got us the following "new" region.



We need to derive the transformation. This seems to be a nearly impossible task at first glance. However, it isn't as difficult as it might appear to be. Or maybe we should say it won't be as difficult as it seems to be once you see how to do it.

Before we start with the derivation process we should point out that there are actually quite a few possible transformations that we could use on the original region and get a rectangle for the new region. They won't however all yield the same rectangle. We are going to get a specific transformation but you should be able how to modify what we're doing here to get a different transformation.

Step 2

The first thing that we're going to do is rewrite all the boundary equations for the original region as follows.

$$\begin{aligned} y &= x + 1 & &\to & y - x &= 1 \\ y &= 4x - 17 & &\to & y - 4x &= -17 \\ y &= x - 2 & &\to & y - x &= -2 \\ y &= 4x - 8 & &\to & y - 4x &= -8 \end{aligned}$$

Now, let's notice that because our original region had two sets of parallel sides our new equations can also be organized into two pairs of equations as follows,

$$\begin{aligned} y - x &= 1 & &\text{AND} & y - x &= -2 \\ y - 4x &= -17 & &\text{AND} & y - 4x &= -8 \end{aligned}$$

The left side of each pair of equations is identical (*i.e.* they came from parallel sides of our original region). This is going to make life easier for us in order to derive a derivation that will give a “nice” region. Of course, in this case, we already know what the new region is going to be but in general we wouldn’t.

Step 3

Now, we want to define new variables u and v in terms of the old variables x and y in such a way that we will get the rectangle we are looking for.

Let’s define u and v as follows (we’ll explain why in a bit),

$$u = y - 4x \qquad\qquad v = y - x$$

With this definition note that we can already see what our “new” region will be. Putting these definitions into the equations we wrote down in Step 2 gives us,

$$\begin{array}{lcl} v = 1 & \text{AND} & v = -2 \\ u = -17 & \text{AND} & u = -8 \end{array}$$

This is exactly the new region that we got in Problem 8. This tells us that we’ll get the transformation in Problem 8 from this definition (we’ll do that work in the next step).

It should also make it a little clearer why we chose the definitions of u and v as we did above. We know that we wanted an easier region to integration over and let’s be honest a rectangle is really easy to integrate over so trying to get a rectangle for the new region seemed like a good idea.

This is where the two pairs of equations comes into play. Because we were able to pair up the equations so the left side was identical in each pair all we had to do was define one of the left sides as u and the other as v and we would know at that point that we’d get a rectangle as our new region.

Note however, that it should also now be clear that there would be another “simple” definition for u and v that we could have used (*i.e.* $u = y - x$, $v = y - 4x$). It would have resulted in a similar, yet different, rectangle and a different transformation. There was no real reason for choosing our definition of u and v as we did over the other possibility other than we knew the region we wanted to get. Back in Problem 8 we basically just chose one at random.

Now, there are probably quite a few questions in your mind at this point and we’ll address as many of them as we can in the last step. Let’s finish the actual problem out first before we do that however just so we don’t forget what we’re doing here.

Step 4

Okay, let's finish the actual problem before we answer some questions in the final step. We need to use our definition of u and v from the last step to get the transformation we used in Problem 8.

First, let's just start off with our definition of u and v again.

$$u = y - 4x$$
$$v = y - x$$

Now, what we have here is a system of two equations. All we need to do to finish this problem out is solve them for x and y.

To do this let's first solve the v definition for y.

$$v = y - x \qquad \to \qquad y = v + x$$

Plug this into the u definition and solve for x.

$$u = y - 4x \qquad \to \qquad u = v + x - 4x = v - 3x \qquad \to \qquad x = \frac{1}{3}(v - u)$$

That should look familiar of course. Now, all we need to do is find the y transformation. To do this we'll just plug our x transformation into $y = v + x$ and do a little simplification.

$$y = v + x = v + \frac{1}{3}(v - u) = \frac{4}{3}v - \frac{1}{3}u = \frac{1}{3}(4v - u)$$

So, there we go. The transformation has been derived and now that you see how to do it we can see that it wasn't really all that bad despite first appearances.

Step 5

Okay, now let's see if we can address at least some of the questions you might have about deriving transformations.

Can we always transform any region into a rectangle?

The short answer is definitely NO. In order to quickly get a transformation that yielded a rectangle here we relied on the fact that we had an original region that had four linear sides and we were able to form them into two pairs of parallel sides. Once that was done it was easy to get a transformation that would yield a rectangle.

Many regions can't be transformed (or at least not easily transformed) into a rectangle.

So, what if we'd had a region with four linear sides but we couldn't form them into two sets of parallel sides?

In this case the problem would still work pretty much the same way but we'd have lots more options for definition of u and v and the result wouldn't be a rectangle.

In a case like this, because all the sides are linear, we can still write each of them in as follows,

$$\begin{aligned} \text{Side 1 : } & a_1 y + b_1 x = c_1 \\ \text{Side 2 : } & a_2 y + b_2 x = c_2 \\ \text{Side 3 : } & a_3 y + b_3 x = c_3 \\ \text{Side 4 : } & a_4 y + b_4 x = c_4 \end{aligned}$$

At this point we could define u to be any of the left sides and v to be any other remaining left sides. This would guarantee one side will be vertical and one side will be horizontal in the new region. The remaining two sides of the original region would then be transformed into new lines and hopefully the resulting region would be easy to work with.

As you can see there would be lots of possible definitions of u and v here which in turn would lead to lots of different transformations.

What if you had more or less than four linear sides?

The process described in this problem should still work to derive a transformation. Whether or not the resulting region is easy to integrate over may be a completely different questions however and there won't necessarily be any way to know until you've derived the transformation and done the work to get the new region.

What if you had at least one side that was not linear?

With these types of regions then the process described in this problem may still work but it also might not. It will depend on just how "messy" the equation of the non-linear boundary is and if we can even solve the definitions of u and v for x and y as we did in Step 4 of this problem.

Unfortunately, there really isn't any good answer to this question. There are many ways of getting transformations. We've simply described one of the easier methods of deriving a transformation for the type of regions that you're most likely to run into in this class. The method will work on occasion for regions that have one or more non-linear boundary but there is no reason to expect it to work with every possible region.

Are all transformations derived in this fashion?

As alluded to in the answer to the previous question, absolutely not. There are many types of regions where this process simply won't work, or at least won't work easily.

A good example is the ellipse we looked at in Problem 4. If you've done that problem then you'll probably already know that you can pretty much get a transformation that will transform the ellipse into a circle by inspection. The process described in this problem simply won't work on that region.

The point of this problem was not to teach you now to derive transformations in general. It was simply to show you how many of the transformations in these problems were derived and to show you one method that, when it can be used, is fairly simple to do.

Are there any variations to this derivation method?

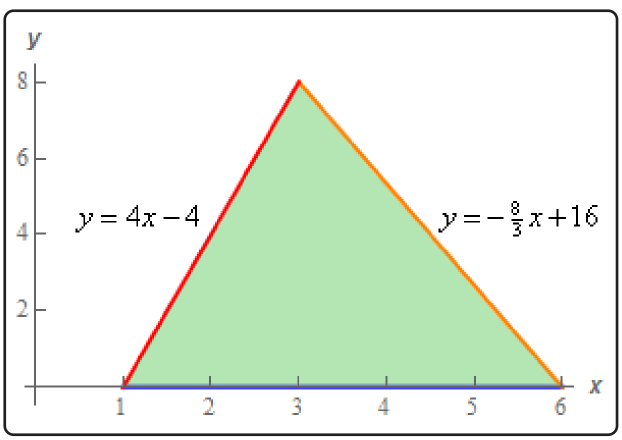
Absolutely. See the next problem for at least one.

11. Derive a transformation that will convert the triangle with vertices $(1, 0)$, $(6, 0)$ and $(3, 8)$ into a right triangle with the right angle occurring at the origin of the uv system.

Step 1

We're going to be using slight modification of the process used in the previous problem here. If you haven't read through that problem you should before proceeding with this problem as we're not going to be going over the explanation in quite as much detail here.

So, let's just start off with a sketch of the triangle.



We'll leave it to you to verify the algebra details of deriving the equations of the two sloped sides. The bottom side is of course just defined by $y = 0$.

Step 2

This region is clearly a little different from the region in the previous problem but the next step is still the same. We start with writing the equations of each of the sides as follows.

$$\begin{aligned} y - 4x &= -4 \\ y + \frac{8}{3}x &= 16 \\ y &= 0 \end{aligned}$$

Again, what we simply put all the variables on one side and the constant on the other. We also aren't going to worry about the fact that the third equation doesn't have an x. Not only will that not be a problem it actually is going to make this problem a little easier.

Step 3

Now, if we followed the process from the last problem we'd define u to be one of the equations and v to be the other. Let's do that real quick and see what we get.

Which equation we use to define u and v technically doesn't matter. However, keep in mind that we're eventually going to have to solve the resulting set of equations for x and y so let's pick definitions with that in mind.

Let's use the following definitions of u and v to get our transformation.

$$u = y - 4x \qquad\qquad v = y$$

The second definition is really just renaming y as v. This isn't a problem and in fact will make the rest of the problem a little easier. Note as well that all were really saying here is that the bottom side of the triangle won't change under this transformation.

Okay, let's think a little bit about what we have here. This will transform $y - 4x = -4$ into $u = -4$ and $y = 0$ into $v = 0$. These two side will form the right angle of the resulting triangle (which will occur where they intersect) and $y + \frac{8}{3}x = 16$ will transform into the hypotenuse of this new triangle. If you aren't sure you believe that then you should actually go through the work to verify it.

So, assuming you either believe us or have done the work to verify that what we've said is true we now have an issue that needs to be addressed.

The problem statement said that the right angle should occur at the origin. Under this transformation however the right angle will occur at $(-4, 0)$ which means this isn't the transformation we were asked to find unfortunately.

Note that we're not saying that this isn't a valid transformation. It absolutely is a valid, and not a particularly bad, transformation. It simply isn't the transformation we were asked to find and so we can't use it for this problem.

Fixing it however isn't hard and if you really followed how we set this up you might see how to quickly "fix" things up.

Step 4

So, we've decided that the definition of u and v we got in the last step isn't the one we want for this problem so we need to fix things up a little bit.

The problem with our definition in the previous step was really that our definition of u transformed $y - 4x = -4$ into $u = -4$ and in order for the right angle to be at the origin we really need to have a side transform into $u = 0$. Note our definition of v is already giving us a side with equation $v = 0$ so we don't need to worry about that one.

So, to fix this let's write the first equation as follows,

$$y - 4x + 4 = 0$$

Seems like a simple enough thing to do here. We simply added 4 to both sides to turn the right side into a zero. With this rewrite we can now see that in order to get a side to transform into the equation $u = 0$ all we need to do is define u as follows,

$$u = y - 4x + 4$$

This will transform the first equation into $u = 0$ as we need and if we keep our definition of v as $v = y$ the third equation will still transform into $v = 0$ and so the right angle of the new triangle will now be at the origin and the hypotenuse of the new triangle will come from the applying the transformation to the third equation.

Step 5

We now have our definitions of u and v, given below for reference purposes, so all we need to do is solve the system of equation for x and y.

$$\begin{aligned} u &= y - 4x + 4 \\ v &= y \end{aligned}$$

In this case, solving for x and y is really simple. We already know that $y = v$ and so from the first equation we get,

$$u = v - 4x + 4 \qquad \rightarrow \qquad x = \frac{1}{4}(v - u + 4)$$

So, it looks like our transformation will be,

$$\begin{aligned} x &= \frac{1}{4}(v - u + 4) \\ y &= v \end{aligned}$$

Step 6

Now, technically we've done what the problem asked us to do. However, it probably wouldn't be a bad thing to verify all the claims we made above by actually applying this transformation to the three equations and making sure we do get a right triangle with the right angle at the origin as we claimed.

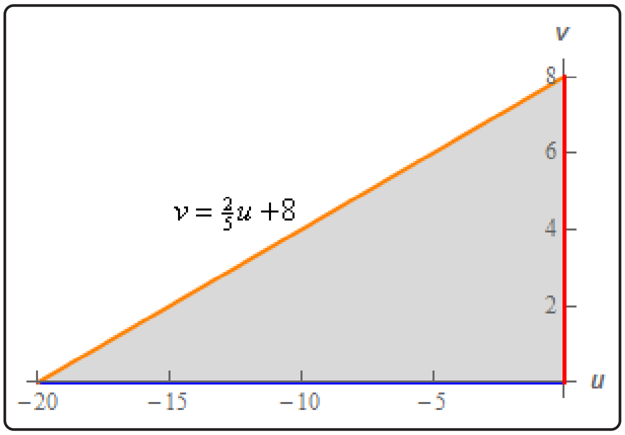
Theoretically we wouldn't need to apply the transformation to the first and third equation since we used those to define u and v that in turn gave us our transformation. However, it wouldn't be a bad idea to apply the transformation to them to verify that we get what we expect to get. If we don't then that would mean we probably made a mistake in the previous step.

So, here is the transformation applied to the three equations.

$$y = 4x - 4: \qquad v = 4\left(\frac{1}{4}\right)(v - u + 4) - 4 \qquad \rightarrow \qquad u = 0$$

$$y = -\frac{8}{3}x + 16: \qquad v = -\frac{8}{3}\left(\frac{1}{4}\right)(v - u + 4) + 16 \qquad \rightarrow \qquad v = \frac{2}{5}u + 8$$

$y = 0: \qquad v = 0$ Okay, so the first and third equations transformed as expected. Adding in the third equation and we get the following region.



So, the resulting region is what we expected it to be. A right triangle with the right angle occurring at the origin.

Note however, that we did get something (maybe) unexpected here. The original triangle was completely in the 1^{st} quadrant and the "new" triangle is completely in the 2^{nd} quadrant. This kind of thing can happen with these types of problems so don't worry about it when it does.

We got a triangle in the 2^{nd} quadrant there because all the u's ended up being negative. This suggests that maybe if we'd defined u to be $u = -(y - 4x + 4) = -y + 4x - 4$ we might get a triangle in the 1^{st} quadrant instead.

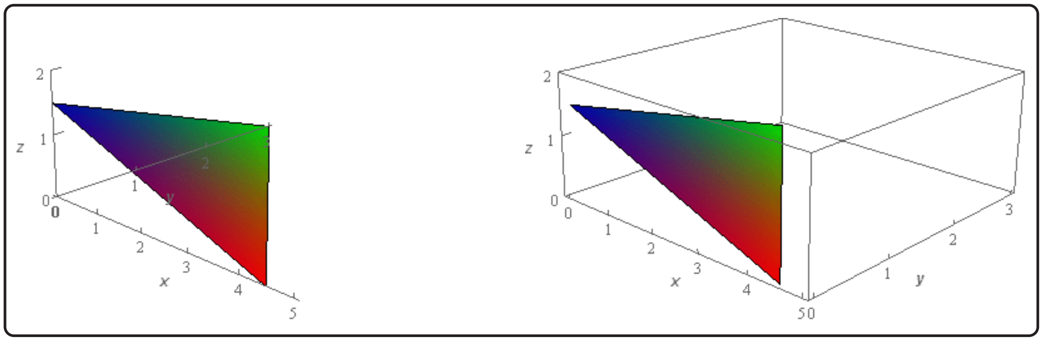
This isn't guaranteed to work in general, but it does work this time. You might want to go through and verify this if you want the practice.

15.9 Surface Area

1. Determine the surface area of the portion of $2x + 3y + 6z = 9$ that is in the 1^{st} octant.

Step 1

Okay, let's start off with a quick sketch of the surface so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface.

Step 2

Let's first set up the integral for the surface area of this surface. This surface doesn't force a region D in any of the coordinates planes so we can work with any of them that we want to.

So, let's work with D in the xy-plane. This, in turn, means we'll first need to solve the equation of the plane for z to get,

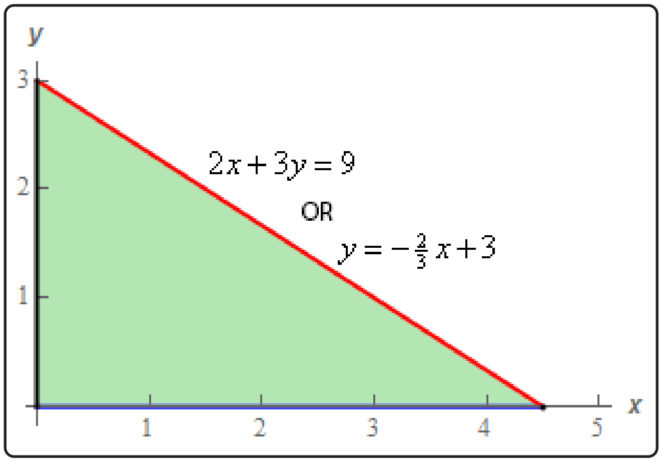
$$z = \frac{3}{2} - \frac{1}{3}x - \frac{1}{2}y$$

Now, the integral for the surface area is,

$$S = \iint\limits_D \sqrt{\left[-\frac{1}{3}\right]^2 + \left[-\frac{1}{2}\right]^2 + 1}\,dA = \iint\limits_D \sqrt{\frac{49}{36}}\,dA = \iint\limits_D \frac{7}{6}\,dA$$

Step 3

Now, we'll need to figure out the region D we'll need to use in our integral. In this case D is simply the triangle in the xy-plane that is directly below the plane. Here is a quick sketch of D.



The equation of the hypotenuse can be simply found by plugging $z = 0$ into the equation for the plane since it is simply nothing more than where the plane intersects with the xy-plane.

Here are a set of limits for this D.

$$0 \le x \le \frac{9}{2}$$
$$0 \le y \le -\frac{2}{3}x + 3$$

Step 4

Now, normally we'd proceed to evaluate the integral at this point and we could do that if we wanted to. However, we don't need to do that in this case.

Let's do a quick rewrite the surface area integral as follows.

$$S = \frac{7}{6}\iint\limits_{D} dA$$

At this point we can see that the integrand of the integral is only 1.

Step 5

Because the integrand is only 1 we can use the fact that the value of this integral is nothing more than the area of D and since D is just a right triangle we can quickly compute the area of D.

The surface area is then,

$$S = \frac{7}{6}\iint\limits_D dA = \frac{7}{6}(\text{Area of } D) = \frac{7}{6}\left(\frac{1}{2}\right)\left(\frac{9}{2}\right)(3) = \boxed{\frac{63}{8}}$$

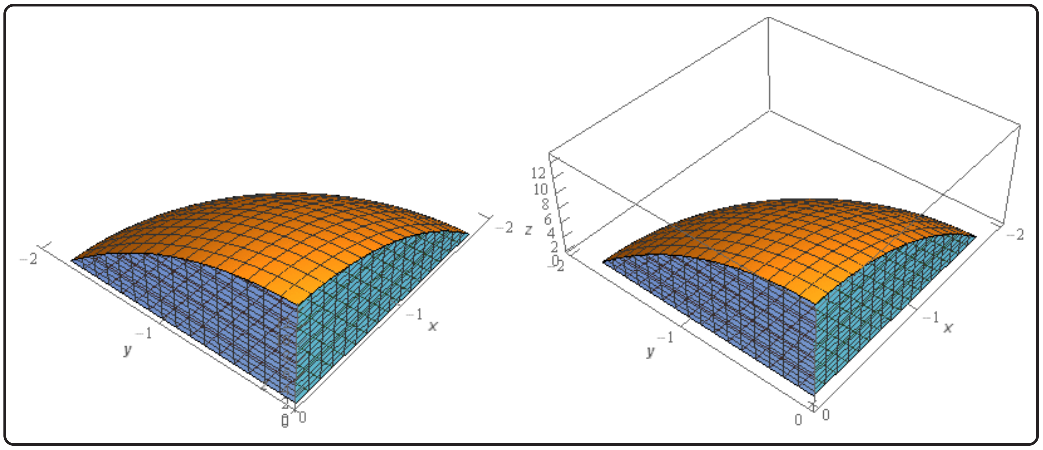
Don't forget this nice little fact about the value of $\iint\limits_D dA$. It doesn't come up often but when it does and D is an easy to compute area it can greatly reduce the amount of work for the integral evaluation.

---

2. Determine the surface area of the portion of $z = 13 - 4x^2 - 4y^2$ that is above $z = 1$ with $x \le 0$ and $y \le 0$.

Step 1

Okay, let's start off with a quick sketch of the surface so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface.

The surface we are after is the portion of the elliptic paraboloid (the orange surface in the sketch) that is above $z = 1$ and in the 3^{rd} quadrant of the xy-plane. The bluish "walls" are simply there to provide a frame of reference to help visualize the surface and are not actually part of the surface we are interested in.

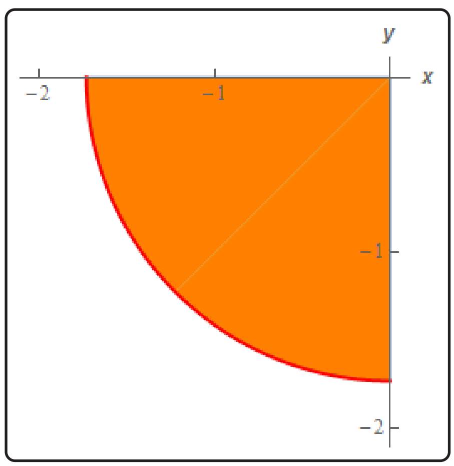
Step 2

The integral for the surface area is,

$$S = \iint\limits_D \sqrt{[-8x]^2 + [-8y]^2 + 1}\, dA = \iint\limits_D \sqrt{64x^2 + 64y^2 + 1}\, dA$$

Step 3

Now, as implied in the last step D must be in the xy-plane and it should be (hopefully) pretty obvious that it will be a circular region. If you look at the surface from directly above you will see the following region.



To determine equation of the circle all we need to do is set the equation of the elliptic paraboloid equal to $z = 1$ to get,

$$1 = 13 - 4x^2 - 4y^2 \quad \to \quad 4x^2 + 4y^2 = 12 \quad \to \quad x^2 + y^2 = 3$$

So, it is the portion of the circle of radius $\sqrt{3}$ in the 3^{rd} quadrant.

At this point it should also be clear that we'll need to evaluate the integral in terms of polar coordinates. In terms of polar coordinates the limits for D are,

$$\pi \le \theta \le \frac{3\pi}{2}$$

$$0 \le r \le \sqrt{3}$$

Step 4

Now, let's convert the integral into polar coordinates.

$$S = \iint\limits_D \sqrt{64x^2 + 64y^2 + 1}\, dA = \int_{\pi}^{\frac{3}{2}\pi} \int_0^{\sqrt{3}} r\sqrt{64r^2 + 1}\, dr\, d\theta$$

Don't forget to pick up the extra r from converting the dA into polar coordinates. If you need a refresher on converting integrals to polar coordinates then you should go back and work some problems from that section.

Step 5

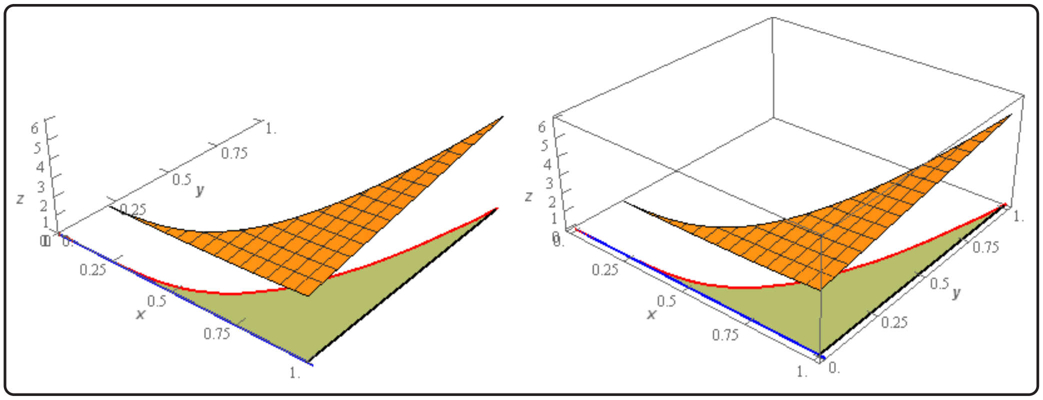
Okay, all we need to do then is evaluate the integral.

$$\begin{aligned} S &= \int_{\pi}^{\frac{3}{2}\pi} \int_0^{\sqrt{3}} r\sqrt{64r^2 + 1}\, dr\, d\theta \\ &= \int_{\pi}^{\frac{3}{2}\pi} \frac{1}{192} \left(64r^2 + 1\right)^{\frac{3}{2}} \Big|_0^{\sqrt{3}}\, d\theta \\ &= \int_{\pi}^{\frac{3}{2}\pi} \frac{1}{192} \left(193^{\frac{3}{2}} - 1\right) d\theta \\ &= \frac{1}{192}\left(193^{\frac{3}{2}} - 1\right)\theta \Big|_{\pi}^{\frac{3}{2}\pi} = \boxed{\frac{\pi}{384}\left(193^{\frac{3}{2}} - 1\right) = 21.9277} \end{aligned}$$

3. Determine the surface area of the portion of $z = 3 + 2y + \frac{1}{4}x^4$ that is above the region in the xy-plane bounded by $y = x^5$, $x = 1$ and the x-axis.

Step 1

Okay, let's start off with a quick sketch of the surface so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface.

The surface we are after is the orange portion that is above the xy-plane and the green-ish region in the xy-plane is the region over which we are graphing the surface, *i.e.* it is the region D we'll use in the integral.

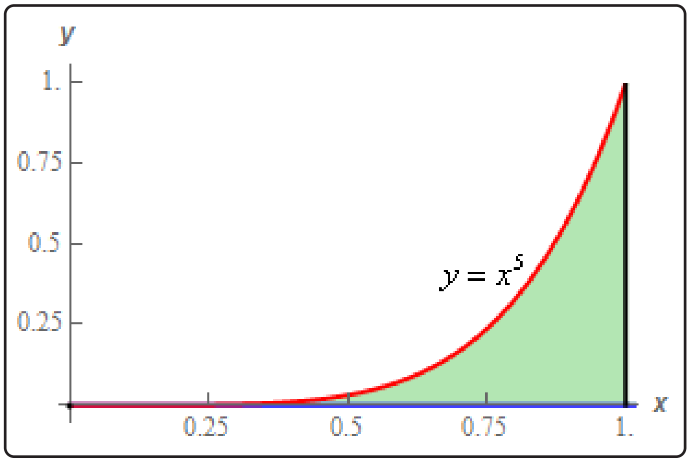
Step 2

The integral for the surface area is,

$$S = \iint\limits_D \sqrt{\left[x^3\right]^2 + \left[2\right]^2 + 1}\, dA = \iint\limits_D \sqrt{x^6 + 5}\, dA$$

Step 3

Now, as mentioned in Step 1 the region D is shown in the sketches of the surface. Here is a 2D sketch of D for the sake of completeness.



The limits for this region are,

$$0 \le x \le 1$$
$$0 \le y \le x^5$$

Note as well that the integrand pretty much requires us to do the integration in this order.

Step 4

With the limits from Step 3 the integral becomes,

$$S = \iint\limits_D \sqrt{x^6 + 5}\, dA = \int_0^1 \int_0^{x^5} \sqrt{x^6 + 5}\, dy\, dx$$

Step 5

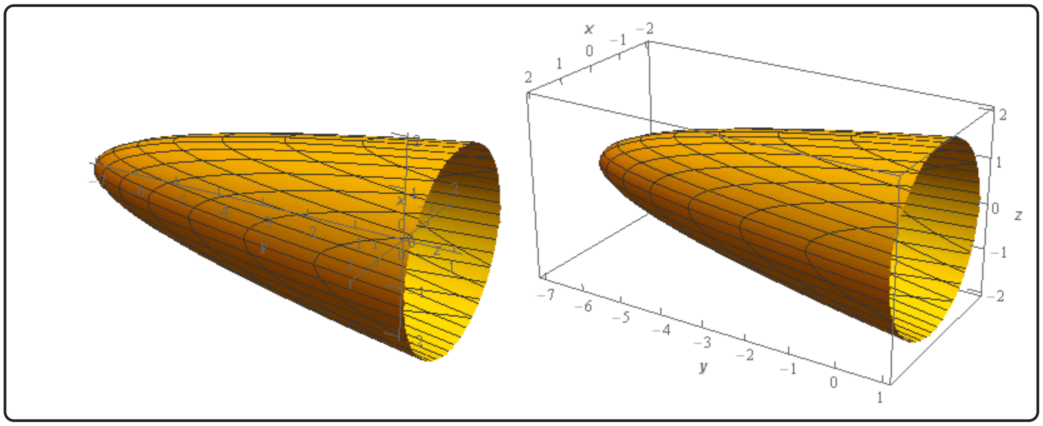
Okay, all we need to do then is evaluate the integral.

$$\begin{aligned} S &= \int_0^1 \int_0^{x^5} \sqrt{x^6 + 5}\, dy\, dx \\ &= \int_0^1 \left(y\sqrt{x^6 + 5} \right)\Big|_0^{x^5} dx \\ &= \int_0^1 x^5 \sqrt{x^6 + 5}\, dx \\ &= \frac{1}{9}\left(x^6 + 5\right)^{\frac{3}{2}}\Big|_0^1 = \boxed{\frac{1}{9}\left(6^{\frac{3}{2}} - 5^{\frac{3}{2}}\right) = 0.3907} \end{aligned}$$

4. Determine the surface area of the portion of $y = 2x^2 + 2z^2 - 7$ that is inside the cylinder $x^2 + z^2 = 4$.

Step 1

Okay, let's start off with a quick sketch of the surface so we can get a feel for what we're dealing with.



We've given the sketches with a set of "traditional" axes as well as a set of "box" axes to help visualize the surface.

So, we have an elliptic paraboloid centered on the y-axis. This also means that the region D for our integral will be in the xz-plane and we'll be needing polar coordinates for the integral.

Step 2

As noted above the region D is going to be in the xz-plane and the surface is given in the form $y = f(x, z)$. The formula for surface area we gave in the notes is only for a region D that is in the xy-plane with the surface given by $z = f(x, y)$.

However, it shouldn't be too difficult to see that all we need to do is modify the formula in the following manner to get one for this setup.

$$S = \iint\limits_D \sqrt{[f_x]^2 + [f_z]^2 + 1}\, dA \qquad y = f(x, z) \qquad D \text{ is in the } xz \text{ - plane}$$

So, the integral for the surface area is,

$$S = \iint\limits_D \sqrt{[4x]^2 + [4z]^2 + 1}\, dA = \iint\limits_D \sqrt{16x^2 + 16z^2 + 1}\, dA$$

Step 3

Now, as we noted in Step 1 the region D is in the xz-plane and because we are after the portion of the elliptical paraboloid that is inside the cylinder given by $x^2 + z^2 = 4$ we can see that the region D must therefore be the disk $x^2 + z^2 \leq 4$.

Also as noted in Step 1 we'll be needing polar coordinates for this integral so here are the limits for the integral in terms of polar coordinates.

$$\begin{gathered} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2 \end{gathered}$$

Step 4

Now, let's convert the integral into polar coordinates. However, they won't be the "standard" polar coordinates. Because D is in the xz-plane we'll need to use the following "modified" polar coordinates.

$$\begin{aligned} x &= r\cos(\theta) \\ z &= r\sin(\theta) \\ x^2 + z^2 &= r^2 \end{aligned}$$

Converting the integral to polar coordinates then gives,

$$S = \iint\limits_D \sqrt{16x^2 + 16z^2 + 1}\, dA = \int_0^{2\pi}\int_0^2 r\sqrt{16r^2 + 1}\, dr\, d\theta$$

Don't forget to pick up the extra r from converting the dA into polar coordinates. It is the same dA as we use for the "standard" polar coordinates.

Step 5

Okay, all we need to do then is evaluate the integral.

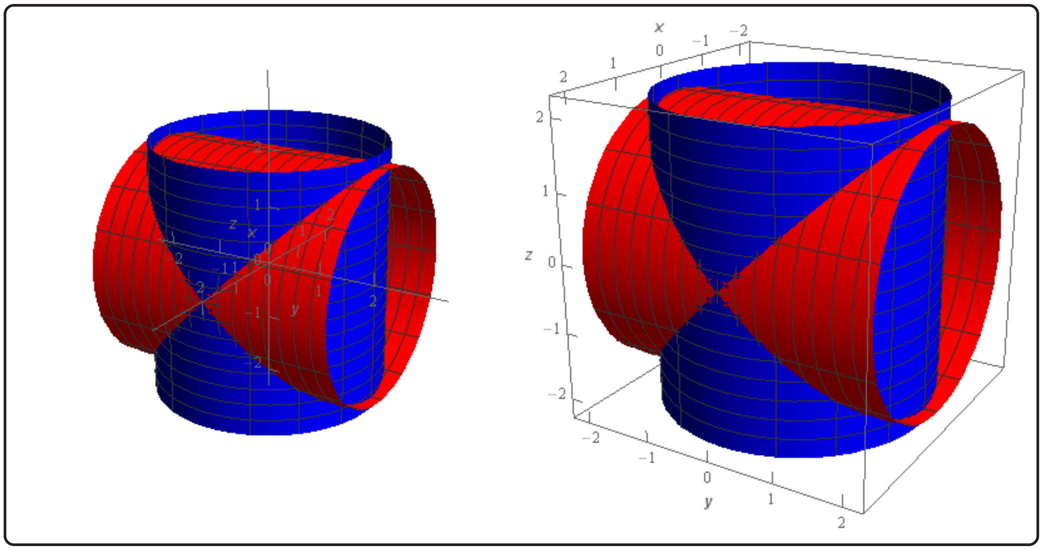
$$\begin{aligned} S &= \int_0^{2\pi} \int_0^2 r\sqrt{16r^2+1}\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{48}\left(16r^2+1\right)^{\frac{3}{2}}\bigg|_0^2 \, d\theta \\ &= \int_0^{2\pi} \frac{1}{48}\left(65^{\frac{3}{2}}-1\right) d\theta \\ &= \frac{1}{48}\left(65^{\frac{3}{2}}-1\right)\theta\bigg|_0^{2\pi} = \boxed{\frac{\pi}{24}\left(65^{\frac{3}{2}}-1\right) = 68.4667} \end{aligned}$$

5. Determine the surface area region formed by the intersection of the two cylinders $x^2+y^2=4$ and $x^2+z^2=4$.

Step 1

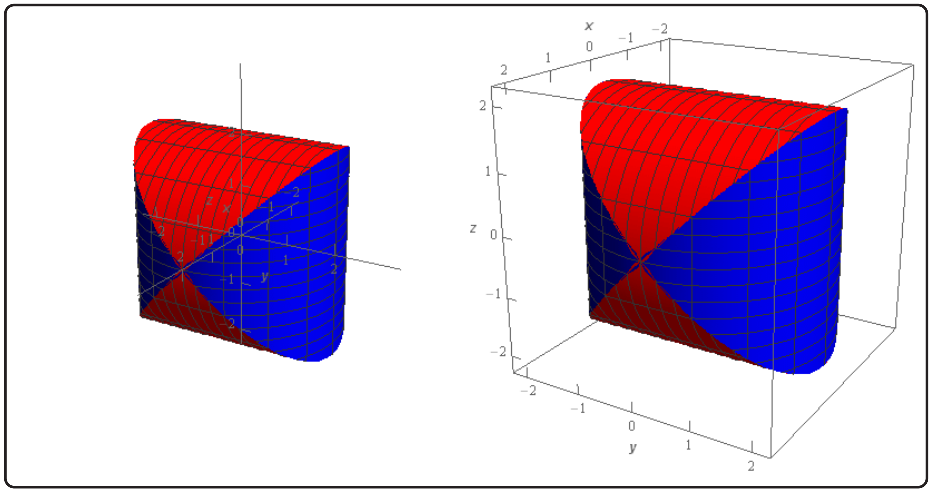
We looked at the volume of this region back in the Double Integral practice problem section. Now we are going to look at the surface area of this region.

Visualizing the surface/region we're looking at is probably one of the harder parts of this problem. So, let's start off with a sketch of the two cylinders.



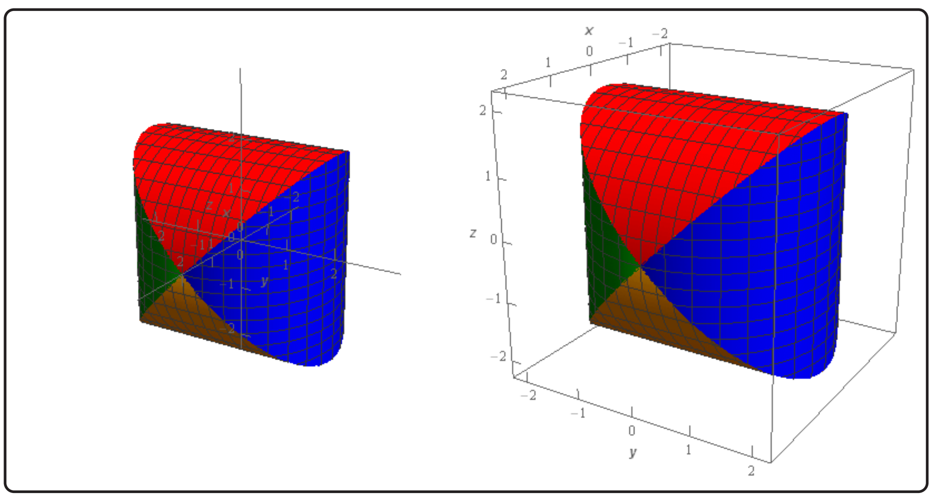
The blue cylinder is the cylinder centered on the z-axis (*i.e.* $x^2 + y^2 = 4$) and the red cylinder is the cylinder centered on the y-axis (*i.e.* $x^2 + z^2 = 4$).

Now, the surface/region we need is the intersection of these two cylinders. So, taking away the "extra" bits of the cylinders that are outside of the intersection we get the following sketches.



The blue part of this sketch is the portion of $x^2 + y^2 = 4$ that is inside of $x^2 + z^2 = 4$ and the red part of this sketch is the portion of $x^2 + z^2 = 4$ that is inside of $x^2 + y^2 = 4$. Taken together we get the surface/region we are interested in.

Now, when we did the volume of this region the sketches above were sufficient for our purposes. This time let's do one more set of sketches.



In this sketch the blue and green parts of the sketch are the right and left portions of $x^2 + y^2 = 4$ respectively and the red and yellow parts of the sketch are the upper and lower portion of $x^2 + z^2 = 4$ respectively.

Step 2

Now, we need to set up an integral that will give the surface area of this surface/region.

Let's first notice that by symmetry the red and yellow surfaces must have the same surface area since they are simply the upper/lower portion of the same cylinder. Likewise the blue and green surfaces will have the same surface area.

Next, because both of the cylinders that make up this surface have the same radius the upper/lower and right/left portions also must have the same surface area.

So, this means that all we need to do is get the surface area of one of the four surfaces shown above and we can them get the full surface area by multiplying that number by four.

So, let's get the surface area of the red portion of the sketch above. To do this we'll need to solve $x^2 + z^2 = 4$ for z to get,

$$z = \pm\sqrt{4 - x^2}$$

The "+" equation will give the red portion of the sketch and the "-" equation will give the yellow portion of the sketch.

So, let's set up the integral that will give the surface area of the red portion. The integral is,

$$\begin{aligned} S &= \iint\limits_D \sqrt{[f_x]^2 + [f_y]^2 + 1}\, dA = \iint\limits_D \sqrt{\left[\frac{-x}{\sqrt{4 - x^2}}\right]^2 + [0]^2 + 1}\, dA \\ &= \iint\limits_D \sqrt{\frac{x^2}{4 - x^2} + 1}\, dA = \iint\limits_D \sqrt{\frac{x^2 + (4 - x^2)}{4 - x^2}}\, dA = \iint\limits_D \sqrt{\frac{4}{4 - x^2}}\, dA \\ &= \iint\limits_D \frac{2}{\sqrt{4 - x^2}}\, dA \end{aligned}$$

Do not get excited about the fact that one of the derivatives in the formula ended up being zero. This will happen on occasion so we don't want to get excited about it when it happens.

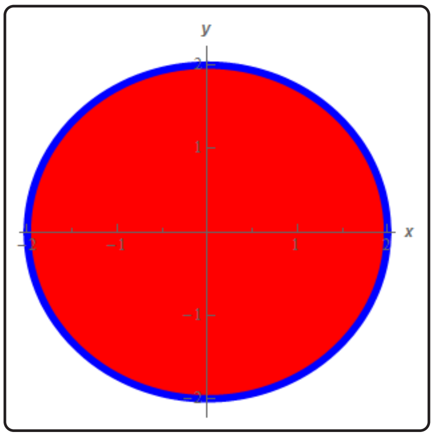
Note as well that we did a fair amount of simplification. Notably, after taking the x derivative and squaring we set up the terms under the radical to have the same dominator.

This in turn gave us a single rational expression under the radical that we could easily take the root of both the numerator and denominator.

As we will eventually see all this simplification will make the integral evaluation work a lot simpler.

Step 3

Now, we need to determine the region D for our integral. If we looked at the surface from directly above (*i.e.* down along the z-axis) we see the following figure.



The blue circle is in fact the cylinder $x^2 + y^2 = 4$ and the red area is the upper portion of $x^2 + z^2 = 4$ that is inside $x^2 + y^2 = 4$.

Now, in general, seeing this region would (and probably should) suggest that we use polar coordinates to do the integral. However, while using polar coordinates to do this integral wouldn't be that difficult, (you might want to do it for the practice) it actually turns out to be easier (in this case!) to use Cartesian coordinates.

In terms of Cartesian coordinates let's use the following limits for the red disk.

$$\begin{gathered} -2 \le x \le 2 \\ -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2} \end{gathered}$$

For the y limits we are just going from the lower portion of the circle to the upper portion of the circle.

Step 4

Okay, let's now set up the integral.

$$S = \iint\limits_D \frac{2}{\sqrt{4-x^2}}\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{2}{\sqrt{4-x^2}}\,dy\,dx$$

Step 5

Next, let's evaluate the integral.

$$\begin{aligned}
S &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{2}{\sqrt{4-x^2}}\,dy\,dx \\
&= \int_{-2}^{2} \frac{2}{\sqrt{4-x^2}} y\bigg|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\,dx \\
&= \int_{-2}^{2} \frac{2}{\sqrt{4-x^2}}\left(\sqrt{4-x^2} - \left(-\sqrt{4-x^2}\right)\right)dx \\
&= \int_{-2}^{2} \frac{2}{\sqrt{4-x^2}}\left(2\sqrt{4-x^2}\right)dx \\
&\int_{-2}^{2} 4\,dx \\
&= 4x\bigg|_{-2}^{2} = \boxed{16}
\end{aligned}$$

We put in a few more steps that absolutely required here to make everything clear but you can see that keeping the integral in terms of Cartesian coordinates here was actually quite simple.

It just goes to show that there really are no hard and fast rules about doing things like converting to polar coordinates versus keeping things in Cartesian coordinates. While we are often taught to see a circle and just immediately convert to polar coordinates this problem has shown that there are the occasional problem that doesn't require that to be done!

Step 6

Okay, finally let's get the surface area. So much of this problem was focused on what we did in the last couple of steps that it is easy to forget about just what we were doing.

Recall that the answer from Step 5 was just the surface area of the red portion of our surface. The full surface area is four times this and so the surface area of the whole surface is 16x4 = **64**.

16 Line Integrals

We now need to move on to a new kind of integral. When doing single variable definite integrals we integrated a function of one variable over an interval. In the last chapter we integrated a function of two variables over a two dimensional region and we integrated a function of three variables over a three dimensional solid. In this chapter we are going to look at Line Integrals. The difference in this chapter versus the last chapter is where the values of the variables will come from. For a line integral of a function of two variables the variables will all be on the graph of a two dimensional curve C. Similarly, for a line integral of a function of three variables, the variables will all be on the graph of a three dimensional curve C.

The other main difference in this chapter versus previous chapters in which we evaluated integrals is that in addition to evaluating line integrals over functions we will also, for the first time, be integrating a vector field (which we'll also define).

Once we have a grasp on line integrals and how to compute them we'll take a look at the Fundamental Theorem of Calculus for Line Integrals and it's relationship with conservative vector fields. We will, in addition, discuss a method for determining if a two dimensional vector field is conservative or not and if it is conservative how to find the potential function for the vector field.

Finally, we'll discuss a very important theorem, Green's Theorem. Green's theorem gives a very important relationship between certain line integrals and double integrals.

The following sections are the practice problems, with solutions, for this material.

16.1 Vector Fields

1. Sketch the vector field for $\vec{F}(x, y) = 2x\,\vec{i} - 2\vec{j}$.

Step 1

Recall that the graph of a vector field is simply sketching the vectors at specific points for a whole bunch of points. This makes sketching vector fields both simple and difficult. It is simple to compute the vectors and sketch them, but it is difficult to know just which points to pick and how many points to pick so we get a good sketch.

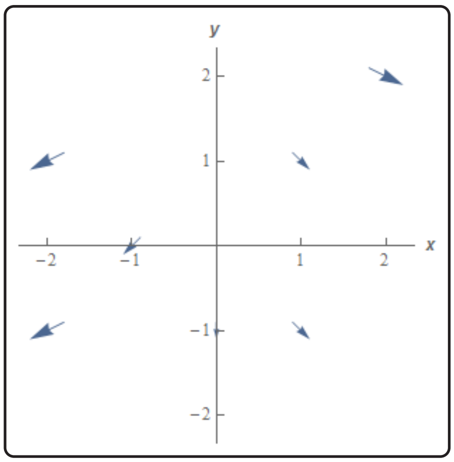
So, let's start off with just computing some vectors at specific points.

$$\vec{F}(0,-1) = -2\vec{j} \qquad \vec{F}(1,1) = 2\vec{i} - 2\vec{j} \qquad \vec{F}(-1,0) = -2\vec{i} - 2\vec{j}$$
$$\vec{F}(-2,-1) = -4\vec{i} - 2\vec{j} \qquad \vec{F}(1,-1) = 2\vec{i} - 2\vec{j} \qquad \vec{F}(2,2) = 4\vec{i} - 2\vec{j}$$
$$\vec{F}(-2,1) = -4\vec{i} - 2\vec{j}$$

Step 2

Now we need to "sketch" each of these vectors at the point that generated them. For example at the point $(0,-1)$ we'll sketch the vector $-2\vec{j}$.

Here is the sketch of these vectors.



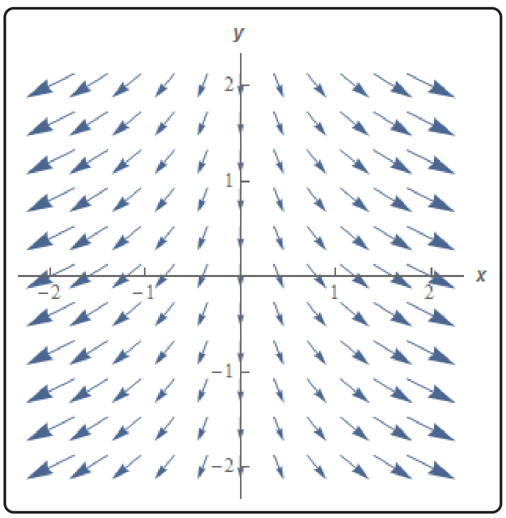
In the sketch above we didn't sketch each of these vectors to scale. In other words we just sketched vectors in the same direction as the indicated vector rather than sketching the vector with "correct" magnitude. The reason for this is to keep the sketch a little easier to see. If we sketched all the vectors to scale we'd just see a mess of overlapping arrows that would be hard to really see what was going on.

Note as well that with the few vectors that we sketched it's difficult to get a real feel for what is going on at any random point. With this sketch we might be able to see some trends but we don't know that those trends will continue into "blank" regions that don't have any vectors sketched in them.

Step 3

Below is a better sketch of the vector field with many more vectors sketched in. We got this sketch by letting a computer just plot quite a few points by itself without actually picking any of them as we did in the previous step.

In general, this is how vector fields are sketched. Computing this number of vectors by hand would so time consuming that it just wouldn't be worth it. Computers however can do all those computations very quickly and so we generally just let them do the sketch.



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2. Sketch the vector field for $\vec{F}(x,y) = (y-1)\,\vec{i} + (x+y)\,\vec{j}$.

Step 1

Recall that the graph of a vector field is simply sketching the vectors at specific points for a whole bunch of points. This makes sketching vector fields both simple and difficult. It is simple to compute the vectors and sketch them, but it is difficult to know just which points to pick and how many points to pick so we get a good sketch.

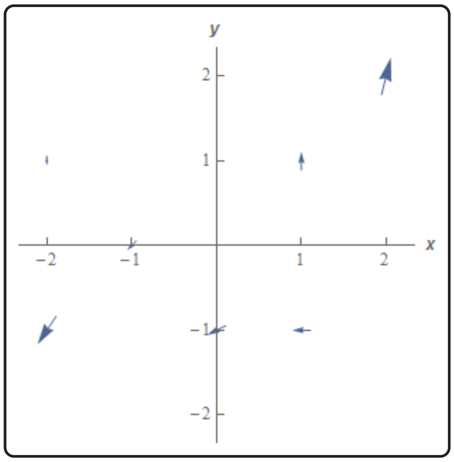
So, let's start off with just computing some vectors at specific points.

$$\vec{F}(0,-1) = -2\,\vec{i} - \vec{j} \qquad \vec{F}(1,1) = 2\vec{j} \qquad \vec{F}(-1,0) = -\vec{i} - \vec{j}$$
$$\vec{F}(-2,-1) = -2\,\vec{i} - 3\vec{j} \qquad \vec{F}(1,-1) = -2\,\vec{i} \qquad \vec{F}(2,2) = \vec{i} + 4\vec{j}$$
$$\vec{F}(-2,1) = -\vec{j}$$

Step 2

Now we need to "sketch" each of these vectors at the point that generated them. For example at the point $(0,-1)$ we'll sketch the vector $-2\vec{i} - \vec{j}$.

Here is the sketch of these vectors.



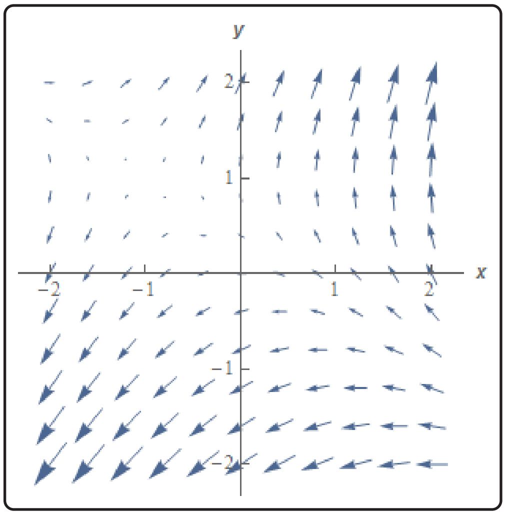
In the sketch above we didn't sketch each of these vectors to scale. In other words we just sketched vectors in the same direction as the indicated vector rather than sketching the vector with "correct" magnitude. The reason for this is to keep the sketch a little easier to see. If we sketched all the vectors to scale we'd just see a mess of overlapping arrows that would be hard to really see what was going on.

Note as well that with the few vectors that we sketched it's difficult to get a real feel for what is going on at any random point let along any trends in the vector field.

Step 3

Below is a better sketch of the vector field with many more vectors sketched in. We got this sketch by letting a computer just plot quite a few points by itself without actually picking any of them as we did in the previous step.

In general, this is how vector fields are sketched. Computing this number of vectors by hand would so time consuming that it just wouldn't be worth it. Computers however can do all those computations very quickly and so we generally just let them do the sketch.



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3. Compute the gradient vector field for $f\left(x,y\right) = y^2\cos\left(2x - y\right)$.

Solution

There really isn't a lot to do for this problem. Here is the gradient vector field for this function.

$$\boxed{\nabla f = \left\langle -2y^2\sin\left(2x - y\right),\, 2y\cos\left(2x - y\right) + y^2\sin\left(2x - y\right)\right\rangle}$$

Don't forget to compute partial derivatives for each of these! The first term is the derivative of the function with respect to x and the second term is the derivative of the function with respect to y.

---

4. Compute the gradient vector field for $f(x,y,z) = z^2\mathbf{e}^{x^2+4y} + \ln\left(\frac{xy}{z}\right)$.

Solution

There really isn't a lot to do for this problem. Here is the gradient vector field for this function.

$$\boxed{\nabla f = \left\langle 2xz^2\mathbf{e}^{x^2+4y} + \frac{1}{x}, 4z^2\mathbf{e}^{x^2+4y} + \frac{1}{y}, 2z\mathbf{e}^{x^2+4y} - \frac{1}{z} \right\rangle}$$

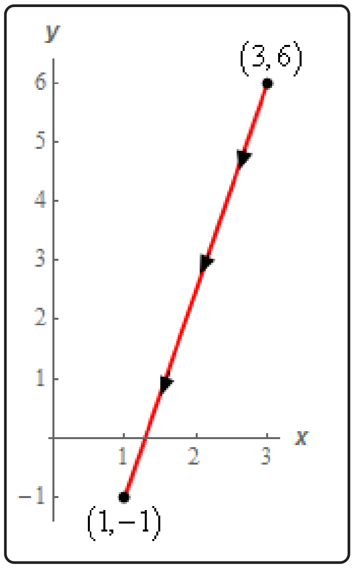
Don't forget to compute partial derivatives for each of these! The first term is the derivative of the function with respect to x, the second term is the derivative of the function with respect to y and the third term is the derivative of the function with respect to z.

16.2 Line Integrals - Part I

1. Evaluate $\int\limits_C 3x^2 - 2y\,ds$ where C is the line segment from $(3,6)$ to $(1,-1)$.

Step 1

Here is a quick sketch of C with the direction specified in the problem statement shown.



Step 2

Now, with the specified direction we can see that x is decreasing as we move along the curve in the specified direction. This means that we can't just determine the equation of the line and use that to work the problem. Using the equation of the line would require us to use increasing x since the limits in the integral must go from smaller to larger value.

We could of course use the fact from the notes that relates the line integral with a specified direction and the line integral with the opposite direction to allow us to use the equation of the line. However, for this problem let's just work with problem without the fact to make sure we can do that type of problem

So, we'll need to parameterize this line and we know how to parameterize the equation of a line between two points. Here is the vector form of the parameterization of the line.

$$\vec{r}(t) = (1-t)\langle 3,6\rangle + t\langle 1,-1\rangle = \langle 3-2t, 6-7t\rangle \qquad 0 \le t \le 1$$

We could also break this up into parameter form as follows.

$$\begin{aligned} x &= 3 - 2t \\ y &= 6 - 7t \end{aligned} \qquad 0 \le t \le 1$$

Either form of the parameterization will work for the problem but we'll use the vector form for the rest of this problem.

Step 3

We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\vec{r}\,'(t) = \langle -2, -7 \rangle \qquad\qquad \left\| \vec{r}\,'(t) \right\| = \sqrt{(-2)^2 + (-7)^2} = \sqrt{53}$$

We'll also need the integrand "evaluated" at the parameterization. Recall all this means is we replace the x/y in the integrand with the x/y from parameterization. Here is the integrand evaluated at the parameterization.

$$3x^2 - 2y = 3(3-2t)^2 - 2(6-7t) = 3(3-2t)^2 - 12 + 14t$$

Step 4

The line integral is then,

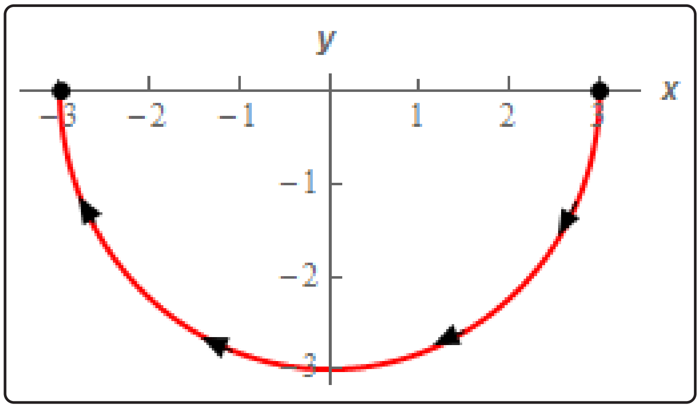
$$\begin{aligned} \int\limits_C 3x^2 - 2y\,ds &= \int_0^1 \left(3(3-2t)^2 - 12 + 14t\right)\sqrt{53}\,dt \\ &= \sqrt{53}\left[-\frac{1}{2}(3-2t)^3 - 12t + 7t^2\right]\Bigg|_0^1 = \boxed{8\sqrt{53}} \end{aligned}$$

Note that we didn't multiply out the first term in the integrand as we could do a quick substitution to do the integral.

2. Evaluate $\displaystyle\int\limits_C 2yx^2 - 4x\,ds$ where C is the lower half of the circle centered at the origin of radius 3 with clockwise rotation.

Step 1

Here is a quick sketch of C with the direction specified in the problem statement shown.



Step 2

Here is a parameterization for this curve.

$$\vec r(t) = \langle 3\cos(t), -3\sin(t)\rangle \qquad 0 \le t \le \pi$$

We could also break this up into parameter form as follows.

$$\begin{aligned} x &= 3\cos(t) \\ y &= -3\sin(t) \end{aligned} \qquad 0 \le t \le \pi$$

Either form of the parameterization will work for the problem but we'll use the vector form for the rest of this problem.

Step 3

We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\vec{r}'(t) = \langle -3\sin(t), -3\cos(t)\rangle$$
$$\|\vec{r}'(t)\| = \sqrt{(-3\sin(t))^2 + (-3\cos(t))^2}$$
$$= \sqrt{9\sin^2(t) + 9\cos^2(t)} = \sqrt{9\left(\sin^2(t) + \cos^2(t)\right)} = \sqrt{9} = 3$$

We'll also need the integrand "evaluated" at the parameterization. Recall all this means is we replace the x/y in the integrand with the x/y from parameterization. Here is the integrand evaluated at the parameterization.

$$2yx^2 - 4x = 2(-3\sin(t))(3\cos(t))^2 - 4(3\cos(t)) = -54\sin(t)\cos^2(t) - 12\cos(t)$$

Step 4

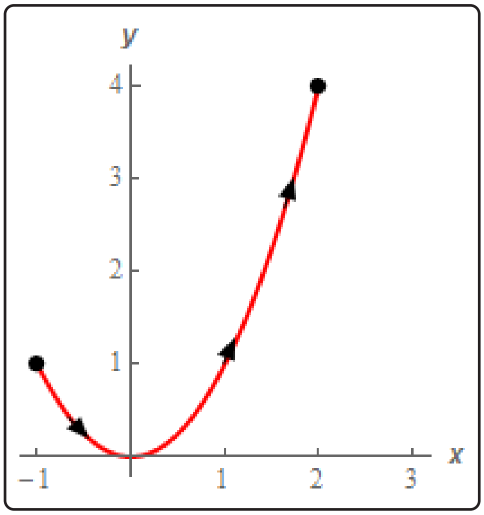
The line integral is then,

$$\int_C 2yx^2 - 4x\,ds = \int_0^{\pi} \left(-54\sin(t)\cos^2(t) - 12\cos(t)\right) 3\,dt$$
$$= 3\left[18\cos^3(t) - 12\sin(t)\right]\Big|_0^{\pi} = \boxed{-108}$$

3. Evaluate $\displaystyle\int_C 6x\, ds$ where C is the portion of $y = x^2$ from $x = -1$ to $x = 2$. The direction of C is in the direction of increasing x.

Step 1

Here is a quick sketch of C with the direction specified in the problem statement shown.



Step 2

In this case we can just use the equation of the curve for the parameterization because the specified direction is going in the direction of increasing x which will give us integral limits from smaller value to larger value as needed. Here is a parameterization for this curve.

$$\vec{r}(t) = \langle t, t^2 \rangle \qquad -1 \le t \le 2$$

We could also break this up into parameter form as follows.

$$\begin{aligned} x &= t \\ y &= t^2 \end{aligned} \qquad -1 \le t \le 2$$

Either form of the parameterization will work for the problem but we'll use the vector form for the rest of this problem.

Step 3

We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\vec{r}\,'(t) = \langle 1, 2t\rangle \qquad \left\|\vec{r}\,'(t)\right\| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1+4t^2}$$

We'll also need the integrand "evaluated" at the parameterization. Recall all this means is we replace the x/y in the integrand with the x/y from parameterization. Here is the integrand evaluated at the parameterization.

$$6x = 6t$$

Step 4

The line integral is then,

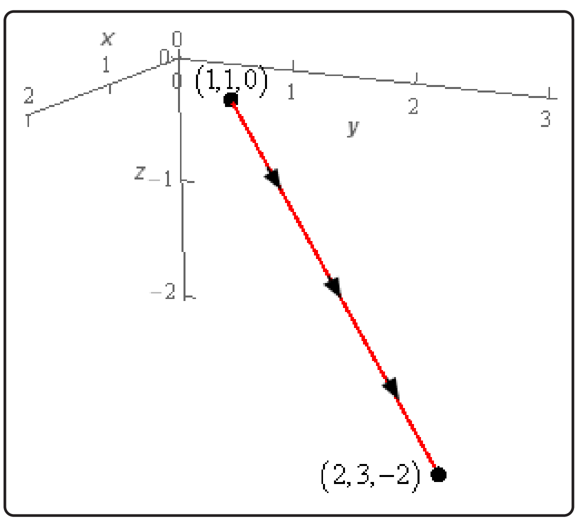
$$\int_C 6x\,ds = \int_{-1}^{2} 6t\sqrt{1+4t^2}\,dt = \frac{1}{2}\left(1+4t^2\right)^{\frac{3}{2}}\Big|_{-1}^{2} = \boxed{\frac{1}{2}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)}$$

---

4. Evaluate $\displaystyle\int_C xy - 4z\,ds$ where C is the line segment from $(1,1,0)$ to $(2,3,-2)$.

Step 1

Here is a quick sketch of C with the direction specified in the problem statement shown.



Step 2

We know how to get the parameterization of a line segment so let's just jump straight into the parameterization of the line segment.

$$\vec{r}(t) = (1-t)\langle 1,1,0\rangle + t\langle 2,3,-2\rangle = \langle 1+t, 1+2t, -2t\rangle \qquad 0 \le t \le 1$$

We could also break this up into parameter form as follows.

$$\begin{aligned} x &= 1+t \\ y &= 1+2t \qquad 0 \le t \le 1 \\ z &= -2t \end{aligned}$$

Either form of the parameterization will work for the problem but we'll use the vector form for the rest of this problem.

Step 3

We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\vec{r}'(t) = \langle 1,2,-2\rangle \qquad \left\|\vec{r}'(t)\right\| = \sqrt{(1)^2 + (2)^2 + (-2)^2} = \sqrt{9} = 3$$

We'll also need the integrand "evaluated" at the parameterization. Recall all this means is we replace the x/y in the integrand with the x/y from parameterization. Here is the integrand evaluated at the parameterization.

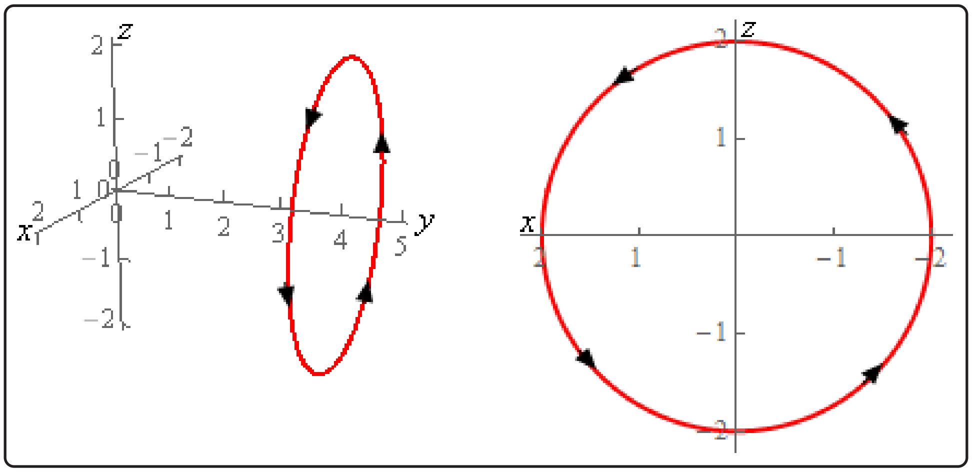
$$xy - 4z = (1+t)(1+2t) - 4(-2t) = 2t^2 + 11t + 1$$

Step 4

The line integral is then,

$$\int_C xy - 4z\,ds = \int_0^1 \left(2t^2 + 11t + 1\right)(3)\,dt = 3\left(\frac{2}{3}t^3 + \frac{11}{2}t^2 + t\right)\Bigg|_0^1 = \boxed{\frac{43}{2}}$$

5. Evaluate $\displaystyle\int\limits_C x^2 y^2\, ds$ where C is the circle centered at the origin of radius 2 centered on the y-axis at $y = 4$. See the sketches below for orientation. Note the "odd" axis orientation on the 2D circle is intentionally that way to match the 3D axis the direction.



Step 1

Before we parameterize the curve note that the "orientation" of the x-axis in the 2D sketch above is backwards from what we are used to. In this sketch the positive x-axis is to the left and the negative x-axis is to the right. This was done to match up with the 3D image.

If we were on the positive y-axis (on the 3D image of course) past $y = 4$ and looking towards the origin we would see the 2D sketch. Generating the 2D sketch in this manner will help to make sure that our parameterization has the correct direction.

Speaking of which, here is the parameterization of the curve.

$$\vec r\,(t) = \langle 2\cos(t)\,, 4, -2\sin(t)\rangle \qquad 0 \le t \le 2\pi$$

If you check the parameterization at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ we can see that we will get the correct (x, z) coordinates for the curve in the 2D sketch and hence we will also get the correct coordinates for the curve in the 3D sketch.

Don't forget that we also need to acknowledge in our parameterization that we are at $y = 4$, *i.e.* the second component of the parameterization. When one of the coordinates on a 3D curve is constant it is often easy to forget to deal with it in the parameterization.

Step 2

We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\vec{r}'(t) = \langle -2\sin(t), 0, -2\cos(t)\rangle$$
$$\|\vec{r}'(t)\| = \sqrt{(-2\sin(t))^2 + (0)^2 + (-2\cos(t))^2}$$
$$= \sqrt{4\sin^2(t) + 4\cos^2(t)} = \sqrt{4\left(\sin^2(t) + \cos^2(t)\right)} = \sqrt{4} = 2$$

We'll also need the integrand "evaluated" at the parameterization. Recall all this means is we replace the x/y/z in the integrand with the x/y/z from parameterization. Here is the integrand evaluated at the parameterization.

$$x^2y^2 = (2\cos(t))^2(4)^2 = 64\cos^2(t)$$

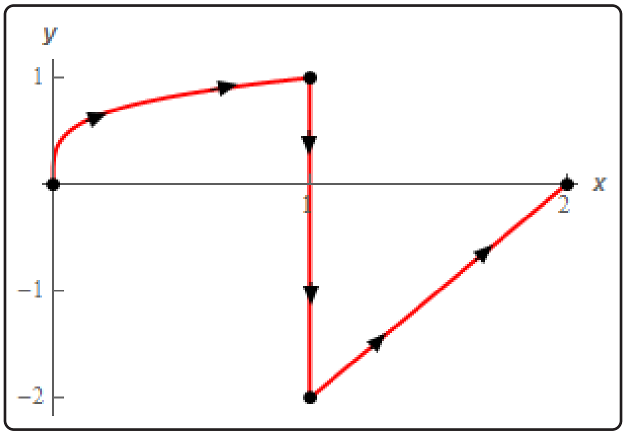
Step 3

The line integral is then,

$$\int_C x^2y^2\,ds = \int_0^{2\pi} \left(64\cos^2(t)\right)(2)\,dt = \int_0^{2\pi} 64\left(1+\cos(2t)\right)\,dt$$
$$= 64\left(t + \frac{1}{2}\sin(2t)\right)\Big|_0^{2\pi} = \boxed{128\pi}$$

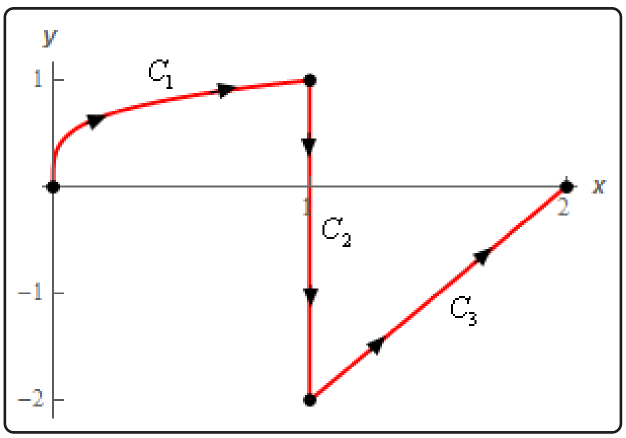
You do recall how to use the double angle formula for cosine to evaluate this integral correct? We'll be seeing a fair number of integrals involving trig functions in this chapter and knowing how to do these kinds of integrals will be important.

6. Evaluate $\displaystyle\int\limits_C 16y^5\,ds$ where C is the portion of $x = y^4$ from $y = 0$ to $y = 1$ followed by the line segment form $(1,1)$ to $(1,-2)$ which in turn is followed by the line segment from $(1,-2)$ to $(2,0)$. See the sketch below for the direction.



Step 1

To help with the problem let's label each of the curves as follows,



Now let's parameterize each of these curves.

$C_1: \ \vec{r}(t) = \langle t^4, t\rangle \qquad 0 \le t \le 1$

$C_2: \ \vec{r}(t) = (1-t)\langle 1,1\rangle + t\langle 1,-2\rangle = \langle 1, 1-3t\rangle \qquad 0 \le t \le 1$

$C_3: \ \vec{r}(t) = (1-t)\langle 1,-2\rangle + t\langle 2,0\rangle = \langle 1+t, 2t-2\rangle \qquad 0 \le t \le 1$

For C_2 we had to use the vector form for the line segment between two points instead of the equation for the line (which is much simpler of course) because the direction was in the decreasing y direction and the limits on our integral must be from smaller to larger. We could have used the fact from the notes that tells us how the line integrals for the two directions related to allow us to use the equation of the line if we'd wanted to. We decided to do it this way just for the practice of dealing with the vector form for the line segment and it's not all that difficult to deal with the result and the limits are "nicer".

Note as well that for C_3 we could have solved for the equation of the line and used that because the direction is in the increasing x direction. However, the vector form for the line segment between two points is just as easy to use so we used that instead.

Step 2

Okay, we now need to compute the line integral along each of these curves. Unlike the first few problems in this section where we found the magnitude and the integrand prior to the integration step we're just going to just straight into the integral and do all the work there.

Here is the integral along each of the curves.

$$\int_{C_1} 16y^5\,ds = \int_0^1 16(t)^5\sqrt{(4t^3)^2+(1)^2}\,dt = \int_0^1 16t^5\sqrt{16t^6+1}\,dt$$

$$= \frac{1}{9}\left(16t^6+1\right)^{\frac{3}{2}}\Big|_0^1 = \underline{\frac{1}{9}\left(17^{\frac{3}{2}}-1\right) = 7.6770}$$

$$\int_{C_2} 16y^5\,ds = \int_0^1 16(1-3t)^5\sqrt{(0)^2+(-3)^2}\,dt = \int_0^1 48(1-3t)^5\,dt$$

$$= -\frac{8}{3}(1-3t)^6\Big|_0^1 = \underline{-168}$$

$$\int_{C_3} 16y^5\,ds = \int_0^1 16(2t-2)^5\sqrt{(1)^2+(2)^2}\,dt = \int_0^1 16\sqrt{5}(2t-2)^5\,dt$$

$$= \frac{4\sqrt{5}}{3}(2t-2)^6\Big|_0^1 = \underline{-\frac{256\sqrt{5}}{3} = -190.8111}$$

Step 3

Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int\limits_C 16y^5\,ds = \left(\frac{1}{9}\left(17^{\frac{3}{2}}+1\right)\right) + (-168) + \left(-\frac{256\sqrt{5}}{3}\right) = \boxed{-351.1341}$$

Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

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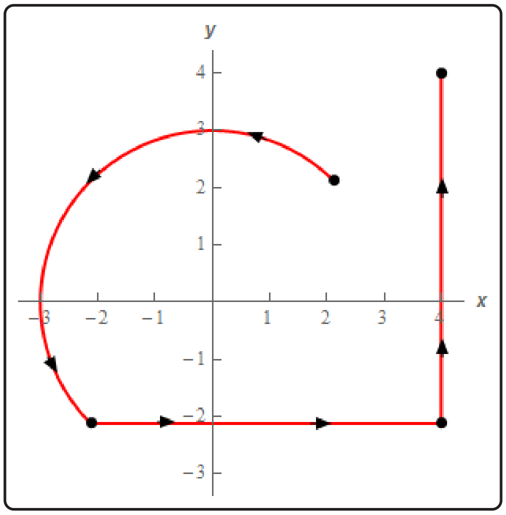
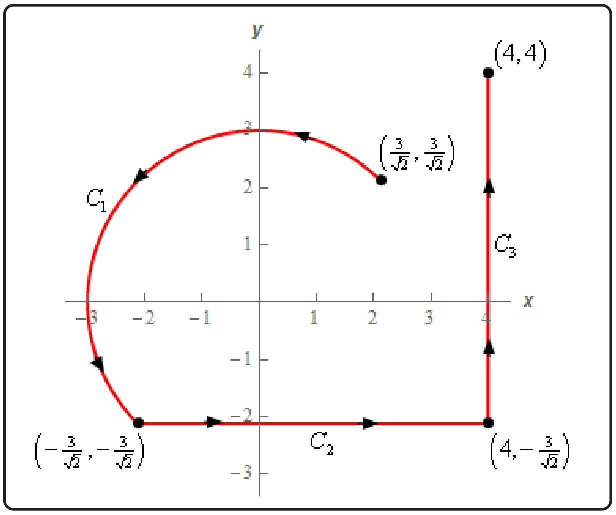
7. Evaluate $\int\limits_C 4y - x\,ds$ where C is the upper portion of the circle centered at the origin of radius 3 from $\left(\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$ to $\left(-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}}\right)$ in the counter clockwise rotation followed by the line segment form $\left(-\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}}\right)$ to $\left(4,-\frac{3}{\sqrt{2}}\right)$ which in turn is followed by the line segment from $\left(4,-\frac{3}{\sqrt{2}}\right)$ to $(4,4)$. See the sketch below for the direction.



Step 1

To help with the problem let's label each of the curves as follows,



Now let's parameterize each of these curves.

$$C_1: \ \vec r(t) = \langle 3\cos(t), 3\sin(t)\rangle \qquad \frac{1}{4}\pi \le t \le \frac{5}{4}\pi$$

$$C_2: \ \vec r(t) = \left\langle t, -\frac{3}{\sqrt{2}}\right\rangle \qquad -\frac{3}{\sqrt{2}} \le t \le 4$$

$$C_3: \ \vec r(t) = \langle 4, t\rangle \qquad -\frac{3}{\sqrt{2}} \le t \le 4$$

The limits for C_1 are actually pretty easy to find. At the starting point, $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, we know that we must have,

$$3\cos(t) = 3\sin(t) = \frac{3}{\sqrt{2}} \qquad \to \qquad \cos(t) = \sin(t) = \frac{1}{\sqrt{2}} \qquad \to \qquad \tan(t) = 1$$

Since the starting point of C_1 is in the 1^{st} quadrant we know that we must have $t = \frac{1}{4}\pi$.

We can do a similar argument of the final point, $\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$. This point is in the 3^{rd} quadrant and again we'll have $\tan(t) = 1$. Therefore we must have $t = \frac{5}{4}\pi$ at this point.

For C_2 and C_3 we just used the equations of the lines since they were horizontal and vertical respectively and followed increasing x and y respectively. The limits for them are a little unpleasant but we can't do anything about the fact that there will be messy numbers in these no matter how we do them.

Step 2

Okay, we now need to compute the line integral along each of these curves. Unlike the first few problems in this section where we found the magnitude and the integrand prior to the integration step we're just going to just straight into the integral and do all the work there.

Here is the integral along each of the curves.

$$\begin{aligned}\int_{C_1} 4y - x\,ds &= \int_{\frac{1}{4}\pi}^{\frac{5}{4}\pi} \left[4\left(3\sin\left(t\right)\right) - 3\cos\left(t\right)\right]\sqrt{\left(-3\sin\left(t\right)\right)^2 + \left(3\cos\left(t\right)\right)^2}\,dt \\ &= \int_{\frac{1}{4}\pi}^{\frac{5}{4}\pi} \left[12\sin\left(t\right) - 3\cos\left(t\right)\right]\sqrt{9\sin^2\left(t\right) + 9\cos^2\left(t\right)}\,dt \\ &= 3\int_{\frac{1}{4}\pi}^{\frac{5}{4}\pi} 12\sin\left(t\right) - 3\cos\left(t\right)\,dt \\ &= 3\left(-12\cos\left(t\right) - 3\sin\left(t\right)\right)\Big|_{\frac{1}{4}\pi}^{\frac{5}{4}\pi} = \underline{\frac{90}{\sqrt{2}}}\end{aligned}$$

$$\begin{aligned}\int_{C_2} 4y - x\,ds &= \int_{-\frac{3}{\sqrt{2}}}^{4} \left[4\left(-\frac{3}{\sqrt{2}}\right) - t\right]\sqrt{\left(1\right)^2 + \left(0\right)^2}\,dt \\ &= \int_{-\frac{3}{\sqrt{2}}}^{4} -\frac{12}{\sqrt{2}} - t\,dt \\ &= \left(-\frac{12}{\sqrt{2}}t - \frac{1}{2}t^2\right)\Big|_{-\frac{3}{\sqrt{2}}}^{4} = \underline{-\frac{95}{4} - \frac{48}{\sqrt{2}}}\end{aligned}$$

$$\begin{aligned}\int_{C_3} 4y - x\,ds &= \int_{-\frac{3}{\sqrt{2}}}^{4} \left[4\left(t\right) - 4\right]\sqrt{\left(0\right)^2 + \left(1\right)^2}\,dt \\ &= \int_{-\frac{3}{\sqrt{2}}}^{4} 4t - 4\,dt \\ &= \left(2t^2 - 4t\right)\Big|_{-\frac{3}{\sqrt{2}}}^{4} = \boxed{7 - \frac{12}{\sqrt{2}}}\end{aligned}$$

Step 3

Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int\limits_C 4y - x\,ds = \left(\frac{90}{\sqrt{2}}\right) + \left(-\frac{95}{4} - \frac{48}{\sqrt{2}}\right) + \left(7 - \frac{12}{\sqrt{2}}\right) = \boxed{-\frac{67}{4} + \frac{30}{\sqrt{2}} = 4.4632}$$

Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

---

8. Evaluate $\int\limits_C y^3 - x^2\,ds$ for each of the following curves.

(a) C is the line segment from $(3,6)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(3,-6)$.

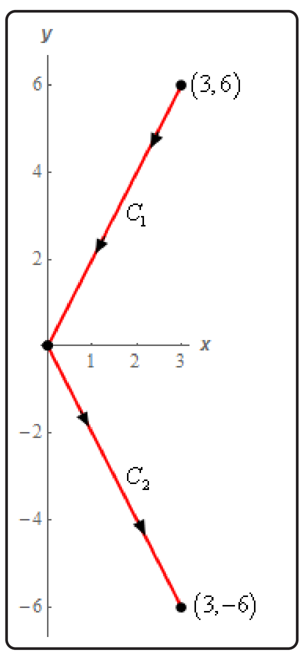
(b) C is the line segment from $(3,6)$ to $(3,-6)$.

Solutions

(a) C is the line segment from $(3,6)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(3,-6)$.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



So, this curve comes in two pieces and we'll need to parameterize each of them so let's take care of that next.

$$C_1: \ \vec{r}(t) = (1-t)\langle 3,6\rangle + t\langle 0,0\rangle = \langle 3-3t, 6-6t\rangle \qquad 0 \le t \le 1$$

$$C_2: \ \vec{r}(t) = (1-t)\langle 0,0\rangle + t\langle 3,-6\rangle = \langle 3t, -6t\rangle \qquad 0 \le t \le 1$$

For each of these curves we just used the vector form of the line segment between two points. In the first case we needed to because the direction was in the decreasing y direction and recall that the integral limits need to be from smaller to larger value. In the second case the equation would have been just as easy to use but we just decided to use the line segment form for the slightly nicer limits.

Step 2

Okay, we now need to compute the line integral along each of these curves. Unlike the first few problems in this section where we found the magnitude and the integrand prior to the integration step we're just going to just straight into the integral and do all the work there.

Here is the integral along each of the curves.

$$\begin{aligned}
\int\limits_{C_1} y^3 - x^2\,ds &= \int_0^1 \left[(6-6t)^3 - (3-3t)^2\right]\sqrt{(-3)^2+(-6)^2}\,dt \\
&= \sqrt{45}\int_0^1 (6-6t)^3 - (3-3t)^2\,dt \\
&= 3\sqrt{5}\left(-\frac{1}{24}(6-6t)^4 + \frac{1}{9}(3-3t)^3\right)\Big|_0^1 = \underline{153\sqrt{5}}
\end{aligned}$$

$$\begin{aligned}
\int\limits_{C_2} y^3 - x^2\,ds &= \int_0^1 \left[(-6t)^3 - (3t)^2\right]\sqrt{(3)^2+(-6)^2}\,dt \\
&= \sqrt{45}\int_0^1 -216t^3 - 9t^2\,dt \\
&= 3\sqrt{5}\left(-54t^4 - 3t^3\right)\Big|_0^1 = \underline{-171\sqrt{5}}
\end{aligned}$$

Step 3

Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

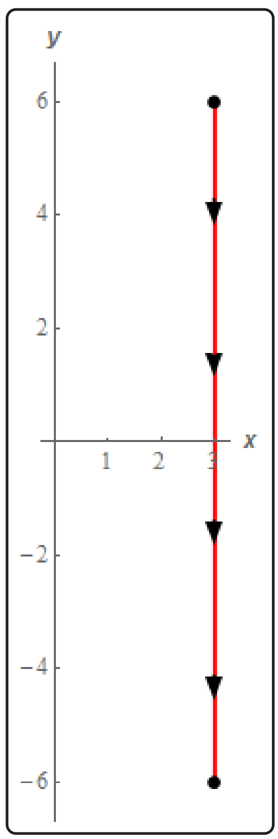
$$\int_C y^3 - x^2\,ds = \left(153\sqrt{5}\right) + \left(-171\sqrt{5}\right) = \boxed{-18\sqrt{5}}$$

Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

(b) C is the line segment from $(3,6)$ to $(3,-6)$.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



So, what we have in this part is a different curve that goes from $(3,6)$ to $(3,-6)$.

Despite the fact that this curve has the same starting and ending point as the curve in the first part there is no reason to expect the line integral to have the same value. Therefore we'll need to go through the work and see what we get from the line integral.

We'll need to parameterize the curve so let's take care of that.

$C: \vec{r}(t) = (1-t)\langle 3,6\rangle + t\langle 3,-6\rangle = \langle 3, 6-12t\rangle \qquad 0 \le t \le 1$ Because this curve is in the direction of decreasing y and the integral needs its limits to go from smaller to larger values we had to use the vector form of the line segment between two points.

Step 2

Now all we need to do is compute the line integral.

$$\begin{aligned}\int\limits_C y^3 - x^2\,ds &= \int_0^1 \left[(6-12t)^3 - (3)^2\right]\sqrt{(0)^2 + (-12)^2}\,dt \\ &= 12\int_0^1 (6-12t)^3 - 9\,dt \\ &= 12\left(-\frac{1}{48}(6-12t)^4 - 9t\right)\Bigg|_0^1 = \boxed{-108}\end{aligned}$$

So, as noted at the start of this part the value of the line integral was not the same as the value of the line integral in the first part despite the same starting and ending points for the curve. Note that it is possible for two line integrals with the same starting and ending points to have the same value but we can't expect that to happen and so need to go through and do the work.

---

9. Evaluate $\displaystyle\int\limits_C 4x^2\,ds$ for each of the following curves.

(a) C is the portion of the circle centered at the origin of radius 2 in the 1^{st} quadrant rotating in the clockwise direction.

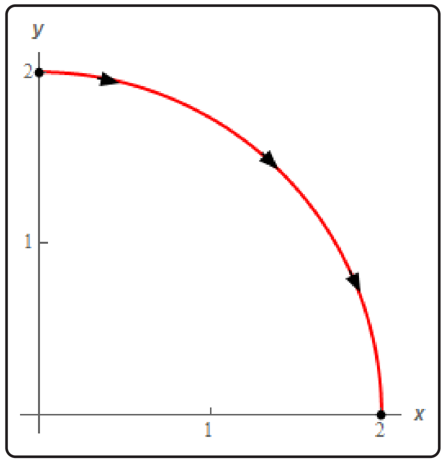
(b) C is the line segment from $(0,2)$ to $(2,0)$.

Solutions

(a) C is the portion of the circle centered at the origin of radius 2 in the 1^{st} quadrant rotating in the clockwise direction.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



Here is the parameterization for this curve.

$$C: \vec{r}(t) = \langle 2\sin(t), 2\cos(t)\rangle \qquad 0 \le t \le \tfrac{1}{2}\pi$$

Step 2

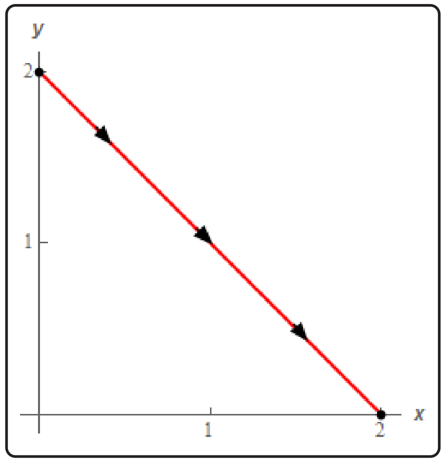
Here is the line integral for this curve.

$$\begin{aligned}\int_C 4x^2\,ds &= \int_0^{\frac{1}{2}\pi} 4(2\sin(t))^2\sqrt{(2\cos(t))^2 + (-2\sin(t))^2}\,dt \\ &= \int_0^{\frac{1}{2}\pi} 16\sin^2(t)\sqrt{4\cos^2(t) + 4\sin^2(t)}\,dt \\ &= \int_0^{\frac{1}{2}\pi} 16\sin^2(t)\sqrt{4}\,dt \\ &= \int_0^{\frac{1}{2}\pi} 32\left(\frac{1}{2}\right)(1 - \cos(2t))\,dt \\ &= 16\left(t - \frac{1}{2}\sin(2t)\right)\Bigg|_0^{\frac{1}{2}\pi} = \boxed{8\pi}\end{aligned}$$

(b) C is the line segment from $(0,2)$ to $(2,0)$.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



So, what we have in this part is a different curve that goes from $(0,2)$ to $(2,0)$. Despite the fact that this curve has the same starting and ending point as the curve in the first part there is no reason to expect the line integral to have the same value. Therefore we'll need to go through the work and see what we get from the line integral.

We'll need to parameterize the curve so let's take care of that.

$$C: \ \vec{r}(t) = (1-t)\langle 0,2\rangle + t\langle 2,0\rangle = \langle 2t, 2-2t\rangle \qquad 0 \le t \le 1$$

Note that we could have just found the equation of this curve but it seemed just as easy to just use the vector form of the line segment between two points.

Step 2

Now all we need to do is compute the line integral.

$$\begin{aligned}\int_C 4x^2\,ds &= \int_0^1 4(2t)^2\sqrt{(2)^2+(-2)^2}\,dt \\ &= \sqrt{8}\int_0^1 16t^2\,dt \\ &= 2\sqrt{2}\left(\frac{16}{3}t^3\right)\Big|_0^1 = \boxed{\frac{32\sqrt{2}}{3}}\end{aligned}$$

So, as noted at the start of this part the value of the line integral was not the same as the value of the line integral in the first part despite the same starting and ending points for the curve. Note that it is possible for two line integrals with the same starting and ending points to have the same value but we can't expect that to happen and so need to go through and do the work.

---

10. Evaluate $\int\limits_C 2x^3\,ds$ for each of the following curves.

(a) C is the portion $y = x^3$ from $x = -1$ to $x = 2$.

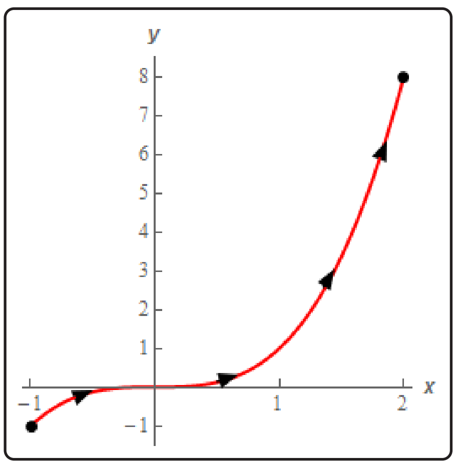
(b) C is the portion $y = x^3$ from $x = 2$ to $x = -1$.

Solutions

(a) C is the portion $y = x^3$ from $x = -1$ to $x = 2$.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



For reasons that will become apparent once we get to the second part of this problem let's call this curve C_1 instead of C. Here then is the parameterization of C_1.

$$C_1 : \vec{r}(t) = \langle t, t^3 \rangle \qquad -1 \le t \le 2$$

Step 2

Here is the line integral for this curve.

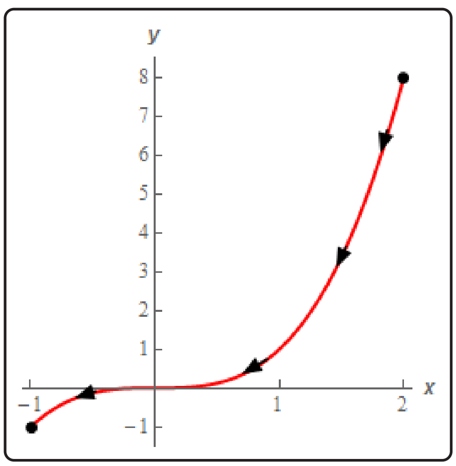
$$\begin{aligned}\int_{C_1} 2x^3\,ds &= \int_{-1}^{2} 2(t)^3\sqrt{(1)^2+(3t^2)^2}\,dt\\ &= \int_{-1}^{2} 2t^3\sqrt{1+9t^4}\,dt\\ &= \frac{1}{27}\left(1+9t^4\right)^{\frac{3}{2}}\Big|_{-1}^{2} = \boxed{\frac{1}{27}\left(145^{\frac{3}{2}}-10^{\frac{3}{2}}\right)=63.4966}\end{aligned}$$

(b) C is the portion $y=x^3$ from $x=2$ to $x=-1$.

Step 1

Now, as we did in the previous part let's "rename" this curve as C_2 instead of C.

Next, note that this curve is just the curve from the first step with opposite direction. In other words what we have here is that $C_2=-C_1$. Here is a quick sketch of C_2 for the sake of completeness.



Step 2

Now, at this point there are two different methods we could use to evaluate the integral.

The first method is use the fact from the notes that if we switch the directions for a curve then the value of this type of line integral doesn't change. Using this fact along with the relationship between the curve from this part and the curve from the first part, *i.e.* $C_2 = -C_1$, the line integral is just,

$$\int\limits_{C_2} 2x^3\,ds = \int\limits_{-C_1} 2x^3\,ds = \int\limits_{C_1} 2x^3\,ds = \boxed{\frac{1}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 63.4966}$$

Note that the first equal sign above was just acknowledging the relationship between the two curves. The second equal sign is where we used the fact from the notes.

This is the "easy" method for doing this problem. Alternatively we could parameterize up the curve and compute the line integral directly. We will do that for the rest of this problem just to show how we would go about doing that.

Step 3

Now, if we are going to parameterize this curve, and follow the indicated direction, we can't just use the parameterization from the first part and then "flip" the limits on the integral to "go backwards". Line integrals just don't work that way. The limits on the line integrals need to go from smaller value to larger value.

We need a new parameterization for this curve that will follow the curve in the indicated direction. Luckily that is actually pretty simple to do in this case. All we need to do is let $x = -t$ as t ranges from $t = -2$ to $t = 1$. In this way as t increases x will go from $x = 2$ to $x = -1$. In other words, at t increases x will decrease as we need it to in order to follow the direction of the curve.

Now that we have x taken care of the y is easy because we know the equation of the curve. To get the parametric equation for y all we need to do is plug in the parametric equation for x into the equation of the curve. Or,

$$y = (-t)^3 = -t^3$$

Putting all of this together we get the following parameterization of the curve.

$$C_2:\ \vec{r}(t) = \langle -t - t^3 \rangle \qquad -2 \le t \le 1$$

Step 4

Now all we need to do is compute the line integral.

$$\begin{aligned}\int\limits_{C_2} 2x^3\,ds &= \int_{-2}^{1} 2(-t)^3\sqrt{(-1)^2+(-3t^2)^2}\,dt\\ &= \int_{-2}^{1} -2t^3\sqrt{1+9t^4}\,dt\\ &= -\frac{1}{27}\left(1+9t^4\right)^{\frac{3}{2}}\Big|_{-2}^{1} = \boxed{-\frac{1}{27}\left(10^{\frac{3}{2}}-145^{\frac{3}{2}}\right) = 63.4966}\end{aligned}$$

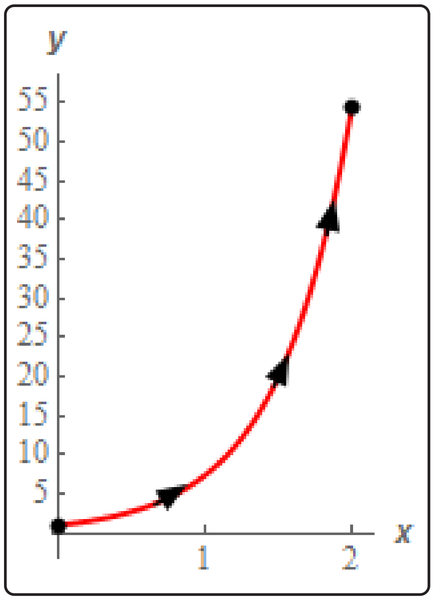
So, the line integral from this part had exactly the same value as the line integral from the first part as we expected it to.

16.3 Line Integrals - Part II

1. Evaluate $\displaystyle\int\limits_C \sqrt{1+y}\,dy$ where C is the portion of $y = \mathbf{e}^{2x}$ from $x = 0$ to $x = 2$.

Step 1

Here is a quick sketch of C with the direction specified in the problem statement shown.



Step 2

Next, we need to parameterize the curve.

$$\vec{r}(t) = \left\langle t, \mathbf{e}^{2t} \right\rangle \qquad 0 \le t \le 2$$

Step 3

Now we need to evaluate the line integral. Be careful with this type line integral. Note that the differential, in this case, is dy and not ds as they were in the previous section.

All we need to do is recall that $dy = y'\,dt$ when we convert the line integral into a "standard" integral.

So, let's evaluate the line integral. Just remember to "plug in" the parameterization into the integrand (*i.e.* replace the x and y in the integrand with the x and y components of the parameterization) and to convert the differential properly.

Here is the line integral.

$$\int_C \sqrt{1+y}\,dy = \int_0^2 \sqrt{1+\mathbf{e}^{2t}}\,\left(2\mathbf{e}^{2t}\right)\,dt$$

$$= \left[\frac{2}{3}\left(1+\mathbf{e}^{2t}\right)^{\frac{3}{2}}\right]\Bigg|_0^2 = \boxed{\frac{2}{3}\left[\left(1+\mathbf{e}^{4}\right)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] = 274.4897}$$

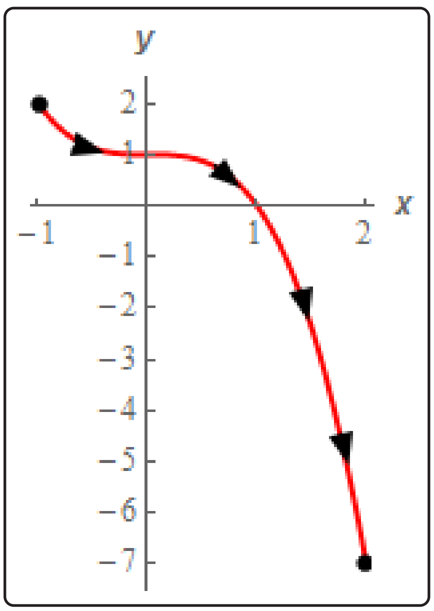
Note that, in this case, the integral ended up being a simple substitution.

---

2. Evaluate $\int_C 2y\,dx + (1-x)\,dy$ where C is portion of $y = 1 - x^3$ from $x = -1$ to $x = 2$.

Step 1

Here is a quick sketch of C with the direction specified in the problem statement shown.



Step 2

Next, we need to parameterize the curve.

$$\vec{r}(t) = \langle t, 1 - t^3 \rangle \qquad -1 \le t \le 2$$

Step 3

Now we need to evaluate the line integral. Be careful with this type of line integral. In this case we have both a dx and a dy in the integrand. Recall that this is just a simplified notation for,

$$\int\limits_C 2y\,dx + (1 - x)\,dy = \int\limits_C 2y\,dx + \int\limits_C 1 - x\,dy$$

Then all we need to do is recall that $dx = x'\,dt$ and $dy = y'\,dt$ when we convert the line integral into a "standard" integral.

So, let's evaluate the line integral. Just remember to "plug in" the parameterization into the integrand (*i.e.* replace the x and y in the integrand with the x and y components of the parameterization) and to convert the differentials properly.

Here is the line integral.

$$\begin{aligned}
\int\limits_C 2y\,dx + (1 - x)\,dy &= \int\limits_C 2y\,dx + \int\limits_C 1 - x\,dy \\
&= \int_{-1}^{2} 2\left(1 - t^3\right)\,(1)\,dt + \int_{-1}^{2} (1 - t)\,\left(-3t^2\right)\,dt \\
&= \int_{-1}^{2} 2\left(1 - t^3\right)\,dt - 3\int_{-1}^{2} t^2 - t^3\,dt \\
&= \int_{-1}^{2} t^3 - 3t^2 + 2\,dt \\
&= \left[\frac{1}{4}t^4 - t^3 + 2t\right]\Bigg|_{-1}^{2} = \boxed{\frac{3}{4}}
\end{aligned}$$

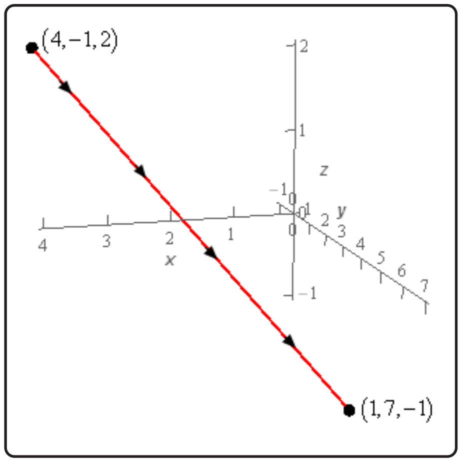
Note that, in this case, we combined the two integrals into a single integral prior to actually evaluating the integral. This doesn't need to be done but can, on occasion, simplify the integrand and hence the evaluation of the integral.

---

3. Evaluate $\displaystyle\int\limits_C x^2\,dy - yz\,dz$ where C is the line segment from $(4, -1, 2)$ to $(1, 7, -1)$.

Step 1

Here is a quick sketch of C with the direction specified in the problem statement shown.



Step 2

Next, we need to parameterize the curve.

$$\vec{r}(t) = (1-t)\langle 4,-1,2\rangle + t\langle 1,7,-1\rangle = \langle 4-3t,-1+8t,2-3t\rangle \qquad 0 \le t \le 1$$

Step 3

Now we need to evaluate the line integral. Be careful with this type of lines integral. In this case we have both a dy and a dz in the integrand. Recall that this is just a simplified notation for,

$$\int\limits_C x^2\,dy - yz\,dz = \int\limits_C x^2\,dy - \int\limits_C yz\,dz$$

Then all we need to do is recall that $dy = y'\,dt$ and $dz = z'\,dt$ when we convert the line integral into a "standard" integral.

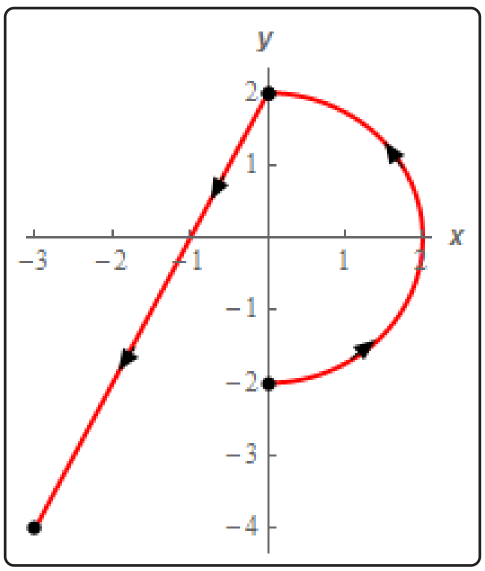
So, let's evaluate the line integral. Just remember to "plug in" the parameterization into the integrand (*i.e.* replace the x, y and z in the integrand with the x, y and z components of the parameterization) and to convert the differentials properly.

Here is the line integral.

$$\begin{aligned}\int_C x^2\,dy - yz\,dz &= \int_C x^2\,dy - \int_C yz\,dz \\ &= \int_0^1 (4-3t)^2(8)\,dt - \int_0^1 (-1+8t)\,(2-3t)\,(-3)\,dt \\ &= \int_0^1 8(4-3t)^2 - 3\left(24t^2 - 19t + 2\,dt\right)\,dt \\ &= \left[-\frac{8}{9}(4-3t)^3 - 3\left(8t^3 - \frac{19}{2}t^2 + 2t\right)\right]\Bigg|_0^1 = \boxed{\frac{109}{2}}\end{aligned}$$

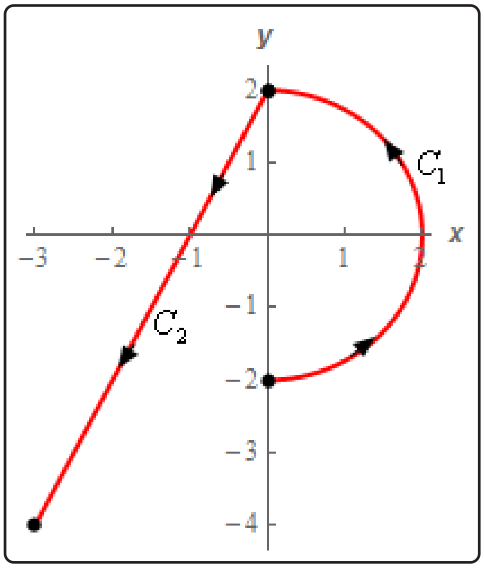
Note that, in this case, we combined the two integrals into a single integral prior to actually evaluating the integral. This doesn't need to be done but can, on occasion, simplify the integrand and hence the evaluation of the integral.

4. Evaluate $\displaystyle\int_C 1 + x^3\,dx$ where C is the right half of the circle of radius 2 with counter clockwise rotation followed by the line segment from $(0,2)$ to $(-3,-4)$. See the sketch below for the direction.



Step 1

To help with the problem let's label each of the curves as follows,



Now let's parameterize each of these curves.

$C_1 \;:\; \vec r(t) = \langle 2\cos(t), 2\sin(t)\rangle \qquad -\frac{1}{2}\pi \le t \le \frac{1}{2}\pi$ $C_2 \;:\; \vec r(t) = (1-t)\langle 0,2\rangle + t\langle -3,-4\rangle = \langle -3t, 2-6t\rangle \qquad 0 \le t \le 1$

Step 2

Now we need to compute the line integral for each of the curves.

$$\begin{aligned}\int_{C_1} 1 + x^3\,dx &= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \left[1 + (2\cos(t))^3\right](-2\sin(t))\,dt \\ &= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} -2\sin(t) - 16\cos^3(t)\sin(t)\,dt \\ &= \left(2\cos(t) + 4\cos^4(t)\right)\Big|_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} = \underline{0}\end{aligned}$$

$$\begin{aligned}\int_{C_2} 1 + x^3\,dx &= \int_0^1 \left[1 + (-3t)^3\right](-3)\,dt \\ &= \int_0^1 -3 + 81t^3\,dt = \left(-3t + \frac{81}{4}t^4\right)\Big|_0^1 = \underline{\frac{69}{4}}\end{aligned}$$

Don't forget to correctly deal with the differentials when converting the line integral into a "standard" integral.

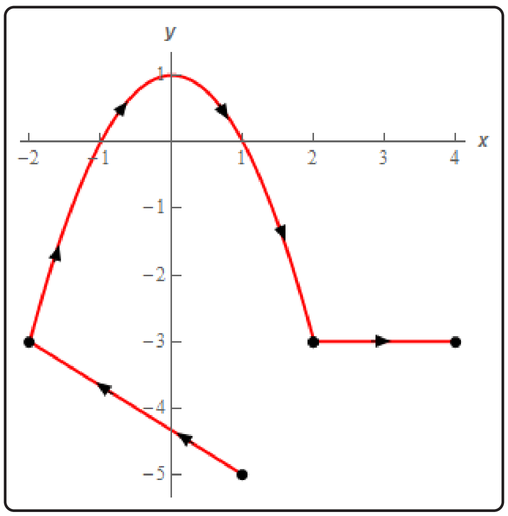
Step 3

Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int\limits_C 1 + x^3\,dx = (0) + \left(\frac{69}{4}\right) = \boxed{\frac{69}{4}}$$

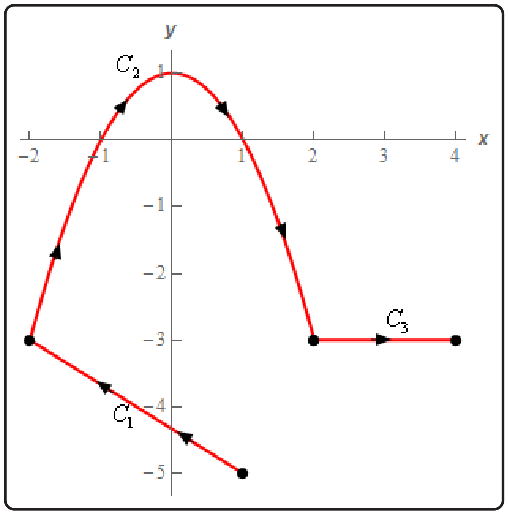
Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

5. Evaluate $\int\limits_C 2x^2\,dy - xy\,dx$ where C is the line segment from $(1,-5)$ to $(-2,-3)$ followed by the portion of $y = 1 - x^2$ from $x = -2$ to $x = 2$ which in turn is followed by the line segment from $(2,-3)$ to $(4,-3)$. See the sketch below for the direction.



Step 1

To help with the problem let's label each of the curves as follows,



Now let's parameterize each of these curves.

C_1 : $\vec{r}(t) = (1-t)\langle 1,-5\rangle + t\langle -2,-3\rangle = \langle 1-3t, -5+2t\rangle \qquad 0 \le t \le 1$ C_2 : $\vec{r}(t) = \langle t, 1-t^2\rangle \qquad -2 \le t \le 2$ C_3 : $\vec{r}(t) = \langle t,-3\rangle \qquad 2 \le t \le 4$

Note that for C_1 we had to use the vector form for the line segment between two points because the specified direction was in the decreasing x direction and so the equation of the line wouldn't work since the limits of the line integral need to go from smaller to larger values.

We did just use the equation of the line for C_3 since it was simple enough to do and the limits were also nice enough.

Step 2

Now we need to compute the line integral for each of the curves.

$$\begin{aligned}\int_{C_1} 2x^2\,dy - xy\,dx &= \int_{C_1} 2x^2\,dy - \int_{C_1} xy\,dx \\ &= \int_0^1 2(1-3t)^2(2)\,dt - \int_0^1 (1-3t)(-5+2t)(-3)\,dt \\ &= \int_0^1 4(1-3t)^2 - 3\left(6t^2-17t+5\right)\,dt \\ &= \left(-\frac{4}{9}(1-3t)^3 - 3\left(2t^3 - \frac{17}{2}t^2 + 5t\right)\right)\Bigg|_0^1 = \underline{\frac{17}{2}}\end{aligned}$$

$$\begin{aligned}\int_{C_2} 2x^2\,dy - xy\,dx &= \int_{C_2} 2x^2\,dy - \int_{C_2} xy\,dx \\ &= \int_{-2}^{2} 2(t)^2\left(-2t\right)\,dt - \int_{-2}^{2}\left(t\right)\left(1-t^2\right)\left(1\right)\,dt \\ &= \int_{-2}^{2} -3t^3 - t\,dt \\ &= \left(-\frac{3}{4}t^4 - \frac{1}{2}t^2\right)\Bigg|_{-2}^{2} = \underline{0}\end{aligned}$$

$$\begin{aligned}\int_{C_3} 2x^2\,dy - xy\,dx &= \int_{C_3} 2x^2\,dy - \int_{C_2} xy\,dx \\ &= \int_{2}^{4} 2(t)^2\left(0\right)\,dt - \int_{2}^{4}\left(t\right)\left(-3\right)\left(1\right)\,dt \\ &= \int_{2}^{4} 3t\,dt \\ &= \left(\frac{3}{2}t^2\right)\Bigg|_{2}^{4} = \underline{18}\end{aligned}$$

Don't forget to correctly deal with the differentials when converting the line integral into a "standard" integral.

Also, don't get excited when one of the differentials "evaluates" to zero as the first one did in the C_3 integral. That will happen on occasion and is not something to get worried about when it does.

Step 3

Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int_C 2x^2\,dy - xy\,dx = \left(\frac{17}{2}\right) + \left(0\right) + \left(18\right) = \boxed{\frac{53}{2}}$$

Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

6. Evaluate $\int\limits_C (x-y)\,dx - yx^2\,dy$ for each of the following curves.

   (a) C is the portion of the circle of radius 6 in the 1^{st}, 2^{nd} and 3^{rd} quadrant with clockwise rotation.

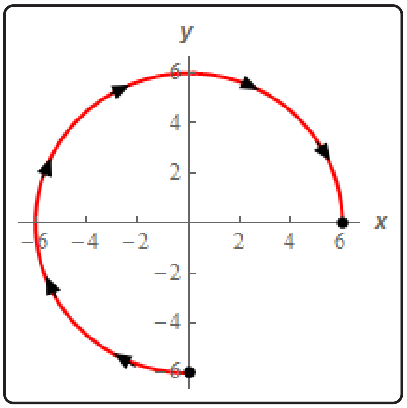
   (b) C is the line segment from $(0,-6)$ to $(6,0)$.

Solutions

(a) C is the portion of the circle of radius 6 in the 1^{st}, 2^{nd} and 3^{rd} quadrant with clockwise rotation.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



Here is the parameterization for this curve.

$$C: \vec{r}(t) = \langle 6\cos(t), -6\sin(t)\rangle \qquad \frac{1}{2}\pi \le t \le 2\pi$$

Step 2

Here is the line integral for this curve.

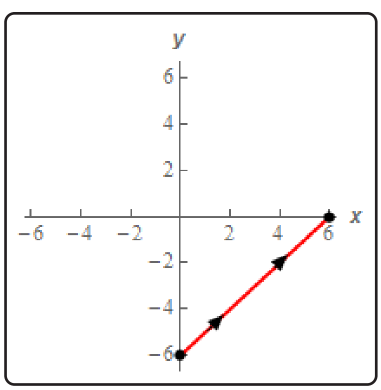
$$\begin{aligned}\int\limits_C (x-y)\,dx - yx^2\,dy &= \int\limits_C (x-y)\,dx - \int\limits_C yx^2\,dy \\ &= \int_{\frac{1}{2}\pi}^{2\pi} [6\cos(t) - (-6\sin(t))]\,(-6\sin(t))\,dt \\ &\qquad - \int_{\frac{1}{2}\pi}^{2\pi} \left[(6\sin(t))\,(6\cos(t))^2\right](-6\cos(t))\,dt \\ &= \int_{\frac{1}{2}\pi}^{2\pi} -36\cos(t)\sin(t) - 36\sin^2(t) - 1296\sin(t)\cos^3(t)\,dt \\ &= \int_{\frac{1}{2}\pi}^{2\pi} -18\sin(2t) - 18(1-\cos(2t)) - 1296\sin(t)\cos^3(t)dt \\ &= \left(9\cos(2t) - 18t + 9\sin(2t) + 324\cos^4(t)\right)\Big|_{\frac{1}{2}\pi}^{2\pi} \\ &= \boxed{342 - 27\pi = 257.1770}\end{aligned}$$

Don't forget to correctly deal with the differentials when converting the line integral into a "standard" integral.

(b) C is the line segment from $(0,-6)$ to $(6,0)$.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



So, what we have in this part is a different curve that goes from $(0,-6)$ to $(6,0)$.

Despite the fact that this curve has the same starting and ending point as the curve in the first part there is no reason to expect the line integral to have the same value. Therefore we'll need to go through the work and see what we get from the line integral.

We'll need to parameterize the curve so let's take care of that.

$$C: \ \vec{r}(t) = (1-t)\langle 0,-6\rangle + t\langle 6,0\rangle = \langle 6t, -6+6t\rangle \qquad 0 \le t \le 1$$

Note that we could have just found the equation of this curve but it seemed just as easy to just use the vector form of the line segment between two points.

Step 2

Now all we need to do is compute the line integral.

$$\begin{aligned}\int_C (x-y)\,dx - yx^2\,dy &= \int_C (x-y)\,dx - \int_C yx^2\,dy \\ &= \int_0^1 [6t-(-6+6t)](6)\,dt - \int_0^1 \left[(-6+6t)(6t)^2\right](6)\,dt \\ &= \int_0^1 36 + 1296t^2 - 1296t^3\,dt \\ &= \left(36t + 432t^3 - 324t^4\right)\Big|_0^1 = \boxed{144}\end{aligned}$$

So, as noted at the start of this part the value of the line integral was not the same as the value of the line integral in the first part despite the same starting and ending points for the curve. Note that it is possible for two line integrals with the same starting and ending points to have the same value but we can't expect that to happen and so need to go through and do the work.

7. Evaluate $\displaystyle\int_C x^3\,dy - (y+1)\,dx$ for each of the following curves.

(a) C is the line segment from $(1,7)$ to $(-2,4)$.

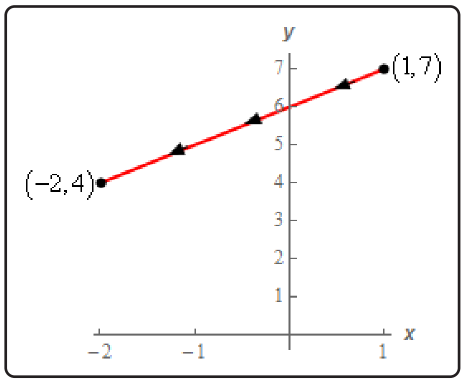
(b) C is the line segment from $(-2,4)$ to $(1,7)$.

Solutions

(a) C is the line segment from $(1,7)$ to $(-2,4)$.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



For reasons that will become apparent once we get to the second part of this problem let's call this curve C_1 instead of C. Here then is the parameterization of C_1.

$$C_1 : \ \vec{r}(t) = (1-t)\langle 1,7\rangle + t\langle -2,4\rangle = \langle 1-3t, 7-3t\rangle \qquad 0 \le t \le 1$$

Step 2

Here is the line integral for this curve.

$$\begin{aligned}\int_{C_1} x^3\,dy - (y+1)\,dx &= \int_{C_1} x^3\,dy - \int_{C_1} y+1\,dx \\ &= \int_0^1 (1-3t)^3(-3)\,dt - \int_0^1 (7-3t+1)(-3)\,dt \\ &= \int_0^1 -3(1-3t)^3 + 3(8-3t)\,dt \\ &= \left[\frac{1}{4}(1-3t)^4 + 24t - \frac{9}{2}t^2\right]\Big|_0^1 = \boxed{\frac{93}{4}}\end{aligned}$$

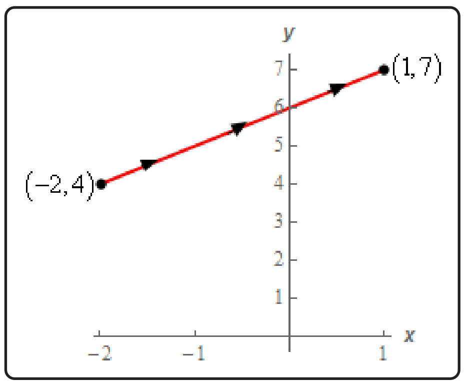
Don't forget to correctly deal with the differentials when converting the line integral into a "standard" integral.

(b) C is the line segment from $(-2, 4)$ to $(1, 7)$.

Step 1

Now, as we did in the previous part let's "rename" this curve as C_2 instead of C.

Next, note that this curve is just the curve from the first step with opposite direction. In other words what we have here is that $C_2 = -C_1$. Here is a quick sketch of C_2 for the sake of completeness.



Step 2

Now, at this point there are two different methods we could use to evaluate the integral.

The first method is use the fact from the notes that if we switch the direction of a curve then the value of this type of line integral will just change signs. Using this fact along with the relationship between the curve from this part and the curve from the first part, *i.e.* $C_2 = -C_1$, the line integral is just,

$$\int_{C_2} x^3\,dy - (y+1)\,dx = \int_{-C_1} x^3\,dy - (y+1)\,dx = -\int_{C_1} x^3\,dy - (y+1)\,dx$$

$$= \boxed{-\frac{93}{4}}$$

Note that the first equal sign above was just acknowledging the relationship between the two curves. The second equal sign is where we used the fact from the notes.

This is the “easy” method for doing this problem. Alternatively, we could parameterize up the curve and compute the line integral directly. We will do that for the rest of this problem just to show how we would go about doing that.

Step 3

Here is the parameterization for this curve.

$$C_2: \ \vec{r}(t) = (1-t)\langle -2, 4\rangle + t\langle 1, 7\rangle = \langle -2+3t, 4+3t\rangle \qquad 0 \le t \le 1$$

Step 4

Now all we need to do is compute the line integral.

$$\begin{aligned}\int_{C_2} x^3\,dy - (y+1)\,dx &= \int_{C_2} x^3\,dy - \int_{C_2} y+1\,dx \\ &= \int_0^1 (-2+3t)^3(3)\,dt - \int_0^1 (4+3t+1)(3)\,dt \\ &= \int_0^1 3(-2+3t)^3 - 3(5+3t)\,dt \\ &= \left[\frac{1}{4}(-2+3t)^4 - 15t - \frac{9}{2}t^2\right]\Big|_0^1 = \boxed{-\frac{93}{4}}\end{aligned}$$

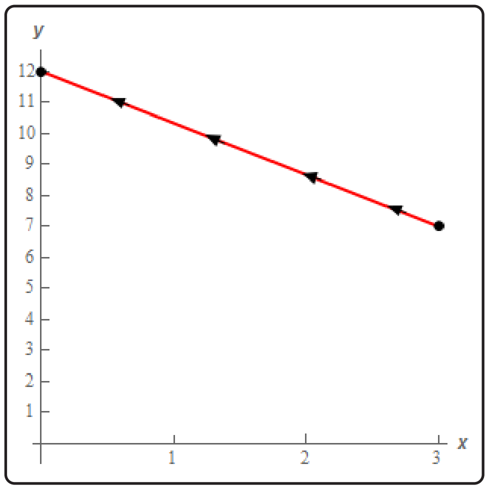
So, the line integral from this part had the same value, except for the sign, as the line integral from the first part as we expected it to.

16.4 Line Integrals of Vector Fields

1. Evaluate $\int\limits_C \vec F \cdot d\vec r$ where $\vec F(x,y) = y^2\,\vec i + (3x - 6y)\,\vec j$ and C is the line segment from $(3,7)$ to $(0,12)$.

Step 1

Here is a quick sketch of C with the direction specified in the problem statement shown.



Step 2

Next, we need to parameterize the curve.

$$\vec r(t) = (1-t)\langle 3,7\rangle + t\langle 0,12\rangle = \langle 3-3t, 7+5t\rangle \qquad 0 \le t \le 1$$

Step 3

In order to evaluate this line integral we'll need the dot product of the vector field (evaluated at the along the curve) and the derivative of the parameterization. Here is the vector field evaluated along the curve (*i.e.* plug in x and y from the parameterization into the vector field).

$$\vec F(\vec r(t)) = (7+5t)^2\,\vec i + (3(3-3t) - 6(7+5t))\,\vec j = (7+5t)^2\,\vec i + (-33-39t)\,\vec j$$

The derivative of the parameterization is,

$$\vec{r}'(t) = \langle -3, 5 \rangle$$

Finally, the dot product of the vector field and the derivative of the parameterization.

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = -3(7+5t)^2 - 5(33+39t)$$

Step 4

Now all we need to do is evaluate the integral.

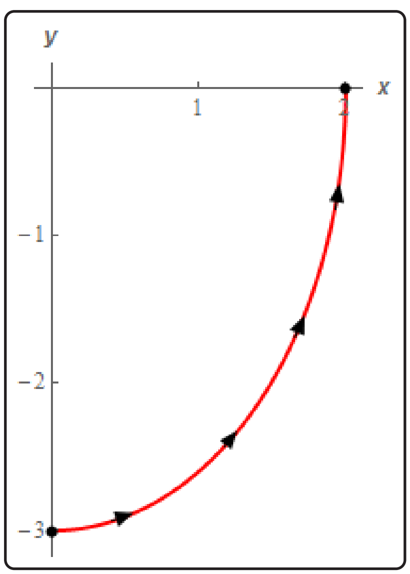
$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 -3(7+5t)^2 - 5(33+39t)\, dt$$

$$= \left[-\frac{1}{5}(7+5t)^3 - 165t - \frac{195}{2}t^2 \right]\Bigg|_0^1 = \boxed{-\frac{1079}{2}}$$

2. Evaluate $\int\limits_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = (x+y)\,\vec{i} + (1-x)\,\vec{j}$ and C is the portion of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that is in the 4^{th} quadrant with the counter clockwise rotation.

Step 1

Here is a quick sketch of C with the direction specified in the problem statement shown.



Step 2

Next, we need to parameterize the curve.

$$\vec r\left(t\right) = \left\langle 2\cos\left(t\right), 3\sin\left(t\right)\right\rangle \qquad \frac{3}{2}\pi \le t \le 2\pi$$

Step 3

In order to evaluate this line integral we'll need the dot product of the vector field (evaluated at the along the curve) and the derivative of the parameterization.

Here is the vector field evaluated along the curve (*i.e.* plug in x and y from the parameterization into the vector field).

$$\vec F\left(\vec r\left(t\right)\right) = \left(2\cos\left(t\right) + 3\sin\left(t\right)\right)\vec i + \left(1 - 2\cos\left(t\right)\right)\vec j$$

The derivative of the parameterization is,

$$\vec r\,'\left(t\right) = \left\langle -2\sin\left(t\right), 3\cos\left(t\right)\right\rangle$$

Finally, the dot product of the vector field and the derivative of the parameterization.

$$\begin{aligned}\vec F\left(\vec r\left(t\right)\right)\cdot\vec r\,'\left(t\right) &= \left(2\cos\left(t\right) + 3\sin\left(t\right)\right)\left(-2\sin\left(t\right)\right) + \left(1 - 2\cos\left(t\right)\right)\left(3\cos\left(t\right)\right)\\ &= -4\cos\left(t\right)\sin\left(t\right) - 6\sin^2\left(t\right) + 3\cos\left(t\right) - 6\cos^2\left(t\right)\\ &= -4\cos\left(t\right)\sin\left(t\right) - 6\left[\sin^2\left(t\right) + \cos^2\left(t\right)\right] + 3\cos\left(t\right)\\ &= -2\sin\left(2t\right) + 3\cos\left(t\right) - 6\end{aligned}$$

Make sure that you simplify the dot product with an eye towards doing the integral! In this case that meant using the double angle formula for sine to "simplify" the first term for the integral.

Step 4

Now all we need to do is evaluate the integral.

$$\int\limits_C \vec F \cdot d\vec r = \int_{\frac{3}{2}\pi}^{2\pi} -2\sin(2t) + 3\cos(t) - 6\,dt$$

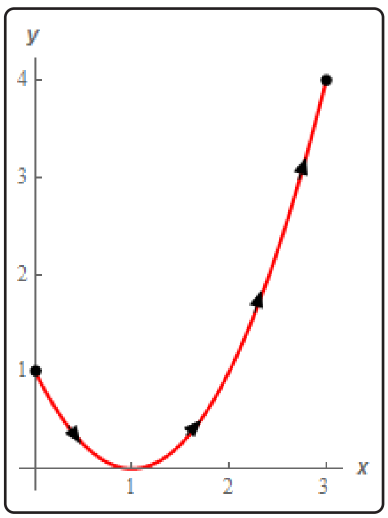
$$= \left[\cos(2t) + 3\sin(t) - 6t\right]\Big|_{\frac{3}{2}\pi}^{2\pi} = \boxed{5 - 3\pi}$$

---

3. Evaluate $\int\limits_C \vec F \cdot d\vec r$ where $\vec F(x,y) = y^2\,\vec i + (x^2 - 4)\,\vec j$ and C is the portion of $y = (x-1)^2$ from $x = 0$ to $x = 3$.

Step 1

Here is a quick sketch of C with the direction specified in the problem statement shown.



Step 2

Next, we need to parameterize the curve.

$$\vec r(t) = \left\langle t, (t-1)^2 \right\rangle \qquad 0 \le t \le 3$$

Step 3

In order to evaluate this line integral we'll need the dot product of the vector field (evaluated at the along the curve) and the derivative of the parameterization.

Here is the vector field evaluated along the curve (*i.e.* plug in x and y from the parameterization into the vector field).

$$\vec{F}\left(\vec{r}\left(t\right)\right)=\left[\left(t-1\right)^{2}\right]^{2}\vec{i}+\left(\left(t\right)^{2}-4\right)\vec{j}=\left(t-1\right)^{4}\vec{i}+\left(t^{2}-4\right)\vec{j}$$

The derivative of the parameterization is,

$$\vec{r}'\left(t\right)=\left\langle 1,2\left(t-1\right)\right\rangle$$

Finally, the dot product of the vector field and the derivative of the parameterization.

$$\begin{aligned}\vec{F}\left(\vec{r}\left(t\right)\right)\cdot\vec{r}'\left(t\right)&=\left(t-1\right)^{4}\left(1\right)+\left(t^{2}-4\right)\left(2t-2\right)\\&=\left(t-1\right)^{4}+2t^{3}-2t^{2}-8t+8\end{aligned}$$

Make sure that you simplify the dot product with an eye towards doing the integral!

Step 4

Now all we need to do is evaluate the integral.

$$\begin{aligned}\int_{C}\vec{F}\cdot d\vec{r}&=\int_{0}^{3}\left(t-1\right)^{4}+2t^{3}-2t^{2}-8t+8\,dt\\&=\left[\frac{1}{5}\left(t-1\right)^{5}+\frac{1}{2}t^{4}-\frac{2}{3}t^{3}-4t^{2}+8t\right]\Bigg|_{0}^{3}=\boxed{\frac{171}{10}}\end{aligned}$$

4. Evaluate $\int_{C}\vec{F}\cdot d\vec{r}$ where $\vec{F}\left(x,y,z\right)=\mathbf{e}^{2x}\,\vec{i}+z\left(y+1\right)\vec{j}+z^{3}\,\vec{k}$ and C is given by $\vec{r}\left(t\right)=t^{3}\,\vec{i}+\left(1-3t\right)\vec{j}+\mathbf{e}^{t}\,\vec{k}$ for $0\leq t\leq 2$.

Step 1

Okay, for this problem we've been given the parameterization of the curve in the problem statement so we don't need to worry about that for this problem and we can jump right into the work needed to evaluate the line integral. This means that we'll need the dot product of the vector field (evaluated at the along the curve) and the derivative of the parameterization.

Here is the vector field evaluated along the curve (*i.e.* plug in x and y from the parameterization into the vector field).

$$\vec{F}\left(\vec{r}\left(t\right)\right) = \mathbf{e}^{2t^3}\,\vec{i} + \mathbf{e}^{t}\left(1 - 3t + 1\right)\vec{j} + \left(\mathbf{e}^{t}\right)^3\vec{k} = \mathbf{e}^{2t^3}\,\vec{i} + \mathbf{e}^{t}\left(2 - 3t\right)\vec{j} + \mathbf{e}^{3t}\,\vec{k}$$

The derivative of the parameterization is,

$$\vec{r}\,'\left(t\right) = 3t^2\,\vec{i} - 3\vec{j} + \mathbf{e}^{t}\,\vec{k}$$

Finally, the dot product of the vector field and the derivative of the parameterization.

$$\begin{aligned}\vec{F}\left(\vec{r}\left(t\right)\right)\cdot\vec{r}\,'\left(t\right) &= \mathbf{e}^{2t^3}\left(3t^2\right) + \mathbf{e}^{t}\left(2 - 3t\right)\left(-3\right) + \mathbf{e}^{3t}\left(\mathbf{e}^{t}\right)\\ &= 3t^2\mathbf{e}^{2t^3} - 3\mathbf{e}^{t}\left(2 - 3t\right) + \mathbf{e}^{4t}\end{aligned}$$

Make sure that you simplify the dot product with an eye towards doing the integral!

Step 2

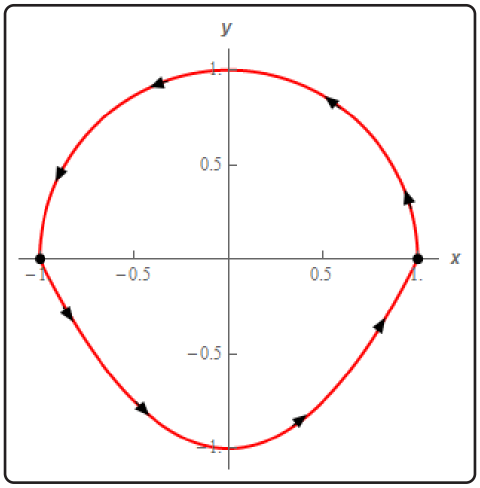
Now all we need to do is evaluate the integral.

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^2 3t^2\mathbf{e}^{2t^3} - 3\mathbf{e}^{t}\left(2 - 3t\right) + \mathbf{e}^{4t}\,dt\\ &= \left[\frac{1}{2}\mathbf{e}^{2t^3} - 3\mathbf{e}^{t}\left(5 - 3t\right) + \frac{1}{4}\mathbf{e}^{4t}\,dt\right]\Bigg|_0^2 = \boxed{\frac{57}{4} + 3\mathbf{e}^2 + \frac{1}{4}\mathbf{e}^8 + \frac{1}{2}\mathbf{e}^{16}}\end{aligned}$$

Note that the second term in the integral involved integration by parts (you do recall how to do that right?). We'll leave the integration by parts details to you to verify and note that we did simplify the results a little bit.

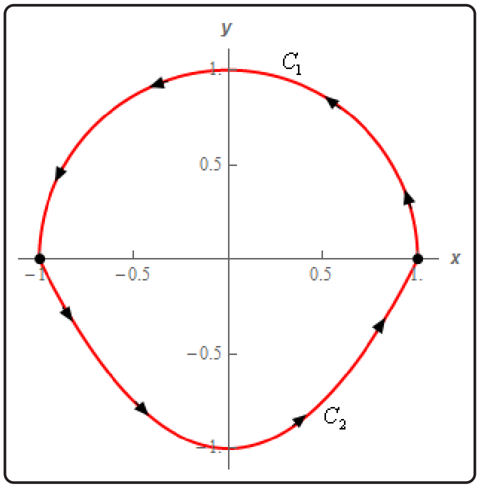
Also, do not get excited about the "messy" answer here! You will get these kinds of answers on occasion.

5. Evaluate $\displaystyle\int\limits_C \vec F \cdot d\vec r$ where $\vec F(x,y) = 3y\,\vec i + (x^2 - y)\,\vec j$ and C is the upper half of the circle centered at the origin of radius 1 with counter clockwise rotation and the portion of $y = x^2 - 1$ from $x = -1$ to $x = 1$. See the sketch below.



Step 1

To help with the problem let's label each of the curves as follows,



The parameterization of each curve is,

$$C_1 : \ \vec r(t) = \langle \cos(t), \sin(t) \rangle \qquad 0 \le t \le \pi \quad C_2 : \ \vec r(t) = \left\langle t, t^2 - 1 \right\rangle \qquad -1 \le t \le 1$$

Step 2

Now we need to compute the line integral for each of the curves. In the first few problems in this section we evaluated the vector function along the curve, took the derivative of the parameterization and computed the dot product separately. For this problem we'll be doing all that work in the integral itself.

Here is the line integral for each of the curves.

$$\begin{aligned}\int_{C_1} \vec{F}\cdot d\vec{r} &= \int_0^{\pi} \left\langle 3\sin(t)\,, \cos^2(t) - \sin(t)\right\rangle \cdot \left\langle -\sin(t)\,, \cos(t)\right\rangle\,dt\\ &= \int_0^{\pi} -3\sin^2(t) + \cos^3(t) - \sin(t)\cos(t)\,dt\\ &= \int_0^{\pi} -\frac{3}{2}\left(1-\cos(2t)\right) + \cos(t)\left(1-\sin^2(t)\right) - \frac{1}{2}\sin(2t)\,dt\\ &= \left(-\frac{3}{2}\left(t - \frac{1}{2}\sin(2t)\right) + \sin(t) - \frac{1}{3}\sin^3(t) + \frac{1}{4}\cos(2t)\,dt\right)\Bigg|_0^{\pi} = \underline{-\frac{3}{2}\pi}\end{aligned}$$

$$\begin{aligned}\int_{C_2} \vec{F}\cdot d\vec{r} &= \int_{-1}^{1} \left\langle 3\left(t^2-1\right), t^2 - \left(t^2-1\right)\right\rangle \cdot \left\langle 1, 2t\right\rangle\,dt\\ &= \int_{-1}^{1} 3\left(t^2-1\right) + 2t\,dt\\ &= \left(t^3 - 3t + t^2\right)\Big|_{-1}^{1} = \underline{-4}\end{aligned}$$

You do recall how to deal with all those trig functions that we saw in the first integral don't you? If not you should go back to the Calculus II material and work some practice problems. You'll be seeing a fair number of integrals involving trig functions from this point on.

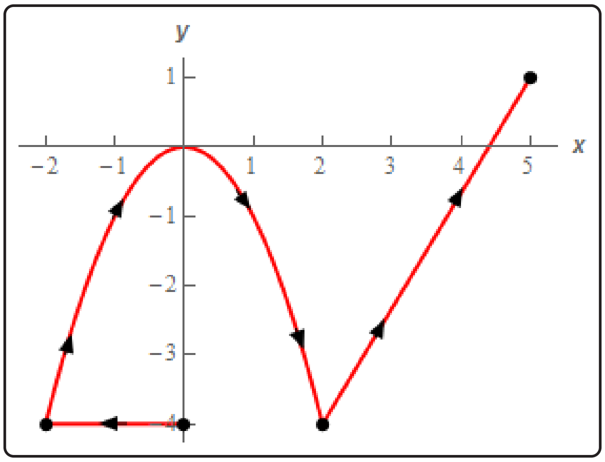
Step 3

Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int_C \vec{F}\cdot d\vec{r} = \left(-\frac{3}{2}\pi\right) + (-4) = \boxed{-4 - \frac{3}{2}\pi = -8.7124}$$

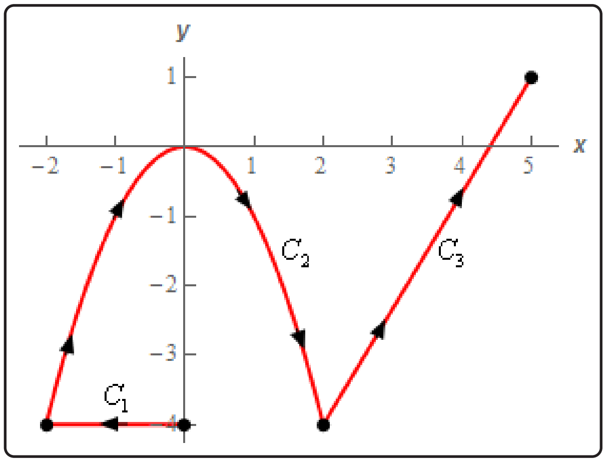
Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

6. Evaluate $\int\limits_C \vec F \cdot d\vec r$ where $\vec F(x,y) = xy\,\vec i + (1+3y)\,\vec j$ and C is the line segment from $(0,-4)$ to $(-2,-4)$ followed by portion of $y=-x^2$ from $x=-2$ to $x=2$ which is in turn followed by the line segment from $(2,-4)$ to $(5,1)$. See the sketch below.



Step 1

To help with the problem let's label each of the curves as follows,



The parameterization of each curve is,

$C_1: \ \vec r(t) = (1-t)\langle 0,-4\rangle + t\langle -2,-4\rangle = \langle -2t,-4\rangle \qquad 0 \le t \le 1$

$C_2: \ \vec r(t) = \langle t,-t^2\rangle \qquad -2 \le t \le 2$

$C_3: \ \vec r(t) = (1-t)\langle 2,-4\rangle + t\langle 5,1\rangle = \langle 2+3t,-4+5t\rangle \qquad 0 \le t \le 1$

Step 2

Now we need to compute the line integral for each of the curves. In the first few problems in this section we evaluated the vector function along the curve, took the derivative of the parameterization and computed the dot product separately. For this problem we'll be doing all that work in the integral itself.

Here is the line integral for each of the curves.

$$\begin{aligned}\int_{C_1} \vec{F}\cdot d\vec{r} &= \int_0^1 \left\langle (-2t)(-4), 1+3(-4)\right\rangle \cdot \langle -2, 0\rangle \, dt \\ &= \int_0^1 -16t\,dt = \left(-8t^2\right)\Big|_0^1 = \underline{-8}\end{aligned}$$

$$\begin{aligned}\int_{C_2} \vec{F}\cdot d\vec{r} &= \int_{-2}^2 \left\langle (t)\left(-t^2\right), 1+3\left(-t^2\right)\right\rangle \cdot \langle 1, -2t\rangle \, dt \\ &= \int_{-2}^2 5t^3 - 2t\,dt = \left(\frac{5}{4}t^4 - t^2\right)\Big|_{-2}^2 = \underline{0}\end{aligned}$$

$$\begin{aligned}\int_{C_3} \vec{F}\cdot d\vec{r} &= \int_0^1 \left\langle (2+3t)(-4+5t), 1+3(-4+5t)\right\rangle \cdot \langle 3, 5\rangle \, dt \\ &= \int_0^1 45t^2 + 69t - 79\,dt = \left(15t^3 + \frac{69}{2}t^2 - 79t\right)\Big|_0^1 = \underline{-\frac{59}{2}}\end{aligned}$$

Step 3

Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int_C \vec{F}\cdot d\vec{r} = (-8) + (0) + \left(-\frac{59}{2}\right) = \boxed{-\frac{75}{2}}$$

Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

7. Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = (6x - 2y)\,\vec{i} + x^2\vec{j}$ for each of the following curves.

(a) C is the line segment from $(6, -3)$ to $(0, 0)$ followed by the line segment from $(0, 0)$ to $(6, 3)$.

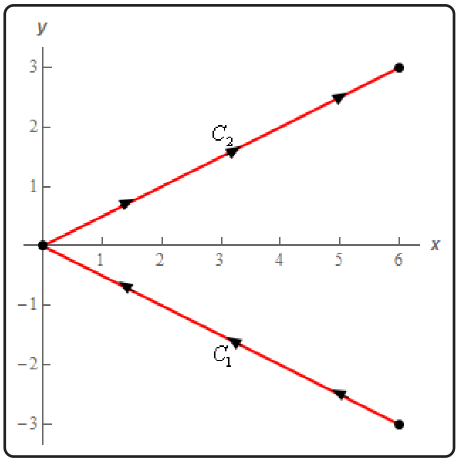
(b) C is the line segment from $(6, -3)$ to $(6, 3)$.

Solutions

(a) C is the line segment from $(6, -3)$ to $(0, 0)$ followed by the line segment from $(0, 0)$ to $(6, 3)$.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



Here is the parameterization for each of these curves.

$$C_1: \;\vec{r}(t) = (1 - t)\langle 6, -3\rangle + t\langle 0, 0\rangle = \langle 6 - 6t, -3 + 3t\rangle \qquad 0 \le t \le 1$$

$$C_2: \;\vec{r}(t) = \left\langle t, \tfrac{1}{2}t\right\rangle \qquad 0 \le t \le 6$$

For C_2 we used the equation of the line to get the parameterization because it gave a slightly nicer form to work with. We couldn't do this with C_1 because the specified direction of the curve was in the decreasing x direction and the limits of the integral need to be from smaller value to larger value.

Step 2

Here is the line integral for each of these curves.

$$\begin{aligned}\int\limits_{C_1} \vec{F}\cdot d\vec{r} &= \int_0^1 \left\langle 6\left(6-6t\right)-2\left(-3+3t\right),\left(6-6t\right)^2\right\rangle\cdot\left\langle -6,3\right\rangle\,dt\\ &= \int_0^1 252t-252+3(6-6t)^2\,dt\\ &= \left(126t^2-252t-\frac{1}{6}(6-6t)^3\right)\Bigg|_0^1 = \underline{-90}\end{aligned}$$

$$\begin{aligned}\int\limits_{C_2} \vec{F}\cdot d\vec{r} &= \int_0^6 \left\langle 6\left(t\right)-2\left(\frac{1}{2}t\right),\left(t\right)^2\right\rangle\cdot\left\langle 1,\frac{1}{2}\right\rangle\,dt\\ &= \int_0^6 5t+\frac{1}{2}t^2\,dt = \left(\frac{5}{2}t^2+\frac{1}{6}t^3\right)\Bigg|_0^6 = \underline{126}\end{aligned}$$

Step 3

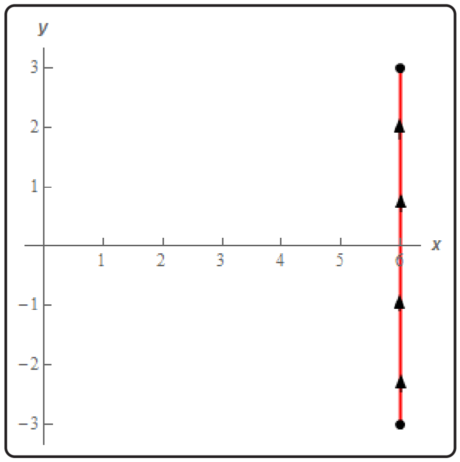
The line integral for this part is then,

$$\int\limits_{C} \vec{F}\cdot d\vec{r} = (-90)+(126) = \boxed{36}$$

(b) C is the line segment from $(6,-3)$ to $(6,3)$.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



So, what we have in this part is a different curve that goes from $(6, -3)$ to $(6, 3)$. Despite the fact that this curve has the same starting and ending point as the curve in the first part there is no reason to expect the line integral to have the same value. Therefore, we'll need to go through the work and see what we get from the line integral.

We'll need to parameterize the curve so let's take care of that.

$$C: \ \vec{r}(t) = \langle 6, t \rangle \qquad -3 \le t \le 3$$

Step 2

Now all we need to do is compute the line integral.

$$\int\limits_C \vec{F} \cdot d\vec{r} = \int_{-3}^{3} \left\langle 6(6) - 2(t), (6)^2 \right\rangle \cdot \langle 0, 1 \rangle \, dt$$

$$= \int_{-3}^{3} 36 \, dt = (36t)\Big|_{-3}^{3} = \boxed{216}$$

So, as noted at the start of this part the value of the line integral was not the same as the value of the line integral in the first part despite the same starting and ending points for the curve. Note that it is possible for two line integrals with the same starting and ending points to have the same value but we can't expect that to happen and so need to go through and do the work.

8. Evaluate $\int\limits_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = 3\,\vec{i} + (xy - 2x)\,\vec{j}$ for each of the following curves.

(a) C is the upper half of the circle centered at the origin of radius 4 with counter clockwise rotation.

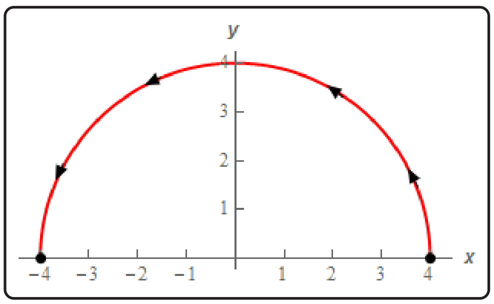
(b) C is the upper half of the circle centered at the origin of radius 4 with clockwise rotation.

Solutions

(a) C is the upper half of the circle centered at the origin of radius 4 with counter clockwise rotation.

Step 1

Let's start off with a quick sketch of the curve for this part of the problem.



For reasons that will become apparent once we get to the second part of this problem let's call this curve C_1 instead of C. Here then is the parameterization of C_1.

$$C_1 : \ \vec{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle \qquad 0 \le t \le \pi$$

Step 2

Here is the line integral for this curve.

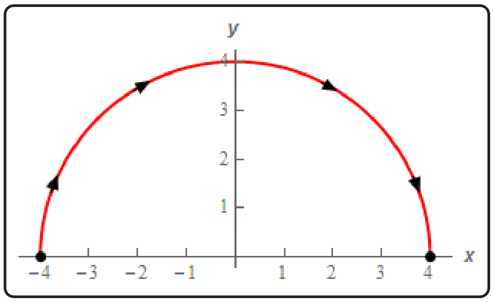
$$\begin{aligned}\int\limits_{C_1} \vec{F} \cdot d\vec{r} &= \int_0^\pi \langle 3, (4\cos(t))(4\sin(t)) - 2(4\cos(t))\rangle \cdot \langle -4\sin(t), 4\cos(t)\rangle \, dt \\ &= \int_0^\pi -12\sin(t) + 64\sin(t)\cos^2(t) - 32\cos^2(t)\, dt \\ &= \int_0^\pi -12\sin(t) + 64\sin(t)\cos^2(t) - 16(1+\cos(2t))\, dt \\ &= \left(12\cos(t) - \frac{64}{3}\cos^3(t) - 16t - 8\sin(2t)\right)\Bigg|_0^\pi \\ &= \boxed{\frac{56}{3} - 16\pi = -31.5988}\end{aligned}$$

(b) C is the upper half of the circle centered at the origin of radius 4 with clockwise rotation.

Step 1

Now, as we did in the previous part let's "rename" this curve as C_2 instead of C.

Next, note that this curve is just the curve from the first step with opposite direction. In other words what we have here is that $C_2 = -C_1$. Here is a quick sketch of C_2 for the sake of completeness.



Step 2

Now, at this point there are two different methods we could use to evaluate the integral.

The first method is use the fact from the notes that if we switch the direction of a curve then the value of this type of line integral will just change signs. Using this fact along with the relationship between the curve from this part and the curve from the first part, *i.e.* $C_2 = -C_1$, the line integral is just,

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{-C_1} \vec{F} \cdot d\vec{r} = -\int_{C_1} \vec{F} \cdot d\vec{r} = \boxed{16\pi - \frac{56}{3} = 31.5988}$$

Note that the first equal sign above was just acknowledging the relationship between the two curves. The second equal sign is where we used the fact from the notes.

This is the "easy" method for doing this problem. Alternatively, we could parameterize up the curve and compute the line integral directly. We will do that for the rest of this problem just to show how we would go about doing that.

Step 3

Here is the parameterization for this curve.

$$C_2: \ \vec{r}(t) = \langle -4\cos(t), 4\sin(t)\rangle \qquad 0 \le t \le \pi$$

Step 4

Now all we need to do is compute the line integral.

$$\begin{aligned}
\int_{C_2} \vec{F}\cdot d\vec{r} &= \int_0^{\pi} \langle 3, (-4\cos(t))(4\sin(t)) - 2(-4\cos(t))\rangle \cdot \langle 4\sin(t), 4\cos(t)\rangle \, dt \\
&= \int_0^{\pi} 12\sin(t) - 64\sin(t)\cos^2(t) + 32\cos^2(t)\, dt \\
&= \int_0^{\pi} 12\sin(t) - 64\sin(t)\cos^2(t) + 16(1+\cos(2t))\, dt \\
&= \left(-12\cos(t) + \frac{64}{3}\cos^3(t) + 16t + 8\sin(2t)\right)\Big|_0^{\pi} \\
&= \boxed{16\pi - \frac{56}{3} = 31.5988}
\end{aligned}$$

So, the line integral from this part had the same value, except for the sign, as the line integral from the first part as we expected it to.

16.5 Fundamental Theorem for Line Integrals

1. Evaluate $\displaystyle\int\limits_C \nabla f \cdot d\vec r$ where $f(x,y) = x^3\left(3 - y^2\right) + 4y$ and C is given by $\vec r(t) = \langle 3 - t^2, 5 - t\rangle$ with $-2 \le t \le 3$.

Step 1

There really isn't all that much to do with this problem. We are integrating over a gradient vector field and so the integral is set up to use the Fundamental Theorem for Line Integrals.

To do that we'll need the following two "points".

$$\vec r(-2) = \langle -1, 7\rangle \qquad\qquad \vec r(3) = \langle -6, 2\rangle$$

Remember that we are thinking of these as the position vector representations of the points $(-1,7)$ and $(-6,2)$ respectively.

Step 2

Now simply apply the Fundamental Theorem to evaluate the integral.

$$\int\limits_C \nabla f \cdot d\vec r = f(\vec r(3)) - f(\vec r(-2)) = f(-6,2) - f(-1,7) = 224 - 74 = \boxed{150}$$

2. Evaluate $\displaystyle\int\limits_C \nabla f \cdot d\vec r$ where $f(x,y) = y\mathbf{e}^{x^2-1} + 4x\sqrt{y}$ and C is given by $\vec r(t) = \langle 1 - t, 2t^2 - 2t\rangle$ with $0 \le t \le 2$.

Step 1

There really isn't all that much to do with this problem. We are integrating over a gradient vector field and so the integral is set up to use the Fundamental Theorem for Line Integrals.

To do that we'll need the following two "points".

$$\vec r(0) = \langle 1, 0\rangle \qquad\qquad \vec r(2) = \langle -1, 4\rangle$$

Remember that we are thinking of these as the position vector representations of the points $(1,0)$ and $(-1,4)$ respectively.

Step 2

Now simply apply the Fundamental Theorem to evaluate the integral.

$$\int_C \nabla f \cdot d\vec{r} = f\left(\vec{r}\left(2\right)\right) - f\left(\vec{r}\left(0\right)\right) = f\left(-1,4\right) - f\left(1,0\right) = -4 - 0 = \boxed{-4}$$

3. Given that $\int_C \vec{F} \cdot d\vec{r}$ is independent of path compute $\int_C \vec{F} \cdot d\vec{r}$ where C is the ellipse given by $\frac{(x-5)^2}{4} + \frac{y^2}{9} = 1$ with the counter clockwise rotation.

Solution

At first glance this problem seems to be impossible since the vector field isn't even given for the problem. However, it's actually quite simple and the vector field is not needed to do the problem.

There are two important things to note in the problem statement.

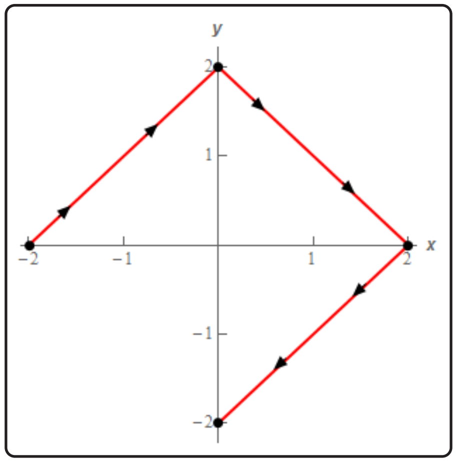
First, and somewhat more importantly, we are told in the problem statement that the integral is independent of path.

Second, we are told that the curve, C, is the full ellipse. It isn't the fact that C is an ellipse that is important. What is important is the fact that C is a closed curve.

Now all we need to do is use Fact 4 from the notes. This tells us that the value of a line integral of this type around a closed path will be zero if the integral is independent of path. Therefore,

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = 0}$$

4. Evaluate $\displaystyle\int\limits_C \nabla f \cdot d\vec{r}$ where $f(x,y) = \mathbf{e}^{x\,y} - x^2 + y^3$ and C is the curve shown below.



Solution

This problem is much simpler than it appears at first. We do not need to compute 3 different line integrals (one for each curve in the sketch).

All we need to do is notice that we are doing a line integral for a gradient vector function and so we can use the Fundamental Theorem for Line Integrals to do this problem.

Using the Fundamental Theorem to evaluate the integral gives the following,

$$\begin{aligned}\int\limits_C \nabla f \cdot d\vec{r} &= f(\text{endpoint}) - f(\text{startpoint}) \\ &= f(0,-2) - f(-2,0) \\ &= -7 - (-3) = \boxed{-4}\end{aligned}$$

Remember that all the Fundamental Theorem requires is the starting and ending point of the curve and the function used to generate the gradient vector field.

16.6 Conservative Vector Fields

1. Determine if the following vector field is conservative.

$$\vec{F} = \left(x^3 - 4xy^2 + 2\right)\vec{i} + \left(6x - 7y + x^3y^3\right)\vec{j}$$

Solution

There really isn't all that much to do with this problem. All we need to do is identify P and Q then run through the test.

So,

$$P = x^3 - 4xy^2 + 2 \qquad P_y = -8xy$$
$$Q = 6x - 7y + x^3y^3 \qquad Q_x = 6 + 3x^2y^3$$

Okay, we can clearly see that $P_y \neq Q_x$ and so the vector field is **not conservative.**

---

2. Determine if the following vector field is conservative.

$$\vec{F} = \left(2x\sin(2y) - 3y^2\right)\vec{i} + \left(2 - 6xy + 2x^2\cos(2y)\right)\vec{j}$$

Solution

There really isn't all that much to do with this problem. All we need to do is identify P and Q then run through the test.

So,

$$P = 2x\sin(2y) - 3y^2 \qquad P_y = 4x\cos(2y) - 6y$$
$$Q = 2 - 6xy + 2x^2\cos(2y) \qquad Q_x = -6y + 4x\cos(2y)$$

Okay, we can clearly see that $P_y = Q_x$ and so the vector field is **conservative.**

---

3. Determine if the following vector field is conservative.

$$\vec{F} = \left(6 - 2xy + y^3\right)\vec{i} + \left(x^2 - 8y + 3xy^2\right)\vec{j}$$

Solution

There really isn't all that much to do with this problem. All we need to do is identify P and Q then run through the test.

So,

$$P = 6 - 2xy + y^3 \qquad P_y = -2x + 3y^2$$
$$Q = x^2 - 8y + 3xy^2 \qquad Q_x = 2x + 3y^2$$

Okay, we can clearly see that $P_y \neq Q_x$ and so the vector field is **not conservative.**

Be careful with these problems. It is easy to get into a hurry and miss a very subtle difference between the two derivatives. In this case, the only difference between the two derivatives is the sign on the first term. That's it. That is also enough for this vector field to not be conservative.

---

4. Find the potential function for the following vector field.

$$\vec{F} = \left(6x^2 - 2xy^2 + \frac{y}{2\sqrt{x}}\right)\vec{i} - \left(2x^2y - 4 - \sqrt{x}\right)\vec{j}$$

Step 1

Now, by assumption from how the problem was asked, we could assume that the vector field is conservative but let's check it anyway just to make sure.

So,

$$P = 6x^2 - 2xy^2 + \frac{y}{2\sqrt{x}} \qquad P_y = -4xy + \frac{1}{2\sqrt{x}}$$
$$Q = -\left(2x^2y - 4 - \sqrt{x}\right) \qquad Q_x = -4xy + \frac{1}{2\sqrt{x}}$$

Okay, we can see that $P_y = Q_x$ and so the vector field is conservative as the problem statement suggested it would be.

Be careful with these problems and watch the signs on the vector components. One of the biggest mistakes that students make with these problems is to miss the minus sign that is in front of the second component of the vector field. There won't always be a minus sign of course, but on occasion there will be one and if we miss it the rest of the problem will be very difficult to do. In fact, if we miss it we won't be able to find a potential function for the vector field!

Step 2

Okay, to find the potential function for this vector field we know that we need to first either integrate P with respect to x or integrate Q with respect to y. It doesn't matter which one we use chose to use and, in this case, it looks like neither will be any harder than the other.

So, let's go with the following integration for this problem.

$$\begin{aligned} f(x,y) &= \int Q\,dy \\ &= \int -2x^2y + 4 + \sqrt{x}\,dy \\ &= -x^2y^2 + 4y + y\sqrt{x} + g(x) \end{aligned}$$

Don't forget that, in this case, because we were integrating with respect to y the "constant of integration" will be a function of x!

Step 3

Next, differentiate the function from the previous step with respect to x and set equal to P since we know the derivative and P should be the same function.

$$f_x = -2xy^2 + \frac{y}{2\sqrt{x}} + g'(x) = 6x^2 - 2xy^2 + \frac{y}{2\sqrt{x}} = P \qquad \Rightarrow \qquad g'(x) = 6x^2$$

Now, recall that because we integrated with respect to y in Step 2 $g(x)$, and hence $g'(x)$, should only be a function of x's (as it is in this case). If there had been any y's in $g'(x)$ we'd know there was something wrong at this point. Either we'd made a mistake somewhere or the vector field was not conservative. Of course, we verified that it was conservative in Step 1 and so this would in fact mean we'd made a mistake somewhere!

Step 4

We can now integrate both sides of the formula for $g'(x)$ above to get,

$$g(x) = 2x^3 + c$$

Don't forget the "$+c$" on this!

Step 5

Finally, putting everything together we get the following potential function for the vector field.

$$\boxed{f(x,y) = -x^2y^2 + 4y + y\sqrt{x} + 2x^3 + c}$$

---

5. Find the potential function for the following vector field.

$$\vec{F} = y^2\left(1 + \cos(x+y)\right)\vec{i} + \left(2xy - 2y + y^2\cos(x+y) + 2y\sin(x+y)\right)\vec{j}$$

Step 1

Now, by assumption from how the problem was asked, we could assume that the vector field is conservative but let's check it anyway just to make sure.

So,

$$\begin{aligned} P &= y^2\left(1 + \cos(x+y)\right) = y^2 + y^2\cos(x+y) \\ P_y &= 2y - y^2\sin(x+y) + 2y\cos(x+y) \end{aligned}$$

$$\begin{aligned} Q &= 2xy - 2y + y^2\cos(x+y) + 2y\sin(x+y) \\ Q_x &= 2y - y^2\sin(x+y) + 2y\cos(x+y) \end{aligned}$$

Okay, we can see that $P_y = Q_x$ and so the vector field is conservative as the problem statement suggested it would be.

Step 2

Okay, to find the potential function for this vector field we know that we need to first either integrate P with respect to x or integrate Q with respect to y. It doesn't matter which one we use chose to use in general, but in in this case integrating Q with respect to y just looks painful (two integration by parts terms!).

So, let's go with the following integration for this problem.

$$\begin{aligned} f(x,y) &= \int P\,dx \\ &= \int y^2 + y^2\cos(x+y)\,dx \\ &= xy^2 + y^2\sin(x+y) + h(y) \end{aligned}$$

Don't forget that, in this case, because we were integrating with respect to x the "constant of integration" will be a function of y!

Note, that as this problem has shown, sometimes one integration order will be significantly easier than the other so be on the lookout for which term might be easier to integrate.

Step 3

Next, differentiate the function from the previous step with respect to y and set equal to Q since we know the derivative and Q should be the same function.

$$\begin{aligned} f_y &= 2xy + 2y\sin(x+y) + y^2\cos(x+y) + h'(y) \\ &= 2xy - 2y + y^2\cos(x+y) + 2y\sin(x+y) = Q \qquad \Rightarrow \qquad h'(y) = -2y \end{aligned}$$

Now, recall that because we integrated with respect to x in Step 2 $h(y)$, and hence $h'(y)$, should only be a function of y's (as it is in this case). If there had been any x's in $h'(y)$ we'd know there was something wrong at this point. Either we'd made a mistake somewhere or the vector field was not conservative. Of course, we verified that it was conservative in Step 1 and so this would in fact mean we'd made a mistake somewhere!

Step 4

We can now integrate both sides of the formula for $h'(y)$ above to get,

$$h(y) = -y^2 + c$$

Don't forget the "$+c$" on this!

Step 5

Finally, putting everything together we get the following potential function for the vector field.

$$\boxed{f(x,y) = xy^2 + y^2\sin(x+y) - y^2 + c}$$

6. Find the potential function for the following vector field.

$$\vec{F} = \left(2z^4 - 2y - y^3\right)\vec{i} + \left(z - 2x - 3xy^2\right)\vec{j} + \left(6 + y + 8xz^3\right)\vec{k}$$

Step 1

Now, by assumption from how the problem was asked, we can assume that the vector field is conservative and because we don't know how to verify this for a 3D vector field we will just need to trust that it is.

Let's start off the problem by labeling each of the components to make the problem easier to deal with as follows.

$$\begin{aligned} P &= 2z^4 - 2y - y^3 \\ Q &= z - 2x - 3xy^2 \\ R &= 6 + y + 8xz^3 \end{aligned}$$

Step 2

To find the potential function for this vector field we know that we need to first either integrate P with respect to x, integrate Q with respect to y or R with respect z. It doesn't matter which one we use chose to use and, in this case, it looks like none of them will be any harder than the others.

So, let's go with the following integration for this problem.

$$\begin{aligned} f(x,y,z) &= \int Q\,dy \\ &= \int z - 2x - 3xy^2\,dy \\ &= zy - 2xy - xy^3 + h(x,z) \end{aligned}$$

Don't forget that, in this case, because we were integrating with respect to y the "constant of integration" will be a function of x and/or z!

Step 3

Next, we can differentiate the function from the previous step with respect to x and set equal to P or differentiate the function with respect to z and set equal to R.

Again, neither looks any more difficult than the other so let's differentiate with respect to z.

$$f_z = y + h_z(x,z) = 6 + y + 8xz^3 = R \qquad \Rightarrow \qquad h_z(x,z) = 6 + 8xz^3$$

Now, recall that because we integrated with respect to y in Step 2 $h(x,z)$, and hence $h_z(x,z)$, should only be a function of x's and z's (as it is in this case). If there had been any y's in $h_z(x,z)$ we'd know there was something wrong at this point. Either we'd made a mistake somewhere or the vector field was not conservative.

Also note that there is no reason to expect $h_z(x,z)$ to have both x's and z's in it. It is completely possible for one (or both) of the variables to not be present!

Step 4

We can now integrate both sides of the formula for $h_z(x,z)$ with respect to z to get,

$$h(x,z) = 6z + 2xz^4 + g(x)$$

Now, because $h(x,z)$ was a function of both x and z and we integrated with respect to z here the "constant of integration" in this case would need to be a function of x, $g(x)$ in this case.

The potential function is now,

$$f(x,y,z) = zy - 2xy - xy^3 + 6z + 2xz^4 + g(x)$$

Step 5

Next, we'll need to differentiate the potential function from Step 4 with respect to x and set equal to P. Doing this gives,

$$f_x = -2y - y^3 + 2z^4 + g'(x) = 2z^4 - 2y - y^3 = P \qquad \Rightarrow \qquad g'(x) = 0$$

Remember, that as in Step 3, we have to recall what variable we are differentiating with respect to here. In this case we are differentiating with respect to x and so $g(x)$ should only be a function of x. Had $g'(x)$ contained either y's or z's we'd know that either we'd made a mistake or the vector field was not conservative.

Also, as shown in this problem, it is completely possible for there to be no x's at all in $g'(x)$.

Step 6

Integrating both sides of the formula for $g'(x)$ from Step 5 and we can see that we must have $g(x) = c$.

Step 7

Finally, putting everything together we get the following potential function for the vector field.

$$\boxed{f(x,y,z) = zy - 2xy - xy^3 + 6z + 2xz^4 + c}$$

7. Find the potential function for the following vector field.

$$\vec{F} = \frac{2xy}{z^3}\vec{i} + \left(2y - z^2 + \frac{x^2}{z^3}\right)\vec{j} - \left(4z^3 + 2yz + \frac{3x^2y}{z^4}\right)\vec{k}$$

Step 1

Now, by assumption from how the problem was asked, we can assume that the vector field is conservative and because we don't know how to verify this for a 3D vector field we will just need to trust that it is.

Let's start off the problem by labeling each of the components to make the problem easier to deal with as follows.

$$P = \frac{2xy}{z^3}$$
$$Q = 2y - z^2 + \frac{x^2}{z^3}$$
$$R = -\left(4z^3 + 2yz + \frac{3x^2y}{z^4}\right) = -4z^3 - 2yz - \frac{3x^2y}{z^4}$$

Be careful with these problems and watch the signs on the vector components. One of the biggest mistakes that students make with these problems is to miss the minus sign that is in front of the third component of the vector field. There won't always be a minus sign of course, but on occasion there will be one and if we miss it the rest of the problem will be very difficult to do. In fact, if we miss it we won't be able to find a potential function for the vector field!

Step 2

To find the potential function for this vector field we know that we need to first either integrate P with respect to x, integrate Q with respect to y or R with respect z. It doesn't matter which one we use chose to use and, in this case, it looks like none of them will be any harder than the other.

In this case the R has quite a few terms in it so let's integrate that one first simply because it might mean less work when dealing with P and Q in later steps.

$$\begin{aligned} f(x,y,z) &= \int R\,dz \\ &= \int -4z^3 - 2yz - 3x^2y\,z^{-4}\,dz \\ &= -z^4 - yz^2 + x^2y\,z^{-3} + h(x,y) \end{aligned}$$

Don't forget that, in this case, because we were integrating with respect to z the "constant of integration" will be a function of x and/or y!

Step 3

Next, we can differentiate the function from the previous step with respect to x and set equal to P or differentiate the function with respect to y and set equal to Q.

Let's differentiate with respect to y in this case.

$$f_y = -z^2 + x^2\,z^{-3} + h_y(x,y) = 2y - z^2 + x^2z^{-3} = Q \qquad \Rightarrow \qquad h_y(x,y) = 2y$$

Now, recall that because we integrated with respect to z in Step 2 $h(x,y)$, and hence $h_y(x,y)$, should only be a function of x's and y's (as it is in this case). If there had been any z's in $h_y(x,y)$ we'd know there was something wrong at this point. Either we'd made a mistake somewhere or the vector field was not conservative.

Also note that there is no reason to expect $h_y(x,y)$ to have both x's and y's in it. It is completely possible for one (or both) of the variables to not be present!

Step 4

We can now integrate both sides of the formula for $h_y(x,y)$ with respect to y to get,

$$h(x,y) = y^2 + g(x)$$

Now, because $h(x, y)$ was a function of both x and y and we integrated with respect to y here the "constant of integration" in this case would need to be a function of x, $g(x)$ in this case.

The potential function is now,

$$f(x, y, z) = -z^4 - yz^2 + x^2 y\, z^{-3} + y^2 + g(x)$$

Step 5

Next, we'll need to differentiate the potential function from Step 4 with respect to x and set equal to P. Doing this gives,

$$f_x = 2xy\, z^{-3} + g'(x) = 2xyz^{-3} = P \qquad \Rightarrow \qquad g'(x) = 0$$

Remember, that as in Step 3, we have to recall what variable we are differentiating with respect to here. In this case we are differentiating with respect to x and $g(x)$ should only be a function of x. Had $g'(x)$ contained either y's or z's we'd know that either we'd made a mistake or the vector field was not conservative.

Also, as shown in this problem it is completely possible for there to be no x's at all in $g'(x)$.

Step 6

Integrating both sides of the formula for $g'(x)$ from Step 5 and we can see that we must have $g(x) = c$.

Step 7

Finally, putting everything together we get the following potential function for the vector field.

$$\boxed{f(x, y, z) = -z^4 - yz^2 + x^2 y\, z^{-3} + y^2 + c}$$

8. Evaluate $\displaystyle\int\limits_C \vec{F}\cdot d\vec{r}$ where C is the portion of the circle centered at the origin with radius 2 in the 1^{st} quadrant with counter clockwise rotation and $\vec{F}(x,y) = \left(2xy - 4 - \frac{1}{2}\sin\left(\frac{1}{2}x\right)\sin\left(\frac{1}{2}y\right)\right)\vec{i} + \left(x^2 + \frac{1}{2}\cos\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}y\right)\right)\vec{j}$.

Step 1

There are two ways to work this problem, a hard way and an easy way. The hard way is to just see a line integral with a curve and a vector field given and just launch into computing the line integral directly (probably very difficult in this case). The easy way is to check and see if the vector field is conservative, and if it is find the potential function and then simply use the Fundamental Theorem for Line Integrals that we saw in the previous section.

So, let's go the easy way and check to see if the vector field is conservative.

$$P = 2xy - 4 - \frac{1}{2}\sin\left(\frac{1}{2}x\right)\sin\left(\frac{1}{2}y\right) \qquad P_y = 2x - \frac{1}{4}\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}y\right)$$
$$Q = x^2 + \frac{1}{2}\cos\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}y\right) \qquad Q_x = 2x - \frac{1}{4}\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}y\right)$$

So, we can see that $P_y = Q_x$ and so the vector field is conservative.

Step 2

Now we just need to find the potential function for the vector field. We'll go through those details a little quicker this time and with less explanation than we did in the some of the previous problems.

First, let's integrate P with respect to x.

$$\begin{aligned} f(x,y) &= \int P\,dx \\ &= \int 2xy - 4 - \frac{1}{2}\sin\left(\frac{1}{2}x\right)\sin\left(\frac{1}{2}y\right)dx \\ &= x^2y - 4x + \cos\left(\frac{1}{2}x\right)\sin\left(\frac{1}{2}y\right) + h(y) \end{aligned}$$

Now, differentiate with respect to y and set equal to Q.

$$f_y = x^2 + \frac{1}{2}\cos\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}y\right) + h'(y) = x^2 + \frac{1}{2}\cos\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}y\right) = Q$$
$$\Rightarrow \qquad h'(y) = 0$$

Solving for $h(y)$ gives $h(y) = c$ and so the potential function for this vector field is,

$$f(x,y) = x^2 y - 4x + \cos\left(\frac{1}{2}x\right)\sin\left(\frac{1}{2}y\right) + c$$

Step 3

Now that we have the potential function we can simply use the Fundamental Theorem for Line Integrals which says,

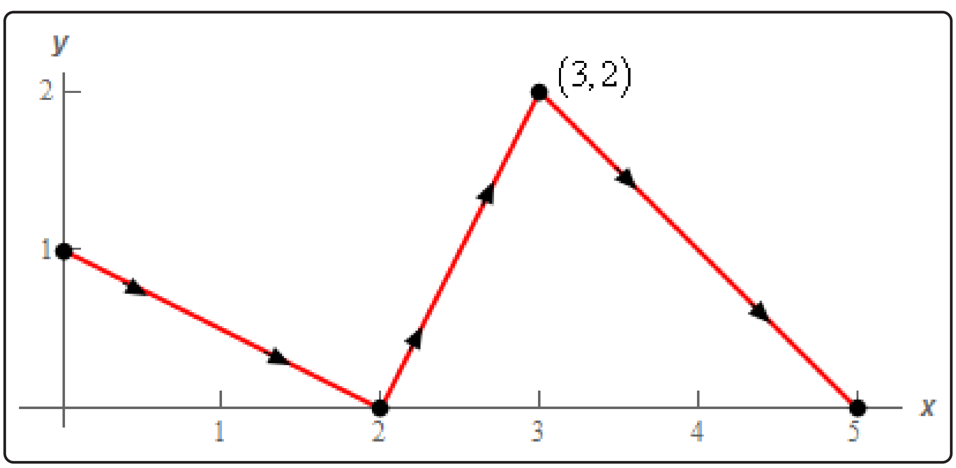
$$\int_C \vec{F}\cdot d\vec{r} = f(\text{end point}) - f(\text{start point})$$

From the problem statement we know that C is the portion of the circle of radius 2 in the 1^{st} quadrant with counter clockwise rotation. Therefore, the starting point of C is $(2,0)$ and the ending point of C is $(0,2)$.

The integral is then,

$$\int_C \vec{F}\cdot d\vec{r} = f(0,2) - f(2,0) = (\sin(1) + c) - (-8 + c) = \boxed{\sin(1) + 8 = 8.8415}$$

9. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = \left(2y\mathbf{e}^{xy} + 2x\mathbf{e}^{x^2-y^2}\right)\vec{i} + \left(2x\mathbf{e}^{xy} - 2y\mathbf{e}^{x^2-y^2}\right)\vec{j}$ and C is the curve shown below.



Step 1

There are two ways to work this problem, a hard way and an easy way. The hard way is to just see a line integral with a curve and a vector field given and just launch into computing the line integral directly (probably quite unpleasant in this case). The easy way is to check and see if the vector field is conservative, and if it is find the potential function and then simply use the Fundamental Theorem for Line Integrals that we saw in the previous section.

So, let's go the easy way and check to see if the vector field is conservative.

$$P = 2y\mathbf{e}^{xy} + 2x\mathbf{e}^{x^2-y^2} \qquad P_y = 2\mathbf{e}^{xy} + 2xy\mathbf{e}^{xy} - 4xy\mathbf{e}^{x^2-y^2}$$
$$Q = 2x\mathbf{e}^{xy} - 2y\mathbf{e}^{x^2-y^2} \qquad Q_x = 2\mathbf{e}^{xy} + 2xy\mathbf{e}^{xy} - 4xy\mathbf{e}^{x^2-y^2}$$

So, we can see that $P_y = Q_x$ and so the vector field is conservative.

Step 2

Now we just need to find the potential function for the vector field. We'll go through those details a little quicker this time and with less explanation than we did in the some of the previous problems.

First, let's integrate Q with respect to y.

$$\begin{aligned} f(x,y) &= \int Q\,dy \\ &= \int 2x\mathbf{e}^{xy} - 2y\mathbf{e}^{x^2-y^2}\,dy \\ &= 2\mathbf{e}^{xy} + \mathbf{e}^{x^2-y^2} + g(x) \end{aligned}$$

Now, differentiate with respect to x and set equal to P.

$$f_x = 2y\mathbf{e}^{xy} + 2x\mathbf{e}^{x^2-y^2} + g'(x) = 2y\mathbf{e}^{xy} + 2x\mathbf{e}^{x^2-y^2} = P \qquad \Rightarrow \qquad g'(x) = 0$$

Solving for $g(x)$ gives $g(x) = c$ and so the potential function for this vector field is,

$$f(x,y) = 2\mathbf{e}^{xy} + \mathbf{e}^{x^2-y^2} + c$$

Step 3

Now that we have the potential function we can simply use the Fundamental Theorem for Line Integrals which says,

$$\int_C \vec{F} \cdot d\vec{r} = f\left(\text{end point}\right) - f\left(\text{start point}\right)$$

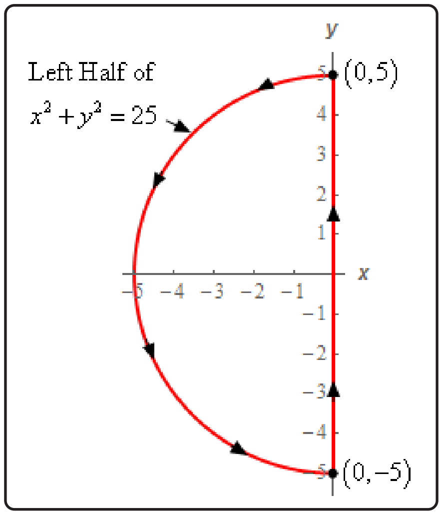
From the graph in the problem statement we can see that the starting point of C is $(0,1)$ and the ending point of C is $(5,0)$.

The integral is then,

$$\int_C \vec{F} \cdot d\vec{r} = f\left(5,0\right) - f\left(0,1\right) = \left(2 + \mathbf{e}^{25} + c\right) - \left(2 + \mathbf{e}^{-1} + c\right) = \boxed{\mathbf{e}^{25} - \mathbf{e}^{-1}}$$

16.7 Green's Theorem

1. Use Green's Theorem to evaluate $\displaystyle\int_C yx^2\,dx - x^2\,dy$ where C is shown below.



Step 1

Okay, first let's notice that if we walk along the path in the direction indicated then our left hand will be over the enclosed area and so this path does have the positive orientation and we can use Green's Theorem to evaluate the integral.

From the integral we have,

$$P = yx^2 \qquad\qquad Q = -x^2$$

Remember that P is multiplied by x and Q is multiplied by y and don't forget to pay attention to signs. It is easy to get in a hurry and miss a sign in front of one of the terms.

Step 2

Using Green's Theorem the line integral becomes,

$$\int_C yx^2\,dx - x^2\,dy = \iint_D -2x - x^2\,dA$$

D is the region enclosed by the curve.

Step 3

Since D is just a half circle it makes sense to use polar coordinates for this problem. The limits for D in polar coordinates are,

$$\frac{1}{2}\pi \le \theta \le \frac{3}{2}\pi$$
$$0 \le r \le 5$$

Step 4

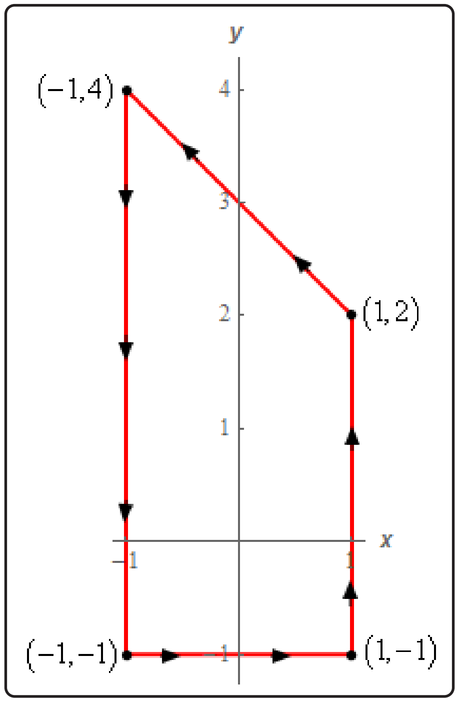
Now all we need to do is evaluate the double integral, after first converting to polar coordinates of course.

Here is the evaluation work.

$$\begin{aligned}
\int_C yx^2\,dx - x^2\,dy &= \iint_D -2x - x^2\,dA \\
&= \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\int_0^5 r\left(-2r\cos(\theta) - r^2\cos^2(\theta)\right)dr\,d\theta \\
&= \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\int_0^5 -2r^2\cos(\theta) - \frac{1}{2}r^3\left(1+\cos(2\theta)\right)dr\,d\theta \\
&= \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\left(-\frac{2}{3}r^3\cos(\theta) - \frac{1}{8}r^4\left(1+\cos(2\theta)\right)\right)\Bigg|_0^5\,d\theta \\
&= \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} -\frac{250}{3}\cos(\theta) - \frac{625}{8}\left(1+\cos(2\theta)\right)\,d\theta \\
&= \left[-\frac{250}{3}\sin(\theta) - \frac{625}{8}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\right]\Bigg|_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \\
&= \boxed{\frac{500}{3} - \frac{625}{8}\pi = -78.7703}
\end{aligned}$$

Don't forget the extra r from converting the A to polar coordinates and make sure you recall all the various tirg identities we need to deal with many of the various trig functions that show up in the integrals.

2. Use Green's Theorem to evaluate $\displaystyle\int_C (6y - 9x)\,dy - (yx - x^3)\,dx$ where C is shown below.



Step 1

Okay, first let's notice that if we walk along the path in the direction indicated then our left hand will be over the enclosed area and so this path does have the positive orientation and we can use Green's Theorem to evaluate the integral.

From the integral we have,

$$P = -(yx - x^3) = x^3 - yx \qquad\qquad Q = 6y - 9x$$

Remember that P is multiplied by x and Q is multiplied by y and don't forget to pay attention to signs. It is easy to get in a hurry and miss a sign in front of one of the terms. It is also easy to get in a hurry and just assume that P is the first term in the integral and Q is the second. That is clearly not the case here so be careful!

Step 2

Using Green's Theorem the line integral becomes,

$$\int_C (6y - 9x)\,dy - \left(yx - x^3\right)\,dx = \iint_D -9 - (-x)\,dA = \iint_D x - 9\,dA$$

D is the region enclosed by the curve.

Step 3

We'll leave it to you to verify that the equation of the line along the top of the region is given by $y = 3 - x$. Once we have this equation the region is then very easy to get limits for. They are,

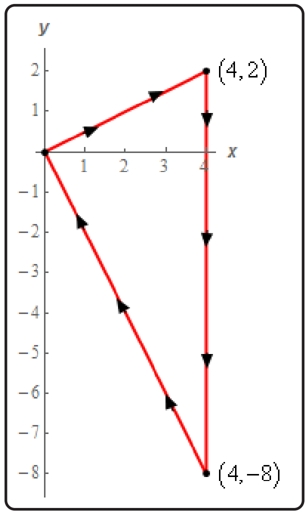
$$-1 \le x \le 1$$
$$-1 \le y \le 3 - x$$

Step 4

Now all we need to do is evaluate the double integral. Here is the evaluation work.

$$\begin{aligned}
\int_C (6y - 9x)\,dy - \left(yx - x^3\right)\,dx &= \iint_D x - 9\,dA \\
&= \int_{-1}^{1}\int_{-1}^{3-x} x - 9\,dy\,dx \\
&= \int_{-1}^{1} (x - 9)\left. y\right|_{-1}^{3-x}\,dx \\
&= \int_{-1}^{1} (x - 9)(4 - x)\,dx \\
&= \int_{-1}^{1} -x^2 + 13x - 36\,dx \\
&= \left.\left[-\frac{1}{3}x^3 + \frac{13}{2}x^2 - 36x\right]\right|_{-1}^{1} = \boxed{-\frac{218}{3}}
\end{aligned}$$

3. Use Green's Theorem to evaluate $\displaystyle\int_C x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy$ where C is shown below.



Step 1

Okay, first let's notice that if we walk along the path in the direction indicated then our left hand will NOT be over the enclosed area and so this path does NOT have the positive orientation. This, in turn, means that we can't actually use Green's Theorem to evaluate the given integral.

However, if C has the negative orientation then $-C$ will have the positive orientation and we know how to relate the values of the line integrals over these two curves. Specifically, we know that,

$$\int_C x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy = -\int_{-C} x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy$$

So, instead of using Green's Theorem to compute the value of the integral in the problem statement we'll use Green's Theorem to compute the value of the following integral.

$$\int_{-C} x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy$$

From this integral we have,

$$P = x^2y^2 \qquad\qquad Q = yx^3 + y^2$$

Remember that P is multiplied by x and Q is multiplied by y.

Step 2

Using Green's Theorem the line integral from over $-C$ becomes,

$$\int\limits_{-C} x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy = \iint\limits_D 3yx^2 - 2yx^2\,dA = \iint\limits_D yx^2\,dA$$

D is the region enclosed by the curve.

Step 3

We'll leave it to you to verify that the equation of the line along the top of the region is given by $y = \frac{1}{2}x$ and the equation of the line along the bottom of the region is given by $y = -2x$. Once we have these equations the region is then very easy to get limits for. They are,

$$\begin{gathered} 0 \le x \le 4 \\ -2x \le y \le \frac{1}{2}x \end{gathered}$$

Step 4

Now all we need to do is evaluate the double integral to get the value of the line integral from $-C$. Here is the evaluation work.

$$\begin{aligned} \int\limits_{-C} x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy &= \iint\limits_D yx^2\,dA \\ &= \int_0^4 \int_{-2x}^{\frac{1}{2}x} yx^2\,dy\,dx \\ &= \int_0^4 \left.\frac{1}{2}y^2x^2\right|_{-2x}^{\frac{1}{2}x}\,dx \\ &= \int_0^4 -\frac{15}{8}x^4\,dx \\ &= \left.-\frac{3x^5}{8}\right|_0^4 = \boxed{-384} \end{aligned}$$

Step 5

Okay, now we can't forget that the integral in Step 4 was not the integral we were asked to find the value of. Using the relationship between the value of the integrals over C and -C we know that the value of the integral we were asked to compute is,

$$\int_C x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy = -\int_{-C} x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy = \boxed{384}$$

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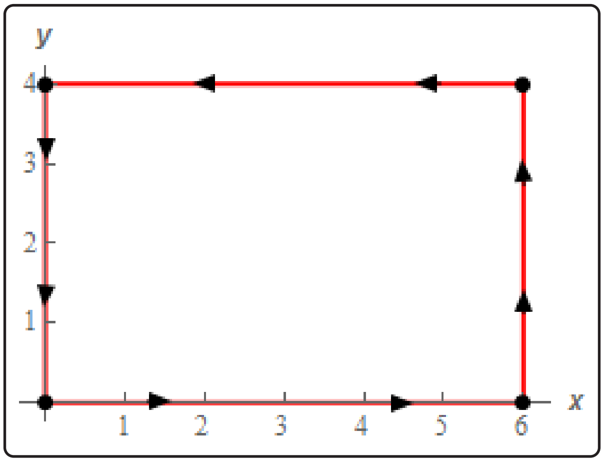
4. Use Green's Theorem to evaluate $\displaystyle\int_C \left(y^4 - 2y\right)\,dx - \left(6x - 4xy^3\right)\,dy$ where C is shown below.



Step 1

Okay, first let's notice that if we walk along the path in the direction indicated then our left hand will be over the enclosed area and so this path does have the positive orientation and we can use Green's Theorem to evaluate the integral.

From the integral we have,

$$P = y^4 - 2y \qquad\qquad Q = -\left(6x - 4xy^3\right) = 4xy^3 - 6x$$

Remember that P is multiplied by x and Q is multiplied by y and don't forget to pay attention to signs. It is easy to get in a hurry and miss a sign in front of one of the terms.

Step 2

Using Green's Theorem the line integral becomes,

$$\int\limits_C \left(y^4 - 2y\right)\,dx - \left(6x - 4xy^3\right)\,dy = \iint\limits_D 4y^3 - 6 - \left(4y^3 - 2\right)\,dA = \iint\limits_D -4\,dA$$

D is the region enclosed by the curve.

Step 3

Okay, we are now pretty much done with this problem. After factoring out the "-4" from the integral we can use the following fact to finish the evaluation of the integral.

$$\begin{aligned}\int\limits_C \left(y^4 - 2y\right)\,dx - \left(6x - 4xy^3\right)\,dy &= -4\iint\limits_D \,dA \\ &= -4\left(\text{Area of } D\right) = -4\left[(6)(4)\right] = \boxed{-96}\end{aligned}$$

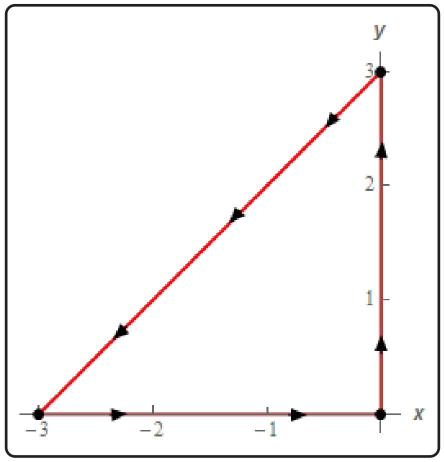
Don't forget the fact that,

$$\iint\limits_D \,dA = \text{Area of } D$$

This is a very useful fact. It doesn't come up often but when it does it can reduce the amount of work required to finish out a problem.

Of course, this fact is really only useful if D is a region that we can easily determine the area for and in this case D was just a rectangle that we could easily get the width, 6, and the height, 4, from the graph in the problem statement.

5. Verify Green's Theorem for $\oint_C (xy^2 + x^2)\,dx + (4x - 1)\,dy$ where C is shown below by **(a)** computing the line integral directly and **(b)** using Green's Theorem to compute the line integral.

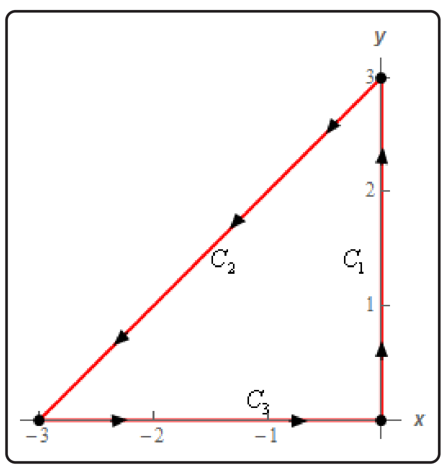


Solutions

(a) computing the line integral directly

Step 1

So, let's start off the problem with labeling the curves as follows,



Following the specified direction for each curve here are the parameterizations for each curve.

$C_1: \; \vec{r}(t) = \langle 0, t \rangle \qquad 0 \le t \le 3$

$C_2: \; \vec{r}(t) = (1-t)\langle 0, 3 \rangle + t\langle -3, 0 \rangle = \langle -3t, 3-3t \rangle \qquad 0 \le t \le 1$

$C_3: \; \vec{r}(t) = \langle t, 0 \rangle \qquad -3 \le t \le 0$

Step 2

Here is the line integral evaluated over each of these curves.

$$\oint_{C_1} \left(xy^2 + x^2\right) dx + (4x-1)\,dy = \int_0^3 \left[(0)\,(t)^2 + (0)^2\right](0)\;dt + \int_0^3 \left[4\,(0) - 1\right](1)\,dt$$
$$= \int_0^3 -1\,dt = -t\Big|_0^3 = \underline{-3}$$

$$\oint_{C_2} \left(xy^2 + x^2\right)\;dx + (4x-1)\;dy = \int_0^1 \left[(-3t)\,(3-3t)^2 + (-3t)^2\right](-3)\;dt$$
$$+ \int_0^1 \left[4\,(-3t) - 1\right](-3)\;dt$$
$$= \int_0^1 81t^3 - 189t^2 + 81t\,dt + \int_0^1 36t + 3\,dt$$
$$= \int_0^1 81t^3 - 189t^2 + 117t + 3\,dt$$
$$= \left(\frac{81}{4}t^4 - 63t^3 + \frac{117}{2}t^2 + 3t\right)\Bigg|_0^1 = \underline{\frac{75}{4}}$$

$$\oint_{C_3} \left(xy^2 + x^2\right) dx + (4x-1)\,dy = \int_{-3}^0 \left[(t)\,(0)^2 + (t)^2\right](1)\,dt + \int_{-3}^0 \left[4\,(t) - 1\right](0)\;dt$$
$$= \int_{-3}^0 t^2\,dt = \frac{1}{3}t^3\Big|_{-3}^0 = \underline{9}$$

Step 3

Now, all we need to do is add up the results from the previous step to get the value of the line integral over the full curve. This gives,

$$\oint_C \left(xy^2 + x^2\right)\;dx + (4x-1)\;dy = (-3) + \left(\frac{75}{4}\right) + (9) = \boxed{\frac{99}{4}}$$

(b) using Green's Theorem to compute the line integral

Step 1

Note that as the circle on the integral implies the curve is in the positive direction and so we can use Green's Theorem on this integral.

From the integral we have,

$$P = xy^2 + x^2 \qquad\qquad Q = 4x - 1$$

Remember that P is multiplied by x and Q is multiplied by y.

Step 2

Using Green's Theorem the line integral becomes,

$$\oint_C \left(xy^2 + x^2\right)\,dx + (4x - 1)\,dy = \iint\limits_D 4 - (2xy)\,dA = \iint\limits_D 4 - 2xy\,dA$$

D is the region enclosed by the curve.

Step 3

We'll leave it to you to verify that the equation of the line along the hypotenuse of the region is given by $y = x + 3$. Once we have this equation the region is then very easy to get limits for. They are,

$$-3 \le x \le 0$$
$$0 \le y \le x + 3$$

Step 4

Now all we need to do is evaluate the double integral. Here is the evaluation work.

$$\begin{aligned}
\oint_C \left(xy^2 + x^2\right)\,dx + (4x-1)\,dy &= \iint\limits_D 4 - 2xy\,dA \\
&= \int_{-3}^{0}\int_{0}^{x+3} 4 - 2xy\,dy\,dx \\
&= \int_{-3}^{0} \left.\left(4y - xy^2\right)\right|_{0}^{x+3} dx \\
&= \int_{-3}^{0} 12 - 5x - 6x^2 - x^3\,dx \\
&= \left.\left(12x - \frac{5}{2}x^2 - 2x^3 - \frac{1}{4}x^4\right)\right|_{-3}^{0} \\
&= \boxed{\frac{99}{4}}
\end{aligned}$$

So, we got the same answer after applying Green's Theorem to the line integral as we got by integrating the line integral directly.

17 Surface Integrals

In this chapter we are going to take a look at surface integrals. In the previous chapter we integrated a line integral of a function of three variables where the variables came from a three dimensional curve. In this chapter we want to integrate a function of three variables but now the variables will come from a three dimensional solid. As with line integrals we will integrate both functions and vector fields.

We will also introduce the concept of the curl and divergence of a vector field. In addition, we will discuss how to write down a set of parametric equations for a surface.

We will close out the chapter by discussing Stokes' Theorem and the Divergence Theorem. Stokes' Theorem will give a nice relationship between line integrals and surface integrals. The Divergence Theorem will give a relationship between surface integrals and triple integrals.

The following sections are the practice problems, with solutions, for this material.

17.1 Curl and Divergence

1. Compute div$\vec{F}$ and curl$\vec{F}$ for $\vec{F} = x^2 y\,\vec{i} - \left(z^3 - 3x\right)\vec{j} + 4y^2\vec{k}$.

Step 1

Let's compute the divergence first and there isn't much to do other than run through the formula.

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(x^2 y\right) + \frac{\partial}{\partial y}\left(3x - z^3\right) + \frac{\partial}{\partial z}\left(4y^2\right) = \boxed{2xy}$$

Be careful to watch for minus signs in front of any of the vector components (2^{nd} component in this case!). It is easy to get in a hurry and miss them.

Step 2

The curl is a little more work but still just formula work so here is the curl.

$$\begin{aligned}
\text{curl}\vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & 3x - z^3 & 4y^2 \end{vmatrix} \\
&= \frac{\partial}{\partial y}\left(4y^2\right)\vec{i} + \frac{\partial}{\partial z}\left(x^2 y\right)\vec{j} + \frac{\partial}{\partial x}\left(3x - z^3\right)\vec{k} - \frac{\partial}{\partial y}\left(x^2 y\right)\vec{k} \\
&\qquad - \frac{\partial}{\partial x}\left(4y^2\right)\vec{j} - \frac{\partial}{\partial z}\left(3x - z^3\right)\vec{i} \\
&= 8y\vec{i} + 3\vec{k} - x^2\vec{k} + 3z^2\vec{i} \\
&= \boxed{\left(8y + 3z^2\right)\vec{i} + \left(3 - x^2\right)\vec{k}}
\end{aligned}$$

Again, don't forget the minus sign on the 2^{nd} component.

2. Compute div$\vec{F}$ and curl$\vec{F}$ for $\vec{F} = \left(3x + 2z^2\right)\vec{i} + \dfrac{x^3y^2}{z}\vec{j} - (z - 7x)\vec{k}$.

Step 1

Let's compute the divergence first and there isn't much to do other than run through the formula.

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(3x + 2z^2\right) + \frac{\partial}{\partial y}\left(\frac{x^3y^2}{z}\right) + \frac{\partial}{\partial z}(7x - z) = \boxed{2 + \frac{2x^3y}{z}}$$

Be careful to watch for minus signs in front of any of the vector components (3^{rd} component in this case!). It is easy to get in a hurry and miss them.

Step 2

The curl is a little more work but still just formula work so here is the curl.

$$\begin{aligned}
\text{curl}\vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x + 2z^2 & \dfrac{x^3y^2}{z} & 7x - z \end{vmatrix} \\
&= \frac{\partial}{\partial y}(7x - z)\vec{i} + \frac{\partial}{\partial z}\left(3x + 2z^2\right)\vec{j} + \frac{\partial}{\partial x}\left(\frac{x^3y^2}{z}\right)\vec{k} \\
&\qquad - \frac{\partial}{\partial y}\left(3x + 2z^2\right)\vec{k} - \frac{\partial}{\partial x}(7x - z)\vec{j} - \frac{\partial}{\partial z}\left(\frac{x^3y^2}{z}\right)\vec{i} \\
&= 4z\vec{j} + \frac{3x^2y^2}{z}\vec{k} - 7\vec{j} + \frac{x^3y^2}{z^2}\vec{i} \\
&= \boxed{\frac{x^3y^2}{z^2}\vec{i} + (4z - 7)\vec{j} + \frac{3x^2y^2}{z}\vec{k}}
\end{aligned}$$

Again, don't forget the minus sign on the 3^{rd} component.

3. Determine if the following vector field is conservative.

$$\vec{F} = \left(4y^2 + \frac{3x^2y}{z^2}\right)\vec{i} + \left(8xy + \frac{x^3}{z^2}\right)\vec{j} + \left(11 - \frac{2x^3y}{z^3}\right)\vec{k}$$

Step 1

We know all we need to do here is compute the curl of the vector field.

$$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4y^2 + \dfrac{3x^2y}{z^2} & 8xy + \dfrac{x^3}{z^2} & 11 - \dfrac{2x^3y}{z^3} \end{vmatrix}$$

$$= \frac{\partial}{\partial y}\left(11 - \frac{2x^3y}{z^3}\right)\vec{i} + \frac{\partial}{\partial z}\left(4y^2 + \frac{3x^2y}{z^2}\right)\vec{j} + \frac{\partial}{\partial x}\left(8xy + \frac{x^3}{z^2}\right)\vec{k}$$
$$- \frac{\partial}{\partial y}\left(4y^2 + \frac{3x^2y}{z^2}\right)\vec{k} - \frac{\partial}{\partial x}\left(11 - \frac{2x^3y}{z^3}\right)\vec{j} - \frac{\partial}{\partial z}\left(8xy + \frac{x^3}{z^2}\right)\vec{i}$$

$$= -\frac{2x^3}{z^3}\vec{i} - \frac{6x^2y}{z^3}\vec{j} + \left(8y + \frac{3x^2}{z^2}\right)\vec{k} - \left(8y + \frac{3x^2}{z^2}\right)\vec{k} + \frac{6x^2y}{z^3}\vec{j} + \frac{2x^3}{z^3}\vec{i}$$

$$= \underline{\vec{0}}$$

Step 2

So, we found that $\text{curl}\vec{F} = \vec{0}$ for this vector field and so the vector field **is conservative**.

4. Determine if the following vector field is conservative.

$$\vec{F} = 6x\,\vec{i} + \left(2y - y^2\right)\vec{j} + \left(6z - x^3\right)\vec{k}$$

Step 1

We know all we need to do here is compute the curl of the vector field.

$$\begin{aligned}
\text{curl}\vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6x & 2y - y^2 & 6z - x^3 \end{vmatrix} \\
&= \frac{\partial}{\partial y}\left(6z - x^3\right)\vec{i} + \frac{\partial}{\partial z}\left(6x\right)\vec{j} + \frac{\partial}{\partial x}\left(2y - y^2\right)\vec{k} \\
&\qquad - \frac{\partial}{\partial y}\left(6x\right)\vec{k} - \frac{\partial}{\partial x}\left(6z - x^3\right)\vec{j} - \frac{\partial}{\partial z}\left(2y - y^2\right)\vec{i} \\
&= \underline{3x^2\vec{j}}
\end{aligned}$$

Step 2

So, we found that $\text{curl}\vec{F} = 3x^2\vec{j} \neq \vec{0}$ for this vector field and so the vector field **is NOT conservative**.

17.2 Parametric Surfaces

1. Write down a set of parametric equations for the plane $7x + 3y + 4z = 15$.

Step 1

There isn't a whole lot to this problem. There are three different acceptable answers here. To get a set of parametric equations for this plane all we need to do is solve for one of the variables and then write down the parametric equations.

For this problem let's solve for z to get,

$$z = \frac{15}{4} - \frac{7}{4}x - \frac{3}{4}y$$

Step 2

The parametric equation for the plane is then,

$$\vec{r}(x, y) = \langle x, y, z\rangle = \boxed{\left\langle x, y, \frac{15}{4} - \frac{7}{4}x - \frac{3}{4}y\right\rangle}$$

Remember that all we need to do to get the parametric equations is plug in the equation for z into the z component of the vector $\langle x, y, z\rangle$.

Also, as noted in Step 1 we could just have easily done either of the following two forms for the parametric equations for this plane.

$$\vec{r}(x, z) = \langle x, g(x, z), z\rangle \qquad \vec{r}(y, z) = \langle h(y, z), y, z\rangle$$

where you solve the equation of the plane for y or x respectively. All three set of parametric equations are all perfectly valid forms for the answer to this problem.

2. Write down a set of parametric equations for the plane $7x + 3y + 4z = 15$ that lies in the 1^{st} octant.

Step 1

This problem is really just an extension of the previous problem so we'll redo the set of parametric equations for the plane a little quicker this time.

First, we need to solve the equation for any of the three variables. We'll solve for z in this case to get,

$$z = \frac{15}{4} - \frac{7}{4}x - \frac{3}{4}y$$

The parametric equation for this plane is then,

$$\vec{r}(x,y) = \langle x, y, z \rangle = \left\langle x, y, \frac{15}{4} - \frac{7}{4}x - \frac{3}{4}y \right\rangle$$

Remember that all we need to do to get the parametric equations is plug in the equation for z into the z component of the vector $\langle x, y, z \rangle$.

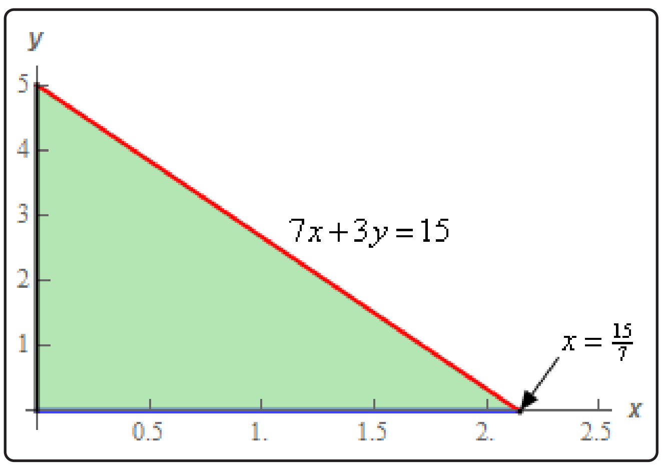
Step 2

Now, the set of parametric equations from above is for the full plane and that isn't what we want in this problem. In this problem we only want the portion of the plane that is in the 1^{st} octant.

So, we'll need to restrict x and y so that the parametric equation from Step 1 will only give the portion of the plane that is in the 1^{st} octant.

If you recall how to get the region D for a triple integral then you know how to do this because it is basically the same idea. In this case we need the region D in the xy-plane that will give the plane in the 1^{st} octant.

Here is a sketch of this region.



The hypotenuse is just where the plane intersects the xy-plane and so we can quickly find the equation of the line by setting $z = 0$ in the equation of the plane.

We can either solve this for x or y to get the ranges for x and y. It doesn't really matter which we solve for here so let's just solve for y to get the following ranges for x and y to describe this triangle.

$$0 \le x \le \frac{15}{7}$$
$$0 \le y \le -\frac{7}{3}x + 5$$

Putting this all together we get the following set of parametric equations for the plane that is in the 1^{st} octant.

$$\boxed{\vec{r}(x,y) = \left\langle x, y, \frac{15}{4} - \frac{7}{4}x - \frac{3}{4}y \right\rangle \qquad 0 \le x \le \frac{15}{7} \;,\; 0 \le y \le -\frac{7}{3}x + 5}$$

---

3. The cylinder $x^2 + y^2 = 5$ for $-1 \le z \le 6$.

Step 1

Because this surface is just a cylinder we just need the cylindrical coordinates conversion formulas with the polar coordinates in the xy-plane (since the cylinder is given in terms of x and y).

The conversion equations are,

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = z$$

However, recall that we are actually on the surface of the cylinder and so we know that $r = \sqrt{5}$. The conversion equations are then,

$$x = \sqrt{5}\cos(\theta) \qquad y = \sqrt{5}\sin(\theta) \qquad z = z$$

Step 2

We can now write down a set of parametric equations for the cylinder. They are,

$$\vec{r}(z,\theta) = \langle x, y, z \rangle = \left\langle \sqrt{5}\cos(\theta), \sqrt{5}\sin(\theta), z \right\rangle$$

Remember that all we do is plug the conversion formulas for x, y, and z into the x, y and z components of the vector $\langle x, y, z \rangle$ and we have a set of parametric equations. Also note that because the resulting vector equation is an equation in terms of z and θ those will also be the variables for our set of parametric equation.

Step 3

Now, the only issue with the set of parametric equations above is that they are for the full cylinder and we don't want that. We only want the cylinder in the given range of z so to finish this problem out all we need to do is add on a set of restrictions or ranges to our variables.

Doing that gives,

$$\boxed{\vec{r}\,(z,\theta) = \left\langle \sqrt{5}\cos(\theta), \sqrt{5}\sin(\theta), z\right\rangle \qquad -1 \le z \le 6\ ,\ 0 \le \theta \le 2\pi}$$

Note that the z range is just the range given in the problem statement and the θ range is the full zero to 2π range since there was no mention of restricting the portion of the cylinder that we wanted with respect to θ (for example, only the top half of the cylinder).

4. The portion of $y = 4 - x^2 - z^2$ that is in front of $y = -6$.

Step 1

Okay, the basic set of parametric equations in this case is pretty easy since we already have the equation in the form of "y = ".

The set of parametric equations that will give the full surface is just,

$$\vec{r}\,(x,z) = \langle x, y, z\rangle = \left\langle x, 4 - x^2 - z^2, z\right\rangle$$

Remember that all we need to do to get the parametric equations is plug in the equation for y into the y component of the vector $\langle x, y, z\rangle$.

Step 2

Finally, all we need to do is restrict x and z to get only the portion of the surface we are looking for. That is pretty simple however since we are given that we only want the portion that is in front of $y = -6$.

This is equivalent to requiring that $y \ge -6$ and we do have the equation of the surface so all we need to do is plug that into the inequality and do a little rewrite. Doing this gives,

$$4 - x^2 - z^2 \geq -6 \quad \rightarrow \quad x^2 + z^2 \leq 10$$

In other words, we only want the points (x, z) that are inside the disk of radius $\sqrt{10}$.

Putting all of this together gives the following set of parametric equations for the portion of the surface we are after.

$$\boxed{\vec{r}(x,z) = \left\langle x, 4 - x^2 - z^2, z \right\rangle \quad x^2 + z^2 \leq 10}$$

---

5. The portion of the sphere of radius 6 with $x \geq 0$.

Step 1

Because we have a portion of a sphere we'll start off with the spherical coordinates conversion formulas.

$$x = \rho \sin(\varphi)\cos(\theta) \qquad y = \rho \sin(\varphi)\sin(\theta) \qquad z = \rho\cos(\varphi)$$

However, we are actually on the surface of the sphere and so we know that $\rho = 6$. With this the conversion formulas become,

$$x = 6\sin(\varphi)\cos(\theta) \qquad y = 6\sin(\varphi)\sin(\theta) \qquad z = 6\cos(\varphi)$$

Step 2

The set of parametric equations that will give the full sphere is then,

$$\vec{r}(\theta,\varphi) = \langle x, y, z\rangle = \langle 6\sin(\varphi)\cos(\theta), 6\sin(\varphi)\sin(\theta), 6\cos(\varphi)\rangle$$

Remember that all we do is plug the conversion formulas for x, y, and z into the x, y and z components of the vector $\langle x, y, z\rangle$ and we have a set of parametric equations. Also note that because the resulting vector equation is an equation in terms of θ and φ those will also be the variables for our set of parametric equation.

Step 3

Finally, we need to deal with the fact that we don't actually want the full sphere here. We only want the portion of the sphere for which $x \ge 0$.

We can restrict x to this range if we restrict θ to the range $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$.

We've not put any restrictions on z and so that means that we'll take the full range of possible φ or $0 \le \varphi \le \pi$. Recall that φ is the angle a point in spherical coordinates makes with the positive z-axis and so that is the quantity we'd need to restrict if we'd wanted to restrict z (for example $z \le 0$).

Putting all of this together gives the following set of parametric equations for the portion of the surface we are after.

$$\boxed{\vec r\left(\theta,\varphi\right) = \left\langle 6\sin\left(\varphi\right)\cos\left(\theta\right), 6\sin\left(\varphi\right)\sin\left(\theta\right), 6\cos\left(\varphi\right)\right\rangle \quad -\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi \quad 0 \le \varphi \le \pi}$$

---

6. The tangent plane to the surface given by the following parametric equation at the point $(8, 14, 2)$.

$$\vec r\left(u,v\right) = \left(u^2 + 2u\right)\vec i + \left(3v - 2u\right)\vec j + \left(6v - 10\right)\vec k$$

Step 1

In order to write down the equation of a plane we need a point, which we have, $(8, 14, 2)$, and a normal vector, which we don't have yet.

However, recall that $\vec r_u \times \vec r_v$ will be normal to the surface. So, let's compute that.

$$\vec r_u = \left(2u+2\right)\vec i - 2\vec j \qquad\qquad \vec r_v = 3\vec j + 6\vec k$$

$$\vec r_u \times \vec r_v = \begin{vmatrix} \vec i & \vec j & \vec k \\ 2u+2 & -2 & 0 \\ 0 & 3 & 6 \end{vmatrix} = -12\vec i - 6\left(2u+2\right)\vec j + 3\left(2u+2\right)\vec k$$

Step 2

Now having $\vec r_u \times \vec r_v$ is all well and good but it is really only useful if we also know the point, (u, v) for which we are at $(8, 14, 2)$ so we next need to set the x, y and z coordinates of our point equal to the x, y and z components of our parametric equation to determine the value of u and v we need.

Here are the equations we get if we do that.

$$\begin{aligned} 8 &= u^2 + 2u \\ 14 &= 3v - 2u \\ 2 &= 6v - 10 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} 0 &= u^2 + 2u - 8 = (u+4)(u-2) \\ 14 &= 3v - 2u \\ 12 &= 6v \end{aligned}$$

Step 3

From the third equation above we can see that we must have $v = 2$ and from the first equation we can see that we must have either $u = -4$ or $u = 2$.

Plugging our only choice for v and both choices for u into the second equation we can see that we must have $u = -4$.

Step 4

Plugging $u = -4$ and $v = 2$ into the equation for $\vec{r}_u \times \vec{r}_v$ we will arrive at the following normal vector to the surface at $(8, 14, 2)$.

$$\vec{n} = \left(\vec{r}_u \times \vec{r}_v\right)\Big|_{u=-4, v=2} = -12\vec{i} + 36\vec{j} - 18\vec{k}$$

Note that, in this case, the normal vector didn't actually depend on the value of v. That won't happen in general, but as we've seen here that kind of thing can happen on occasion so don't get excited about it when it does.

The equation of the tangent plane to the surface at $(8, 14, 2)$ with normal vector $\vec{n} = -12\vec{i} + 36\vec{j} - 18\vec{k}$ is,

$$-12(x-8) + 36(y-14) - 18(z-2) = 0 \qquad \rightarrow \qquad -12x + 36y - 18z = 372$$

Step 5

To get a set of parametric equations for the tangent plane all we need to do is solve the equation for z to get,

$$z = -\frac{62}{3} - \frac{2}{3}x + 2y$$

We can then plug this into the vector $\langle x, y, z\rangle$ to get the following set of parametric equations for the tangent plane.

$$\boxed{\vec{r}(x,y) = \left\langle x, y, -\frac{62}{3} - \frac{2}{3}x + 2y \right\rangle}$$

Note that there will be no restrictions on x and y because we wanted the full tangent plane.

7. Determine the surface area of the portion of $2x+3y+6z=9$ that is inside the cylinder $x^2+y^2=7$.

Step 1

We first need to parameterize the surface. Because we are wanting the portion that is inside the cylinder centered on the z-axis it makes sense to first solve the equation of the plane for z to get,

$$z = \frac{3}{2} - \frac{1}{3}x - \frac{1}{2}y$$

The parameterization for the full plane is then,

$$\vec{r}(x,y) = \left\langle x, y, \frac{3}{2} - \frac{1}{3}x - \frac{1}{2}y \right\rangle$$

We only want the portion that is inside the cylinder given in the problem statement so we'll also need to restrict x and y to those in the disk $x^2+y^2 \le 7$. This will now give only the portion of the plane that is inside the cylinder.

Step 2

Next, we need to compute $\vec{r}_x \times \vec{r}_y$. Here is that work.

$$\vec{r}_x = \left\langle 1, 0, -\frac{1}{3} \right\rangle \qquad \vec{r}_y = \left\langle 0, 1, -\frac{1}{2} \right\rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{2} \end{vmatrix} = \frac{1}{3}\vec{i} + \frac{1}{2}\vec{j} + \vec{k}$$

Now, we what we really need is,

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{2}\right)^2 + 1} = \sqrt{\frac{49}{36}} = \frac{7}{6}$$

Step 3

The integral for the surface area is then,

$$A = \iint\limits_D \frac{7}{6}\,dA$$

In this case D is just the restriction on x and y that we noted in Step 1. So, D is just the disk $x^2 + y^2 \le 7$.

Step 4

Computing the integral in this case is very simple. All we need to do is take advantage of the fact that,

$$\iint\limits_D dA = \text{Area of } D$$

So, the surface area is simply,

$$A = \iint\limits_D \frac{7}{6}\,dA = \frac{7}{6}\iint\limits_D dA = \frac{7}{6}\left[\text{Area of } D\right] = \frac{7}{6}\left[\pi\left(\sqrt{7}\right)^2\right] = \boxed{\frac{49}{6}\pi}$$

---

8. Determine the surface area of the portion of $x^2 + y^2 + z^2 = 25$ with $z \le 0$.

Step 1

We first need to parameterize the sphere and we've already done a sphere in this problem set so we won't go into great detail with the parameterization here.

The parameterization for the full sphere is,

$$\vec{r}(\theta, \varphi) = \langle 5\sin(\varphi)\cos(\theta), 5\sin(\varphi)\sin(\theta), 5\cos(\varphi)\rangle$$

We don't want the full sphere of course. We only want the lower half of the sphere, *i.e.* the portion with $z \le 0$. This means that we'll need to restrict φ to $\frac{1}{2}\pi \le \varphi \le \pi$. Recall that φ is the angle points make with the positive z-axis and because we only want points below the xy-plane we'll need the range of $\frac{1}{2}\pi \le \varphi \le \pi$.

We want the full lower half and so we'll use $0 \le \theta \le 2\pi$ for our θ range.

Step 2

Next, we need to compute $\vec{r}_\theta \times \vec{r}_\varphi$. Here is that work.

$$\begin{aligned} \vec{r}_\theta &= \langle -5\sin(\varphi)\sin(\theta), 5\sin(\varphi)\cos(\theta), 0\rangle \\ \vec{r}_\varphi &= \langle 5\cos(\varphi)\cos(\theta), 5\cos(\varphi)\sin(\theta), -5\sin(\varphi)\rangle \end{aligned}$$

$$\begin{aligned} \vec{r}_\theta \times \vec{r}_\varphi &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5\sin(\varphi)\sin(\theta) & 5\sin(\varphi)\cos(\theta) & 0 \\ 5\cos(\varphi)\cos(\theta) & 5\cos(\varphi)\sin(\theta) & -5\sin(\varphi) \end{vmatrix} \\ &= -25\sin^2(\varphi)\cos(\theta)\,\vec{i} - 25\sin(\varphi)\cos(\varphi)\sin^2(\theta)\vec{k} \\ &\qquad - 25\sin(\varphi)\cos(\varphi)\cos^2(\theta)\vec{k} - 25\sin^2(\varphi)\sin(\theta)\vec{j} \\ &= -25\sin^2(\varphi)\cos(\theta)\,\vec{i} - 25\sin^2(\varphi)\sin(\theta)\vec{j} \\ &\qquad - 25\sin(\varphi)\cos(\varphi)\left(\sin^2(\theta) + \cos^2(\theta)\right)\vec{k} \\ &= -25\sin^2(\varphi)\cos(\theta)\,\vec{i} - 25\sin^2(\varphi)\sin(\theta)\vec{j} - 25\sin(\varphi)\cos(\varphi)\vec{k} \end{aligned}$$

Now, we what we really need is,

$$\begin{aligned} \|\vec{r}_\theta \times \vec{r}_\varphi\| &= \sqrt{\left(-25\sin^2(\varphi)\cos(\theta)\right)^2 + \left(-25\sin^2(\varphi)\sin(\theta)\right)^2 + \left(-25\sin(\varphi)\cos(\varphi)\right)^2} \\ &= \sqrt{625\sin^4(\varphi)\left(\cos^2(\theta) + \sin^2(\theta)\right) + 625\sin^2(\varphi)\cos^2(\varphi)} \\ &= \sqrt{625\sin^2(\varphi)\left(\sin^2(\varphi) + \cos^2(\varphi)\right)} \\ &= 25\left|\sin(\varphi)\right| \\ &= 25\sin(\varphi) \end{aligned}$$

Note that we can drop the absolute value bars on the sine because we know that sine will be positive in $\frac{1}{2}\pi \le \varphi \le \pi$.

Step 3

The integral for the surface area is then,

$$A = \iint\limits_D 25\sin(\varphi)\,dA = \int_0^{2\pi}\int_{\frac{1}{2}\pi}^{\pi} 25\sin(\varphi)\,d\varphi\,d\theta$$

As noted in the integral above D is just the ranges of θ and φ we found in Step 1.

Step 4

Now we just need to evaluate the integral to get the surface area.

$$A = \int_0^{2\pi}\int_{\frac{1}{2}\pi}^{\pi} 25\sin(\varphi)\,d\varphi\,d\theta = \int_0^{2\pi} -25\cos(\varphi)\Big|_{\frac{1}{2}\pi}^{\pi}\,d\theta = \int_0^{2\pi} 25\,d\theta = \boxed{50\pi}$$

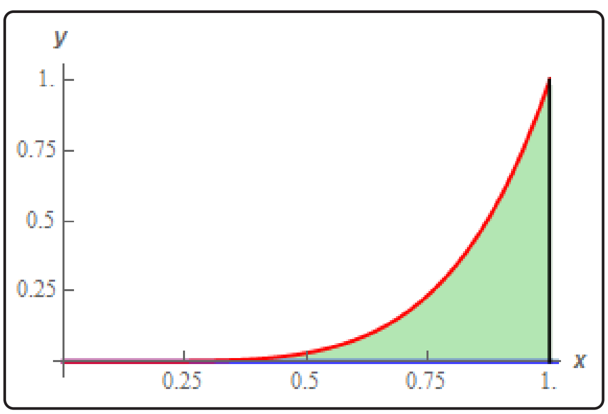
9. Determine the surface area of the portion of $z = 3 + 2y + \frac{1}{4}x^4$ that is above the region in the xy-plane bounded by $y = x^5$, $x = 1$ and the x-axis.

Step 1

Parameterizing this surface is pretty simple. We have the equation of the surface in the form $z = f(x, y)$ and so the parameterization of the surface is,

$$\vec{r}(x,y) = \left\langle x, y, 3 + 2y + \frac{1}{4}x^4 \right\rangle$$

Now, this is the parameterization of the full surface and we only want the portion that lies over the following region.



So, to get only the portion of the surface we'll need to restrict x and y to the following ranges,

$$\begin{aligned} 0 \leq &x \leq 1 \\ 0 \leq &y \leq x^5 \end{aligned}$$

On a side note we can see that we are in the 1^{st} quadrant here and so we know that $x \geq 0$ and $y \geq 0$. Therefore, we can see that the surface in the 1^{st} quadrant is always above the xy-plane and so will in fact always be above the region above as suggested in the problem statement.

Step 2

Next, we need to compute $\vec{r}_x \times \vec{r}_y$. Here is that work.

$$\vec{r}_x = \langle 1, 0, x^3 \rangle \qquad \vec{r}_y = \langle 0, 1, 2 \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & x^3 \\ 0 & 1 & 2 \end{vmatrix} = -x^3\vec{i} - 2\vec{j} + \vec{k}$$

Now, we what we really need is,

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{(-x^3)^2 + (-2)^2 + (1)^2} = \sqrt{x^6 + 5}$$

Step 3

The integral for the surface area is then,

$$A = \iint\limits_D \sqrt{x^6 + 5}\, dA = \int_0^1 \int_0^{x^5} \sqrt{x^6 + 5}\, dy\, dx$$

As noted in the integral above D is just the ranges of x and y we found in Step 1.

Step 4

Now we just need to evaluate the integral to get the surface area.

$$A = \int_0^1 \int_0^{x^5} \sqrt{x^6+5}\,dy\,dx = \int_0^1 y\sqrt{x^6+5}\Big|_0^{x^5}\,dx$$
$$= \int_0^1 x^5\sqrt{x^6+5}\,dx = \frac{1}{9}\left(x^6+5\right)^{\frac{3}{2}}\Big|_0^1 = \boxed{\frac{1}{9}\left(6^{\frac{3}{2}} - 5^{\frac{3}{2}}\right) = 0.3907}$$

---

10. Determine the surface area of the portion of the surface given by the following parametric equation that lies inside the cylinder $u^2+v^2=4$.

$$\vec{r}(u,v) = \langle 2u, vu, 1-2v\rangle$$

Step 1

We've already been given the parameterization of the surface in the problem statement so we don't need to worry about that for this problem. All we really need to do yet is to acknowledge that we'll need to restrict u and v to the disk $u^2+v^2 \le 4$.

Step 2

Next, we need to compute $\vec{r}_u \times \vec{r}_v$. Here is that work.

$$\vec{r}_u = \langle 2, v, 0\rangle \qquad \vec{r}_v = \langle 0, u, -2\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & v & 0 \\ 0 & u & -2 \end{vmatrix} = -2v\vec{i} + 4\vec{j} + 2u\vec{k}$$

Now, we what we really need is,

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{(-2v)^2 + (4)^2 + (2u)^2} = \sqrt{4u^2+4v^2+16} = 2\sqrt{u^2+v^2+4}$$

Step 3

The integral for the surface area is then,

$$A = \iint_D 2\sqrt{u^2 + v^2 + 4}\, dA$$

Where D is the disk $u^2 + v^2 \le 4$.

Step 4

Because D is a disk the best bet for this integral is to use the following "version" of polar coordinates.

$$u = r\cos(\theta) \qquad v = r\sin(\theta) \qquad u^2 + v^2 = r^2 \qquad dA = r\,dr\,d\theta$$

The polar coordinate limits for this D is,

$$\begin{gathered} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{gathered}$$

So, the integral to converting to polar coordinates gives,

$$A = \iint_D 2\sqrt{u^2 + v^2 + 4}\, dA = \int_0^{2\pi}\int_0^2 2r\sqrt{r^2 + 4}\, dr\, d\theta$$

Step 5

Now we just need to evaluate the integral to get the surface area.

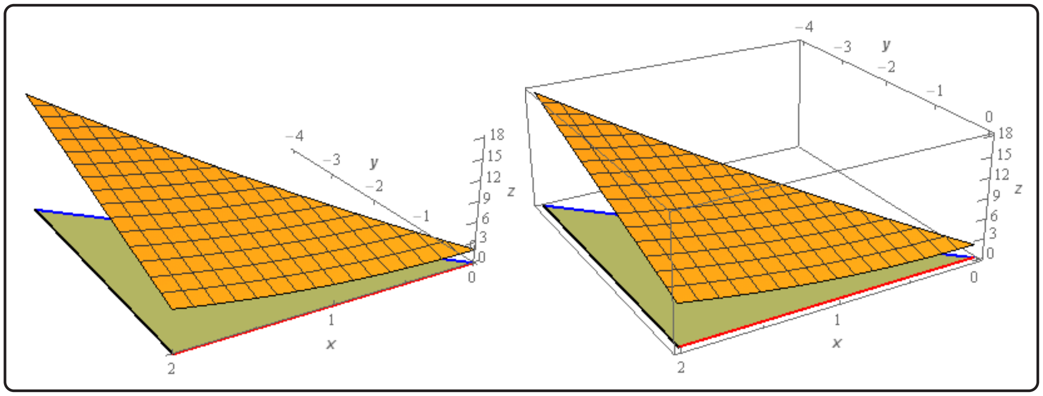
$$\begin{aligned} A &= \int_0^{2\pi}\int_0^2 2r\sqrt{r^2 + 4}\, dr\, d\theta = \int_0^{2\pi} \left. \frac{2}{3}\left(r^2 + 4\right)^{\frac{3}{2}} \right|_0^2 d\theta \\ &= \int_0^{2\pi} \frac{2}{3}\left(8^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) d\theta = \left. \frac{2}{3}\left(8^{\frac{3}{2}} - 8\right)\theta \right|_0^{2\pi} = \boxed{\frac{32}{3}\pi\left(\sqrt{8} - 1\right) = 61.2712} \end{aligned}$$

17.3 Surface Integrals

1. Evaluate $\displaystyle\iint\limits_S z + 3y - x^2\, dS$ where S is the portion of $z = 2 - 3y + x^2$ that lies over the triangle in the xy-plane with vertices $(0, 0)$, $(2, 0)$ and $(2, -4)$.

Step 1

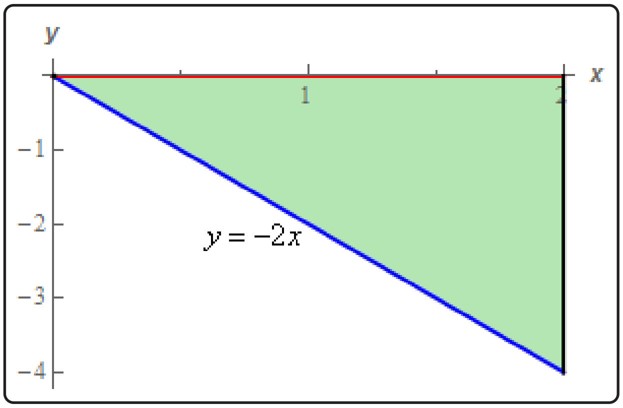
Let's start off with a quick sketch of the surface we are working with in this problem.



We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

The orange surface is the sketch of $z = 2 - 3y + x^2$ that we are working with in this problem. The greenish triangle below the surface is the triangle referenced in the problem statement that lies below the surface. This triangle will be the region D for this problem.

Here is a quick sketch of D just to get a better view of it than the mostly obscured view in the sketch above.



We could use either of the following sets of limits to describe D.

$$0 \le x \le 2 \qquad\qquad -4 \le y \le 0$$
$$-2x \le y \le 0 \qquad\qquad -\frac{1}{2}y \le x \le 2$$

We'll decide which set to use in the integral once we get that set up.

Step 2

Let's get the integral set up now. In this case the surface is in the form,

$$z = g(x, y) = 2 - 3y + x^2$$

so we'll use the following formula for the surface integral.

$$\iint\limits_S f(x, y, z)\, dS = \iint\limits_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dA$$

The integral is then,

$$\begin{aligned} \iint\limits_S z + 3y - x^2\, dS &= \iint\limits_D \left[\left(2 - 3y + x^2\right) + 3y - x^2\right] \sqrt{(2x)^2 + (-3)^2 + 1}\, dA \\ &= \iint\limits_D 2\sqrt{4x^2 + 10}\, dA \end{aligned}$$

Don't forget to plug the equation of the surface into z in the integrand and recall that D is the triangle sketched in Step 1.

Step 3

Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

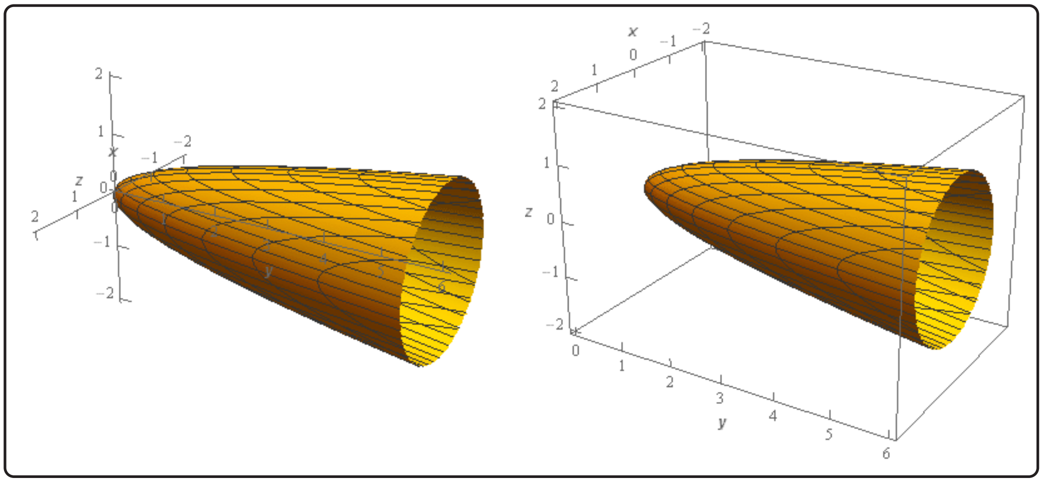
First note that from the integrand it should be pretty clear that we'll want to integrate with respect to y first (unless you want to do a trig substitution of course....). So, the integral becomes,

$$\begin{aligned}\iint\limits_S z + 3y - x^2\,dS &= \iint\limits_D 2\sqrt{4x^2+10}\,dA \\ &= \int_0^2 \int_{-2x}^0 2\sqrt{4x^2+10}\,dy\,dx \\ &= \int_0^2 \left(2y\sqrt{4x^2+10}\right)\Big|_{-2x}^0\,dx \\ &= \int_0^2 4x\sqrt{4x^2+10}\,dx \\ &= \frac{1}{3}\left(4x^2+10\right)^{\frac{3}{2}}\Big|_0^2 = \boxed{\frac{1}{3}\left(26^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 33.6506}\end{aligned}$$

2. Evaluate $\displaystyle\iint\limits_S 40y\,dS$ where S is the portion of $y = 3x^2 + 3z^2$ that lies behind $y = 6$.

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



Note that the surface in this problem is only the elliptic paraboloid and does not include the "cap" at $y = 6$. We would only include the "cap" if the problem had specified that in some manner to make it clear.

In this case D will be the circle/disk we get by setting the two equations equal or,

$$6 = 3x^2 + 3z^2 \qquad \Rightarrow \qquad x^2 + z^2 = 2$$

So, D will be the disk $x^2 + z^2 \le 2$.

Step 2

Let's get the integral set up now. In this case the surface is in the form,

$$y = g(x,z) = 3x^2 + 3z^2$$

so we'll use the following formula for the surface integral.

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(x, g(x,z), z)\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + 1 + \left(\frac{\partial g}{\partial z}\right)^2}\,dA$$

The integral is then,

$$\begin{aligned}\iint\limits_S 40y\,dS &= \iint\limits_D 40\left(3x^2 + 3z^2\right)\sqrt{(6x)^2 + 1 + (6z)^2}\,dA \\ &= \iint\limits_D 120\left(x^2 + z^2\right)\sqrt{36\left(x^2 + z^2\right) + 1}\,dA\end{aligned}$$

Don't forget to plug the equation of the surface into y in the integrand and recall that D is the disk we found in Step 1.

Step 3

Now, for this problem it should be pretty clear that we'll want to use polar coordinates to do the integral. We'll use the following set of polar coordinates.

$$x = r\cos(\theta) \qquad z = r\sin(\theta) \qquad x^2 + z^2 = r^2$$

Also, because D is the disk $x^2 + z^2 \le 2$ the limits for the integral will be,

$$\begin{gathered}0 \le \theta \le 2\pi \\ 0 \le r \le \sqrt{2}\end{gathered}$$

Converting the integral to polar coordinates gives,

$$\iint\limits_S 40y\,dS = \int_0^{2\pi}\int_0^{\sqrt{2}} 120r^2\sqrt{36r^2+1}\,(r)\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\sqrt{2}} 120r^3\sqrt{36r^2+1}\,dr\,d\theta$$

Don't forget to pick up the extra r when converting the dA into polar coordinates.

Step 4

Now all that we need to do is evaluate the double integral and this one can be a little tricky unless you've seen this kind of integral done before.

We'll use the following substitution to do the integral.

$$u = 36r^2+1 \qquad \to \qquad du = 72r\,dr \qquad \to \qquad \frac{1}{72}du = r\,dr$$

The problem is that this doesn't seem to work at first glance because the differential will only get rid of one of the three r's in front of the root. However, we can also solve the substitution for r^2 to get,

$$r^2 = \frac{1}{36}(u-1)$$

and we can now convert the remaining two r's into u's.

So, using the substitution the integral becomes,

$$\iint\limits_S 40y\,dS = \int_0^{2\pi}\int_1^{73} 120\left(\frac{1}{72}\right)\left(\frac{1}{36}\right)(u-1)\,u^{\frac{1}{2}}\,du\,d\theta$$
$$= \int_0^{2\pi}\int_1^{73} \frac{5}{108}\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\,du\,d\theta$$

Note that we also converted the r limits in the original integral into u limits simply by plugging the "old" r limits into the substitution to get "new" u limits.

We can now easily finish evaluating the integral.

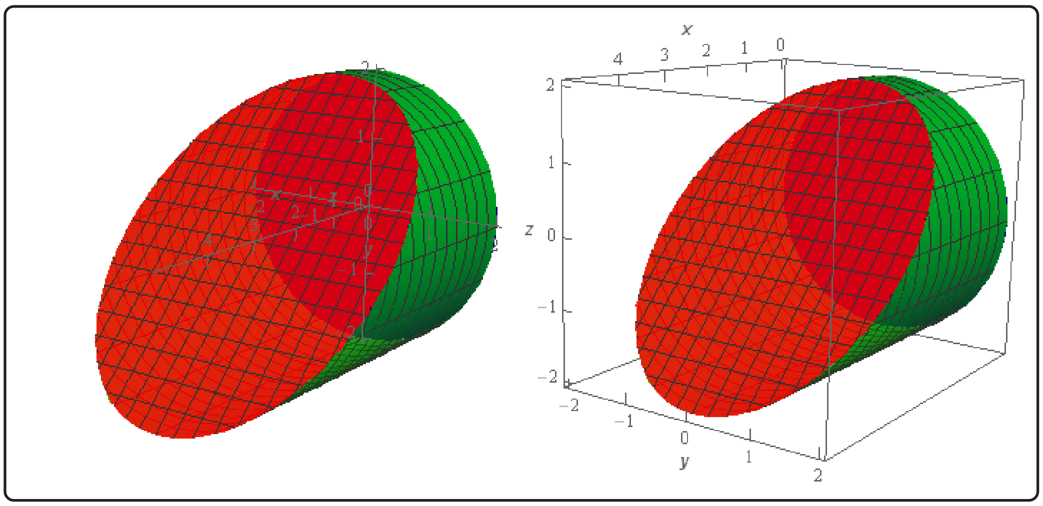
$$\begin{aligned}\iint\limits_S 40y\,dS &= \int_0^{2\pi} \frac{5}{108}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_1^{73} d\theta \\ &= \int_0^{2\pi} \frac{5}{108}\left[\frac{2}{5}\left(73^{\frac{5}{2}}\right) - \frac{2}{3}\left(73^{\frac{3}{2}}\right) - \left(-\frac{4}{15}\right)\right] d\theta \\ &= \boxed{\frac{5\pi}{54}\left[\frac{2}{5}\left(73^{\frac{5}{2}}\right) - \frac{2}{3}\left(73^{\frac{3}{2}}\right) + \frac{4}{15}\right] = 5176.8958}\end{aligned}$$

Kind of messy integral with a messy answer but that will happen on occasion so we shouldn't get too excited about it when that does happen.

3. Evaluate $\displaystyle\iint\limits_S 2y\,dS$ where S is the portion of $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$.

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



Note that the surface in this problem is only the cylinder itself. The "caps" of the cylinder are not part of this surface despite the red "cap" in the sketch. That was included in the sketch to make the front edge of the cylinder clear in the sketch. We would only include the "caps" if the problem had specified that in some manner to make it clear.

Step 2

Now because our surface is a cylinder we'll need to parameterize it and use the following formula for the surface integral.

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(\vec{r}(u,v))\,\|\vec{r}_u \times \vec{r}_v\|\,dA$$

where u and v will be chosen as needed when doing the parameterization.

We saw how to parameterize a cylinder in the previous section so we won't go into detail for the parameterization. The parameterization is,

$$\vec{r}(x,\theta) = \langle x, 2\sin(\theta), 2\cos(\theta)\rangle \qquad 0 \le \theta \le 2\pi, \quad 0 \le x \le 3 - z = 3 - 2\cos(\theta)$$

We'll use the full range of θ since we are allowing it to rotate all the way around the *x*-axis. The x limits come from the two planes that "bound" the cylinder and we'll need to convert the upper limit using the parameterization.

Next, we'll need to compute the cross product.

$$\vec{r}_x = \langle 1,0,0\rangle \qquad \vec{r}_\theta = \langle 0, 2\cos(\theta), -2\sin(\theta)\rangle$$

$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 2\cos(\theta) & -2\sin(\theta) \end{vmatrix} = 2\sin(\theta)\vec{j} + 2\cos(\theta)\vec{k}$$

The magnitude of the cross product is,

$$\|\vec{r}_x \times \vec{r}_\theta\| = \sqrt{4\sin^2(\theta) + 4\cos^2(\theta)} = 2$$

The integral is then,

$$\iint\limits_S 2y\,dS = \iint\limits_D 2\,(2\sin(\theta))\,(2)\;dA = \iint\limits_D 8\sin(\theta)\,dA$$

Don't forget to plug the y component of the surface parametrization into the integrand and D is just the limits on x and θ we noted above in the parameterization.

Step 3

Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

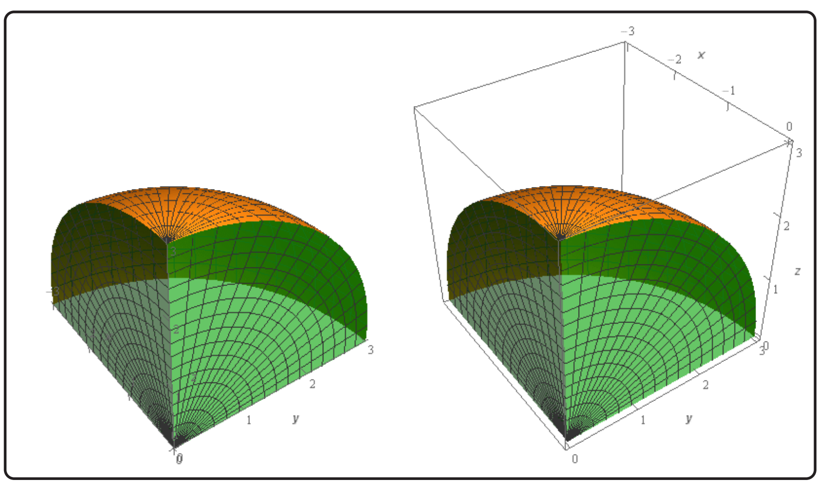
The integral is then,

$$\begin{aligned}\iint\limits_S 2y\,dS &= \int_0^{2\pi}\int_0^{3-2\cos(\theta)} 8\sin(\theta)\,dx\,d\theta \\ &= \int_0^{2\pi} \left(8x\sin(\theta)\right)\Big|_0^{3-2\cos(\theta)}\,d\theta \\ &= \int_0^{2\pi} 8\left(3-2\cos(\theta)\right)\sin(\theta)\,d\theta \\ &= \int_0^{2\pi} 24\sin(\theta) - 16\sin(\theta)\cos(\theta)\,d\theta \\ &= \int_0^{2\pi} 24\sin(\theta) - 8\sin\left(2\theta\right)\,d\theta \\ &= \left(-24\cos(\theta) + 4\cos\left(2\theta\right)\right)\Big|_0^{2\pi} = \boxed{0}\end{aligned}$$

4. Evaluate $\displaystyle\iint\limits_S xz\,dS$ where S is the portion of the sphere of radius 3 with $x \le 0$, $y \ge 0$ and $z \ge 0$.

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



Note that the surface in this problem is only the part of the sphere itself. The "edges" (the greenish portions on the right/left) are not part of this surface despite the fact that they are in the sketch. They were included in the sketch to try and make the surface a little clearer in the sketch. We would only include the "edges" if the problem had specified that in some manner to make it clear.

Step 2

Now because our surface is a sphere we'll need to parameterize it and use the following formula for the surface integral.

$$\iint\limits_{S} f(x,y,z)\, dS = \iint\limits_{D} f(\vec{r}(u,v))\, \|\vec{r}_u \times \vec{r}_v\|\, dA$$

where u and v will be chosen as needed when doing the parameterization.

We saw how to parameterize a sphere in the previous section so we won't go into detail for the parameterization. The parameterization is,

$$\vec{r}(\theta,\varphi) = \langle 3\sin(\varphi)\cos(\theta), 3\sin(\varphi)\sin(\theta), 3\cos(\varphi)\rangle \qquad \frac{1}{2}\pi \le \theta \le \pi, \quad 0 \le \varphi \le \frac{1}{2}\pi$$

We needed the restriction on φ to make sure that we only get a portion of the upper half of the sphere (*i.e.* $z \ge 0$). Likewise the restriction on θ was needed to get only the portion that was in the 2^{nd} quadrant of the xy-plane (*i.e.* $x \le 0$ and $y \ge 0$).

Next, we'll need to compute the cross product.

$$\vec{r}_\theta = \langle -3\sin(\varphi)\sin(\theta), 3\sin(\varphi)\cos(\theta), 0\rangle$$
$$\vec{r}_\varphi = \langle 3\cos(\varphi)\cos(\theta), 3\cos(\varphi)\sin(\theta), -3\sin(\varphi)\rangle$$

$$\begin{aligned}
\vec{r}_\theta \times \vec{r}_\varphi &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin(\varphi)\sin(\theta) & 3\sin(\varphi)\cos(\theta) & 0 \\ 3\cos(\varphi)\cos(\theta) & 3\cos(\varphi)\sin(\theta) & -3\sin(\varphi) \end{vmatrix} \\
&= -9\sin^2(\varphi)\cos(\theta)\,\vec{i} - 9\sin(\varphi)\cos(\varphi)\sin^2(\theta)\vec{k} - 9\sin(\varphi)\cos(\varphi)\cos^2(\theta)\vec{k} \\
&\qquad - 9\sin^2(\varphi)\sin(\theta)\vec{j} \\
&= -9\sin^2(\varphi)\cos(\theta)\,\vec{i} - 9\sin^2(\varphi)\sin(\theta)\vec{j} - 9\sin(\varphi)\cos(\varphi)\left(\sin^2(\theta) + \cos^2(\theta)\right)\vec{k} \\
&= -9\sin^2(\varphi)\cos(\theta)\,\vec{i} - 9\sin^2(\varphi)\sin(\theta)\vec{j} - 9\sin(\varphi)\cos(\varphi)\vec{k}
\end{aligned}$$

The magnitude of the cross product is,

$$\begin{aligned}\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\| &= \sqrt{\left(-9\sin^2(\varphi)\cos(\theta)\right)^2 + \left(-9\sin^2(\varphi)\sin(\theta)\right)^2 + \left(-9\sin(\varphi)\cos(\varphi)\right)^2} \\ &= \sqrt{81\sin^4(\varphi)\left(\cos^2(\theta)+\sin^2(\theta)\right) + 81\sin^2(\varphi)\cos^2(\varphi)} \\ &= \sqrt{81\sin^2(\varphi)\left(\sin^2(\varphi)+\cos^2(\varphi)\right)} \\ &= 9\left|\sin(\varphi)\right| \\ &= 9\sin(\varphi)\end{aligned}$$

The integral is then,

$$\begin{aligned}\iint\limits_S xz\,dS &= \iint\limits_D \left(3\sin(\varphi)\cos(\theta)\right)\left(3\cos(\varphi)\right)\left(9\sin(\varphi)\right)\,dA \\ &= \iint\limits_D 81\cos(\varphi)\sin^2(\varphi)\cos(\theta)\,dA\end{aligned}$$

Don't forget to plug the x and z component of the surface parameterization into the integrand and D is just the limits on θ and φ we noted above in the parameterization.

Step 3

Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

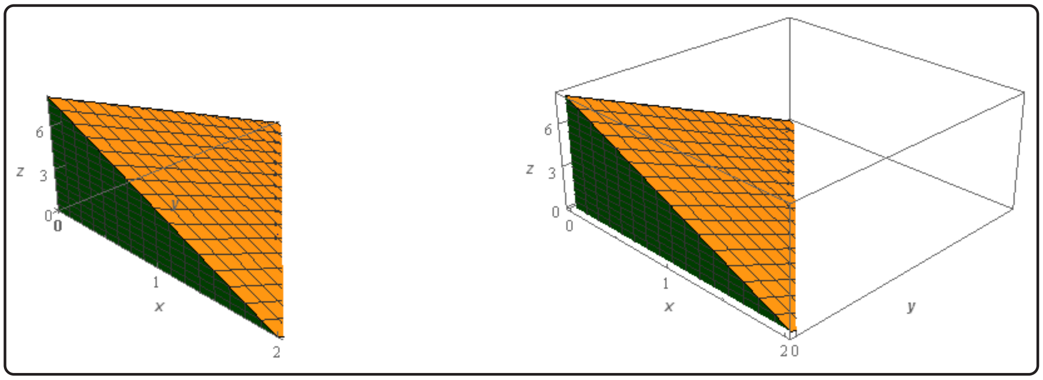
The integral is then,

$$\begin{aligned}\iint\limits_S xz\,dS &= \iint\limits_D 81\cos(\varphi)\sin^2(\varphi)\cos(\theta)\,dA \\ &= \int_{\frac{1}{2}\pi}^{\pi}\int_0^{\frac{1}{2}\pi} 81\cos(\varphi)\sin^2(\varphi)\cos(\theta)\,d\varphi\,d\theta \\ &= \int_{\frac{1}{2}\pi}^{\pi}\left(27\sin^3(\varphi)\cos(\theta)\,d\varphi\right)\Big|_0^{\frac{1}{2}\pi}\,d\theta \\ &= \int_{\frac{1}{2}\pi}^{\pi} 27\cos(\theta)\,d\theta \\ &= \left(27\sin(\theta)\right)\Big|_{\frac{1}{2}\pi}^{\pi} = \boxed{-27}\end{aligned}$$

5. Evaluate $\displaystyle\iint\limits_S yz + 4xy\,dS$ where S is the surface of the solid bounded by $4x + 2y + z = 8$, $z = 0$, $y = 0$ and $x = 0$. Note that all four surfaces of this solid are included in S.

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



Okay, as noted in the problem statement all four surfaces in the sketch (two not shown) are part of S so let's define each of them as follows.

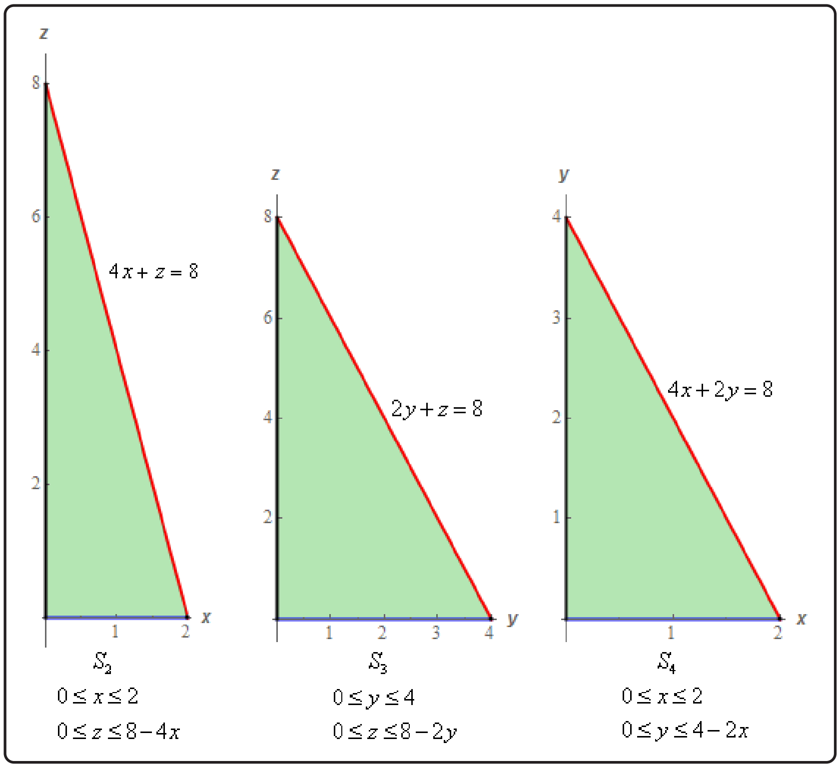
S_1 : Plane given by $4x + 2y + z = 8$ ($i.e$ the top of the solid)

S_2 : Plane given by $y = 0$ ($i.e$ the triangle on right side of the solid)

S_3 : Plane given by $x = 0$ ($i.e$ the triangle at back of the solid - not shown in sketch)

S_4 : Plane given by $z = 0$ ($i.e$ the triangle on bottom of the solid - not shown in sketch)

As noted in the definitions above the first two surfaces are shown in the sketch but the last two are not actually shown due to the orientation of the solid. Below are sketches of each of the three surfaces that correspond to the coordinates planes.



With each of the sketches we gave limits on the variables for each of them since we'll eventually need that when we start doing the surface integral along each surface.

Now we need to go through and do the integral for each of these surfaces and we're going to go through these a little quicker than we did for the first few problems in this section.

Step 2

Let's start with S_1. In this case the surface can easily be solved for z to get,

$$z = 8 - 4x - 2y$$

With the equation of the surface written in this manner the region D will be in the *xy*-plane and if you think about it you'll see that in fact D is nothing more than S_4!

The integral in this case is,

$$\iint\limits_{S_1} yz + 4xy\,dS = \iint\limits_{D} \left[y\left(8-4x-2y\right)+4xy\right]\sqrt{\left(-4\right)^2+\left(-2\right)^2+1}\,dA$$
$$= \sqrt{21}\iint\limits_{D} 8y - 2y^2\,dA$$

Don't forget to plug the equation of the surface into z in the integrand and don't forget to use the equation of the surface in the computation of the root!

Now, as noted above D for this surface is nothing more than S_4 and so we can use the limits from the sketch of S_4 in Step 1.

Now let's compute the integral for this surface.

$$\iint\limits_{S_1} yz + 4xy\,dS = \sqrt{21}\int_0^2\int_0^{4-2x} 8y - 2y^2\,dy\,dx$$
$$= \sqrt{21}\int_0^2 \left[4y^2 - \frac{2}{3}y^3\right]\bigg|_0^{4-2x}\,dx$$
$$= \sqrt{21}\int_0^2 4(4-2x)^2 - \frac{2}{3}(4-2x)^3\,dx$$
$$= \sqrt{21}\left[-\frac{2}{3}(4-2x)^3 + \frac{1}{12}(4-2x)^4\right]\bigg|_0^2 = \underline{\frac{64\sqrt{21}}{3} = 97.7616}$$

Step 3

Next we'll take care of S_2. In this case the equation for the surface is simply $y = 0$ and D is given in the sketch of S_2 in Step 1.

The integral in this case is,

$$\iint\limits_{S_2} yz + 4xy\,dS = \iint\limits_{D} 0\sqrt{\left(0\right)^2+1+\left(0\right)^2}\,dA = \iint\limits_{D} 0\,dA = \underline{0}$$

So, in this case we didn't need to actually compute the integral. Sometimes we'll get lucky like this, although it probably won't happen all that often.

Step 4

Now we can take care of S_3. In this case the equation for the surface is simply $x = 0$ and D is given in the sketch of S_3 in Step 1.

The integral in this case is,

$$\iint\limits_{S_3} yz + 4xy\,dS = \iint\limits_{D} [yz + 4(0)\,y]\sqrt{1 + (0)^2 + (0)^2}\,dA = \iint\limits_{D} yz\,dA$$

Don't forget to plug the equation of the surface into x in the integrand and don't forget to use the equation of the surface in the computation of the root (although in this case the root just evaluates to one)!

Using the limits for D from the sketch in Step 1 we can quickly evaluate the integral for this surface.

$$\begin{aligned}\iint\limits_{S_3} yz + 4xy\,dS &= \int_0^4\int_0^{8-2y} yz\,dz\,dy \\ &= \int_0^4 \left[\frac{1}{2}yz^2\right]\Bigg|_0^{8-2y} dy \\ &= \int_0^4 32y - 16y^2 + 2y^3\,dy \\ &= \left[16y^2 - \frac{16}{3}y^3 + \frac{1}{2}y^4\right]\Bigg|_0^4 = \underline{\frac{128}{3} = 42.6667}\end{aligned}$$

Step 5

Finally let's take care of S_4. In this case the equation for the surface is simply $z = 0$ and D is given in the sketch of S_4 in Step 1.

The integral in this case is,

$$\iint\limits_{S_4} yz + 4xy\,dS = \iint\limits_{D} [y(0) + 4xy]\sqrt{(0)^2 + (0)^2 + 1}\,dA = \iint\limits_{D} 4xy\,dA$$

Don't forget to plug the equation of the surface into z in the integrand and don't forget to use the equation of the surface in the computation of the root (although in this case the root just evaluates to one)!

Using the limits for D from the sketch in Step 1 we can quickly evaluate the integral for this surface.

$$\iint\limits_{S_4} yz + 4xy\,dS = \int_0^2 \int_0^{4-2x} 4xy\,dy\,dx$$
$$= \int_0^2 \left[2xy^2\right]\Big|_0^{4-2x} dx$$
$$= \int_0^2 32x - 32x^2 + 8x^3\,dx$$
$$= \left[16x^2 - \frac{32}{3}x^3 + 2x^4\right]\Big|_0^2 = \underline{\frac{32}{3} = 10.6667}$$

Step 6

Now, to get the value of the integral over the full surface all we need to do is sum up the values of each of the integrals over the four surfaces above. Doing this gives,

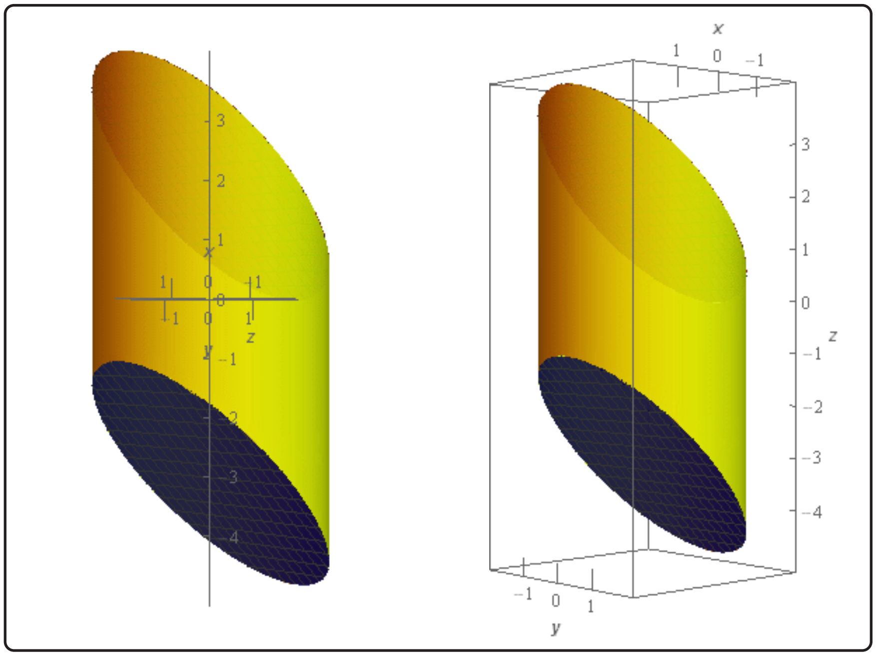
$$\iint\limits_S yz + 4xy\,dS = \left(\frac{64\sqrt{21}}{3}\right) + (0) + \left(\frac{128}{3}\right) + \left(\frac{32}{3}\right) = \frac{64\sqrt{21}}{3} + \frac{160}{3} = \boxed{151.0949}$$

We put parenthesis around each of the individual integral values just to indicate where each came from. In general, these aren't needed of course.

6. Evaluate $\displaystyle\iint\limits_S x - z\,dS$ where S is the surface of the solid bounded by $x^2 + y^2 = 4$, $z = x - 3$, and $z = x + 2$. Note that all three surfaces of this solid are included in S.

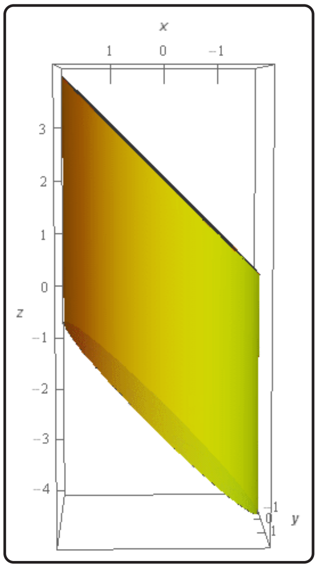
Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



As noted in the problem statement there are three surfaces here. The "top" of the cylinder is a little hard to see. We made the walls of the cylinder slightly transparent and the top of the cylinder can be seen as a darker ellipse along the top of the surface.

To help visualize the relationship between the top and bottom of the cylinder here is a different view of the surface.



From this view we can see that the top and bottom planes that "cap" the cylinder are parallel.

Let's define the three surfaces in the sketch as follows.

S_1 : Cylinder given by $x^2 + y^2 = 4$ $(i.e$ the walls of the solid$)$

S_2 : Plane given by $z = x + 2$ $(i.e$ the top cap of the cylinder$)$

S_3 : Plane given by $z = x - 3$ $(i.e$ the bottom cap of the cylinder$)$

Now we need to go through and do the integral for each of these surfaces and we're going to go through these a little quicker than we did for the first few problems in this section.

Step 2

Let's start with S_1. The surface in this case is a cylinder and so we'll need to parameterize it. The parameterization of the surface is,

$$\vec{r}(z,\theta) = \langle 2\cos(\theta), 2\sin(\theta), z\rangle$$

The limits on z and θ are,

$$0 \le \theta \le 2\pi, \quad 2\cos(\theta) - 3 = x - 3 \le z \le x + 2 = 2\cos(\theta) + 2$$

With the z limits we'll need to make sure that we convert the x's into their parameterized form.

In order to evaluate the integral in this case we'll need the cross product $\vec{r}_z \times \vec{r}_\theta$ so here is that work.

$$\vec{r}_z = \langle 0, 0, 1\rangle \qquad \vec{r}_\theta = \langle -2\sin(\theta), 2\cos(\theta), 0\rangle$$

$$\vec{r}_z \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -2\sin(\theta) & 2\cos(\theta) & 0 \end{vmatrix} = -2\cos(\theta)\vec{i} - 2\sin(\theta)\vec{j}$$

Next, we'll need the magnitude of the cross product so here is that.

$$\|\vec{r}_z \times \vec{r}_\theta\| = \sqrt{4\cos^2(\theta) + 4\sin^2(\theta)} = 2$$

The integral in this case is,

$$\iint\limits_{S_1} x - z\,dS = \iint\limits_{D} [2\cos(\theta) - z]\,(2)\,dA = \iint\limits_{D} 4\cos(\theta) - 2z\,dA$$

Don't forget to plug the parameterization of the surface into the integrand and don't forget to add in the magnitude of the cross product!

Now, D for this surface is nothing more than the limits on z and θ we gave above.

Now let's compute the integral for this surface.

$$\begin{aligned}\iint\limits_{S_1} x - z\,dS &= \int_0^{2\pi}\int_{2\cos(\theta)-3}^{2\cos(\theta)+2} 4\cos(\theta) - 2z\,dz\,d\theta \\ &= \int_0^{2\pi} \left(4z\cos(\theta) - z^2\right)\Big|_{2\cos(\theta)-3}^{2\cos(\theta)+2} d\theta \\ &= \int_0^{2\pi} 4\cos(\theta)\left[2\cos(\theta)+2-(2\cos(\theta)-3)\right] - \\ &\qquad\qquad \left[(2\cos(\theta)+2)^2 - (2\cos(\theta)-3)^2\right] d\theta \\ &= \int_0^{2\pi} 5\,d\theta = \underline{10\pi}\end{aligned}$$

Do not forget to simplify! As we saw with this problem after the z integration the integrand looked really messy but after some pretty simple simplification it reduced down to an incredibly simple integrand.

Step 3

Next we'll take care of S_2. In this case the equation for the surface is simply $z = x + 2$ and D is the disk $x^2 + y^2 \le 4$.

The integral in this case is,

$$\iint\limits_{S_2} x - z\,dS = \iint\limits_{D} \left[x - (x+2)\right]\sqrt{(1)^2 + (0)^2 + 1}\,dA = \iint\limits_{D} -2\sqrt{2}\,dA = -2\sqrt{2}\iint\limits_{D} dA$$

Okay, in this case we don't need to actually do the evaluation of the integral because we know that,

$$\iint\limits_{D} dA = \text{Area of } D$$

and in this case D is just a disk and we can quickly determine its area without any evaluation.

So, the integral for this surface is then just,

$$\iint\limits_{S_2} x - z\,dS = -2\sqrt{2}\,(\text{Area of } D) = -2\sqrt{2}\left[(2)^2\pi\right] = \underline{-8\sqrt{2}\pi}$$

Step 4

Finally, let's integrate over S_3. In this case the equation for the surface is simply $z = x-3$ and D is the disk $x^2 + y^2 \le 4$.

The integral in this case is,

$$\begin{aligned}\iint\limits_{S_3} x - z\,dS &= \iint\limits_{D} [x - (x-3)]\sqrt{(1)^2 + (0)^2 + 1}\,dA \\ &= \iint\limits_{D} 3\sqrt{2}\,dA = 3\sqrt{2}\iint\limits_{D} dA = 3\sqrt{2}\,(4\pi) = \underline{12\sqrt{2}\pi}\end{aligned}$$

So, the integral in this case ended up being every similar to the integral in Step 3 and so we didn't put in any of the explanation here.

Step 5

Now, to get the value of the integral over the full surface all we need to do is sum up the values of each of the integrals over the three surfaces above. Doing this gives,

$$\iint\limits_{S} x - z\,dS = (10\pi) + \left(-8\sqrt{2}\pi\right) + \left(12\sqrt{2}\pi\right) = \left(10 + 4\sqrt{2}\right)\pi = \boxed{49.1875}$$

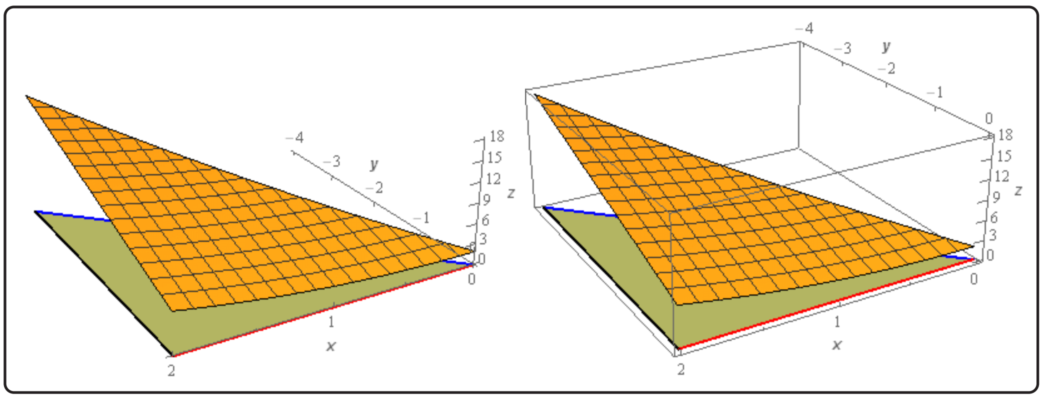
We put parenthesis around each of the individual integral values just to indicate where each came from. In general, these aren't needed of course.

17.4 Surface Integrals of Vector Fields

1. Evaluate $\displaystyle\iint\limits_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = 3x\,\vec{i} + 2z\,\vec{j} + \left(1 - y^2\right)\vec{k}$ and S is the portion of $z = 2 - 3y + x^2$ that lies over the triangle in the xy-plane with vertices $(0,0)$, $(2,0)$ and $(2,-4)$ oriented in the negative z-axis direction.

Step 1

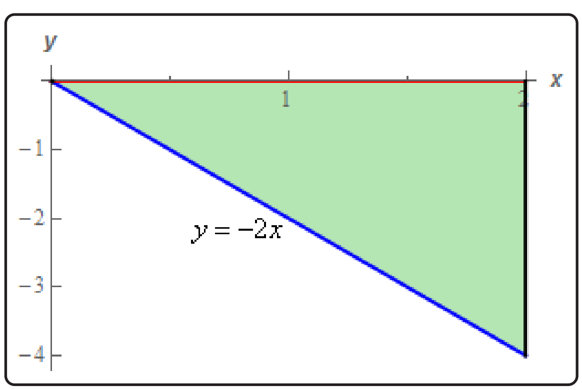
Let's start off with a quick sketch of the surface we are working with in this problem.



We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

The orange surface is the sketch of $z = 2 - 3y + x^2$ that we are working with in this problem. The greenish triangle below the surface is the triangle referenced in the problem statement that lies below the surface. This triangle will be the region D for this problem.

Here is a quick sketch of D just to get a better view of it than the mostly obscured view in the sketch above.



We could use either of the following sets of limits to describe D.

$$\begin{aligned} 0 \leq &x \leq 2 \\ -2x \leq &y \leq 0 \end{aligned} \qquad\qquad \begin{aligned} -4 \leq &y \leq 0 \\ -\frac{1}{2}y \leq &x \leq 2 \end{aligned}$$

We'll decide which set to use in the integral once we get that set up.

Step 2

Let's get the integral set up now. In this case the we can write the equation of the surface as follows,

$$f(x, y, z) = 2 - 3y + x^2 - z = 0$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle 2x, -3, -1\rangle}{\|\nabla f\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that, in this case, the normal vector we computed above has the correct orientation. We were told in the problem statement that the orientation was in the negative z-axis direction and this means that the normal vector should always have a downwards direction (*i.e.* a negative z component) and this one does.

Step 3

Next, we'll need to compute the following dot product.

$$\begin{aligned} \vec{F}\left(x, y, 2 - 3y + x^2\right) \cdot \vec{n} &= \left\langle 3x, 2\left(2 - 3y + x^2\right), 1 - y^2\right\rangle \cdot \frac{\langle 2x, -3, -1\rangle}{\|\nabla f\|} \\ &= \frac{1}{\|\nabla f\|}\left(6x^2 - 6\left(2 - 3y + x^2\right) - \left(1 - y^2\right)\right) \\ &= \frac{1}{\|\nabla f\|}\left(y^2 + 18y - 13\right) \end{aligned}$$

Remember that we needed to plug in the equation of the surface, $z = 2 - 3y + x^2$, into z in the vector field!

The integral is then,

$$\begin{aligned}\iint\limits_S \vec{F}\cdot d\vec{S} &= \iint\limits_S \frac{1}{\|\nabla f\|}\left(y^2+18y-13\right)\,dS \\ &= \iint\limits_D \frac{1}{\|\nabla f\|}\left(y^2+18y-13\right)\|\nabla f\|\,dA \\ &= \iint\limits_D y^2+18y-13\,dA\end{aligned}$$

As noted above we didn't need to compute the magnitude of the gradient since it would just cancel out when we converted the surface integral into a "normal" double integral.

Also, recall that D was given in Step 1. We had two sets of limits to use here but it seems like the first set is probably just as easy to use so we'll use that one in the integral.

Step 4

Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

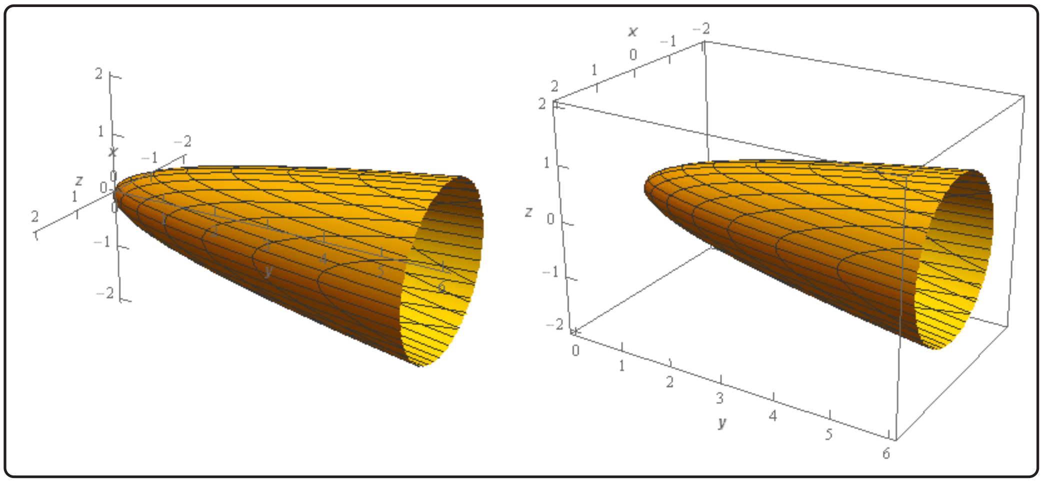
Here is the integral,

$$\begin{aligned}\iint\limits_S \vec{F}\cdot d\vec{S} &= \iint\limits_D y^2+18y-13\,dA \\ &= \int_0^2\int_{-2x}^0 y^2+18y-13\,dy\,dx \\ &= \int_0^2 \left.\left(\frac{1}{3}y^3+9y^2-13y\right)\right|_{-2x}^0 dx \\ &= \int_0^2 \frac{8}{3}x^3-36x^2-26x\,dx \\ &= \left.\left(\frac{2}{3}x^4-12x^3-13x^2\right)\right|_0^2 = \boxed{-\frac{412}{3}}\end{aligned}$$

2. Evaluate $\iint\limits_S \vec F \cdot d\vec S$ where $\vec F = -x\,\vec i + 2y\,\vec j - z\,\vec k$ and S is the portion of $y = 3x^2 + 3z^2$ that lies behind $y = 6$ oriented in the positive y-axis direction.

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



Note that the surface in this problem is only the elliptic paraboloid and does not include the "cap" at $y = 6$. We would only include the "cap" if the problem had specified that in some manner to make it clear.

In this case D will be the circle/disk we get by setting the two equations equal or,

$$6 = 3x^2 + 3z^2 \qquad \Rightarrow \qquad x^2 + z^2 = 2$$

So, D will be the disk $x^2 + z^2 \le 2$.

Step 2

Let's get the integral set up now. In this case the we can write the equation of the surface as follows,

$$f\left(x, y, z\right) = 3x^2 + 3z^2 - y = 0$$

A unit normal vector for the surface is then,

$$\vec n = \frac{\nabla f}{\left\| \nabla f \right\|} = \frac{\left\langle 6x, -1, 6z \right\rangle}{\left\| \nabla f \right\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that, in this case, the normal vector we computed above does not have the correct orientation. We were told in the problem statement that the orientation was in the positive y-axis direction and this means that the normal vector should always point in the general direction of the positive y-axis (*i.e.* a positive y component) and this one does not.

That is easy to fix however. All we need to do is multiply the above normal vector by minus one and we'll get what we need. So, here is the normal vector we need for this problem.

$$\vec{n} = -\frac{\nabla f}{\|\nabla f\|} = \frac{\langle -6x, 1, -6z\rangle}{\|\nabla f\|}$$

As we can see this normal vector does in fact have a positive y component as we need.

Step 3

Next, we'll need to compute the following dot product.

$$\begin{aligned}\vec{F}\left(x, 3x^2+3z^2, z\right)\cdot\vec{n} &= \left\langle -x, 2\left(3x^2+3z^2\right), -z\right\rangle\cdot\frac{\langle -6x, 1, -6z\rangle}{\|\nabla f\|}\\ &= \frac{1}{\|\nabla f\|}\left(6x^2+2\left(3x^2+3z^2\right)+6z^2\right)\\ &= \frac{1}{\|\nabla f\|}\left[12\left(x^2+z^2\right)\right]\end{aligned}$$

Remember that we needed to plug in the equation of the surface, $y = 3x^2+3z^2$, into y in the vector field!

The integral is then,

$$\begin{aligned}\iint\limits_S \vec{F}\cdot d\vec{S} &= \iint\limits_S \frac{1}{\|\nabla f\|}\left[12\left(x^2+z^2\right)\right]\,dS\\ &= \iint\limits_D \frac{1}{\|\nabla f\|}\left[12\left(x^2+z^2\right)\right]\|\nabla f\|\,dA\\ &= \iint\limits_D 12\left(x^2+z^2\right)\,dA\end{aligned}$$

As noted above we didn't need to compute the magnitude of the gradient since it would just cancel out when we converted the surface integral into a "normal" double integral.

Also, recall that D was given in Step 1 and is just the disk $x^2+z^2 \le 2$

Step 4

Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

Note as well that we'll want to use polar coordinates in the double integral. We'll use the following set of polar coordinates.

$$x = r\cos(\theta) \qquad z = r\sin(\theta) \qquad x^2 + z^2 = r^2$$

The polar limits for D are,

$$0 \le \theta \le 2\pi$$
$$0 \le r \le \sqrt{2}$$

The integral is then,

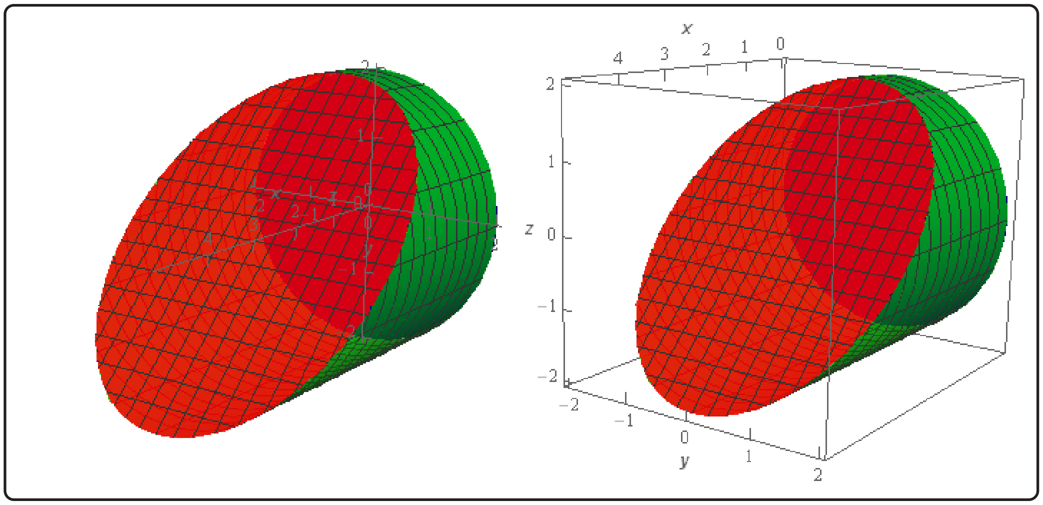
$$\begin{aligned}\iint\limits_S \vec{F} \cdot d\vec{S} &= \iint\limits_D 12\left(x^2 + z^2\right)\, dA \\ &= \int_0^{2\pi}\int_0^{\sqrt{2}} 12r^3\, dr\, d\theta \\ &= \int_0^{2\pi} 3r^4\Big|_0^{\sqrt{2}}\, d\theta \\ &= \int_0^{2\pi} 12\, d\theta = \boxed{24\pi}\end{aligned}$$

Don't forget that we pick up an extra r from the dA when converting to polar coordinates.

3. Evaluate $\displaystyle\iint\limits_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = x^2\,\vec{i} + 2z\,\vec{j} - 3y\,\vec{k}$ and S is the portion of $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$ oriented outwards (*i.e.* away from the x-axis).

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



Note that the surface in this problem is only the cylinder itself. The “caps” of the cylinder are not part of this surface despite the red “cap” in the sketch. That was included in the sketch to make the front edge of the cylinder clear in the sketch. We would only include the “caps” if the problem had specified that in some manner to make it clear.

Step 2

Let’s get the integral set up now. In this case the we are integrating over a cylinder and so we’ll need to set up a parameterization for the surface.

We saw how to parameterize a cylinder in the first section of this chapter so we won’t go into detail for the parameterization. The parameterization is,

$$\vec{r}\left(x,\theta\right) = \left\langle x, 2\sin\left(\theta\right), 2\cos\left(\theta\right)\right\rangle \qquad 0 \le \theta \le 2\pi, \quad 0 \le x \le 3 - z = 3 - 2\cos\left(\theta\right)$$

We’ll use the full range of θ since we are allowing it to rotate all the way around the *x*-axis. The x limits come from the two planes that “bound” the cylinder and we’ll need to convert the upper limit using the parameterization.

Next, we’ll need to compute the cross product.

$$\vec{r}_x = \left\langle 1,0,0\right\rangle \qquad \vec{r}_\theta = \left\langle 0, 2\cos\left(\theta\right), -2\sin\left(\theta\right)\right\rangle$$

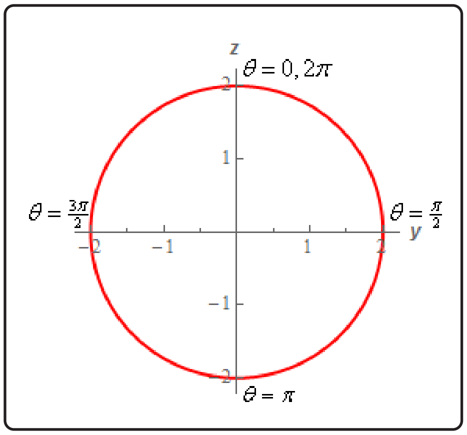
$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 2\cos\left(\theta\right) & -2\sin\left(\theta\right) \end{vmatrix} = 2\sin\left(\theta\right)\vec{j} + 2\cos\left(\theta\right)\vec{k}$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_\theta}{\|\vec{r}_x \times \vec{r}_\theta\|} = \frac{\langle 0, 2\sin(\theta), 2\cos(\theta)\rangle}{\|\vec{r}_x \times \vec{r}_\theta\|}$$

We didn't compute the magnitude of the cross product since we know that it will just cancel out when we start working with the integral.

Now we need to determine if this vector has the correct orientation. First let's look at the cylinder from in front of the cylinder and directly along the x-axis. This is what we'd see.



In this sketch the x axis will be coming straight out of the sketch at the origin. Plugging in a few value of θ into the parameterization we can see that we'll be at the points listed above.

Now, in the range $0 \le \theta \le \frac{1}{2}\pi$ we know that sine and cosine are both positive and so in the normal vector both the y and z components will be positive. This means that in the 1^{st} quadrant above the normal vector would need to be pointing out away from the origin. This is exactly what we need to see since the orientation was given as pointing away from the x-axis and recall that the x-axis is coming straight out of the sketch from the origin.

Next, if we look at $\frac{1}{2}\pi \le \theta \le \pi$ (so we're in the 4^{th} quadrant of the graph above....) we know that in this range sine is still positive but cosine is now negative. From our unit vector above this means that the y component is positive (so pointing in positive y direction) and the z component is negative (so pointing in negative z direction). Together this again means that we have to be pointing away from the origin in the 4^{th} quadrant which is again the orientation we want.

We could continue in this fashion looking at the remaining two quadrants but once we've done a couple and gotten the correct orientation we know we'll continue to get the correct orientation for the rest.

Step 3

Next, we'll need to compute the following dot product.

$$\begin{aligned}\vec{F}\left(\vec{r}\left(x,\theta\right)\right)\cdot\vec{n} &= \left\langle x^2, 2\left(2\cos(\theta)\right), -3\left(2\sin(\theta)\right)\right\rangle \cdot \frac{\langle 0, 2\sin(\theta), 2\cos(\theta)\rangle}{\|\vec{r}_x \times \vec{r}_\theta\|} \\ &= \frac{1}{\|\vec{r}_x \times \vec{r}_\theta\|}\left(-4\sin(\theta)\cos(\theta)\right)\end{aligned}$$

Remember that we needed to plug in the parameterization for the surface into the vector field!

The integral is then,

$$\begin{aligned}\iint\limits_S \vec{F}\cdot d\vec{S} &= \iint\limits_S \frac{1}{\|\vec{r}_x \times \vec{r}_\theta\|}\left(-4\sin(\theta)\cos(\theta)\right)\,dS \\ &= \iint\limits_D \frac{1}{\|\vec{r}_x \times \vec{r}_\theta\|}\left(-4\sin(\theta)\cos(\theta)\right)\|\vec{r}_x \times \vec{r}_\theta\|\,dA \\ &= \iint\limits_D -4\sin(\theta)\cos(\theta)\,dA\end{aligned}$$

As noted above we didn't need to compute the magnitude of the cross product since it would just cancel out when we converted the surface integral into a "normal" double integral.

Also, recall that D is given by the limits on x and θ we found at the start of Step 2.

Step 4

Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

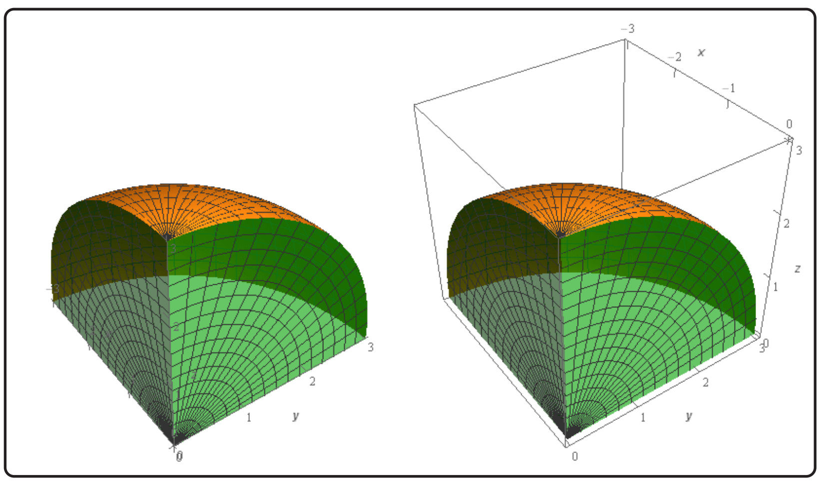
The integral is then,

$$\begin{aligned}\iint\limits_S \vec F \cdot d\vec S &= \iint\limits_D -4\sin(\theta)\cos(\theta)\,dA\\ &= \int_0^{2\pi}\int_0^{3-2\cos(\theta)} -4\sin(\theta)\cos(\theta)\,dx\,d\theta\\ &= \int_0^{2\pi} -4x\sin(\theta)\cos(\theta)\Big|_0^{3-2\cos(\theta)}\,d\theta\\ &= \int_0^{2\pi} -4\left(3-2\cos(\theta)\right)\sin(\theta)\cos(\theta)\,d\theta\\ &= \int_0^{2\pi} -12\sin(\theta)\cos(\theta)+8\sin(\theta)\cos^2(\theta)\,d\theta\\ &= \int_0^{2\pi} -6\sin(2\theta)+8\sin(\theta)\cos^2(\theta)\,d\theta\\ &= \left(3\cos(2\theta)-\frac{8}{3}\cos^3(\theta)\right)\Big|_0^{2\pi} = \boxed{0}\end{aligned}$$

4. Evaluate $\displaystyle\iint\limits_S \vec F\cdot d\vec S$ where $\vec F = \vec i + z\,\vec j + 6x\,\vec k$ and S is the portion of the sphere of radius 3 with $x \le 0$, $y \ge 0$ and $z \ge 0$ oriented inward (*i.e.* towards the origin).

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



Note that the surface in this problem is only the part of the sphere itself. The “edges” (the greenish portions on the right/left) are not part of this surface despite the fact that they are in the sketch. They were included in the sketch to try and make the surface a little clearer in the sketch. We would only include the “edges” if the problem had specified that in some manner to make it clear.

Step 2

Let’s get the integral set up now. In this case the we are integrating over a sphere and so we’ll need to set up a parameterization for the surface.

We saw how to parameterize a sphere in the first section of this chapter so we won’t go into detail for the parameterization. The parameterization is,

$$\vec{r}(\theta,\varphi) = \langle 3\sin(\varphi)\cos(\theta), 3\sin(\varphi)\sin(\theta), 3\cos(\varphi)\rangle \qquad \frac{1}{2}\pi \le \theta \le \pi, \quad 0 \le \varphi \le \frac{1}{2}\pi$$

We needed the restriction on φ to make sure that we only get a portion of the upper half of the sphere (*i.e.* $z \ge 0$). Likewise the restriction on θ was needed to get only the portion that was in the 2^{nd} quadrant of the xy-plane (*i.e.* $x \le 0$ and $y \ge 0$).

Next, we’ll need to compute the cross product.

$$\begin{aligned}\vec{r}_\theta &= \langle -3\sin(\varphi)\sin(\theta), 3\sin(\varphi)\cos(\theta), 0\rangle\\ \vec{r}_\varphi &= \langle 3\cos(\varphi)\cos(\theta), 3\cos(\varphi)\sin(\theta), -3\sin(\varphi)\rangle\end{aligned}$$

$$\begin{aligned}\vec{r}_\theta \times \vec{r}_\varphi &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin(\varphi)\sin(\theta) & 3\sin(\varphi)\cos(\theta) & 0 \\ 3\cos(\varphi)\cos(\theta) & 3\cos(\varphi)\sin(\theta) & -3\sin(\varphi)\end{vmatrix}\\ &= -9\sin^2(\varphi)\cos(\theta)\,\vec{i} - 9\sin(\varphi)\cos(\varphi)\sin^2(\theta)\vec{k} - 9\sin(\varphi)\cos(\varphi)\cos^2(\theta)\vec{k}\\ &\qquad - 9\sin^2(\varphi)\sin(\theta)\vec{j}\\ &= -9\sin^2(\varphi)\cos(\theta)\,\vec{i} - 9\sin^2(\varphi)\sin(\theta)\vec{j} - 9\sin(\varphi)\cos(\varphi)\left(\sin^2(\theta)+\cos^2(\theta)\right)\vec{k}\\ &= -9\sin^2(\varphi)\cos(\theta)\,\vec{i} - 9\sin^2(\varphi)\sin(\theta)\vec{j} - 9\sin(\varphi)\cos(\varphi)\vec{k}\end{aligned}$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\vec{r}_\theta \times \vec{r}_\varphi}{\|\vec{r}_\theta \times \vec{r}_\varphi\|} = \frac{\langle -9\sin^2(\varphi)\cos(\theta), -9\sin^2(\varphi)\sin(\theta), -9\sin(\varphi)\cos(\varphi)\rangle}{\|\vec{r}_\theta \times \vec{r}_\varphi\|}$$

We didn't compute the magnitude of the cross product since we know that it will just cancel out when we start working with the integral.

Now we need to determine if this vector has the correct orientation. We know that the normal vector needs to point in towards the origin. Let's think about what that would mean for a normal vector on the upper half of a sphere and it won't matter which quadrant in the xy-plane we are in.

If we are on the upper half of a sphere and the normal vectors must point towards the origin then we know that they will all need to point downwards. They could point in the positive or negative x (or y) direction depending on which quadrant from the xy-plane we are on but they will have to all point downwards. Or in other words, the z component must be negative.

So, the z component of the normal vector above is $-9\sin(\varphi)\cos(\varphi)$ and we know that we are restricted to $0 \le \varphi \le \frac{1}{2}\pi$ for the portion of the sphere we are working on in this problem. In this range of φ we know that both sine and cosine are positive and so the z component must always be negative. This means that the normal vector above has the correct orientation for this problem.

Note that if we were on the lower half of a sphere (not relevant for this problem but useful to think about anyway) and the normal vector would be pointing towards the origin and so they would have to all be pointing upwards.

Also note that if the normal vectors were all pointing out away from the origin then we'd just need to multiply the normal vector above by minus one to get the normal vector we'd need.

Step 3

Next, we'll need to compute the following dot product.

$$\begin{aligned}\vec{F}\left(\vec{r}\left(\theta,\varphi\right)\right)\cdot\vec{n} &= \left\langle 1, 3\cos(\varphi), 18\sin(\varphi)\cos(\theta)\right\rangle \cdot \\ &\qquad \frac{\left\langle -9\sin^2(\varphi)\cos(\theta), -9\sin^2(\varphi)\sin(\theta), -9\sin(\varphi)\cos(\varphi)\right\rangle}{\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\|} \\ &= \frac{1}{\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\|}\left(-9\sin^2(\varphi)\cos(\theta) - 27\sin^2(\varphi)\cos(\varphi)\sin(\theta)\right. \\ &\qquad \left. -162\sin^2(\varphi)\cos(\varphi)\cos(\theta)\right) \\ &= \frac{1}{\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\|}\left(-\frac{9}{2}\left(1-\cos\left(2\varphi\right)\right)\cos(\theta)\right. \\ &\qquad \left. -\sin^2(\varphi)\cos(\varphi)\left(27\sin(\theta) + 162\cos(\theta)\right)\right)\end{aligned}$$

Note that we did a little simplification for the integration process in the last step above.

The integral is then,

$$\begin{aligned}\iint\limits_S \vec{F}\cdot d\vec{S} &= \iint\limits_S \frac{1}{\|\vec{r}_\theta\times\vec{r}_\varphi\|}\left(-\frac{9}{2}\left(1-\cos\left(2\varphi\right)\right)\cos(\theta)\right.\\ &\qquad\qquad \left.-\sin^2(\varphi)\cos(\varphi)\left(27\sin(\theta)+162\cos(\theta)\right)\right)dS\\ &= \iint\limits_D \frac{1}{\|\vec{r}_\theta\times\vec{r}_\varphi\|}\left(-\frac{9}{2}\left(1-\cos\left(2\varphi\right)\right)\cos(\theta)\right.\\ &\qquad\qquad \left.-\sin^2(\varphi)\cos(\varphi)\left(27\sin(\theta)+162\cos(\theta)\right)\right)\|\vec{r}_\theta\times\vec{r}_\varphi\|\,dA\\ &= \iint\limits_D -\frac{9}{2}\left(1-\cos\left(2\varphi\right)\right)\cos(\theta)-\sin^2(\varphi)\cos(\varphi)\left(27\sin(\theta)+162\cos(\theta)\right)dA\end{aligned}$$

As noted above we didn't need to compute the magnitude of the cross product since it would just cancel out when we converted the surface integral into a "normal" double integral.

Also, recall that D is given by the limits on θ and φ we found at the start of Step 2.

Step 4

Now all that we need to do is evaluate the double integral and that shouldn't be too difficult at this point.

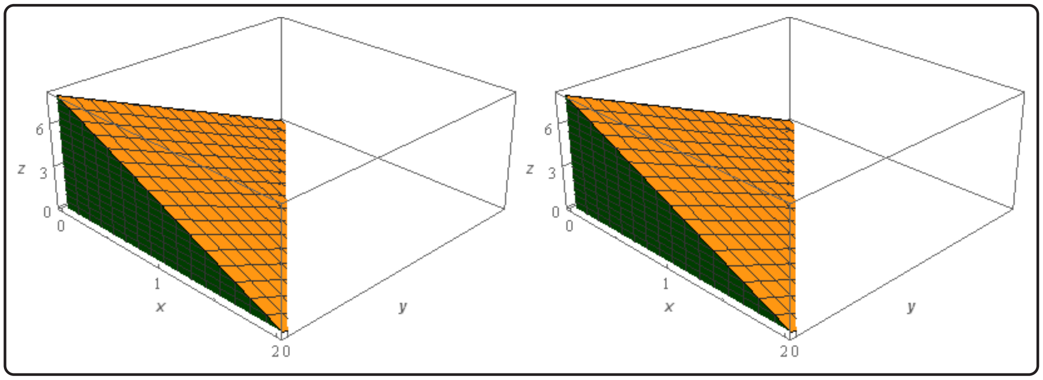
The integral is then,

$$\begin{aligned}\iint\limits_S \vec{F}\cdot d\vec{S} &= \iint\limits_D -\frac{9}{2}\left(1-\cos\left(2\varphi\right)\right)\cos(\theta)-\sin^2(\varphi)\cos(\varphi)\left(27\sin(\theta)+162\cos(\theta)\right)dA\\ &= \int_{\frac{1}{2}\pi}^{\pi}\int_0^{\frac{1}{2}\pi} -\frac{9}{2}\left(1-\cos\left(2\varphi\right)\right)\cos(\theta)\\ &\qquad\qquad -\sin^2(\varphi)\cos(\varphi)\left(27\sin(\theta)+162\cos(\theta)\right)\,d\varphi\,d\theta\\ &= \int_{\frac{1}{2}\pi}^{\pi} -\frac{9}{2}\left(\varphi-\frac{1}{2}\sin\left(2\varphi\right)\right)\cos(\theta)-\frac{1}{3}\sin^3(\varphi)\left(27\sin(\theta)+162\cos(\theta)\right)\Big|_0^{\frac{1}{2}\pi}\,d\theta\\ &= \left(-\frac{9}{4}\pi\sin(\theta)+9\cos(\theta)-54\sin(\theta)\right)\Big|_{\frac{1}{2}\pi}^{\pi} = \boxed{\frac{9}{4}\pi+45}\end{aligned}$$

5. Evaluate $\displaystyle\iint\limits_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = y\,\vec{i} + 2x\,\vec{j} + (z-8)\,\vec{k}$ and S is the surface of the solid bounded by $4x + 2y + z = 8$, $z = 0$, $y = 0$ and $x = 0$ with the positive orientation. Note that all four surfaces of this solid are included in S.

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



Okay, as noted in the problem statement all four surfaces in the sketch (two not shown) are part of S so let's define each of them as follows.

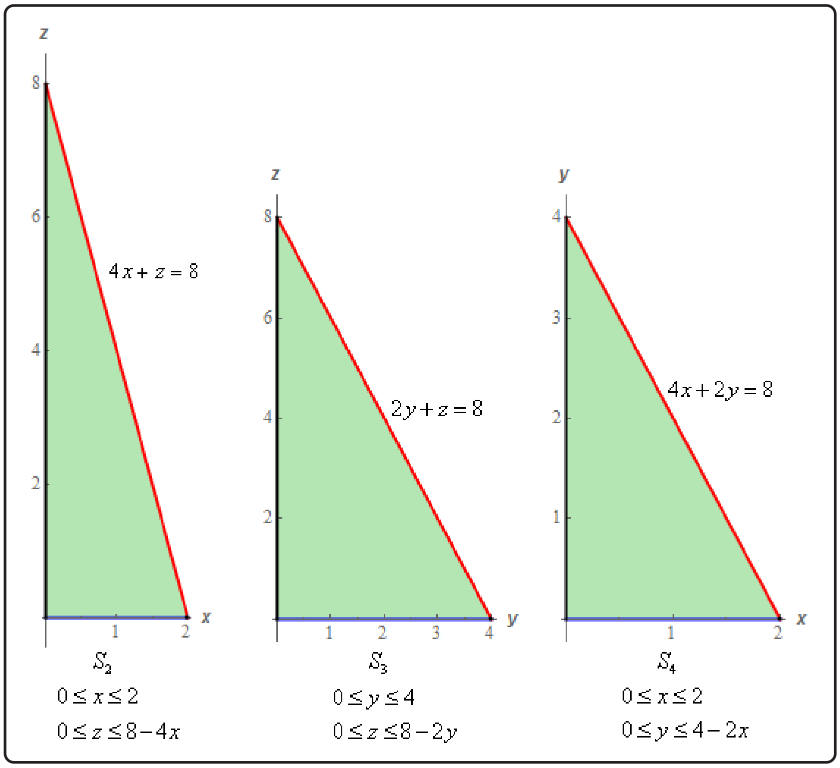
S_1 : Plane given by $4x + 2y + z = 8\,(i.e$ the top of the solid$)$

S_2 : Plane given by $y = 0\,(i.e$ the triangle on right side of the solid$)$

S_3 : Plane given by $x = 0\,(i.e$ the triangle at back of the solid - not shown in sketch$)$

S_4 : Plane given by $z = 0\,(i.e$ the triangle on bottom of the solid - not shown in sketch$)$

As noted in the definitions above the first two surfaces are shown in the sketch but the last two are not actually shown due to the orientation of the solid. Below are sketches of each of the three surfaces that correspond to the coordinates planes.



With each of the sketches we gave limits on the variables for each of them since we'll eventually need that when we start doing the surface integral along each surface.

Now we need to go through and do the integral for each of these surfaces and we're going to go through these a little quicker than we did for the first few problems in this section.

Step 2

Let's start with S_1. In this case we can write the equation of the plane as follows,

$$f(x, y, z) = 4x + 2y + z - 8 = 0$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle 4, 2, 1\rangle}{\|\nabla f\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

The surface has the positive orientation and so must point outwards from the region enclosed by the surface. This means that normal vectors on this plane will need to be pointing generally upwards (*i.e.* have a positive z component) which this normal vector does.

Now we'll need the following dot product and don't forget to plug in the equation of the plane (solved for z of course) into z in the vector field.

$$\begin{aligned}\vec{F}\left(x,y,8-4x-2y\right)\cdot\vec{n}&=\langle y,2x,-4x-2y\rangle\cdot\frac{\langle 4,2,1\rangle}{\|\nabla f\|}\\&=\frac{1}{\|\nabla f\|}\left(2y\right)\end{aligned}$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_1}\vec{F}\cdot d\vec{S}&=\iint\limits_{S_1}\frac{1}{\|\nabla f\|}\left(2y\right)\,dS\\&=\iint\limits_{D}\frac{1}{\|\nabla f\|}\left(2y\right)\|\nabla f\|\,dA=\iint\limits_{D}2y\,dA\end{aligned}$$

In this case D is just S_4 and so we can now finish out the integral.

$$\begin{aligned}\iint\limits_{S_1}\vec{F}\cdot d\vec{S}&=\iint\limits_{D}2y\,dA\\&=\int_0^2\int_0^{4-2x}2y\,dy\,dx\\&=\int_0^2\left.y^2\right|_0^{4-2x}dx\\&=\int_0^2\left(4-2x\right)^2dx\\&=\left.-\frac{1}{6}\left(4-2x\right)^3\right|_0^2=\underline{\frac{32}{3}}\end{aligned}$$

Step 3

Next, we'll take care of S_2. In this case the equation for the surface is simply $y=0$ and D is given in the sketch of S_2 in Step 1.

In this case S_2 is simply a portion of the xz-plane and we have the positive orientation and so the normal vector must point away from the region enclosed by the surface. That means that, in this case, the normal vector is simply $\vec{n}=-\vec{j}=\langle 0,-1,0\rangle$.

The dot product for this surface is,

$$\vec{F}(x,0,z)\cdot\vec{n}=\langle 0,2x,z-8\rangle\cdot\frac{\langle 0,-1,0\rangle}{1}=-2x$$

Don't forget to plug $y=0$ into the vector field and note that the magnitude of the gradient is,

$$\|\nabla f\|=\sqrt{(0)^2+(1)^2+(0)^2}=1$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_2}\vec{F}\cdot d\vec{S}&=\iint\limits_{S_2}-2x\,dS\\&=\iint\limits_{D}-2x\,(1)\;dA\\&=\int_0^2\int_0^{8-4x}-2x\,dz\,dx\\&=\int_0^2-2xz\Big|_0^{8-4x}dx\\&=\int_0^2 8x^2-16x\,dx\\&=\left(\frac{8}{3}x^3-8x^2\right)\Big|_0^2=\underline{-\frac{32}{3}}\end{aligned}$$

Step 4

Now we can deal with S_3. In this case the equation for the surface is simply $x=0$ and D is given in the sketch of S_3 in Step 1.

In this case S_3 is simply a portion of the yz-plane and we have the positive orientation and so the normal vector must point away from the region enclosed by the surface. That means that, in this case, the normal vector is simply $\vec{n}=-\vec{i}=\langle -1,0,0\rangle$.

The dot product for this surface is,

$$\vec{F}(0,y,z)\cdot\vec{n}=\langle y,0,z-8\rangle\cdot\frac{\langle -1,0,0\rangle}{1}=-y$$

Don't forget to plug $x=0$ into the vector field and note that the magnitude of the gradient is,

$$\|\nabla f\|=\sqrt{(1)^2+(0)^2+(0)^2}=1$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_3} \vec{F} \cdot d\vec{S} &= \iint\limits_{S_3} -y\,dS \\ &= \iint\limits_{D} -y\,(1)\;dA \\ &= \int_0^4 \int_0^{8-2y} -y\,dz\,dy \\ &= \int_0^4 -yz\Big|_0^{8-2y}\,dy \\ &= \int_0^4 2y^2 - 8y\,dy \\ &= \left(\frac{2}{3}y^3 - 4y^2\right)\Big|_0^4 = \underline{-\frac{64}{3}}\end{aligned}$$

Step 5

Finally let's take care of S_4. In this case the equation for the surface is simply $z = 0$ and D is given in the sketch of S_4 in Step 1.

In this case S_4 is simply a portion of the xy-plane and we have the positive orientation and so the normal vector must point away from the region enclosed by the surface. That means that, in this case, the normal vector is simply $\vec{n} = -\vec{k} = \langle 0, 0, -1\rangle$.

The dot product for this surface is,

$$\vec{F}\,(x, y, 0) \cdot \vec{n} = \langle y, 2x, -8\rangle \cdot \frac{\langle 0, 0, -1\rangle}{1} = 8$$

Don't forget to plug $z = 0$ into the vector field and note that the magnitude of the gradient is,

$$\|\nabla f\| = \sqrt{(0)^2 + (0)^2 + (1)^2} = 1$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_4} \vec{F}\cdot d\vec{S} &= \iint\limits_{S_4} 8\,dS \\ &= \iint\limits_{D} 8\,dA \\ &= 8\iint\limits_{D} dA \\ &= 8\,(\text{Area of } D) \\ &= 8\left(\frac{1}{2}\right)(2)(4) = \underline{32}\end{aligned}$$

In this case notice that we didn't have to actually compute the double integral since D was just a right triangle and we can easily compute its area.

Step 6

Now, to get the value of the integral over the full surface all we need to do is sum up the values of each of the integrals over the four surfaces above. Doing this gives,

$$\iint\limits_{S} \vec{F}\cdot d\vec{S} = \left(\frac{32}{3}\right) + \left(-\frac{32}{3}\right) + \left(-\frac{64}{3}\right) + (32) = \boxed{\frac{32}{3}}$$

We put parenthesis around each of the individual integral values just to indicate where each came from. In general, these aren't needed of course.

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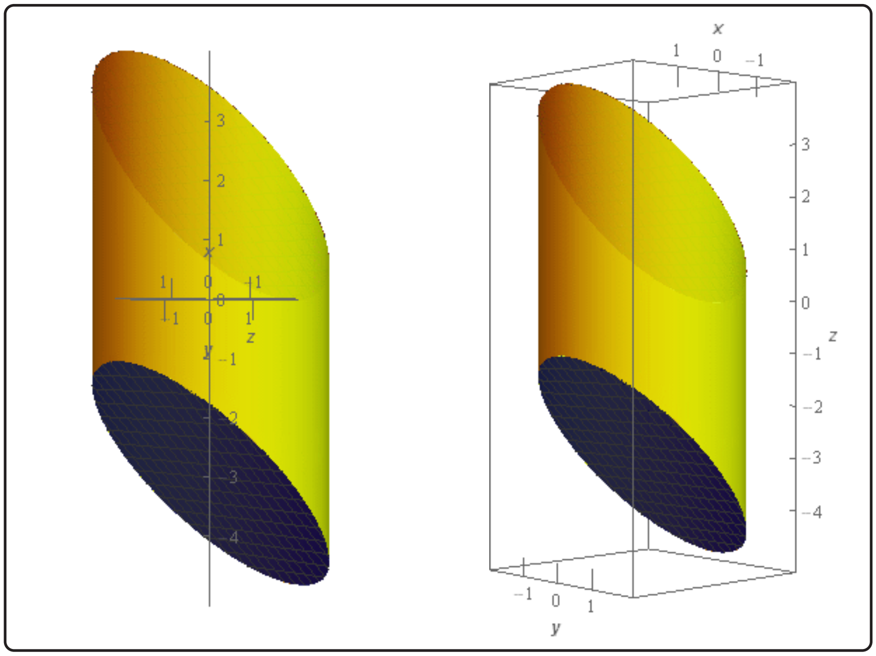
6. Evaluate $\displaystyle\iint\limits_{S} \vec{F}\cdot d\vec{S}$ where $\vec{F} = yz\,\vec{i} + x\,\vec{j} + 3y^2\,\vec{k}$ and S is the surface of the solid bounded by $x^2+y^2=4$, $z=x-3$, and $z=x+2$ with the negative orientation. Note that all three surfaces of this solid are included in S.

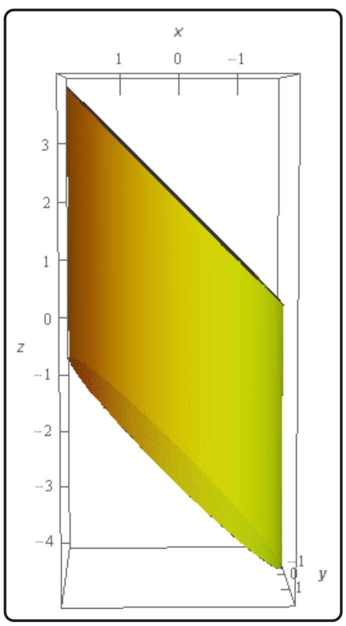
Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



As noted in the problem statement there are three surfaces here. The "top" of the cylinder is a little hard to see. We made the walls of the cylinder slightly transparent and the top of the cylinder can be seen as a darker ellipse along the top of the surface.

To help visualize the relationship between the top and bottom of the cylinder here is a different view of the surface.



From this view we can see that the top and bottom planes that "cap" the cylinder are parallel.

Let's define the three surfaces in the sketch as follows.

S_1 : Cylinder given by $x^2 + y^2 = 4$ $(i.e$ the walls of the solid)

S_2 : Plane given by $z = x + 2$ $(i.e$ the top cap of the cylinder)

S_3 : Plane given by $z = x - 3$ $(i.e$ the bottom cap of the cylinder)

Now we need to go through and do the integral for each of these surfaces and we're going to go through these a little quicker than we did for the first few problems in this section.

Step 2

Let's start with S_1. The surface in this case is a cylinder and so we'll need to parameterize it. The parameterization of the surface is,

$$\vec{r}(z,\theta) = \langle 2\cos(\theta), 2\sin(\theta), z\rangle$$

The limits on z and θ are,

$$0 \le \theta \le 2\pi, \quad 2\cos(\theta) - 3 = x - 3 \le z \le x + 2 = 2\cos(\theta) + 2$$

With the z limits we'll need to make sure that we convert the x's into their parameterized form.

In order to evaluate the integral in this case we'll need the cross product $\vec{r}_z \times \vec{r}_\theta$ so here is that work.

$$\vec{r}_z = \langle 0, 0, 1\rangle \qquad \vec{r}_\theta = \langle -2\sin(\theta), 2\cos(\theta), 0\rangle$$

$$\vec{r}_z \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -2\sin(\theta) & 2\cos(\theta) & 0 \end{vmatrix} = -2\cos(\theta)\vec{i} - 2\sin(\theta)\vec{j}$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\vec{r}_z \times \vec{r}_\theta}{\|\vec{r}_z \times \vec{r}_\theta\|} = \frac{\langle -2\cos(\theta), -2\sin(\theta), 0\rangle}{\|\vec{r}_z \times \vec{r}_\theta\|}$$

We didn't compute the magnitude of the cross product since we know that it will just cancel out when we start working with the integral.

The surface has the negative orientation and so must point in towards the region enclosed by the surface. This means that normal vectors on cylinder will need to point in towards the z-axis and this vector does point in that direction.

To see that this vector points in towards the z-axis consider the $0 \le \theta \le \frac{\pi}{2}$. In this range both sine and cosine are positive and so the x and y component of the normal vector will be negative and so will point in towards the z-axis.

Now we'll need the following dot product and don't forget to plug in the parameterization of the surface in the vector field.

$$\begin{aligned}\vec{F}\left(\vec{r}\left(z,\theta\right)\right)\cdot\vec{n} &= \left\langle 2z\sin(\theta), 2\cos(\theta), 12\sin^2(\theta)\right\rangle \cdot \frac{\langle -2\cos(\theta), -2\sin(\theta), 0\rangle}{\|\vec{r}_z \times \vec{r}_\theta\|} \\ &= \frac{1}{\|\vec{r}_z \times \vec{r}_\theta\|}\left(-4z\sin(\theta)\cos(\theta) - 4\sin(\theta)\cos(\theta)\right) \\ &= \frac{1}{\|\vec{r}_z \times \vec{r}_\theta\|}\left[-4\sin(\theta)\cos(\theta)\left(z+1\right)\right]\end{aligned}$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_1} \vec{F}\cdot d\vec{S} &= \iint\limits_{S_1} \frac{1}{\|\vec{r}_z \times \vec{r}_\theta\|}\left[-2\sin\left(2\theta\right)\left(z+1\right)\right]\,dS \\ &= \iint\limits_{D} \frac{1}{\|\vec{r}_z \times \vec{r}_\theta\|}\left[-2\sin\left(2\theta\right)\left(z+1\right)\right]\|\vec{r}_z \times \vec{r}_\theta\|\,dA \\ &= \iint\limits_{D} -2\sin\left(2\theta\right)\left(z+1\right)\,dA\end{aligned}$$

In this case D is is nothing more than the limits on z and θ we gave above and so we can now finish out the integral.

$$\begin{aligned}\iint\limits_{S_1} \vec{F}\cdot d\vec{S} &= \iint\limits_{D} -4\sin(\theta)\cos(\theta)\left(z+1\right)\,dA \\ &= \int_0^{2\pi}\int_{2\cos(\theta)-3}^{2\cos(\theta)+2} -4\sin(\theta)\cos(\theta)\left(z+1\right)\,dz\,d\theta \\ &= \int_0^{2\pi} -4\sin(\theta)\cos(\theta)\left(\frac{1}{2}z^2 + z\right)\Bigg|_{2\cos(\theta)-3}^{2\cos(\theta)+2} d\theta \\ &= \int_0^{2\pi} -10\sin(\theta)\cos(\theta) - 40\sin(\theta)\cos^2(\theta)\,d\theta \\ &= \int_0^{2\pi} -5\sin\left(2\theta\right) - 40\sin(\theta)\cos^2(\theta)\,d\theta \\ &= \left(\frac{5}{2}\cos\left(2\theta\right) + \frac{40}{3}\cos^3(\theta)\right)^3\Bigg|_0^{2\pi} = \underline{0}\end{aligned}$$

Step 3

Next we'll take care of S_2. In this case the equation for the surface is can be written as $z - x - 2 = 0$ and D is the disk $x^2 + y^2 \le 4$.

A unit normal vector for S_2 is then,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle -1, 0, 1\rangle}{\|\nabla f\|}$$

The region has the negative orientation and so must point into the enclosed region and so must point downwards (since this is the top "cap" of the cylinder). The normal vector above points upwards (it has a positive z component) and so we'll need to multiply this by minus one to get the normal vector we need for this surface.

The correct normal vector is then,

$$\vec{n} = -\frac{\nabla f}{\|\nabla f\|} = \frac{\langle 1, 0, -1\rangle}{\|\nabla f\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

The dot product we'll need for this surface is,

$$\begin{aligned}\vec{F}(x, y, x+2) \cdot \vec{n} &= \left\langle y(x+2), x, 3y^2\right\rangle \cdot \frac{\langle 1, 0, -1\rangle}{\|\nabla f\|} \\ &= \frac{1}{\|\nabla f\|}\left(xy + 2y - 3y^2\right)\end{aligned}$$

Don't forget to plug the equation of the surface into z in the vector field.

The integral is then,

$$\begin{aligned}\iint\limits_{S_2} \vec{F} \cdot d\vec{S} &= \iint\limits_{S_2} \frac{1}{\|\nabla f\|}\left(xy + 2y - 3y^2\right)\, dS \\ &= \iint\limits_{D} \frac{1}{\|\nabla f\|}\left(xy + 2y - 3y^2\right)\|\nabla f\|\, dA \\ &= \iint\limits_{D} xy + 2y - 3y^2\, dA\end{aligned}$$

Note that we'll need to finish this integral with polar coordinates and the polar limits will be,

$$\begin{gathered}0 \le \theta \le 2\pi \\ 0 \le r \le 2\end{gathered}$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_2} \vec{F}\cdot d\vec{S} &= \iint\limits_{D} xy + 2y - 3y^2\,dA\\ &= \int_0^{2\pi}\int_0^2 \left(r^2\sin(\theta)\cos(\theta) + 2r\sin(\theta) - 3r^2\sin^2(\theta)\right)(r)\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^2 \frac{1}{2}r^3\sin(2\theta) + 2r^2\sin(\theta) - \frac{3}{2}r^3\left(1-\cos(2\theta)\right)dr\,d\theta\\ &= \int_0^{2\pi} \frac{1}{8}r^4\sin(2\theta) + \frac{2}{3}r^3\sin(\theta) - \frac{3}{8}r^4\left(1-\cos(2\theta)\right)\Big|_0^2\,d\theta\\ &= \int_0^{2\pi} 2\sin(2\theta) + \frac{16}{3}\sin(\theta) - 6\left(1-\cos(2\theta)\right)d\theta\\ &= \left(-\cos(2\theta) - \frac{16}{3}\cos(\theta) - 6\left(\theta - \frac{1}{2}\sin(2\theta)\right)\right)\Bigg|_0^{2\pi} = \underline{-12\pi}\end{aligned}$$

Step 4

Finally, let's integrate over S_3. In this case the equation for the surface is can be written as $z - x + 3 = 0$ and D is the disk $x^2 + y^2 \le 4$.

A unit normal vector for S_2 is then,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle -1, 0, 1\rangle}{\|\nabla f\|}$$

The region has the negative orientation and so must point into the enclosed region and so must point upwards (since this is the bottom "cap" of the cylinder). The normal vector above does point upwards (it has a positive z component) and so is the normal vector we'll need.

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

The dot product we'll need for this surface is,

$$\begin{aligned}\vec{F}(x,y,x-3)\cdot\vec{n} &= \left\langle y(x-3), x, 3y^2\right\rangle \cdot \frac{\langle -1,0,1\rangle}{\|\nabla f\|}\\ &= \frac{1}{\|\nabla f\|}\left(-xy + 3y + 3y^2\right)\end{aligned}$$

Don't forget to plug the equation of the surface into z in the vector field.

The integral is then,

$$\begin{aligned}\iint\limits_{S_3} \vec{F} \cdot d\vec{S} &= \iint\limits_{S_3} \frac{1}{\|\nabla f\|}\left(-xy + 3y + 3y^2\right)\, dS \\ &= \iint\limits_{D} \frac{1}{\|\nabla f\|}\left(-xy + 3y + 3y^2\right)\|\nabla f\|\, dA \\ &= \iint\limits_{D} -xy + 3y + 3y^2\, dA\end{aligned}$$

Note that we'll need to finish this integral with polar coordinates and the polar limits will be,

$$\begin{aligned}0 \le \theta \le 2\pi \\ 0 \le r \le 2\end{aligned}$$

The integral is then,

$$\begin{aligned}\iint\limits_{S_3} \vec{F} \cdot d\vec{S} &= \iint\limits_{D} -xy + 3y + 3y^2\, dA \\ &= \int_0^{2\pi}\int_0^2 \left(-r^2 \sin(\theta)\cos(\theta) + 3r\sin(\theta) + 3r^2\sin^2(\theta)\right)(r)\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^2 -\frac{1}{2}r^3\sin(2\theta) + 3r^2\sin(\theta) + \frac{3}{2}r^3\left(1 - \cos(2\theta)\right)\, dr\, d\theta \\ &= \int_0^{2\pi} -\frac{1}{8}r^4\sin(2\theta) + r^3\sin(\theta) + \frac{3}{8}r^4\left(1 - \cos(2\theta)\right)\Big|_0^2\, d\theta \\ &= \int_0^{2\pi} -2\sin(2\theta) + 16\sin(\theta) + 6\left(1 - \cos(2\theta)\right)\, d\theta \\ &= \left(\cos(2\theta) - 16\cos(\theta) + 6\left(\theta - \frac{1}{2}\sin(2\theta)\right)\right)\Big|_0^{2\pi} = \underline{12\pi}\end{aligned}$$

Step 5

Now, to get the value of the integral over the full surface all we need to do is sum up the values of each of the integrals over the three surfaces above. Doing this gives,

$$\iint\limits_{S} \vec{F} \cdot d\vec{S} = (0) + (-12\pi) + (12\pi) = \boxed{0}$$

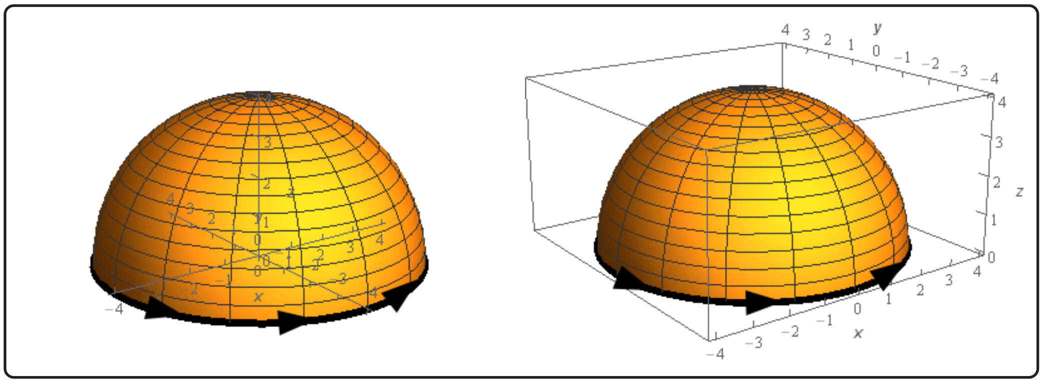
We put parenthesis around each of the individual integral values just to indicate where each came from. In general, these aren't needed of course.

17.5 Stokes' Theorem

1. Use Stokes' Theorem to evaluate $\iint\limits_S \text{curl}\vec{F}\cdot d\vec{S}$ where $\vec{F} = y\,\vec{i} - x\,\vec{j} + yx^3\,\vec{k}$ and S is the portion of the sphere of radius 4 with $z \geq 0$ and the upwards orientation.

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

Because the orientation of the surface is upwards then all the normal vectors will be pointing outwards. So, if we walk along the edge of the surface, *i.e.* the curve C, in the direction indicated with our head pointed away from the surface (*i.e.* in the same direction as the normal vectors) then our left hand will be over the region. Therefore the direction indicated in the sketch is the positive orientation of C.

If you have trouble visualizing the direction of the curve simply get a cup or bowl and put it upside down on a piece of paper on a table. Sketch a set of axis on the piece of paper that will represent the plane the cup/bowl is sitting on to really help with the visualization. Then cut out a little stick figure and put a face on the "front" side of it and color the left hand a bright color so you can quickly see it. Now, on the edge of the cup/bowl/whatever you place the stick figure with it's head pointing in the direction of the normal vectors (out away from the sphere/cup/bowl in our case) with its left hand over the surface. The direction that the "face" on the stick figure is facing is the direction you'd need to walk along the surface to get the positive orientation for C.

Step 2

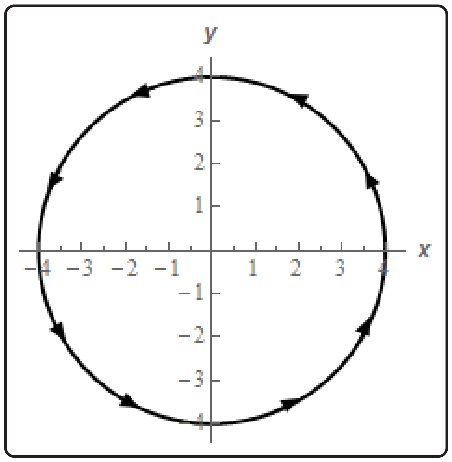
We are going to use Stokes' Theorem in the following direction.

$$\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

We've been given the vector field in the problem statement so we don't need to worry about that. We will need to deal with C.

Because C is just the curve along the bottom of the upper half of the sphere we can see that C in fact will be the intersection of the sphere and the xy-plane (*i.e.* $z = 0$). Therefore, C is just the circle of radius 4.

If we look at the sphere from above we get the following sketch of C.



The parameterization of C is given by,

$$\vec{r}(t) = \langle 4\cos(t), 4\sin(t), 0\rangle \qquad 0 \le t \le 2\pi$$

The z component of the parameterization is zero because C lies in the xy-plane.

Step 3

Since we know we'll need to eventually do the line integral we know we'll need the following dot product.

$$\begin{aligned}\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) &= \left\langle 4\sin(t), -4\cos(t), 256\sin(t)\cos^3(t)\right\rangle \cdot \langle -4\sin(t), 4\cos(t), 0\rangle \\ &= -16\sin^2(t) - 16\cos^2(t) \\ &= -16\end{aligned}$$

Don't forget to plug the parameterization of C into the vector field!

Step 4

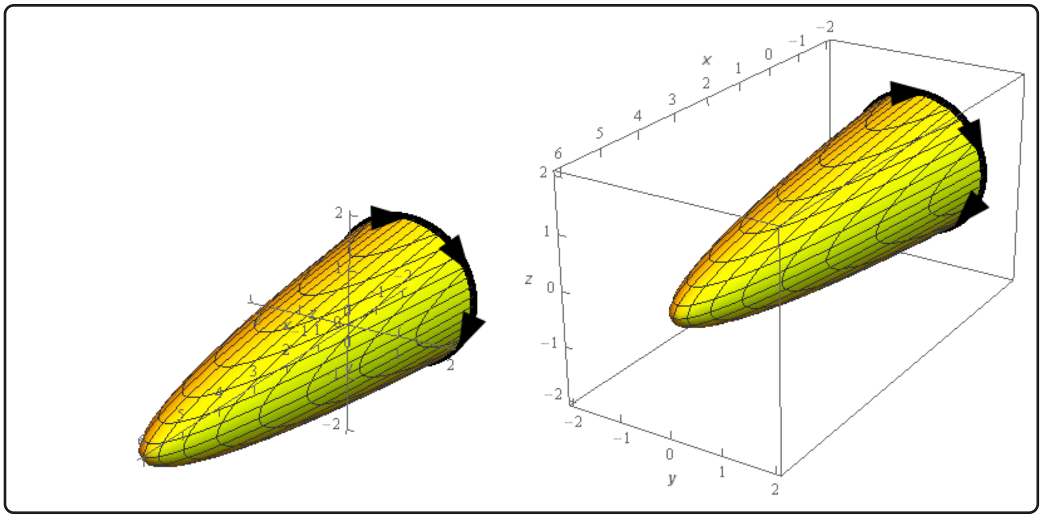
Okay, let's go ahead and evaluate the integral using Stokes' Theorem.

$$\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$
$$= \int_0^{2\pi} -16\, dt = \boxed{-32\pi}$$

2. Use Stokes' Theorem to evaluate $\displaystyle\iint_S \text{curl}\vec{F} \cdot d\vec{S}$ where $\vec{F} = (z^2 - 1)\,\vec{i} + (z + xy^3)\,\vec{j} + 6\,\vec{k}$ and S is the portion of $x = 6 - 4y^2 - 4z^2$ in front of $x = -2$ with orientation in the negative x-axis direction.

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

Because the orientation of the surface is towards the negative x-axis all the normal vectors will be pointing into the region enclosed by the surface. So, if we walk along the edge of the surface, *i.e.* the curve C, in the direction indicated with our head pointed into the region enclosed by the surface (*i.e.* in the same direction as the normal vectors) then our left hand will be over the region. Therefore, the direction indicated in the sketch is the positive orientation of C.

If you have trouble visualizing the direction of the curve simply get a cup or bowl and put it on its side with a piece of paper behind it. Sketch a set of axes on the piece of paper that will represent the plane the cup/bowl is sitting in front of to really help with the visualization. Then cut out a little stick figure and put a face on the "front" side of it and color the left hand a bright color so you can quickly see it. Now, on the edge of the cup/bowl/whatever you place the stick figure with its head pointing in the direction of the normal vectors (into the paraboloid/cup/bowl in our case) with its left hand over the surface. The direction that the "face" on the stick figure is facing is the direction you'd need to walk along the surface to get the positive orientation for C.

Step 2

We are going to use Stokes' Theorem in the following direction.

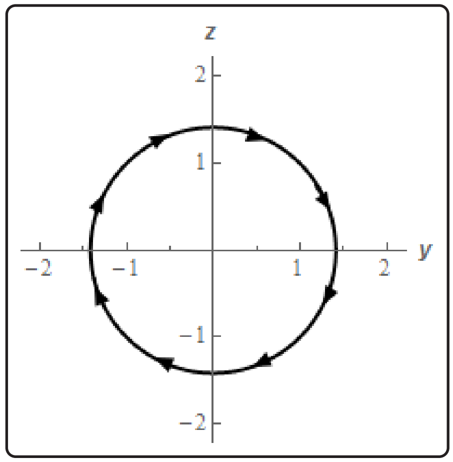
$$\iint\limits_S \text{curl}\vec{F} \cdot d\vec{S} = \int\limits_C \vec{F} \cdot d\vec{r}$$

We've been given the vector field in the problem statement so we don't need to worry about that. We will need to deal with C.

In this case C is the curve we get by setting the two equations in the problem statement equal. Doing this gives,

$$-2 = 6 - 4y^2 - 4z^2 \qquad \to \qquad 4y^2 + 4z^2 = 8 \qquad \Rightarrow \qquad y^2 + z^2 = 2$$

We will see following sketch of C if we are in front of the paraboloid and look directly along the x-axis.



One possible parameterization of C is given by,

$$\vec r\left(t\right) = \left\langle -2, \sqrt{2}\,\sin(t), \sqrt{2}\,\cos(t)\right\rangle \qquad\qquad 0 \le t \le 2\pi$$

The x component of the parameterization is -2 because C lies at $x = -2$.

Step 3

Since we know we'll need to eventually do the line integral we know we'll need the following dot product.

$$\begin{aligned}\vec F\left(\vec r\left(t\right)\right) \cdot \vec r\,'\left(t\right) &= \left\langle 2\cos^2(t) - 1, \sqrt{2}\cos(t) - 4\sqrt{2}\sin^3(t), 6\right\rangle \cdot \left\langle 0, \sqrt{2}\cos(t), -\sqrt{2}\sin(t)\right\rangle \\ &= \sqrt{2}\cos(t)\left(\sqrt{2}\cos(t) - 4\sqrt{2}\sin^3(t)\right) - 6\sqrt{2}\sin(t) \\ &= 2\cos^2(t) - 8\cos(t)\sin^3(t) - 6\sqrt{2}\sin(t) \\ &= \left(1 + \cos\left(2t\right)\right) - 8\cos(t)\sin^3(t) - 6\sqrt{2}\sin(t)\end{aligned}$$

Don't forget to plug the parameterization of C into the vector field!

We also did a little simplification on the first term with an eye towards the integration.

Step 4

Okay, let's go ahead and evaluate the integral using Stokes' Theorem.

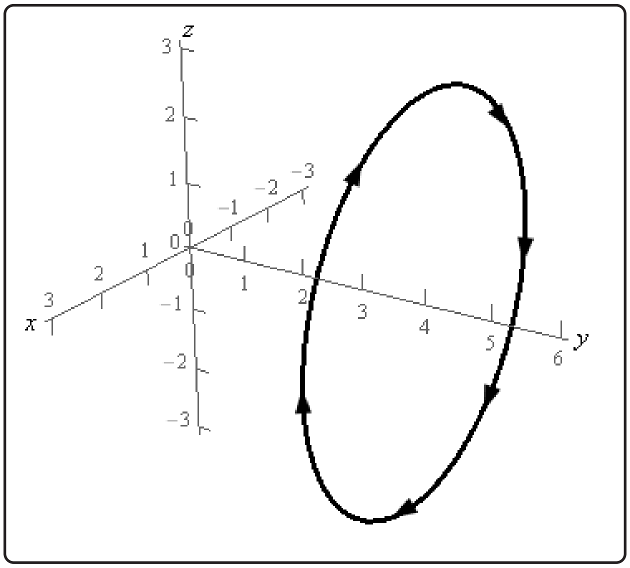
$$\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} (1 + \cos(2t)) - 8\cos(t)\sin^3(t) - 6\sqrt{2}\sin(t)\,dt$$

$$= \left(t + \frac{1}{2}\sin(2t) - 2\sin^4(t) + 6\sqrt{2}\cos(t)\,dt \right)\Big|_0^{2\pi} = \boxed{2\pi}$$

---

3. Use Stokes' Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = -yz\,\vec{i} + (4y+1)\,\vec{j} + xy\,\vec{k}$ and C is is the circle of radius 3 at $y = 4$ and perpendicular to the y-axis. C has a clockwise rotation if you are looking down the y-axis from the positive y-axis to the negative y-axis. See the figure below for a sketch of the curve.



Step 1

Okay, we are going to use Stokes' Theorem in the following direction.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{S}$$

So, let's first compute curl$\vec{F}$ since that is easy enough to compute and might be useful to have when we go to determine the surface S we're going to integrate over.

The curl of the vector field is then,

$$\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -yz & 4y+1 & xy \end{vmatrix} = x\vec{i} - y\vec{j} + z\vec{k} - y\vec{j} = \underline{x\vec{i} - 2y\vec{j} + z\vec{k}}$$

Step 2

Now we need to find a surface S with an orientation that will have a boundary curve that is the curve shown in the problem statement, including the correct orientation. This can seem to be a daunting task at times but it's not as bad as it might appear to be. First we know that the boundary curve needs to be a circle. This means that we're going to be looking for a surface whose cross section is a circle and we know of several surfaces that meet this requirement. We know that spheres, cones and elliptic paraboloids all have circles as cross sections.

The question becomes which of these surfaces would be best for us in this problem. To make this decision remember that we'll eventually need to plug this surface into the vector field and then take the dot product of this with normal vector (which will also come from the surface of course).

In general, it won't be immediately clear from the curl of the vector field by itself which surface we should use and that is the case here. The curl of the vector field has all three components and none of them are that difficult to deal with but there isn't anything that suggests one surface might be easier than the other.

So, let's consider a sphere first. The issue with spheres is that its parameterization and normal vector are lengthy and many lead to messy integrands. So, because the curl of the vector field does have all three components to it which may well lead to long and/or messy integrands we'll not work with a sphere for this problem.

Now let's think about a cone. Equations of cones aren't that bad but they will involve a square root and in this case would need to be in the form $y = \sqrt{ax^2 + bz^2}$ because the boundary curve is centered on the y-axis. The normal vector will also contain roots and this will often lead to messy integrands. So, let's not work with a cone either in this problem.

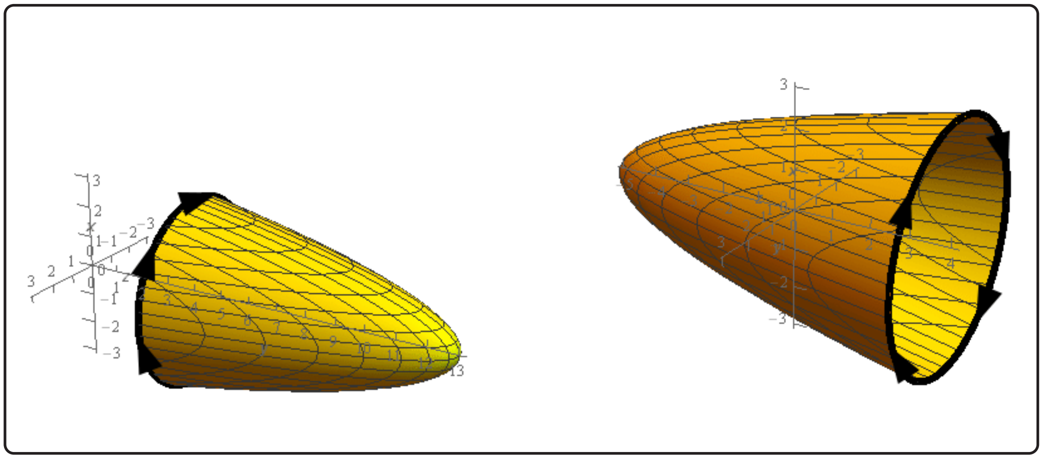
That leaves elliptic paraboloids and we probably should have considered them first. The equations are simple and the normal vectors are even simpler so they seem like a good choice of surface for this problem.

Note that we're not saying that spheres and cones are never good choices for the surface. For some vector fields the curl may end up being very simple with one of these surfaces and so they would be perfectly good choices.

Step 3

We have two possibilities for elliptic paraboloids that we could use here. Both will be centered on the y-axis but one will open in the negative y direction while the other will open in the positive y direction.

Here is a couple of sketches of possible elliptic paraboloids we could use here.



Let's get an equation for each of these. Note that for each of these if we set the equation of the paraboloid and the plane $y = 4$ equal we need to get the circle $x^2 + z^2 = 9$ since this is the boundary curve that should occur at $y = 4$.

Let's get the equation of the first paraboloid (the one that opens in the negative y direction. We know that the equation of this paraboloid should be $y = a - x^2 - z^2$ for some value of a. As noted if we set this equal to $y = 4$ and do some simplification we know what equation we should get. So, let's set the two equations equal.

$$4 = a - x^2 - z^2 \qquad \to \qquad x^2 + z^2 = a - 4 = 9 \qquad \to \qquad a = 13$$

As shown we know that the $a - 4$ should be 9 and so we must have $a = 13$. Therefore, the equation of the paraboloid that open in the negative y direction is,

$$y = 13 - x^2 - z^2$$

Next, let's get the equation of the paraboloid that opens in the positive y direction. The equation of this paraboloid will be in the form $y = x^2 + z^2 + a$ for some a. Setting this equal to $y = 4$ gives,

$$4 = x^2 + z^2 + a \qquad \to \qquad x^2 + z^2 = 4 - a = 9 \qquad \to \qquad a = -5$$

The equation of the paraboloid that opens in the positive y direction is then,

$$y = x^2 + z^2 - 5$$

Either of these surfaces could be used to do this problem.

Step 4

We now need to determine the orientation of the normal vectors that will induce a positive orientation of the boundary curve, C, that matches the orientation that was given in the problem statement.

We'll find the normal vectors for each surface despite the fact that we really only need to do it for one of them since we only need one of the surfaces to do the problem as noted in the previous step. Determining the orientation of the surface can be a little tricky for some folks so doing an extra one might help see what's going on here.

Remember that what we want to do here is think of ourselves as walking along the boundary curve of the surface in the direction indicated while our left hand is over the surface itself. We now need to determine if we are walking along the outside of the surface or the inside of the surface.

If we are walking along the outside of the surface then our heads, and hence the normal vectors, will be pointing away from the region enclosed by the surface. On the other hand, if we are walking along the inside of the surface then our heads, and hence the normal vectors, will be pointing into the region enclosed by the surface.

To help visualize this for our two surfaces it might help to get a cup or bowl that we can use to represent the surface. The edge of the cup/bowl will then represent the boundary curve. Next cut out a stick figure and put a face on one side so we know which direction we'll be walking and brightly color the left hand to make it really clear which side is the left side.

Now, put the cup/bowl on its side so it looks vaguely like the surface we're working with and put the stick figure on the edge with the face pointing in the direction the curve is moving and the left hand over the cup/bowl. Do we need to put the stick figure on the inside or outside of the cup/bowl to do this?

Okay, let's do this for the first surface, $y = 13 - x^2 - z^2$. In this case our stick figure would need to be standing on the inside of the cup/bowl/surface. Therefore, the normal vectors on the surface would all need to be point in towards the region enclosed by the surface. This also will mean that all the normal vectors will need to have a negative y component. Again, to visualize this take the stick figure and move it into the region and toward the end of cup/bowl/surface and you'll see it start to point more and more in the negative y direction (and hence will have a negative y component). Note that the x and z component can be either positive or negative depending on just where we are on the interior of the surface.

Now, let's take a look at the first surface, $y = x^2 + z^2 - 5$. For this surface our stick figure would need to be standing on the outside of the cup/bowl/surface. So, in this case, the normal vectors would point out away from the region enclosed by the surface. These will also have a negative y component and you can use the method we discussed in the above paragraph to help visualize this.

Step 5

We now need to start thinking about actually computing the integral. We'll write the equation of the surface as,

$$f(x, y, z) = 13 - x^2 - z^2 - y = 0$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle -2x, -1, -2z \rangle}{\|\nabla f\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that this does have the correct orientation because the y component is negative.

Next, we'll need to compute the following dot product.

$$\begin{aligned} \text{curl}\vec{F} \cdot \vec{n} &= \left\langle x, -2\left(13 - x^2 - z^2\right), z \right\rangle \cdot \frac{\langle -2x, -1, -2z \rangle}{\|\nabla f\|} \\ &= \frac{1}{\|\nabla f\|}\left(-2x^2 + 2\left(13 - x^2 - z^2\right) - 2z^2\right) \\ &= \frac{1}{\|\nabla f\|}\left(26 - 4x^2 - 4z^2\right) \end{aligned}$$

Step 6

Now, applying Stokes' Theorem to the integral and converting to a "normal" double integral gives,

$$\begin{aligned}\int\limits_C \vec{F}\cdot d\vec{r} &= \iint\limits_S \text{curl}\vec{F}\cdot d\vec{S}\\ &= \iint\limits_S \frac{1}{\|\nabla f\|}\left(26-4x^2-4z^2\right)dS\\ &= \iint\limits_D \frac{1}{\|\nabla f\|}\left(26-4x^2-4z^2\right)\|\nabla f\|\,dA\\ &= \iint\limits_D 26-4x^2-4z^2 dA\end{aligned}$$

Step 7

To finish this integral out then we'll need to convert to polar coordinates using the following polar coordinates.

$$x = r\cos(\theta) \qquad z = r\sin(\theta) \qquad x^2+z^2=r^2$$

In this case D is just the disk $x^2+z^2 \le 9$ and so the limits for the integral are,

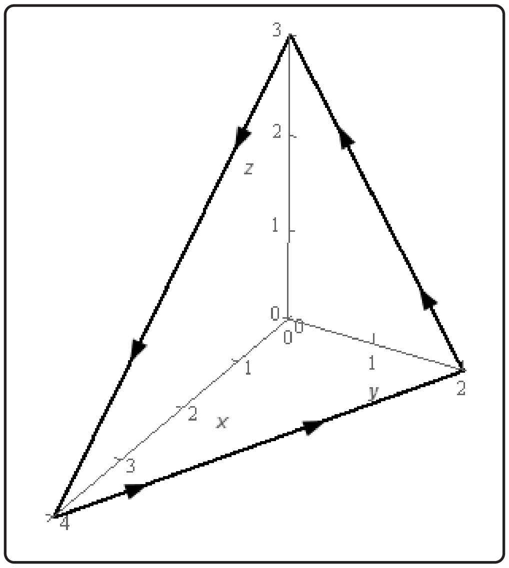
$$\begin{gathered}0 \le \theta \le 2\pi\\ 0 \le r \le 3\end{gathered}$$

The integral is then,

$$\begin{aligned}\int\limits_C \vec{F}\cdot d\vec{r} &= \iint\limits_S \text{curl}\vec{F}\cdot d\vec{S}\\ &= \int_0^{2\pi}\int_0^3 \left(26-4r^2\right) r\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^3 26r-4r^3\,dr\,d\theta\\ &= \int_0^{2\pi}\left.\left(13r^2-r^4\right)\right|_0^3 d\theta\\ &= \int_0^{2\pi} 36\,d\theta\\ &= \boxed{72\pi}\end{aligned}$$

Don't forget to pick up an extra r from converting the dA to polar coordinates.

4. Use Stokes' Theorem to evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left(3yx^2+z^3\right)\vec{i} + y^2\,\vec{j} + 4yx^2\,\vec{k}$ and C is is triangle with vertices $(0,0,3)$, $(0,2,0)$ and $(4,0,0)$. C has a counter clockwise rotation if you are above the triangle and looking down towards the xy-plane. See the figure below for a sketch of the curve.



Step 1

Okay, we are going to use Stokes' Theorem in the following direction.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot d\vec{S}$$

So, let's first compute curl$\vec{F}$ since that is easy enough to compute and might be useful to have when we go to determine the surface S we're going to integrate over.

The curl of the vector field is then,

$$\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3yx^2+z^3 & y^2 & 4yx^2 \end{vmatrix} = 4x^2\vec{i} + 3z^2\vec{j} - 3x^2\vec{k} - 8yx\vec{j} = \underline{4x^2\vec{i} + \left(3z^2-8yx\right)\vec{j} - 3x^2\vec{k}}$$

Step 2

Now we need to find a surface S with an orientation that will have a boundary curve that is the curve shown in the problem statement, including the correct orientation.

In this case we can see that the triangle looks like the portion of a plane and so it makes sense that we use the equation of the plane containing the three vertices for the surface here.

The curl of the vector field looks a little messy so using a plane here might be the best bet from this perspective as well. It will (hopefully) not make the curl of the vector field any messier and the normal vector, which we'll get from the equation of the plane, will be simple and so shouldn't make the curl of the vector field any worse.

Step 3

Determining the equation of the plane is pretty simple. We have three points on the plane, the vertices, and so we can quickly determine the equation.

First, let's "label" the points as follows,

$$P = (4,0,0) \qquad Q = (0,2,0) \qquad R = (0,0,3)$$

Then two vectors that must lie in the plane are,

$$\overrightarrow{QP} = \langle 4,-2,0\rangle \qquad \overrightarrow{QR} = \langle 0,-2,3\rangle$$

To write the equation of a plane recall that we need a normal vector to the plane. Now, we know that the cross product of these two vectors will be orthogonal to both of the vectors. Also, since both of the vectors lie in the plane the cross product will also be orthogonal, or normal, to the plane. In other words, we can use the cross product of these two vectors as the normal vector to the plane.

The cross product is,

$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & 0 \\ 0 & -2 & 3 \end{vmatrix} = -6\vec{i} - 12\vec{j} - 8\vec{k}$$

Now we can use any of the points in our equation. We'll use Q for our point. The equation of the plane is then,

$$-6(x-0) - 12(y-2) - 8(z-0) = 0 \qquad \to \qquad \begin{aligned} -6x - 12y - 8z &= -24 \\ \underline{3x + 6y + 4z = 12} \end{aligned}$$

Note that we divided the equation by -2 to make the equation a little "nicer" to work with.

Step 4

We now need to determine the orientation of the normal vectors that will induce a positive orientation of the boundary curve, C, that matches the orientation that was given in the problem statement.

Remember that what we want to do here is think of ourselves as walking along the boundary curve of the surface in the direction indicated while our left hand is over the plane. We now need to determine if we are walking along the top or bottom of the plane.

If we are walking along the top of the plane then our heads, and hence the normal vectors, will be pointing in a generally upwards direction. On the other hand, if we are walking along the bottom of the plane then our heads, and hence the normal vectors, will be pointing generally downwards.

To help visualize this for our plane it might help to cut out a triangular piece of paper that we can use to represent the plane. The edge of the piece of paper will then represent the boundary curve. Next cut out a stick figure and put a face on one side so we know which direction we'll be walking and brightly color the left hand to make it really clear which side is the left side.

Now, hold the piece of paper so that it looks vaguely like the surface we're working with and put the stick figure on the edge with the face pointing in the direction the curve is moving and the left hand over the cup/bowl. Doing this we'll quickly see that we must be walking along the top of the surface. Therefore, the normal vectors on the surface need to be pointing in a generally upwards direction (and hence will have a positive z component).

Step 5

We now need to start thinking about actually computing the integral. We'll write the equation of the surface as,

$$z = 3 - \frac{3}{4}x - \frac{3}{2}y$$

Recall that if we aren't going to parameterize the surface we need it to be written as $z = g\left(x, y\right)$ so that the magnitude of the normal vector will eventually cancel.

Now, that we have the surface written in the "proper" form let's define,

$$f\left(x, y, z\right) = z - 3 + \frac{3}{4}x + \frac{3}{2}y = 0$$

The unit normal vector for the surface is then,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\left\langle \frac{3}{4}, \frac{3}{2}, 1 \right\rangle}{\|\nabla f\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that this does have the correct orientation because the z component is positive.

Next, we'll need to compute the following dot product.

$$\begin{aligned}
\text{curl}\vec{F} \cdot \vec{n} &= \left\langle 4x^2, \left(3\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 8yx\right), -3x^2 \right\rangle \cdot \frac{\left\langle \frac{3}{4}, \frac{3}{2}, 1 \right\rangle}{\|\nabla f\|} \\
&= \frac{1}{\|\nabla f\|}\left(3x^2 + \frac{9}{2}\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xy - 3x^2\right) \\
&= \frac{1}{\|\nabla f\|}\left[\frac{9}{2}\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xy\right]
\end{aligned}$$

Don't forget to plug the equation of the surface into the curl of the vector field.

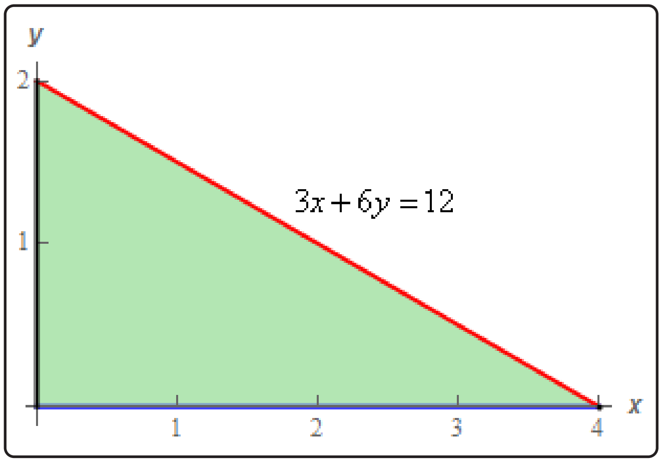
Step 6

Now, applying Stokes' Theorem to the integral and converting to a "normal" double integral gives,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \iint_S \text{curl}\vec{F} \cdot d\vec{S} \\
&= \iint_S \frac{1}{\|\nabla f\|}\left[\frac{9}{2}\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xy\right] dS \\
&= \iint_D \frac{1}{\|\nabla f\|}\left[\frac{9}{2}\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xy\right] \|\nabla f\|\, dA \\
&= \iint_D \frac{9}{2}\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xy\, dA
\end{aligned}$$

Step 7

To finish this integral we just need to determine D. In this case D just the triangle in the xy-plane that lies below the plane. Here is a quick sketch of D.



The integral doesn't seem to suggest one integration order over the other so let's use the following set of limits for our integral.

$$\begin{gathered} 0 \le x \le 4 \\ 0 \le y \le 2 - \tfrac{1}{2}x \end{gathered}$$

The integral is then,

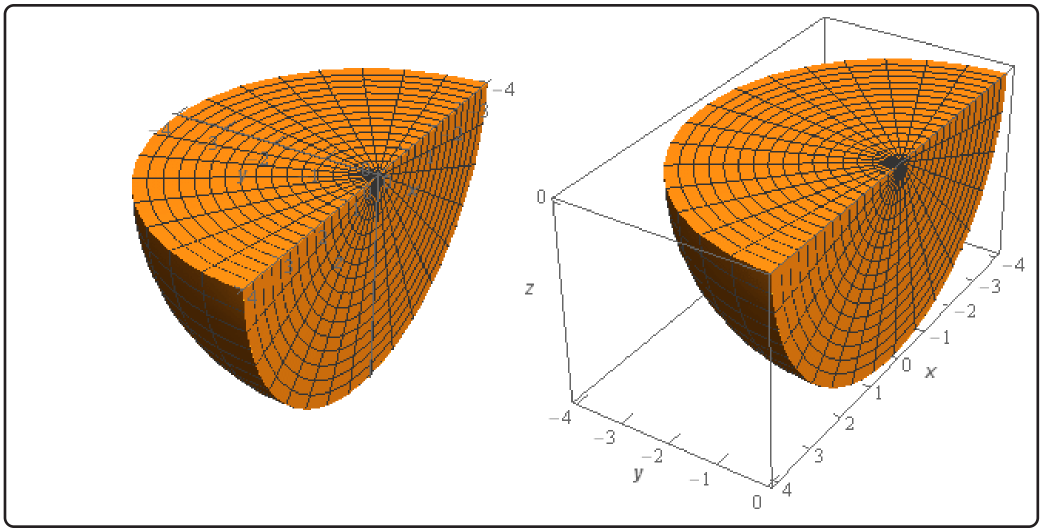
$$\begin{aligned} \int\limits_C \vec{F} \cdot d\vec{r} &= \iint\limits_S \text{curl}\vec{F} \cdot d\vec{S} \\ &= \int_0^4 \int_0^{2-\frac{1}{2}x} \frac{9}{2}\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xy\, dy\, dx \\ &= \int_0^4 \left(-\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^3 - 6xy^2\right)\Bigg|_0^{2-\frac{1}{2}x} dx \\ &= \int_0^4 \left(3 - \frac{3}{4}x\right)^3 - 6x\left(2 - \frac{1}{2}x\right)^2 dx \\ &= \int_0^4 \left(3 - \frac{3}{4}x\right)^3 - 24x + 12x^2 - \frac{3}{2}x^3\, dx \\ &= \left(-\frac{1}{3}\left(3 - \frac{3}{4}x\right)^4 - 12x^2 + 4x^3 - \frac{3}{8}x^4\right)\Bigg|_0^4 \\ &= \boxed{-5} \end{aligned}$$

17.6 Divergence Theorem

1. Use the Divergence Theorem to evaluate $\displaystyle\iint\limits_S \vec F\cdot d\vec S$ where $\vec F = yx^2\,\vec i + \left(xy^2 - 3z^4\right)\vec j + \left(x^3+y^2\right)\vec k$ and S is the surface of the sphere of radius 4 with $z \le 0$ and $y \le 0$. Note that all three surfaces of this solid are included in S.

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

Note as well here that because we are including all three surfaces shown above that the surface does enclose (or is the boundary curve if you want to use that terminology) the portion of the sphere shown above.

Step 2

We are going to use the Divergence Theorem in the following direction.

$$\iint\limits_S \vec F\cdot d\vec S = \iiint\limits_E \mathrm{div}\vec F\,dV$$

where E is just the solid shown in the sketches from Step 1.

Because E is a portion of a sphere we'll be wanting to use spherical coordinates for the integration. Here are the spherical limits we'll need to use for this region.

$$\begin{gathered}\pi \le \theta \le 2\pi \\ \frac{\pi}{2} \le \varphi \le \pi \\ 0 \le \rho \le 4\end{gathered}$$

One of the restrictions on the region in the problem statement was $y \le 0$. This means that if we look at this from above we'd see the portion of the circle of radius 4 that is below the x axis and so we need the given range of θ above to cover this region.

We were also told in the problem statement that $z \le 0$ and so we only want the portion of the sphere that is below the xy-plane. We therefore need the given range of φ to make sure we are only below the xy-plane.

We'll also need the divergence of the vector field so here is that.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(yx^2\right) + \frac{\partial}{\partial y}\left(xy^2 - 3z^4\right) + \frac{\partial}{\partial z}\left(x^3 + y^2\right) = 4xy$$

Step 3

Now let's apply the Divergence Theorem to the integral and get it converted to spherical coordinates while we're at it.

$$\begin{aligned}\iint\limits_S \vec{F}\cdot d\vec{S} &= \iiint\limits_E \operatorname{div}\vec{F}\,dV \\ &= \iiint\limits_E 4xy\,dV \\ &= \int_{\pi}^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_0^4 4\left(\rho\sin(\varphi)\cos(\theta)\right)\left(\rho\sin(\varphi)\sin(\theta)\right)\left(\rho^2\sin(\varphi)\right)\,d\rho\,d\varphi\,d\theta \\ &= \int_{\pi}^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_0^4 4\rho^4\sin^3(\varphi)\cos(\theta)\sin(\theta)\,d\rho\,d\varphi\,d\theta\end{aligned}$$

Don't forget to pick up the $\rho^2\sin(\varphi)$ when converting the dV to spherical coordinates.

Step 4

All we need to do then in evaluate the integral.

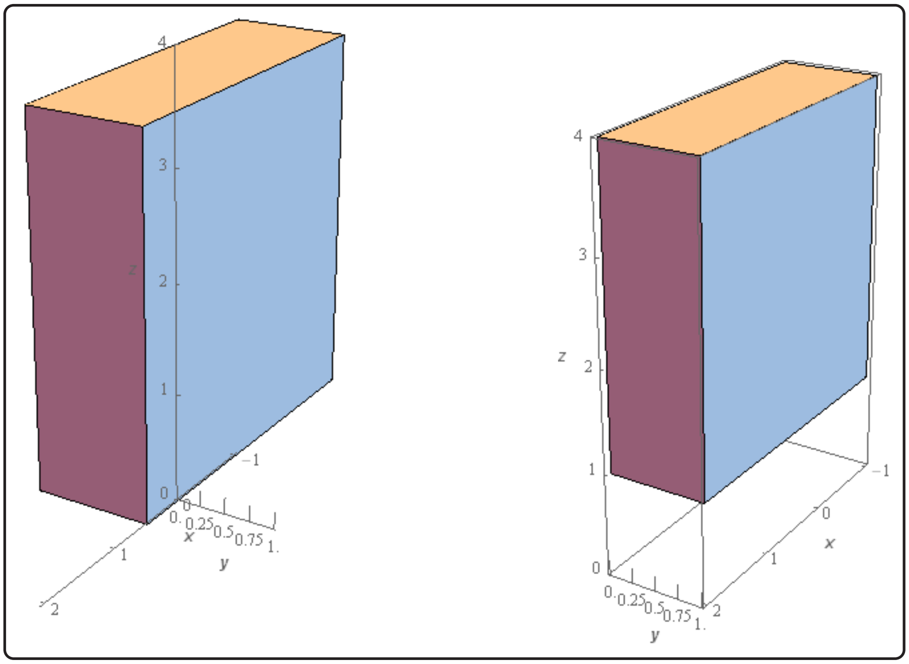
$$\begin{aligned}\iint\limits_S \vec{F}\cdot d\vec{S} &= \int_{\pi}^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_0^4 4\rho^4\sin^3(\varphi)\cos(\theta)\sin(\theta)\,d\rho\,d\varphi\,d\theta\\
&= \int_{\pi}^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\left(\frac{4}{5}\rho^5\sin^3(\varphi)\cos(\theta)\sin(\theta)\right)\Big|_0^4\,d\varphi\,d\theta\\
&= \int_{\pi}^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\frac{4096}{5}\sin(\varphi)\left(1-\cos^2(\varphi)\right)\cos(\theta)\sin(\theta)\,d\varphi\,d\theta\\
&= \int_{\pi}^{2\pi}\left(-\frac{4096}{5}\left(\cos(\varphi)-\frac{1}{3}\cos^3(\varphi)\right)\cos(\theta)\sin(\theta)\right)\Big|_{\frac{1}{2}\pi}^{\pi}\,d\theta\\
&= \int_{\pi}^{2\pi}\frac{8192}{15}\cos(\theta)\sin(\theta)\,d\theta\\
&= \int_{\pi}^{2\pi}\frac{4096}{15}\sin(2\theta)\,d\theta\\
&= -\frac{2048}{15}\cos(2\theta)\Big|_{\pi}^{2\pi} = \boxed{0}\end{aligned}$$

Make sure you can do use your trig formulas as we did here to deal with these kinds of integrals!

2. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \sin(\pi x)\,\vec{i} + zy^3\,\vec{j} + \left(z^2+4x\right)\,\vec{k}$ and S is the surface of the box with $-1\le x\le 2$, $0\le y\le 1$ and $1\le z\le 4$. Note that all six sides of the box are included in S.

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

Note as well here that because we are including all six sides of the box shown above that the surface does enclose (or is the boundary curve if you want to use that terminology) for the box.

Step 2

We are going to use the Divergence Theorem in the following direction.

$$\iint\limits_{S} \vec{F} \cdot d\vec{S} = \iiint\limits_{E} \text{div}\vec{F}\, dV$$

where E is just the solid shown in the sketches from Step 1.

E is just a box and the limits defining it where given in the problem statement. The limits for our integral will then be,

$$\begin{gathered} -1 \le x \le 2 \\ 0 \le y \le 1 \\ 1 \le z \le 4 \end{gathered}$$

We'll also need the divergence of the vector field so here is that.

$$\text{div}\vec{F} = \frac{\partial}{\partial x}\left(\sin\left(\pi x\right)\right) + \frac{\partial}{\partial y}\left(zy^3\right) + \frac{\partial}{\partial z}\left(z^2 + 4x\right) = \pi\cos\left(\pi x\right) + 3zy^2 + 2z$$

Step 3

Now let's apply the Divergence Theorem to the integral and get it converted to a triple integral.

$$\begin{aligned}\iint\limits_S \vec{F}\cdot d\vec{S} &= \iiint\limits_E \text{div}\vec{F}\,dV \\ &= \iiint\limits_E \pi\cos\left(\pi x\right) + 3zy^2 + 2z\,dV \\ &= \int_{-1}^{2}\int_{0}^{1}\int_{1}^{4} \pi\cos\left(\pi x\right) + 3zy^2 + 2z\,dz\,dy\,dx\end{aligned}$$

Step 4

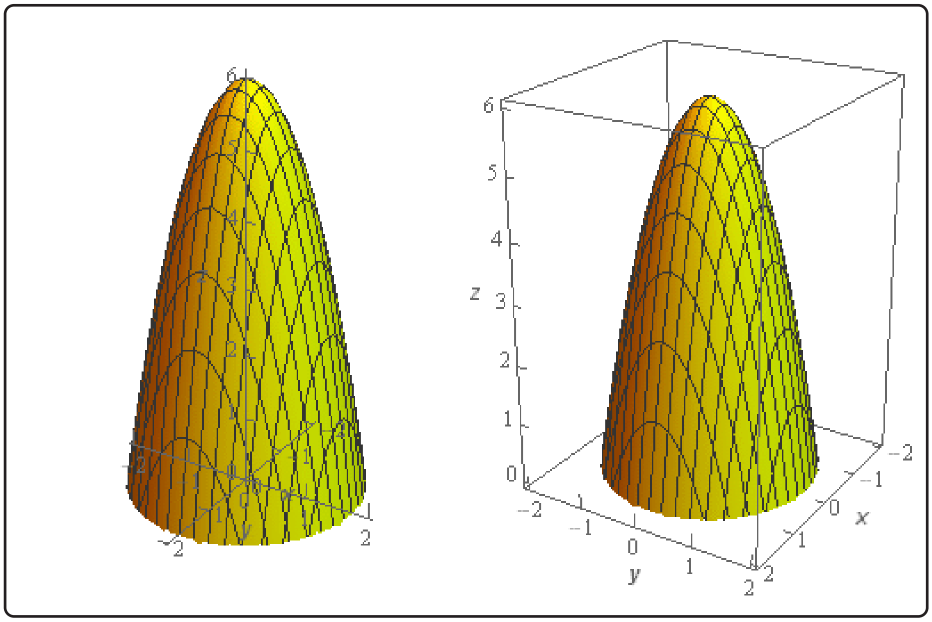
All we need to do then in evaluate the integral.

$$\begin{aligned}\iint\limits_S \vec{F}\cdot d\vec{S} &= \int_{-1}^{2}\int_{0}^{1}\int_{1}^{4} \pi\cos\left(\pi x\right) + 3zy^2 + 2z\,dz\,dy\,dx \\ &= \int_{-1}^{2}\int_{0}^{1} \left.\left(\pi z\cos\left(\pi x\right) + \frac{3}{2}z^2y^2 + z^2\right)\right|_1^4\,dy\,dx \\ &= \int_{-1}^{2}\int_{0}^{1} 3\pi\cos\left(\pi x\right) + \frac{45}{2}y^2 + 15\,dy\,dx \\ &= \int_{-1}^{2} \left.\left(3y\pi\cos\left(\pi x\right) + \frac{15}{2}y^3 + 15y\right)\right|_0^1\,dx \\ &= \int_{-1}^{2} 3\pi\cos\left(\pi x\right) + \frac{45}{2}\,dx \\ &= \left.\left(3\sin\left(\pi x\right) + \frac{45}{2}x\right)\right|_{-1\pi}^{2} = \boxed{\frac{135}{2}}\end{aligned}$$

3. Use the Divergence Theorem to evaluate $\displaystyle\iint\limits_S \vec F \cdot d\vec S$ where $\vec F = 2xz\vec i + \left(1 - 4xy^2\right)\vec j + \left(2z - z^2\right)\vec k$ and S is the surface of the solid bounded by $z = 6 - 2x^2 - 2y^2$ and the plane $z = 0$. Note that both of the surfaces of this solid included in S.

Step 1

Let's start off with a quick sketch of the surface we are working with in this problem.



We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface. The bottom "cap" of the elliptic paraboloid is also included in the surface but isn't shown.

Note as well here that because we are including both of the surfaces shown above that the surface does enclose (or is the boundary curve if you want to use that terminology) the region.

Step 2

We are going to use Stokes' Theorem in the following direction.

$$\iint\limits_S \vec{F} \cdot d\vec{S} = \iiint\limits_E \text{div}\vec{F}\, dV$$

where E is just the solid shown in the sketches from Step 1.

The region D for that we'll need in converting the triple integral into iterated integrals is just the intersection of the two surfaces from the problem statement. This is,

$$0 = 6 - 2x^2 - 2y^2 \qquad \rightarrow \qquad x^2 + y^2 = 3$$

So, D is a disk and so we'll eventually be doing cylindrical coordinates for this integral. Here are the cylindrical limits for the region E.

$$\begin{gathered} 0 \le \theta \le 2\pi \\ 0 \le r \le \sqrt{3} \\ 0 \le z \le 6 - 2x^2 - 2y^2 = 6 - 2r^2 \end{gathered}$$

Don't forget to convert the z limits into cylindrical coordinates as well!

We'll also need the divergence of the vector field so here is that.

$$\text{div}\vec{F} = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}\left(1 - 4xy^2\right) + \frac{\partial}{\partial z}\left(2z - z^2\right) = 2 - 8xy$$

Step 3

Now let's apply the Divergence Theorem to the integral and get it converted to cylindrical coordinates while we're at it.

$$\begin{aligned} \iint\limits_S \vec{F} \cdot d\vec{S} &= \iiint\limits_E \text{div}\vec{F}\, dV \\ &= \iiint\limits_E 2 - 8xy\, dV \\ &= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} \left(2 - 8r^2\cos(\theta)\sin(\theta)\right) r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} 2r - 8r^3\cos(\theta)\sin(\theta)\, dz\, dr\, d\theta \end{aligned}$$

Don't forget to pick up the r when converting the dV to cylindrical coordinates.

Step 4

All we need to do then in evaluate the integral.

$$\begin{aligned}
\iint\limits_{S} \vec{F}\cdot d\vec{S} &= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} 2r - 8r^3\cos(\theta)\sin(\theta)\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(2r - 8r^3\cos(\theta)\sin(\theta)\right) z\Big|_0^{6-2r^2}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(2r - 8r^3\cos(\theta)\sin(\theta)\right)\left(6-2r^2\right)\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} 12r - 4r^3 - \left(48r^3 - 16r^5\right)\cos(\theta)\sin(\theta)\,dr\,d\theta \\
&= \int_0^{2\pi} \left(6r^2 - r^4 - \left(12r^4 - \frac{8}{3}r^6\right)\cos(\theta)\sin(\theta)\right)\Bigg|_0^{\sqrt{3}}\,d\theta \\
&= \int_0^{2\pi} 9 - 36\cos(\theta)\sin(\theta)\,d\theta \\
&= \int_0^{2\pi} 9 - 18\sin\left(2\theta\right)\,d\theta \\
&= \left(9\theta - 9\cos\left(2\theta\right)\right)\Big|_0^{2\pi} = \boxed{18\pi}
\end{aligned}$$

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