

CALCULUS I

Solutions to Practice Problems

Limits

Paul Dawkins

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Preface

Here are the solutions to the practice problems for the Calculus I notes.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Chapter 2 : Limits

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Tangent Lines and Rates of Change – In this section we will introduce two problems that we will see time and again in this course : Rate of Change of a function and Tangent Lines to functions. Both of these problems will be used to introduce the concept of limits, although we won't formally give the definition or notation until the next section.

The Limit – In this section we will introduce the notation of the limit. We will also take a conceptual look at limits and try to get a grasp on just what they are and what they can tell us. We will be estimating the value of limits in this section to help us understand what they tell us. We will actually start computing limits in a couple of sections.

One-Sided Limits – In this section we will introduce the concept of one-sided limits. We will discuss the differences between one-sided limits and limits as well as how they are related to each other.

Limit Properties – In this section we will discuss the properties of limits that we'll need to use in computing limits (as opposed to estimating them as we've done to this point). We will also compute a couple of basic limits in this section.

Computing Limits – In this section we will look at several types of limits that require some work before we can use the limit properties to compute them. We will also look at computing limits of piecewise functions and use of the Squeeze Theorem to compute some limits.

Infinite Limits – In this section we will look at limits that have a value of infinity or negative infinity. We'll also take a brief look at vertical asymptotes.

Limits At Infinity, Part I – In this section we will start looking at limits at infinity, i.e. limits in which the variable gets very large in either the positive or negative sense. We will concentrate on polynomials and rational expressions in this section. We'll also take a brief look at horizontal asymptotes.

Limits At Infinity, Part II – In this section we will continue covering limits at infinity. We'll be looking at exponentials, logarithms and inverse tangents in this section.

Continuity – In this section we will introduce the concept of continuity and how it relates to limits. We will also see the Intermediate Value Theorem in this section and how it can be used to determine if functions have solutions in a given interval.

The Definition of the Limit – In this section we will give a precise definition of several of the limits covered in this section. We will work several basic examples illustrating how to use this precise definition to compute a limit. We'll also give a precise definition of continuity.

Section 2-1 : Tangent Lines and Rates of Change

1. For the function $f(x) = 3(x+2)^2$ and the point P given by $x = -3$ answer each of the following questions.

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ} , of the secant line through points P and Q .

(i) -3.5 (ii) -3.1 (iii) -3.01 (iv) -3.001 (v) -3.0001
(vi) -2.5 (vii) -2.9 (viii) -2.99 (ix) -2.999 (x) -2.9999

(b) Use the information from (a) to estimate the slope of the tangent line to $f(x)$ at $x = -3$ and write down the equation of the tangent line.

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ} , of the secant line through points P and Q .

(i) -3.5 (ii) -3.1 (iii) -3.01 (iv) -3.001 (v) -3.0001
(vi) -2.5 (vii) -2.9 (viii) -2.99 (ix) -2.999 (x) -2.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$m_{PQ} = \frac{f(x) - f(-3)}{x - (-3)} = \frac{3(x+2)^2 - 3}{x+3}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x . All of the values in the table below are accurate to 8 decimal places, but in this case the values terminated prior to 8 decimal places and so the "trailing" zeros are not shown.

x	m_{PQ}	x	m_{PQ}
-3.5	-7.5	-2.5	-4.5
-3.1	-6.3	-2.9	-5.7
-3.01	-6.03	-2.99	-5.97
-3.001	-6.003	-2.999	-5.997
-3.0001	-6.0003	-2.9999	-5.9997

(b) Use the information from (a) to estimate the slope of the tangent line to $f(x)$ at $x = -3$ and write down the equation of the tangent line.

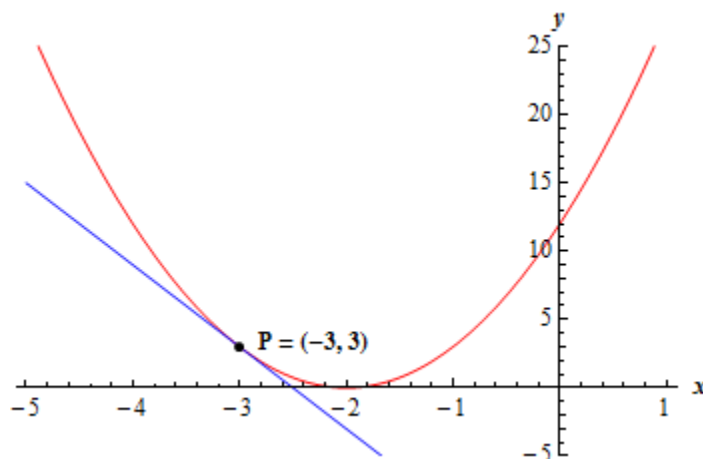
Solution

From the table of values above we can see that the slope of the secant lines appears to be moving towards a value of -6 from both sides of $x = -3$ and so we can estimate that the slope of the tangent line is : $\boxed{m = -6}$.

The equation of the tangent line is then,

$$y = f(-3) + m(x - (-3)) = 3 - 6(x + 3) \quad \Rightarrow \quad \boxed{y = -6x - 15}$$

Here is a graph of the function and the tangent line.



2. For the function $g(x) = \sqrt{4x+8}$ and the point P given by $x = 2$ answer each of the following questions.

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ} , of the secant line through points P and Q .

- | | | | | |
|-----------------|------------------|--------------------|-------------------|-------------------|
| (i) 2.5 | (ii) 2.1 | (iii) 2.01 | (iv) 2.001 | (v) 2.0001 |
| (vi) 1.5 | (vii) 1.9 | (viii) 1.99 | (ix) 1.999 | (x) 1.9999 |

(b) Use the information from **(a)** to estimate the slope of the tangent line to $g(x)$ at $x = 2$ and write down the equation of the tangent line.

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ} , of the secant line through points P and Q .

- | | | | | |
|-----------------|------------------|--------------------|-------------------|-------------------|
| (i) 2.5 | (ii) 2.1 | (iii) 2.01 | (iv) 2.001 | (v) 2.0001 |
| (vi) 1.5 | (vii) 1.9 | (viii) 1.99 | (ix) 1.999 | (x) 1.9999 |

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$m_{PQ} = \frac{g(x) - g(2)}{x - 2} = \frac{\sqrt{4x+8} - 4}{x - 2}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x . All of the values in the table below are accurate to 8 decimal places.

x	m_{PQ}	x	m_{PQ}
2.5	0.48528137	1.5	0.51668523
2.1	0.49691346	1.9	0.50316468
2.01	0.49968789	1.99	0.50031289
2.001	0.49996875	1.999	0.50003125
2.0001	0.49999688	1.9999	0.50000313

(b) Use the information from **(a)** to estimate the slope of the tangent line to $g(x)$ at $x = 2$ and write down the equation of the tangent line.

Solution

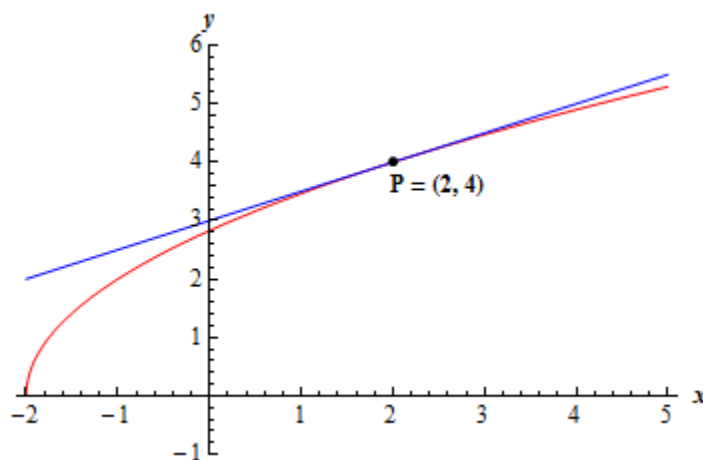
From the table of values above we can see that the slope of the secant lines appears to be moving towards a value of 0.5 from both sides of $x = 2$ and so we can estimate that the slope of the tangent

line is : $m = 0.5 = \frac{1}{2}$.

The equation of the tangent line is then,

$$y = g(2) + m(x - 2) = 4 + \frac{1}{2}(x - 2) \quad \Rightarrow \quad y = \frac{1}{2}x + 3$$

Here is a graph of the function and the tangent line.



3. For the function $W(x) = \ln(1+x^4)$ and the point P given by $x = 1$ answer each of the following questions.

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ} , of the secant line through points P and Q .

(i) 1.5	(ii) 1.1	(iii) 1.01	(iv) 1.001	(v) 1.0001
(vi) 0.5	(vii) 0.9	(viii) 0.99	(ix) 0.999	(x) 0.9999

(b) Use the information from **(a)** to estimate the slope of the tangent line to $W(x)$ at $x = 1$ and write down the equation of the tangent line.

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ} , of the secant line through points P and Q .

(i) 1.5	(ii) 1.1	(iii) 1.01	(iv) 1.001	(v) 1.0001
(vi) 0.5	(vii) 0.9	(viii) 0.99	(ix) 0.999	(x) 0.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$m_{PQ} = \frac{W(x) - W(1)}{x - 1} = \frac{\ln(1 + x^4) - \ln(2)}{x - 1}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x . All of the values in the table below are accurate to 8 decimal places.

x	m_{PQ}	x	m_{PQ}
1.5	2.21795015	0.5	1.26504512
1.1	2.08679449	0.9	1.88681740
1.01	2.00986668	0.99	1.98986668
1.001	2.00099867	0.999	1.99899867
1.0001	2.00009999	0.9999	1.99989999

(b) Use the information from **(a)** to estimate the slope of the tangent line to $W(x)$ at $x = 1$ and write down the equation of the tangent line.

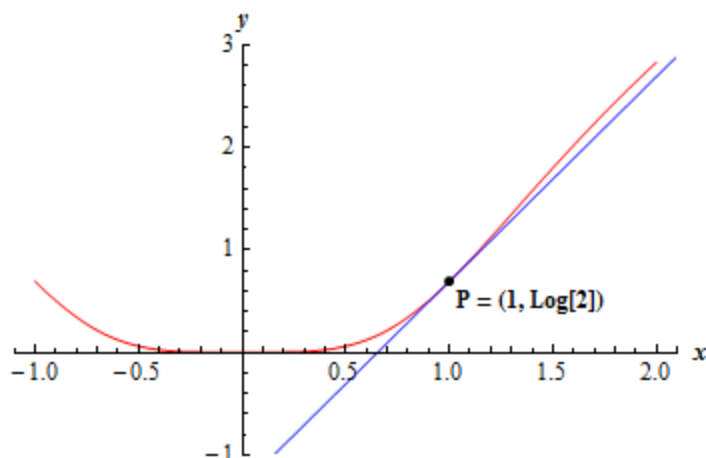
Solution

From the table of values above we can see that the slope of the secant lines appears to be moving towards a value of 2 from both sides of $x = 1$ and so we can estimate that the slope of the tangent line is : $\boxed{m = 2}$.

The equation of the tangent line is then,

$$y = W(1) + m(x - 1) = \boxed{\ln(2) + 2(x - 1)}$$

Here is a graph of the function and the tangent line.



4. The volume of air in a balloon is given by $V(t) = \frac{6}{4t+1}$ answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the volume of air in the balloon between $t = 0.25$ and the following values of t .

- (i) 1 (ii) 0.5 (iii) 0.251 (iv) 0.2501 (v) 0.25001
 (vi) 0 (vii) 0.1 (viii) 0.249 (ix) 0.2499 (x) 0.24999

(b) Use the information from **(a)** to estimate the instantaneous rate of change of the volume of air in the balloon at $t = 0.25$.

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the volume of air in the balloon between $t = 0.25$ and the following values of t .

- (i) 1 (ii) 0.5 (iii) 0.251 (iv) 0.2501 (v) 0.25001
 (vi) 0 (vii) 0.1 (viii) 0.249 (ix) 0.2499 (x) 0.24999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$ARC = \frac{V(t) - V(0.25)}{t - 0.25} = \frac{\frac{6}{4t+1} - 3}{t - 0.25}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x . All of the values in the table below are accurate to 8 decimal places. In several of the initial values in the table the values terminated and so the “trailing” zeroes were not shown.

x	ARC	x	ARC
1	-2.4	0	-12
0.5	-4	0.1	-8.57142857

0.251	-5.98802395	0.249	-6.01202405
0.2501	-5.99880024	0.2499	-6.00120024
0.25001	-5.99988000	0.24999	-6.00012000

(b) Use the information from **(a)** to estimate the instantaneous rate of change of the volume of air in the balloon at $t = 0.25$.

Solution

From the table of values above we can see that the average rate of change of the volume of air is moving towards a value of -6 from both sides of $t = 0.25$ and so we can estimate that the instantaneous rate of change of the volume of air in the balloon is -6 .

5. The population (in hundreds) of fish in a pond is given by $P(t) = 2t + \sin(2t - 10)$ answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the population of fish between $t = 5$ and the following values of t . Make sure your calculator is set to radians for the computations.

- | | | | | |
|-----------------|------------------|--------------------|-------------------|-------------------|
| (i) 5.5 | (ii) 5.1 | (iii) 5.01 | (iv) 5.001 | (v) 5.0001 |
| (vi) 4.5 | (vii) 4.9 | (viii) 4.99 | (ix) 4.999 | (x) 4.9999 |

(b) Use the information from **(a)** to estimate the instantaneous rate of change of the population of the fish at $t = 5$.

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the population of fish between $t = 5$ and the following values of t . Make sure your calculator is set to radians for the computations.

- | | | | | |
|-----------------|------------------|--------------------|-------------------|-------------------|
| (i) 5.5 | (ii) 5.1 | (iii) 5.01 | (iv) 5.001 | (v) 5.0001 |
| (vi) 4.5 | (vii) 4.9 | (viii) 4.99 | (ix) 4.999 | (x) 4.9999 |

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$ARC = \frac{P(t) - P(5)}{t - 5} = \frac{2t + \sin(2t - 10) - 10}{t - 5}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x . All of the values in the table below are accurate to 8 decimal places.

x	ARC	x	ARC
5.5	3.68294197	4.5	3.68294197
5.1	3.98669331	4.9	3.98669331
5.01	3.99986667	4.99	3.99986667
5.001	3.99999867	4.999	3.99999867

$$5.0001 \mid 3.99999999 \mid 4.9999 \mid 3.99999999$$

(b) Use the information from **(a)** to estimate the instantaneous rate of change of the population of the fish at $t = 5$.

Solution

From the table of values above we can see that the average rate of change of the population of fish is moving towards a value of 4 from both sides of $t = 5$ and so we can estimate that the instantaneous rate of change of the population of the fish is 400 (remember the population is in hundreds).

6. The position of an object is given by $s(t) = \cos^2\left(\frac{3t-6}{2}\right)$ answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 2$ and the following values of t . Make sure your calculator is set to radians for the computations.

(i) 2.5 (ii) 2.1 (iii) 2.01 (iv) 2.001 (v) 2.0001
(vi) 1.5 (vii) 1.9 (viii) 1.99 (ix) 1.999 (x) 1.9999

(b) Use the information from **(a)** to estimate the instantaneous velocity of the object at $t = 2$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 2$ and the following values of t . Make sure your calculator is set to radians for the computations.

(i) 2.5 (ii) 2.1 (iii) 2.01 (iv) 2.001 (v) 2.0001
(vi) 1.5 (vii) 1.9 (viii) 1.99 (ix) 1.999 (x) 1.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$AV = \frac{s(t) - s(2)}{t - 2} = \frac{\cos^2\left(\frac{3t-6}{2}\right) - 1}{t - 2}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x . All of the values in the table below are accurate to 8 decimal places.

t	AV	t	AV
2.5	-0.92926280	1.5	0.92926280
2.1	-0.22331755	1.9	0.22331755
2.01	-0.02249831	1.99	0.02249831
2.001	-0.00225000	1.999	0.00225000
2.0001	-0.00022500	1.9999	0.00022500

(b) Use the information from **(a)** to estimate the instantaneous velocity of the object at $t = 2$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

Solution

From the table of values above we can see that the average velocity of the object is moving towards a value of 0 from both sides of $t = 2$ and so we can estimate that the instantaneous velocity is 0 and so the object will not be moving at $t = 2$.

7. The position of an object is given by $s(t) = (8-t)(t+6)^{\frac{3}{2}}$. Note that a negative position here simply means that the position is to the left of the “zero position” and is perfectly acceptable. Answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 10$ and the following values of t .

(i) 10.5 (ii) 10.1 (iii) 10.01 (iv) 10.001 (v) 10.0001
(vi) 9.5 (vii) 9.9 (viii) 9.99 (ix) 9.999 (x) 9.9999

(b) Use the information from **(a)** to estimate the instantaneous velocity of the object at $t = 10$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 10$ and the following values of t .

(i) 10.5 (ii) 10.1 (iii) 10.01 (iv) 10.001 (v) 10.0001
(vi) 9.5 (vii) 9.9 (viii) 9.99 (ix) 9.999 (x) 9.9999

Solution

The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$AV = \frac{s(t) - s(10)}{t - 10} = \frac{(8-t)(t+6)^{\frac{3}{2}} + 128}{t - 10}$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x . All of the values in the table below are accurate to 8 decimal places.

t	AV	t	AV
10.5	-79.11658419	9.5	-72.92931693
10.1	-76.61966704	9.9	-75.38216890
10.01	-76.06188418	9.99	-75.93813418
10.001	-76.00618759	9.999	-75.99381259

$$10.0001 \mid -76.00061875 \parallel 9.9999 \mid -75.99938125$$

(b) Use the information from **(a)** to estimate the instantaneous velocity of the object at $t = 10$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

Solution

From the table of values above we can see that the average velocity of the object is moving towards a value of -76 from both sides of $t = 10$ and so we can estimate that the instantaneous velocity is -76 and so the object will be moving to the left at $t = 10$.

Section 2-2 : The Limit

1. For the function $f(x) = \frac{8-x^3}{x^2-4}$ answer each of the following questions.

(a) Evaluate the function the following values of x compute (accurate to at least 8 decimal places).

- (i) 2.5 (ii) 2.1 (iii) 2.01 (iv) 2.001 (v) 2.0001
 (vi) 1.5 (vii) 1.9 (viii) 1.99 (ix) 1.999 (x) 1.9999

(b) Use the information from (a) to estimate the value of $\lim_{x \rightarrow 2} \frac{8-x^3}{x^2-4}$.

(a) Evaluate the function the following values of x compute (accurate to at least 8 decimal places).

- (i) 2.5 (ii) 2.1 (iii) 2.01 (iv) 2.001 (v) 2.0001
 (vi) 1.5 (vii) 1.9 (viii) 1.99 (ix) 1.999 (x) 1.9999

Solution

Here is a table of values of the function at the given points accurate to 8 decimal places.

x	$f(x)$	x	$f(x)$
2.5	-3.38888889	1.5	-2.64285714
2.1	-3.07560976	1.9	-2.92564103
2.01	-3.00750623	1.99	-2.99250627
2.001	-3.00075006	1.999	-2.99925006
2.0001	-3.00007500	1.9999	-2.99992500

(b) Use the information from (a) to estimate the value of $\lim_{x \rightarrow 2} \frac{8-x^3}{x^2-4}$.

Solution

From the table of values above it looks like we can estimate that,

$$\lim_{x \rightarrow 2} \frac{8-x^3}{x^2-4} = -3$$

2. For the function $R(t) = \frac{2-\sqrt{t^2+3}}{t+1}$ answer each of the following questions.

(a) Evaluate the function the following values of t compute (accurate to at least 8 decimal places).

- (i) -0.5 (ii) -0.9 (iii) -0.99 (iv) -0.999 (v) -0.9999
 (vi) -1.5 (vii) -1.1 (viii) -1.01 (ix) -1.001 (x) -1.0001

(b) Use the information from **(a)** to estimate the value of $\lim_{t \rightarrow -1} \frac{2 - \sqrt{t^2 + 3}}{t + 1}$.

(a) Evaluate the function the following values of t compute (accurate to at least 8 decimal places).

(i) -0.5 (ii) -0.9 (iii) -0.99 (iv) -0.999 (v) -0.9999
 (vi) -1.5 (vii) -1.1 (viii) -1.01 (ix) -1.001 (x) -1.0001

Solution

Here is a table of values of the function at the given points accurate to 8 decimal places.

t	$R(t)$	t	$R(t)$
-0.5	0.39444872	-1.5	0.58257569
-0.9	0.48077870	-1.1	0.51828453
-0.99	0.49812031	-1.01	0.50187032
-0.999	0.49981245	-1.001	0.50018745
-0.9999	0.49998125	-1.0001	0.50001875

(b) Use the information from **(a)** to estimate the value of $\lim_{t \rightarrow -1} \frac{2 - \sqrt{t^2 + 3}}{t + 1}$.

Solution

From the table of values above it looks like we can estimate that,

$$\lim_{t \rightarrow -1} \frac{2 - \sqrt{t^2 + 3}}{t + 1} = \frac{1}{2}$$

3. For the function $g(\theta) = \frac{\sin(7\theta)}{\theta}$ answer each of the following questions.

(a) Evaluate the function the following values of θ compute (accurate to at least 8 decimal places).
 Make sure your calculator is set to radians for the computations.

(i) 0.5 (ii) 0.1 (iii) 0.01 (iv) 0.001 (v) 0.0001
 (vi) -0.5 (vii) -0.1 (viii) -0.01 (ix) -0.001 (x) -0.0001

(b) Use the information from **(a)** to estimate the value of $\lim_{\theta \rightarrow 0} \frac{\sin(7\theta)}{\theta}$.

(a) Evaluate the function the following values of x compute (accurate to at least 8 decimal places).

(i) 0.5 (ii) 0.1 (iii) 0.01 (iv) 0.001 (v) 0.0001
 (vi) -0.5 (vii) -0.1 (viii) -0.01 (ix) -0.001 (x) -0.0001

Solution

Here is a table of values of the function at the given points accurate to 8 decimal places.

θ	$g(\theta)$	θ	$g(\theta)$
0.5	-0.70156646	-0.5	-0.70156646
0.1	6.44217687	-0.1	6.44217687
0.01	6.99428473	-0.01	6.99428473
0.001	6.99994283	-0.001	6.99994283
0.0001	6.99999943	-0.0001	6.99999943

(b) Use the information from (a) to estimate the value of $\lim_{\theta \rightarrow 0} \frac{\sin(7\theta)}{\theta}$.

Solution

From the table of values above it looks like we can estimate that,

$$\lim_{\theta \rightarrow 0} \frac{\sin(7\theta)}{\theta} = 7$$

4. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim_{x \rightarrow a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -3$

(b) $a = -1$

(c) $a = 2$

(d) $a = 4$



(a) $a = -3$

From the graph we can see that,

$$\boxed{f(-3) = 4}$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = -3$ from both sides the graph is approaching different values (4 from the left and -2 from the right). Because of this we get,

$$\lim_{x \rightarrow -3} f(x) \text{ does not exist}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(b) $a = -1$

From the graph we can see that,

$$f(-1) = 3$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -1$ from both sides the graph is approaching the same value, 1, and so we get,

$$\lim_{x \rightarrow -1} f(x) = 1$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(c) $a = 2$

Because there is no closed dot for $x = 2$ we can see that,

$$f(2) \text{ does not exist}$$

We can also see that as we approach $x = 2$ from both sides the graph is approaching the same value, 1, and so we get,

$$\lim_{x \rightarrow 2} f(x) = 1$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Therefore, even though the function doesn't exist at this point the limit can still have a value.

(d) $a = 4$

From the graph we can see that,

$$f(4) = 5$$

because the closed dot is at the value of $y = 5$.

We can also see that as we approach $x = 4$ from both sides the graph is approaching the same value, 5, and so we get,

$$\lim_{x \rightarrow 4} f(x) = 5$$

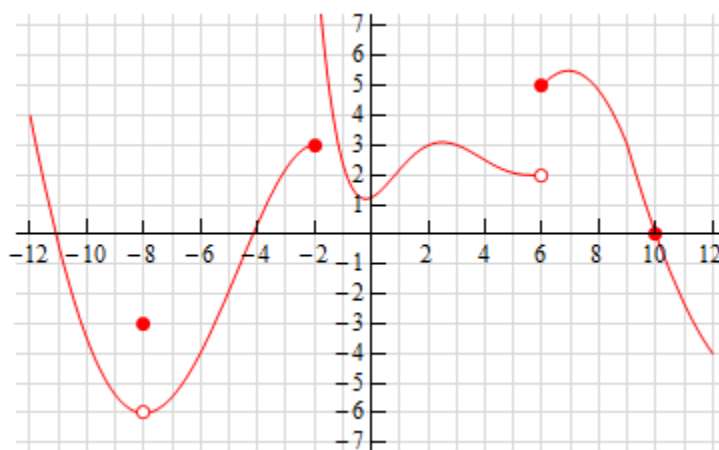
5. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim_{x \rightarrow a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -8$

(b) $a = -2$

(c) $a = 6$

(d) $a = 10$



(a) $a = -8$

From the graph we can see that,

$$\boxed{f(-8) = -3}$$

because the closed dot is at the value of $y = -3$.

We can also see that as we approach $x = -8$ from both sides the graph is approaching the same value, -6, and so we get,

$$\boxed{\lim_{x \rightarrow -8} f(x) = -6}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(b) $a = -2$

From the graph we can see that,

$$\boxed{f(-2) = 3}$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -2$ from both sides the graph is approaching different values (3 from the left and doesn't approach any value from the right). Because of this we get,

$$\lim_{x \rightarrow -2} f(x) \text{ does not exist}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(c) $a = 6$

From the graph we can see that,

$$f(6) = 5$$

because the closed dot is at the value of $y = 5$.

We can also see that as we approach $x = 6$ from both sides the graph is approaching different values (2 from the left and 5 from the right). Because of this we get,

$$\lim_{x \rightarrow 6} f(x) \text{ does not exist}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(d) $a = 10$

From the graph we can see that,

$$f(10) = 0$$

because the closed dot is at the value of $y = 0$.

We can also see that as we approach $x = 10$ from both sides the graph is approaching the same value, 0, and so we get,

$$\lim_{x \rightarrow 10} f(x) = 0$$

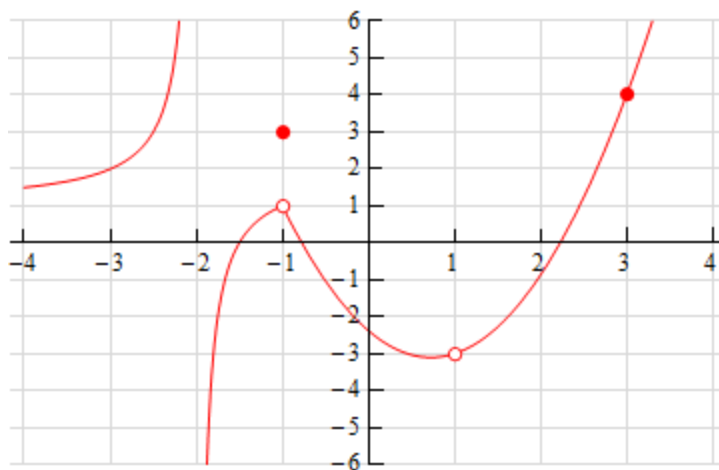
6. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim_{x \rightarrow a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -2$

(b) $a = -1$

(c) $a = 1$

(d) $a = 3$



(a) $a = -2$

Because there is no closed dot for $x = -2$ we can see that,

$$\boxed{f(-2) \text{ does not exist}}$$

We can also see that as we approach $x = -2$ from both sides the graph is not approaching a value from either side and so we get,

$$\boxed{\lim_{x \rightarrow -2} f(x) \text{ does not exist}}$$

(b) $a = -1$

From the graph we can see that,

$$\boxed{f(-1) = 3}$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -1$ from both sides the graph is approaching the same value, 1, and so we get,

$$\boxed{\lim_{x \rightarrow -1} f(x) = 1}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(c) $a = 1$

Because there is no closed dot for $x = 1$ we can see that,

$$\boxed{f(1) \text{ does not exist}}$$

We can also see that as we approach $x = 1$ from both sides the graph is approaching the same value, -3, and so we get,

$$\lim_{x \rightarrow 1} f(x) = -3$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Therefore, even though the function doesn't exist at this point the limit can still have a value.

(d) $a = 3$

From the graph we can see that,

$$f(3) = 4$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = 3$ from both sides the graph is approaching the same value, 4, and so we get,

$$\lim_{x \rightarrow 3} f(x) = 4$$

Section 2-3 : One-Sided Limits

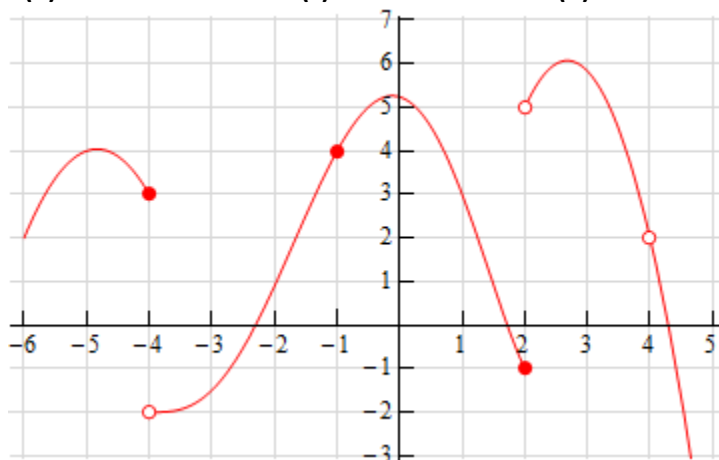
1. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$, $\lim_{x \rightarrow a^-} f(x)$, $\lim_{x \rightarrow a^+} f(x)$, and $\lim_{x \rightarrow a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -4$

(b) $a = -1$

(c) $a = 2$

(d) $a = 4$



(a) $a = -4$

From the graph we can see that,

$$f(-4) = 3$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -4$ from the left the graph is approaching a value of 3 and as we approach from the right the graph is approaching a value of -2. Therefore, we get,

$$\lim_{x \rightarrow -4^-} f(x) = 3 \quad \& \quad \lim_{x \rightarrow -4^+} f(x) = -2$$

Now, because the two one-sided limits are different we know that,

$$\lim_{x \rightarrow -4} f(x) \text{ does not exist}$$

(b) $a = -1$

From the graph we can see that,

$$f(-1) = 4$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = -1$ from both sides the graph is approaching the same value, 4, and so we get,

$$\lim_{x \rightarrow -1^-} f(x) = 4 \quad \& \quad \lim_{x \rightarrow -1^+} f(x) = 4$$

The two one-sided limits are the same so we know,

$$\lim_{x \rightarrow -1} f(x) = 4$$

(c) $a = 2$

From the graph we can see that,

$$f(2) = -1$$

because the closed dot is at the value of $y = -1$.

We can also see that as we approach $x = 2$ from the left the graph is approaching a value of -1 and as we approach from the right the graph is approaching a value of 5. Therefore, we get,

$$\lim_{x \rightarrow 2^-} f(x) = -1 \quad \& \quad \lim_{x \rightarrow 2^+} f(x) = 5$$

Now, because the two one-sided limits are different we know that,

$$\lim_{x \rightarrow 2} f(x) \text{ does not exist}$$

(d) $a = 4$

Because there is no closed dot for $x = 4$ we can see that,

$$f(4) \text{ does not exist}$$

We can also see that as we approach $x = 4$ from both sides the graph is approaching the same value, 2, and so we get,

$$\lim_{x \rightarrow 4^-} f(x) = 2 \quad \& \quad \lim_{x \rightarrow 4^+} f(x) = 2$$

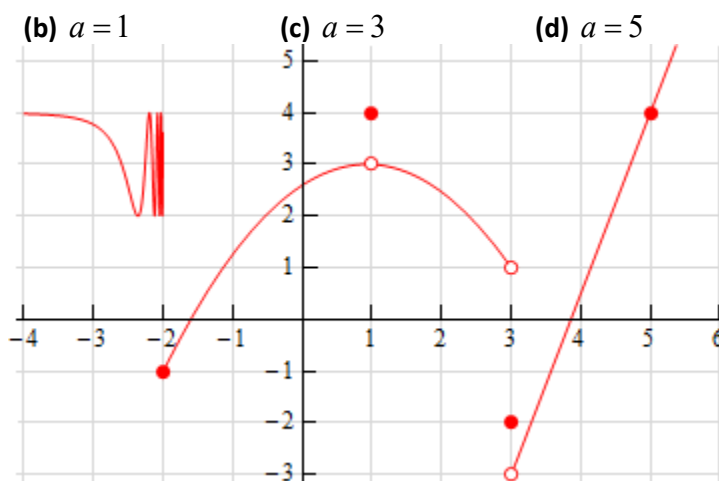
The two one-sided limits are the same so we know,

$$\lim_{x \rightarrow 4} f(x) = 2$$

Always recall that the value of a limit (including one-sided limits) does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Therefore, even though the function doesn't exist at this point the limit and one-sided limits can still have a value.

2. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$, $\lim_{x \rightarrow a^-} f(x)$, $\lim_{x \rightarrow a^+} f(x)$, and $\lim_{x \rightarrow a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -2$



(a) $a = -2$

From the graph we can see that,

$$\boxed{f(-2) = -1}$$

because the closed dot is at the value of $y = -1$.

We can also see that as we approach $x = -2$ from the left the graph is not approaching a single value, but instead oscillating wildly, and as we approach from the right the graph is approaching a value of -1. Therefore, we get,

$$\boxed{\lim_{x \rightarrow -2^-} f(x) \text{ does not exist} \quad \& \quad \lim_{x \rightarrow -2^+} f(x) = -1}$$

Recall that in order for limit to exist the function must be approaching a single value and so, in this case, because the graph to the left of $x = -2$ is not approaching a single value the left-hand limit will not exist. This does not mean that the right-hand limit will not exist. In this case the graph to the right of $x = -2$ is approaching a single value the right-hand limit will exist.

Now, because the two one-sided limits are different we know that,

$$\boxed{\lim_{x \rightarrow -2} f(x) \text{ does not exist}}$$

(b) $a = 1$

From the graph we can see that,

$$\boxed{f(1) = 4}$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = 1$ from both sides the graph is approaching the same value, 3, and so we get,

$$\lim_{x \rightarrow 1^-} f(x) = 3 \quad \& \quad \lim_{x \rightarrow 1^+} f(x) = 3$$

The two one-sided limits are the same so we know,

$$\lim_{x \rightarrow 1} f(x) = 3$$

(c) $a = 3$

From the graph we can see that,

$$f(3) = -2$$

because the closed dot is at the value of $y = -2$.

We can also see that as we approach $x = 2$ from the left the graph is approaching a value of 1 and as we approach from the right the graph is approaching a value of -3. Therefore, we get,

$$\lim_{x \rightarrow 3^-} f(x) = 1 \quad \& \quad \lim_{x \rightarrow 3^+} f(x) = -3$$

Now, because the two one-sided limits are different we know that,

$$\lim_{x \rightarrow 3} f(x) \text{ does not exist}$$

(d) $a = 5$

From the graph we can see that,

$$f(5) = 4$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = 5$ from both sides the graph is approaching the same value, 4, and so we get,

$$\lim_{x \rightarrow 5^-} f(x) = 4 \quad \& \quad \lim_{x \rightarrow 5^+} f(x) = 4$$

The two one-sided limits are the same so we know,

$$\lim_{x \rightarrow 5} f(x) = 4$$

3. Sketch a graph of a function that satisfies each of the following conditions.

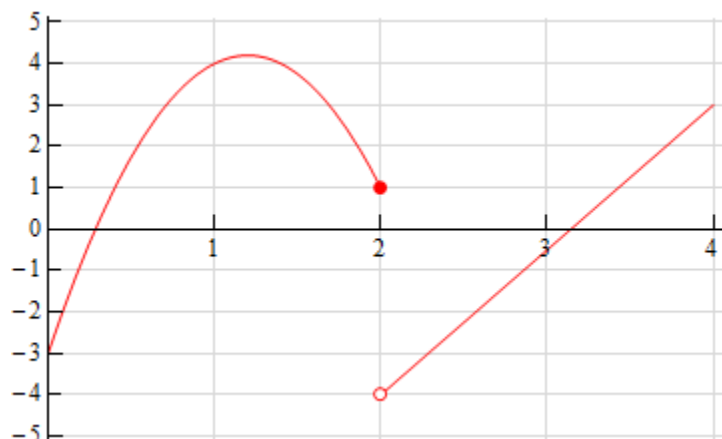
$$\lim_{x \rightarrow 2^-} f(x) = 1 \qquad \lim_{x \rightarrow 2^+} f(x) = -4 \qquad f(2) = 1$$

Solution

There are literally an infinite number of possible graphs that we could give here for an answer.

However, all of them must have a closed dot on the graph at the point $(2, 1)$, the graph must be approaching a value of 1 as it approaches $x = 2$ from the left (as indicated by the left-hand limit) and it must be approaching a value of -4 as it approaches $x = 2$ from the right (as indicated by the right-hand limit).

Here is a sketch of one possible graph that meets these conditions.



4. Sketch a graph of a function that satisfies each of the following conditions.

$$\lim_{x \rightarrow 3^-} f(x) = 0$$

$$\lim_{x \rightarrow 3^+} f(x) = 4$$

$$f(3) \text{ does not exist}$$

$$\lim_{x \rightarrow -1} f(x) = -3$$

$$f(-1) = 2$$

Solution

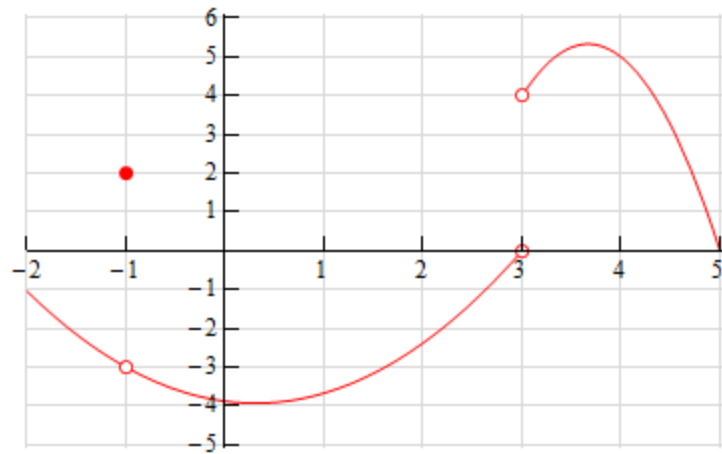
There are literally an infinite number of possible graphs that we could give here for an answer.

However, all of them must the following two sets of criteria.

First, at $x = 3$ there cannot be a closed dot on the graph (as indicated by the fact that the function does not exist here), the graph must be approaching a value of 0 as it approaches $x = 3$ from the left (as indicated by the left-hand limit) and it must be approaching a value of 4 as it approaches $x = 3$ from the right (as indicated by the right-hand limit).

Next, the graph must have a closed dot at the point $(-1, 2)$ and the graph must be approaching a value of -3 as it approaches $x = -1$ from both sides (as indicated by the fact that value of the overall limit is -3 at this point).

Here is a sketch of one possible graph that meets these conditions.



Section 2-4 : Limit Properties

1. Given $\lim_{x \rightarrow 8} f(x) = -9$, $\lim_{x \rightarrow 8} g(x) = 2$ and $\lim_{x \rightarrow 8} h(x) = 4$ use the limit properties given in this section to compute each of the following limits. If it is not possible to compute any of the limits clearly explain why not.

(a) $\lim_{x \rightarrow 8} [2f(x) - 12h(x)]$

(b) $\lim_{x \rightarrow 8} [3h(x) - 6]$

(c) $\lim_{x \rightarrow 8} [g(x)h(x) - f(x)]$

(d) $\lim_{x \rightarrow 8} [f(x) - g(x) + h(x)]$

Hint : For each of these all we need to do is use the limit properties on the limit until the given limits appear and we can then compute the value.

(a) $\lim_{x \rightarrow 8} [2f(x) - 12h(x)]$

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned} \lim_{x \rightarrow 8} [2f(x) - 12h(x)] &= \lim_{x \rightarrow 8} [2f(x)] - \lim_{x \rightarrow 8} [12h(x)] && \text{Property 2} \\ &= 2 \lim_{x \rightarrow 8} f(x) - 12 \lim_{x \rightarrow 8} h(x) && \text{Property 1} \\ &= 2(-9) - 12(4) && \text{Plug in values of limits} \\ &= \boxed{-66} \end{aligned}$$

(b) $\lim_{x \rightarrow 8} [3h(x) - 6]$

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned} \lim_{x \rightarrow 8} [3h(x) - 6] &= \lim_{x \rightarrow 8} [3h(x)] - \lim_{x \rightarrow 8} 6 && \text{Property 2} \\ &= 3 \lim_{x \rightarrow 8} h(x) - \lim_{x \rightarrow 8} 6 && \text{Property 1} \\ &= 3(4) - 6 && \text{Plug in value of limits \& Property 7} \\ &= \boxed{6} \end{aligned}$$

(c) $\lim_{x \rightarrow 8} [g(x)h(x) - f(x)]$

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
\lim_{x \rightarrow 8} [g(x)h(x) - f(x)] &= \lim_{x \rightarrow 8} [g(x)h(x)] - \lim_{x \rightarrow 8} f(x) && \text{Property 2} \\
&= \left[\lim_{x \rightarrow 8} g(x) \right] \left[\lim_{x \rightarrow 8} h(x) \right] - \lim_{x \rightarrow 8} f(x) && \text{Property 3} \\
&= (2)(4) - (-9) && \text{Plug in values of limits} \\
&= \boxed{17}
\end{aligned}$$

$$(d) \lim_{x \rightarrow 8} [f(x) - g(x) + h(x)]$$

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
\lim_{x \rightarrow 8} [f(x) - g(x) + h(x)] &= \lim_{x \rightarrow 8} f(x) - \lim_{x \rightarrow 8} g(x) + \lim_{x \rightarrow 8} h(x) && \text{Property 2} \\
&= -9 - 2 + 4 && \text{Plug in values of limits} \\
&= \boxed{-7}
\end{aligned}$$

2. Given $\lim_{x \rightarrow -4} f(x) = 1$, $\lim_{x \rightarrow -4} g(x) = 10$ and $\lim_{x \rightarrow -4} h(x) = -7$ use the limit properties given in this section to compute each of the following limits. If it is not possible to compute any of the limits clearly explain why not.

$$\begin{aligned}
(a) \lim_{x \rightarrow -4} \left[\frac{f(x)}{g(x)} - \frac{h(x)}{f(x)} \right] & \qquad (b) \lim_{x \rightarrow -4} [f(x)g(x)h(x)] \\
(c) \lim_{x \rightarrow -4} \left[\frac{1}{h(x)} + \frac{3 - f(x)}{g(x) + h(x)} \right] & \qquad (d) \lim_{x \rightarrow -4} \left[2h(x) - \frac{1}{h(x) + 7f(x)} \right]
\end{aligned}$$

Hint : For each of these all we need to do is use the limit properties on the limit until the given limits appear and we can then compute the value.

$$(a) \lim_{x \rightarrow -4} \left[\frac{f(x)}{g(x)} - \frac{h(x)}{f(x)} \right]$$

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
 \lim_{x \rightarrow -4} \left[\frac{f(x)}{g(x)} - \frac{h(x)}{f(x)} \right] &= \lim_{x \rightarrow -4} \frac{f(x)}{g(x)} - \lim_{x \rightarrow -4} \frac{h(x)}{f(x)} && \text{Property 2} \\
 &= \frac{\lim_{x \rightarrow -4} f(x)}{\lim_{x \rightarrow -4} g(x)} - \frac{\lim_{x \rightarrow -4} h(x)}{\lim_{x \rightarrow -4} f(x)} && \text{Property 4} \\
 &= \frac{1}{10} - \frac{-7}{1} && \text{Plug in values of limits} \\
 &= \boxed{\frac{71}{10}}
 \end{aligned}$$

Note that we were able to use Property 4 in the second step only because after we evaluated the limit of the denominators (both of them) we found that the limits of the denominators were not zero.

(b) $\lim_{x \rightarrow -4} [f(x)g(x)h(x)]$

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
 \lim_{x \rightarrow -4} [f(x)g(x)h(x)] &= \left[\lim_{x \rightarrow -4} f(x) \right] \left[\lim_{x \rightarrow -4} g(x) \right] \left[\lim_{x \rightarrow -4} h(x) \right] && \text{Property 3} \\
 &= (1)(10)(-7) && \text{Plug in value of limits} \\
 &= \boxed{-70}
 \end{aligned}$$

Note that the properties 2 & 3 in this section were only given with two functions but they can easily be extended out to more than two functions as we did here for property 3.

(c) $\lim_{x \rightarrow -4} \left[\frac{1}{h(x)} + \frac{3-f(x)}{g(x)+h(x)} \right]$

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
\lim_{x \rightarrow -4} \left[\frac{1}{h(x)} + \frac{3-f(x)}{g(x)+h(x)} \right] &= \lim_{x \rightarrow -4} \frac{1}{h(x)} + \lim_{x \rightarrow -4} \frac{3-f(x)}{g(x)+h(x)} && \text{Property 2} \\
&= \frac{\lim_{x \rightarrow -4} 1}{\lim_{x \rightarrow -4} h(x)} + \frac{\lim_{x \rightarrow -4} [3-f(x)]}{\lim_{x \rightarrow -4} [g(x)+h(x)]} && \text{Property 4} \\
&= \frac{\lim_{x \rightarrow -4} 1}{\lim_{x \rightarrow -4} h(x)} + \frac{\lim_{x \rightarrow -4} 3 - \lim_{x \rightarrow -4} f(x)}{\lim_{x \rightarrow -4} g(x) + \lim_{x \rightarrow -4} h(x)} && \text{Property 2} \\
&= \frac{1}{-7} + \frac{3-1}{10-7} && \text{Plug in values of limits} \\
&&& \text{\& Property 1} \\
&= \boxed{\frac{11}{21}}
\end{aligned}$$

Note that we were able to use Property 4 in the second step only because after we evaluated the limit of the denominators (both of them) we found that the limits of the denominators were not zero.

$$(d) \lim_{x \rightarrow -4} \left[2h(x) - \frac{1}{h(x)+7f(x)} \right]$$

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
\lim_{x \rightarrow -4} \left[2h(x) - \frac{1}{h(x)+7f(x)} \right] &= \lim_{x \rightarrow -4} 2h(x) - \lim_{x \rightarrow -4} \frac{1}{h(x)+7f(x)} && \text{Property 2} \\
&= \lim_{x \rightarrow -4} 2h(x) - \frac{\lim_{x \rightarrow -4} 1}{\lim_{x \rightarrow -4} [h(x)+7f(x)]} && \text{Property 4}
\end{aligned}$$

At this point let's step back a minute. In the previous parts we didn't worry about using property 4 on a rational expression. However, in this case let's be a little more careful. We can only use property 4 if the limit of the denominator is not zero. Let's check that limit and see what we get.

$$\begin{aligned}
\lim_{x \rightarrow -4} [h(x) + 7f(x)] &= \lim_{x \rightarrow -4} h(x) + \lim_{x \rightarrow -4} [7f(x)] && \text{Property 2} \\
&= \lim_{x \rightarrow -4} h(x) + 7 \lim_{x \rightarrow -4} f(x) && \text{Property 1} \\
&= -7 + 7(1) && \text{Plug in values of limits \& Property 1} \\
&= 0
\end{aligned}$$

Okay, we can see that the limit of the denominator in the second term will be zero so we cannot actually use property 4 on that term. This means that this limit cannot be done and note that the fact that we

could determine a value for the limit of the first term will not change this fact. This limit cannot be done.

3. Given $\lim_{x \rightarrow 0} f(x) = 6$, $\lim_{x \rightarrow 0} g(x) = -4$ and $\lim_{x \rightarrow 0} h(x) = -1$ use the limit properties given in this section to compute each of the following limits. If it is not possible to compute any of the limits clearly explain why not.

(a) $\lim_{x \rightarrow 0} [f(x) + h(x)]^3$

(b) $\lim_{x \rightarrow 0} \sqrt{g(x)h(x)}$

(c) $\lim_{x \rightarrow 0} \sqrt[3]{11 + [g(x)]^2}$

(d) $\lim_{x \rightarrow 0} \sqrt{\frac{f(x)}{h(x) - g(x)}}$

Hint : For each of these all we need to do is use the limit properties on the limit until the given limits appear and we can then compute the value.

(a) $\lim_{x \rightarrow 0} [f(x) + h(x)]^3$

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned} \lim_{x \rightarrow 0} [f(x) + h(x)]^3 &= \left[\lim_{x \rightarrow 0} (f(x) + h(x)) \right]^3 && \text{Property 5} \\ &= \left[\lim_{x \rightarrow 0} f(x) + \lim_{x \rightarrow 0} h(x) \right]^3 && \text{Property 2} \\ &= [6 - 1]^3 && \text{Plug in values of limits} \\ &= \boxed{125} \end{aligned}$$

(b) $\lim_{x \rightarrow 0} \sqrt{g(x)h(x)}$

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned} \lim_{x \rightarrow 0} \sqrt{g(x)h(x)} &= \sqrt{\lim_{x \rightarrow 0} g(x)h(x)} && \text{Property 6} \\ &= \sqrt{\left[\lim_{x \rightarrow 0} g(x) \right] \left[\lim_{x \rightarrow 0} h(x) \right]} && \text{Property 3} \\ &= \sqrt{(-4)(-1)} && \text{Plug in value of limits} \\ &= \boxed{2} \end{aligned}$$

(c) $\lim_{x \rightarrow 0} \sqrt[3]{11 + [g(x)]^2}$

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
 \lim_{x \rightarrow 0} \sqrt[3]{11 + [g(x)]^2} &= \sqrt[3]{\lim_{x \rightarrow 0} (11 + [g(x)]^2)} && \text{Property 6} \\
 &= \sqrt[3]{\lim_{x \rightarrow 0} 11 + \lim_{x \rightarrow 0} [g(x)]^2} && \text{Property 2} \\
 &= \sqrt[3]{\lim_{x \rightarrow 0} 11 + \left[\lim_{x \rightarrow 0} g(x) \right]^2} && \text{Property 5} \\
 &= \sqrt[3]{11 + (-4)^2} && \text{Plug in values of limits \& Property 7} \\
 &= \boxed{3}
 \end{aligned}$$

(d) $\lim_{x \rightarrow 0} \sqrt{\frac{f(x)}{h(x) - g(x)}}$

Here is the work for this limit. At each step the property (or properties) used are listed and note that in some cases the properties may have been used more than once in the indicated step.

$$\begin{aligned}
 \lim_{x \rightarrow 0} \sqrt{\frac{f(x)}{h(x) - g(x)}} &= \sqrt{\lim_{x \rightarrow 0} \frac{f(x)}{h(x) - g(x)}} && \text{Property 6} \\
 &= \sqrt{\frac{\lim_{x \rightarrow 0} f(x)}{\lim_{x \rightarrow 0} (h(x) - g(x))}} && \text{Property 4} \\
 &= \sqrt{\frac{\lim_{x \rightarrow 0} f(x)}{\lim_{x \rightarrow 0} h(x) - \lim_{x \rightarrow 0} g(x)}} && \text{Property 2} \\
 &= \sqrt{\frac{6}{-1 - (-4)}} && \text{Plug in values of limits} \\
 &= \sqrt{2}
 \end{aligned}$$

Note that we were able to use Property 4 in the second step only because after we evaluated the limit of the denominators (both of them) we found that the limits of the denominators were not zero.

4. Use the limit properties given in this section to compute the following limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

$$\lim_{t \rightarrow -2} (14 - 6t + t^3)$$

Hint : All we need to do is use the limit properties on the limit until we can use Properties 7, 8 and/or 9 from this section to compute the limit.

$$\begin{aligned}
 \lim_{t \rightarrow -2} (14 - 6t + t^3) &= \lim_{t \rightarrow -2} 14 - \lim_{t \rightarrow -2} 6t + \lim_{t \rightarrow -2} t^3 && \text{Property 2} \\
 &= \lim_{t \rightarrow -2} 14 - 6 \lim_{t \rightarrow -2} t + \lim_{t \rightarrow -2} t^3 && \text{Property 1} \\
 &= 14 - 6(-2) + (-2)^3 && \text{Properties 7, 8, \& 9} \\
 &= \boxed{18}
 \end{aligned}$$

5. Use the limit properties given in this section to compute the following limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

$$\lim_{x \rightarrow 6} (3x^2 + 7x - 16)$$

Hint : All we need to do is use the limit properties on the limit until we can use Properties 7, 8 and/or 9 from this section to compute the limit.

$$\begin{aligned}
 \lim_{x \rightarrow 6} (3x^2 + 7x - 16) &= \lim_{x \rightarrow 6} 3x^2 + \lim_{x \rightarrow 6} 7x - \lim_{x \rightarrow 6} 16 && \text{Property 2} \\
 &= 3 \lim_{x \rightarrow 6} x^2 + 7 \lim_{x \rightarrow 6} x - \lim_{x \rightarrow 6} 16 && \text{Property 1} \\
 &= 3(6^2) + 7(6) - 16 && \text{Properties 7, 8, \& 9} \\
 &= \boxed{134}
 \end{aligned}$$

6. Use the limit properties given in this section to compute the following limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

$$\lim_{w \rightarrow 3} \frac{w^2 - 8w}{4 - 7w}$$

Hint : All we need to do is use the limit properties on the limit until we can use Properties 7, 8 and/or 9 from this section to compute the limit.

$$\begin{aligned}
\lim_{w \rightarrow 3} \frac{w^2 - 8w}{4 - 7w} &= \frac{\lim_{w \rightarrow 3} (w^2 - 8w)}{\lim_{w \rightarrow 3} (4 - 7w)} && \text{Property 4} \\
&= \frac{\lim_{w \rightarrow 3} w^2 - \lim_{w \rightarrow 3} 8w}{\lim_{w \rightarrow 3} 4 - \lim_{w \rightarrow 3} 7w} && \text{Property 2} \\
&= \frac{\lim_{w \rightarrow 3} w^2 - 8 \lim_{w \rightarrow 3} w}{\lim_{w \rightarrow 3} 4 - 7 \lim_{w \rightarrow 3} w} && \text{Property 1} \\
&= \frac{3^2 - 8(3)}{4 - 7(3)} && \text{Properties 7, 8, \& 9} \\
&= \boxed{\frac{15}{17}}
\end{aligned}$$

Note that we were able to use property 4 in the first step because after evaluating the limit in the denominator we found that it wasn't zero.

7. Use the limit properties given in this section to compute the following limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

$$\lim_{x \rightarrow -5} \frac{x + 7}{x^2 + 3x - 10}$$

Hint : All we need to do is use the limit properties on the limit until we can use Properties 7, 8 and/or 9 from this section to compute the limit.

$$\lim_{x \rightarrow -5} \frac{x + 7}{x^2 + 3x - 10} = \frac{\lim_{x \rightarrow -5} (x + 7)}{\lim_{x \rightarrow -5} (x^2 + 3x - 10)} \quad \text{Property 4}$$

Okay, at this point let's step back a minute. We used property 4 here and we know that we can only do that if the limit of the denominator is not zero. So, let's check that out and see what we get.

$$\begin{aligned}
\lim_{x \rightarrow -5} (x^2 + 3x - 10) &= \lim_{x \rightarrow -5} x^2 + \lim_{x \rightarrow -5} 3x - \lim_{x \rightarrow -5} 10 && \text{Property 2} \\
&= \lim_{x \rightarrow -5} x^2 + 3 \lim_{x \rightarrow -5} x - \lim_{x \rightarrow -5} 10 && \text{Property 1} \\
&= (-5)^2 + 3(-5) - 10 && \text{Properties 7, 8, \& 9} \\
&= 0
\end{aligned}$$

So, the limit of the denominator is zero so we couldn't use property 4 in this case. Therefore, we cannot do this limit at this point (note that it will be possible to do this limit after the next section).

8. Use the limit properties given in this section to compute the following limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

$$\lim_{z \rightarrow 0} \sqrt{z^2 + 6}$$

Hint : All we need to do is use the limit properties on the limit until we can use Properties 7, 8 and/or 9 from this section to compute the limit.

$$\begin{aligned} \lim_{z \rightarrow 0} \sqrt{z^2 + 6} &= \sqrt{\lim_{z \rightarrow 0} (z^2 + 6)} && \text{Property 6} \\ &= \sqrt{\lim_{z \rightarrow 0} z^2 + \lim_{z \rightarrow 0} 6} && \text{Property 2} \\ &= \sqrt{0^2 + 6} && \text{Properties 7 \& 9} \\ &= \sqrt{6} \end{aligned}$$

9. Use the limit properties given in this section to compute the following limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

$$\lim_{x \rightarrow 10} (4x + \sqrt[3]{x-2})$$

Hint : All we need to do is use the limit properties on the limit until we can use Properties 7, 8 and/or 9 from this section to compute the limit.

$$\begin{aligned} \lim_{x \rightarrow 10} (4x + \sqrt[3]{x-2}) &= \lim_{x \rightarrow 10} 4x + \lim_{x \rightarrow 10} \sqrt[3]{x-2} && \text{Property 2} \\ &= \lim_{x \rightarrow 10} 4x + \sqrt[3]{\lim_{x \rightarrow 10} (x-2)} && \text{Property 6} \\ &= \lim_{x \rightarrow 10} 4x + \sqrt[3]{\lim_{x \rightarrow 10} x - \lim_{x \rightarrow 10} 2} && \text{Property 2} \\ &= 4 \lim_{x \rightarrow 10} x + \sqrt[3]{\lim_{x \rightarrow 10} x - \lim_{x \rightarrow 10} 2} && \text{Property 1} \\ &= 4(10) + \sqrt[3]{10-2} && \text{Properties 7 \& 8} \\ &= 42 \end{aligned}$$

Section 2-5 : Computing Limits

1. Evaluate $\lim_{x \rightarrow 2} (8 - 3x + 12x^2)$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. We know that the first thing that we should try to do is simply plug in the value and see if we can compute the limit.

$$\lim_{x \rightarrow 2} (8 - 3x + 12x^2) = 8 - 3(2) + 12(4) = \boxed{50}$$

2. Evaluate $\lim_{t \rightarrow -3} \frac{6 + 4t}{t^2 + 1}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. We know that the first thing that we should try to do is simply plug in the value and see if we can compute the limit.

$$\lim_{t \rightarrow -3} \frac{6 + 4t}{t^2 + 1} = \frac{-6}{10} = \boxed{-\frac{3}{5}}$$

3. Evaluate $\lim_{x \rightarrow -5} \frac{x^2 - 25}{x^2 + 2x - 15}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get 0/0. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{x \rightarrow -5} \frac{x^2 - 25}{x^2 + 2x - 15} = \lim_{x \rightarrow -5} \frac{(x-5)(x+5)}{(x-3)(x+5)} = \lim_{x \rightarrow -5} \frac{x-5}{x-3} = \boxed{\frac{5}{4}}$$

4. Evaluate $\lim_{z \rightarrow 8} \frac{2z^2 - 17z + 8}{8 - z}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get $0/0$. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{z \rightarrow 8} \frac{2z^2 - 17z + 8}{8 - z} = \lim_{z \rightarrow 8} \frac{(2z-1)(z-8)}{-(z-8)} = \lim_{z \rightarrow 8} \frac{2z-1}{-1} = \boxed{-15}$$

5. Evaluate $\lim_{y \rightarrow 7} \frac{y^2 - 4y - 21}{3y^2 - 17y - 28}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get $0/0$. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{y \rightarrow 7} \frac{y^2 - 4y - 21}{3y^2 - 17y - 28} = \lim_{y \rightarrow 7} \frac{(y-7)(y+3)}{(3y+4)(y-7)} = \lim_{y \rightarrow 7} \frac{y+3}{3y+4} = \frac{10}{25} = \boxed{\frac{2}{5}}$$

6. Evaluate $\lim_{h \rightarrow 0} \frac{(6+h)^2 - 36}{h}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get $0/0$. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. All we need to do here is some simplification and then we'll reach a point where we can plug in the value.

$$\lim_{h \rightarrow 0} \frac{(6+h)^2 - 36}{h} = \lim_{h \rightarrow 0} \frac{36 + 12h + h^2 - 36}{h} = \lim_{h \rightarrow 0} \frac{h(12+h)}{h} = \lim_{h \rightarrow 0} (12+h) = \boxed{12}$$

7. Evaluate $\lim_{z \rightarrow 4} \frac{\sqrt{z} - 2}{z - 4}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get $0/0$. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. If you're really good at factoring you can factor this and simplify. Another method that can be used however is to rationalize the numerator, so let's do that for this problem.

$$\lim_{z \rightarrow 4} \frac{\sqrt{z} - 2}{z - 4} = \lim_{z \rightarrow 4} \frac{(\sqrt{z} - 2)(\sqrt{z} + 2)}{(z - 4)(\sqrt{z} + 2)} = \lim_{z \rightarrow 4} \frac{z - 4}{(z - 4)(\sqrt{z} + 2)} = \lim_{z \rightarrow 4} \frac{1}{\sqrt{z} + 2} = \boxed{\frac{1}{4}}$$

8. Evaluate $\lim_{x \rightarrow -3} \frac{\sqrt{2x+22} - 4}{x+3}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get $0/0$. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. Simply factoring will not do us much good here so in this case it looks like we'll need to rationalize the numerator.

$$\begin{aligned} \lim_{x \rightarrow -3} \frac{\sqrt{2x+22} - 4}{x+3} &= \lim_{x \rightarrow -3} \frac{(\sqrt{2x+22} - 4)(\sqrt{2x+22} + 4)}{(x+3)(\sqrt{2x+22} + 4)} = \lim_{x \rightarrow -3} \frac{2x+22-16}{(x+3)(\sqrt{2x+22} + 4)} \\ &= \lim_{x \rightarrow -3} \frac{2(x+3)}{(x+3)(\sqrt{2x+22} + 4)} = \lim_{x \rightarrow -3} \frac{2}{\sqrt{2x+22} + 4} = \frac{2}{8} = \boxed{\frac{1}{4}} \end{aligned}$$

9. Evaluate $\lim_{x \rightarrow 0} \frac{x}{3 - \sqrt{x+9}}$, if it exists.

Solution

There is not really a lot to this problem. Simply recall the basic ideas for computing limits that we looked at in this section. In this case we see that if we plug in the value we get $0/0$. Recall that this DOES NOT mean that the limit doesn't exist. We'll need to do some more work before we make that conclusion. Simply factoring will not do us much good here so in this case it looks like we'll need to rationalize the denominator.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x}{3 - \sqrt{x+9}} &= \lim_{x \rightarrow 0} \frac{x}{(3 - \sqrt{x+9})(3 + \sqrt{x+9})} \frac{(3 + \sqrt{x+9})}{(3 + \sqrt{x+9})} = \lim_{x \rightarrow 0} \frac{x(3 + \sqrt{x+9})}{9 - (x+9)} \\ &= \lim_{x \rightarrow 0} \frac{x(3 + \sqrt{x+9})}{-x} = \lim_{x \rightarrow 0} \frac{3 + \sqrt{x+9}}{-1} = \boxed{-6}\end{aligned}$$

10. Given the function

$$f(x) = \begin{cases} 7 - 4x & x < 1 \\ x^2 + 2 & x \geq 1 \end{cases}$$

Evaluate the following limits, if they exist.

(a) $\lim_{x \rightarrow -6} f(x)$ (b) $\lim_{x \rightarrow 1} f(x)$

Hint : Recall that when looking at overall limits (as opposed to one-sided limits) we need to make sure that the value of the function must be approaching the same value from both sides. In other words, the two one sided limits must both exist and be equal.

(a) $\lim_{x \rightarrow -6} f(x)$ Solution

For this part we know that $-6 < 1$ and so there will be values of x on both sides of -6 in the range $x < 1$ and so we can assume that, in the limit, we will have $x < 1$. This will allow us to use the piece of the function in that range and then just use standard limit techniques to compute the limit.

$$\lim_{x \rightarrow -6} f(x) = \lim_{x \rightarrow -6} (7 - 4x) = \boxed{31}$$

(b) $\lim_{x \rightarrow 1} f(x)$ Solution

This part is going to be different from the previous part. We are looking at the limit at $x = 1$ and that is the “cut-off” point in the piecewise functions. Recall from the discussion in the section, that this means that we are going to have to look at the two one sided limits.

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (7 - 4x) = \underline{3} && \text{because } x \rightarrow 1^- \text{ implies that } x < 1 \\ \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} (x^2 + 2) = \underline{3} && \text{because } x \rightarrow 1^+ \text{ implies that } x > 1\end{aligned}$$

So, in this case, we can see that,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 3$$

and so we know that the overall limit must exist and,

$$\lim_{x \rightarrow 1} f(x) = \boxed{3}$$

11. Given the function

$$h(z) = \begin{cases} 6z & z \leq -4 \\ 1-9z & z > -4 \end{cases}$$

Evaluate the following limits, if they exist.

(a) $\lim_{z \rightarrow 7} h(z)$ (b) $\lim_{z \rightarrow -4} h(z)$

Hint : Recall that when looking at overall limits (as opposed to one-sided limits) we need to make sure that the value of the function must be approaching the same value from both sides. In other words, the two one sided limits must both exist and be equal.

(a) $\lim_{z \rightarrow 7} h(z)$ Solution

For this part we know that $7 > -4$ and so there will be values of z on both sides of 7 in the range $z > -4$ and so we can assume that, in the limit, we will have $z > -4$. This will allow us to use the piece of the function in that range and then just use standard limit techniques to compute the limit.

$$\lim_{z \rightarrow 7} h(z) = \lim_{z \rightarrow 7} (1-9z) = \boxed{-62}$$

(b) $\lim_{z \rightarrow -4} h(z)$ Solution

This part is going to be different from the previous part. We are looking at the limit at $z = -4$ and that is the “cut-off” point in the piecewise functions. Recall from the discussion in the section, that this means that we are going to have to look at the two one sided limits.

$$\lim_{z \rightarrow -4^-} h(z) = \lim_{z \rightarrow -4^-} 6z = \underline{-24} \quad \text{because } z \rightarrow -4^- \text{ implies that } z < -4$$

$$\lim_{z \rightarrow -4^+} h(z) = \lim_{z \rightarrow -4^+} (1-9z) = \underline{37} \quad \text{because } z \rightarrow -4^+ \text{ implies that } z > -4$$

So, in this case, we can see that,

$$\lim_{z \rightarrow -4^-} h(z) = -24 \neq 37 = \lim_{z \rightarrow -4^+} h(z)$$

and so we know that the overall limit **does not exist**.

12. Evaluate $\lim_{x \rightarrow 5} (10 + |x-5|)$, if it exists.

Hint : Recall the mathematical definition of the absolute value function and that it is in fact a piecewise function.

Solution

Recall the definition of the absolute value function.

$$|p| = \begin{cases} p & p \geq 0 \\ -p & p < 0 \end{cases}$$

So, because the function inside the absolute value is zero at $x = 5$ we can see that,

$$|x-5| = \begin{cases} x-5 & x \geq 5 \\ -(x-5) & x < 5 \end{cases}$$

This means that we are being asked to compute the limit at the “cut-off” point in a piecewise function and so, as we saw in this section, we’ll need to look at two one-sided limits in order to determine if this limit exists (and its value if it does exist).

$$\lim_{x \rightarrow 5^-} (10 + |x-5|) = \lim_{x \rightarrow 5^-} (10 - (x-5)) = \lim_{x \rightarrow 5^-} (15-x) = 10 \quad \text{recall } x \rightarrow 5^- \text{ implies } x < 5$$

$$\lim_{x \rightarrow 5^+} (10 + |x-5|) = \lim_{x \rightarrow 5^+} (10 + (x-5)) = \lim_{x \rightarrow 5^+} (5+x) = 10 \quad \text{recall } x \rightarrow 5^+ \text{ implies } x > 5$$

So, for this problem, we can see that,

$$\lim_{x \rightarrow 5} (10 + |x-5|) = \lim_{x \rightarrow 5^+} (10 + |x-5|) = 10$$

and so the overall limit must exist and,

$$\lim_{x \rightarrow 5} (10 + |x-5|) = \boxed{10}$$

13. Evaluate $\lim_{t \rightarrow -1} \frac{t+1}{|t+1|}$, if it exists.

Hint : Recall the mathematical definition of the absolute value function and that it is in fact a piecewise function.

Solution

Recall the definition of the absolute value function.

$$|p| = \begin{cases} p & p \geq 0 \\ -p & p < 0 \end{cases}$$

So, because the function inside the absolute value is zero at $t = -1$ we can see that,

$$|t+1| = \begin{cases} t+1 & t \geq -1 \\ -(t+1) & t < -1 \end{cases}$$

This means that we are being asked to compute the limit at the “cut-off” point in a piecewise function and so, as we saw in this section, we’ll need to look at two one-sided limits in order to determine if this limit exists (and its value if it does exist).

$$\lim_{t \rightarrow -1^-} \frac{t+1}{|t+1|} = \lim_{t \rightarrow -1^-} \frac{t+1}{-(t+1)} = \lim_{t \rightarrow -1^-} -1 = -1 \quad \text{recall } t \rightarrow -1^- \text{ implies } t < -1$$

$$\lim_{t \rightarrow -1^+} \frac{t+1}{|t+1|} = \lim_{t \rightarrow -1^+} \frac{t+1}{t+1} = \lim_{t \rightarrow -1^+} 1 = 1 \quad \text{recall } t \rightarrow -1^+ \text{ implies } t > -1$$

So, for this problem, we can see that,

$$\lim_{t \rightarrow -1^-} \frac{t+1}{|t+1|} = -1 \neq 1 = \lim_{t \rightarrow -1^+} \frac{t+1}{|t+1|}$$

and so the overall limit **does not exist**.

14. Given that $7x \leq f(x) \leq 3x^2 + 2$ for all x determine the value of $\lim_{x \rightarrow 2} f(x)$.

Hint : Recall the Squeeze Theorem.

Solution

This problem is set up to use the Squeeze Theorem. First, we already know that $f(x)$ is always between two other functions. Now all that we need to do is verify that the two “outer” functions have the same limit at $x = 2$ and if they do we can use the Squeeze Theorem to get the answer.

$$\lim_{x \rightarrow 2} 7x = 14$$

$$\lim_{x \rightarrow 2} (3x^2 + 2) = 14$$

So, we have,

$$\lim_{x \rightarrow 2} 7x = \lim_{x \rightarrow 2} (3x^2 + 2) = 14$$

and so by the Squeeze Theorem we must also have,

$$\lim_{x \rightarrow 2} f(x) = 14$$

15. Use the Squeeze Theorem to determine the value of $\lim_{x \rightarrow 0} x^4 \sin\left(\frac{\pi}{x}\right)$.

Hint : Recall how we worked the Squeeze Theorem problem in this section to find the lower and upper functions we need in order to use the Squeeze Theorem.

Solution

We first need to determine lower/upper functions. We'll start off by acknowledging that provided $x \neq 0$ (which we know it won't be because we are looking at the limit as $x \rightarrow 0$) we will have,

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

Now, simply multiply through this by x^4 to get,

$$-x^4 \leq x^4 \sin\left(\frac{\pi}{x}\right) \leq x^4$$

Before proceeding note that we can only do this because we know that $x^4 > 0$ for $x \neq 0$. Recall that if we multiply through an inequality by a negative number we would have had to switch the signs. So, for

instance, had we multiplied through by x^3 we would have had issues because this is positive if $x > 0$ and negative if $x < 0$.

Now, let's get back to the problem. We have a set of lower/upper functions and clearly,

$$\lim_{x \rightarrow 0} x^4 = \lim_{x \rightarrow 0} (-x^4) = 0$$

Therefore, by the Squeeze Theorem we must have,

$$\lim_{x \rightarrow 0} x^4 \sin\left(\frac{\pi}{x}\right) = 0$$

Section 2-6 : Infinite Limits

1. For $f(x) = \frac{9}{(x-3)^5}$ evaluate the indicated limits, if they exist.

(a) $\lim_{x \rightarrow 3^-} f(x)$

(b) $\lim_{x \rightarrow 3^+} f(x)$

(c) $\lim_{x \rightarrow 3} f(x)$

(a) $\lim_{x \rightarrow 3^-} f(x)$

Let's start off by acknowledging that for $x \rightarrow 3^-$ we know $x < 3$.

For the numerator we can see that, in the limit, it will just be 9.

The denominator takes a little more work. Clearly, in the limit, we have,

$$x - 3 \rightarrow 0$$

but we can actually go a little farther. Because we know that $x < 3$ we also know that,

$$x - 3 < 0$$

More compactly, we can say that in the limit we will have,

$$x - 3 \rightarrow 0^-$$

Raising this to the fifth power will not change this behavior and so, in the limit, the denominator will be,

$$(x - 3)^5 \rightarrow 0^-$$

We can now do the limit of the function. In the limit, the numerator is a fixed positive constant and the denominator is an increasingly small negative number. In the limit, the quotient must then be an increasing large negative number or,

$$\boxed{\lim_{x \rightarrow 3^-} \frac{9}{(x-3)^5} = -\infty}$$

Note that this also means that there is a vertical asymptote at $x = 3$.

(b) $\lim_{x \rightarrow 3^+} f(x)$

Let's start off by acknowledging that for $x \rightarrow 3^+$ we know $x > 3$.

As in the first part the numerator, in the limit, it will just be 9.

The denominator will also work similarly to the first part. In the limit, we have,

$$x - 3 \rightarrow 0$$

and because we know that $x > 3$ we also know that,

$$x - 3 > 0$$

More compactly, we can say that in the limit we will have,

$$x - 3 \rightarrow 0^+$$

Raising this to the fifth power will not change this behavior and so, in the limit, the denominator will be,

$$(x - 3)^5 \rightarrow 0^+$$

We can now do the limit of the function. In the limit, the numerator is a fixed positive constant and the denominator is an increasingly small positive number. In the limit, the quotient must then be an increasing large positive number or,

$$\lim_{x \rightarrow 3^+} \frac{9}{(x - 3)^5} = \infty$$

Note that this also means that there is a vertical asymptote at $x = 3$, which we already knew from the first part.

(c) $\lim_{x \rightarrow 3} f(x)$

In this case we can see from the first two parts that,

$$\lim_{x \rightarrow 3^-} f(x) \neq \lim_{x \rightarrow 3^+} f(x)$$

and so, from our basic limit properties we can see that $\lim_{x \rightarrow 3} f(x)$ **does not exist**.

For the sake of completeness and to verify the answers for this problem here is a quick sketch of the function.



2. For $h(t) = \frac{2t}{6+t}$ evaluate the indicated limits, if they exist.

$$(a) \lim_{t \rightarrow -6^-} h(t)$$

$$(b) \lim_{t \rightarrow -6^+} h(t)$$

$$(c) \lim_{t \rightarrow -6} h(t)$$

$$(a) \lim_{t \rightarrow -6^-} h(t)$$

Let's start off by acknowledging that for $t \rightarrow -6^-$ we know $t < -6$.

For the numerator we can see that, in the limit, we will get -12.

The denominator takes a little more work. Clearly, in the limit, we have,

$$6 + t \rightarrow 0$$

but we can actually go a little farther. Because we know that $t < -6$ we also know that,

$$6 + t < 0$$

More compactly, we can say that in the limit we will have,

$$6 + t \rightarrow 0^-$$

So, in the limit, the numerator is approaching a negative number and the denominator is an increasingly small negative number. The quotient must then be an increasing large positive number or,

$$\boxed{\lim_{t \rightarrow -6^-} \frac{2t}{6+t} = \infty}$$

Note that this also means that there is a vertical asymptote at $t = -6$.

$$(b) \lim_{t \rightarrow -6^+} h(t)$$

Let's start off by acknowledging that for $t \rightarrow -6^+$ we know $t > -6$.

For the numerator we can see that, in the limit, we will get -12.

The denominator will also work similarly to the first part. In the limit, we have,

$$6 + t \rightarrow 0$$

but we can actually go a little farther. Because we know that $t > -6$ we also know that,

$$6 + t > 0$$

More compactly, we can say that in the limit we will have,

$$6 + t \rightarrow 0^+$$

So, in the limit, the numerator is approaching a negative number and the denominator is an increasingly small positive number. The quotient must then be an increasing large negative number or,

$$\boxed{\lim_{t \rightarrow -6^+} \frac{2t}{6+t} = -\infty}$$

Note that this also means that there is a vertical asymptote at $t = -6$, which we already knew from the first part.

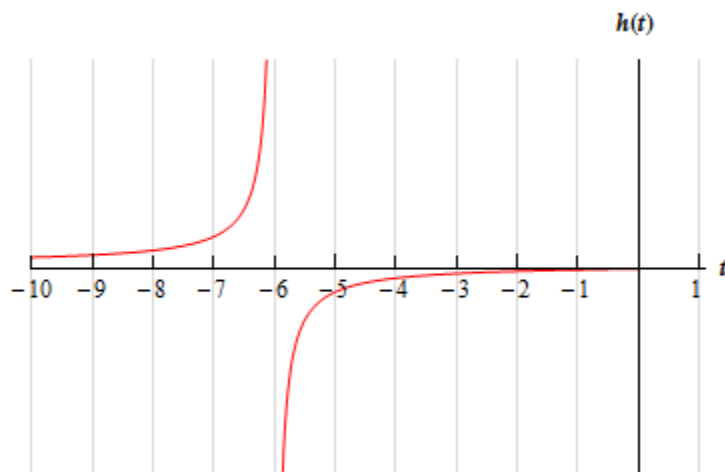
(c) $\lim_{t \rightarrow -6} h(t)$

In this case we can see from the first two parts that,

$$\lim_{t \rightarrow -6^-} h(t) \neq \lim_{t \rightarrow -6^+} h(t)$$

and so, from our basic limit properties we can see that $\lim_{t \rightarrow -6} h(t)$ **does not exist**.

For the sake of completeness and to verify the answers for this problem here is a quick sketch of the function.



3. For $g(z) = \frac{z+3}{(z+1)^2}$ evaluate the indicated limits, if they exist.

(a) $\lim_{z \rightarrow -1^-} g(z)$

(b) $\lim_{z \rightarrow -1^+} g(z)$

(c) $\lim_{z \rightarrow -1} g(z)$

(a) $\lim_{z \rightarrow -1^-} g(z)$

Let's start off by acknowledging that for $z \rightarrow -1^-$ we know $z < -1$.

For the numerator we can see that, in the limit, we will get 2.

Now let's take care of the denominator. In the limit, we will have,

$$z+1 \rightarrow 0^-$$

and upon squaring the $z+1$ we see that, in the limit, we will have,

$$(z+1)^2 \rightarrow 0^+$$

So, in the limit, the numerator is approaching a positive number and the denominator is an increasingly small positive number. The quotient must then be an increasing large positive number or,

$$\lim_{z \rightarrow -1^-} \frac{z+3}{(z+1)^2} = \infty$$

Note that this also means that there is a vertical asymptote at $z = -1$.

(b) $\lim_{z \rightarrow -1^+} g(z)$

Let's start off by acknowledging that for $z \rightarrow -1^+$ we know $z > -1$.

For the numerator we can see that, in the limit, we will get 2.

Now let's take care of the denominator. In the limit, we will have,

$$z+1 \rightarrow 0^+$$

and upon squaring the $z+1$ we see that, in the limit, we will have,

$$(z+1)^2 \rightarrow 0^+$$

So, in the limit, the numerator is approaching a positive number and the denominator is an increasingly small positive number. The quotient must then be an increasing large positive number or,

$$\lim_{z \rightarrow -1^+} \frac{z+3}{(z+1)^2} = \infty$$

Note that this also means that there is a vertical asymptote at $z = -1$, which we already knew from the first part.

(c) $\lim_{z \rightarrow -1} g(z)$

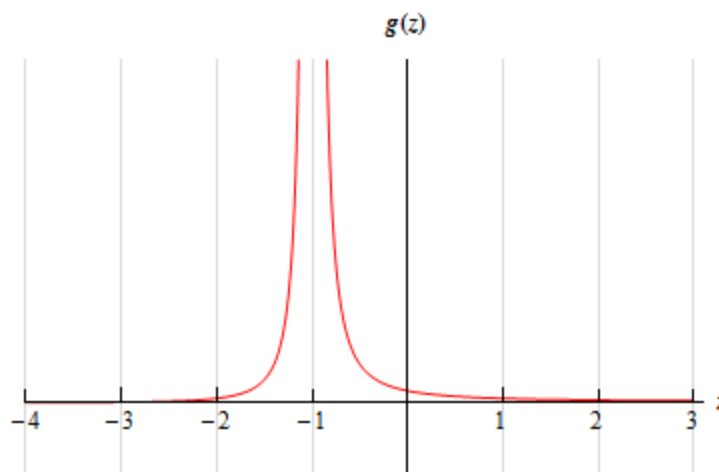
In this case we can see from the first two parts that,

$$\lim_{z \rightarrow -1^-} g(z) = \lim_{z \rightarrow -1^+} g(z) = \infty$$

and so, from our basic limit properties we can see that,

$$\lim_{z \rightarrow -1} g(z) = \infty$$

For the sake of completeness and to verify the answers for this problem here is a quick sketch of the function.



4. For $g(x) = \frac{x+7}{x^2-4}$ evaluate the indicated limits, if they exist.

(a) $\lim_{x \rightarrow 2^-} g(x)$

(b) $\lim_{x \rightarrow 2^+} g(x)$

(c) $\lim_{x \rightarrow 2} g(x)$

(a) $\lim_{x \rightarrow 2^-} g(x)$

Let's start off by acknowledging that for $x \rightarrow 2^-$ we know $x < 2$.

For the numerator we can see that, in the limit, we will get 9.

Now let's take care of the denominator. First, we know that if we square a number less than 2 (and greater than -2, which it is safe to assume we have here because we're doing the limit) we will get a number that is less than 4 and so, in the limit, we will have,

$$x^2 - 4 \rightarrow 0^-$$

So, in the limit, the numerator is approaching a positive number and the denominator is an increasingly small negative number. The quotient must then be an increasing large negative number or,

$$\boxed{\lim_{x \rightarrow 2^-} \frac{x+7}{x^2-4} = -\infty}$$

Note that this also means that there is a vertical asymptote at $x = 2$.

(b) $\lim_{x \rightarrow 2^+} g(x)$

Let's start off by acknowledging that for $x \rightarrow 2^+$ we know $x > 2$.

For the numerator we can see that, in the limit, we will get 9.

Now let's take care of the denominator. First, we know that if we square a number greater than 2 we will get a number that is greater than 4 and so, in the limit, we will have,

$$x^2 - 4 \rightarrow 0^+$$

So, in the limit, the numerator is approaching a positive number and the denominator is an increasingly small positive number. The quotient must then be an increasing large positive number or,

$$\lim_{x \rightarrow 2^+} \frac{x+7}{x^2-4} = \infty$$

Note that this also means that there is a vertical asymptote at $x = 2$, which we already knew from the first part.

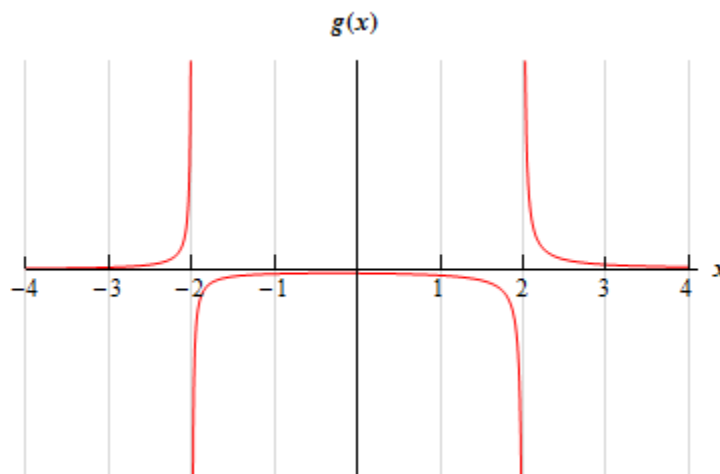
(c) $\lim_{x \rightarrow 2} g(x)$

In this case we can see from the first two parts that,

$$\lim_{x \rightarrow 2^-} g(x) \neq \lim_{x \rightarrow 2^+} g(x)$$

and so, from our basic limit properties we can see that $\lim_{x \rightarrow 2} g(x)$ **does not exist**.

For the sake of completeness and to verify the answers for this problem here is a quick sketch of the function.



As we're sure that you had already noticed there would be another vertical asymptote at $x = -2$ for this function. For the practice you might want to make sure that you can also do the limits for that point.

5. For $h(x) = \ln(-x)$ evaluate the indicated limits, if they exist.

(a) $\lim_{x \rightarrow 0^-} h(x)$

(b) $\lim_{x \rightarrow 0^+} h(x)$

(c) $\lim_{x \rightarrow 0} h(x)$

Hint : Do not get excited about the $-x$ inside the logarithm. Just recall what you know about natural logarithms, where they exist and don't exist and the limits of the natural logarithm at $x = 0$.

(a) $\lim_{x \rightarrow 0^-} h(x)$

Okay, let's start off by acknowledging that for $x \rightarrow 0^-$ we know $x < 0$ and so $-x > 0$ or,
 $-x \rightarrow 0^+$

What this means for us is that this limit *does* make sense! We know that we can't have negative arguments in a logarithm, but because of the negative sign in this particular logarithm that means that we *can* use negative x 's in this function (positive x 's on the other hand will now cause problems of course...).

By Example 6 in the notes for this section we know that as the argument of a logarithm approaches zero from the right (as ours does in this limit) then the logarithm will approach $-\infty$.

Therefore, the answer for this part is,

$$\lim_{x \rightarrow 0^-} \ln(-x) = -\infty$$

(b) $\lim_{x \rightarrow 0^+} h(x)$

In this part we know that for $x \rightarrow 0^+$ we have $x > 0$ and so $-x < 0$. At this point we can stop. We know that we can't have negative arguments in a logarithm and for this limit that is exactly what we'll get and so $\lim_{x \rightarrow 0^+} h(x)$ **does not exist**.

(c) $\lim_{x \rightarrow 0} h(x)$

The answer for this part is $\lim_{x \rightarrow 0} h(x)$ **does not exist**. We can use two lines of reasoning to justify this.

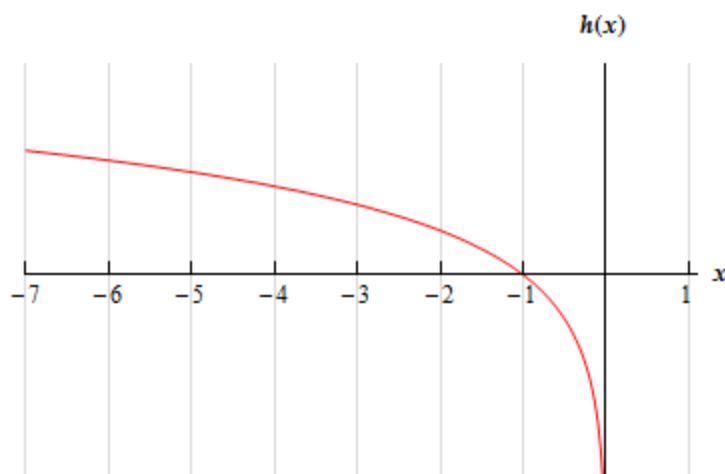
First, we are unable to look at both sides of the point in question and so there is no possible way for the limit to exist.

The second line of reasoning is really the same as the first but put in different terms. From the first two parts that,

$$\lim_{x \rightarrow 0^-} h(x) \neq \lim_{x \rightarrow 0^+} h(x)$$

and so, from our basic limit properties we can see that $\lim_{x \rightarrow 0} h(x)$ **does not exist**.

For the sake of completeness and to verify the answers for this problem here is a quick sketch of the function.



6. For $R(y) = \tan(y)$ evaluate the indicated limits, if they exist.

(a) $\lim_{y \rightarrow \frac{3\pi}{2}^-} R(y)$

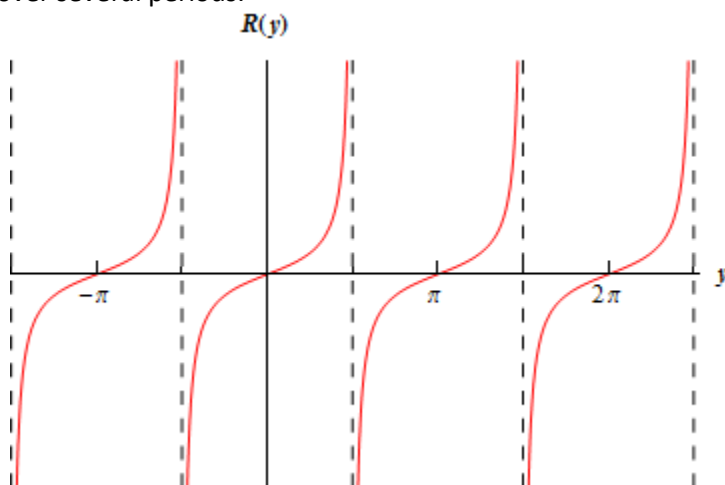
(b) $\lim_{y \rightarrow \frac{3\pi}{2}^+} R(y)$

(c) $\lim_{y \rightarrow \frac{3\pi}{2}} R(y)$

Hint : Don't forget the graph of the tangent function.

(a) $\lim_{y \rightarrow \frac{3\pi}{2}^-} R(y)$

The easiest way to do this problem is from the graph of the tangent function so here is a quick sketch of the tangent function over several periods.



From the sketch we can see that,

$$\lim_{y \rightarrow \frac{3\pi}{2}^-} \tan(y) = \infty$$

(b) $\lim_{y \rightarrow \frac{3\pi}{2}^+} R(y)$

From the graph in the first part we can see that,

$$\lim_{y \rightarrow \frac{3\pi}{2}^+} \tan(y) = -\infty$$

(c) $\lim_{y \rightarrow \frac{3\pi}{2}} R(y)$

From the first two parts that,

$$\lim_{y \rightarrow \frac{3\pi}{2}^-} R(y) \neq \lim_{y \rightarrow \frac{3\pi}{2}^+} R(y)$$

and so, from our basic limit properties we can see that $\lim_{y \rightarrow \frac{3\pi}{2}} R(y)$ **does not exist**.

7. Find all the vertical asymptotes of $f(x) = \frac{7x}{(10-3x)^4}$.

Hint : Remember how vertical asymptotes are defined and use the examples above to help determine where they are liable to be for the given function. Once you have the locations for the possible vertical asymptotes verify that they are in fact vertical asymptotes.

Solution

Recall that vertical asymptotes will occur at $x = a$ if any of the limits (one-sided or overall limit) at $x = a$ are plus or minus infinity.

From previous examples we can see that for rational expressions vertical asymptotes will occur where there is division by zero. Therefore, it looks like the only possible vertical asymptote will be at $x = \frac{10}{3}$.

Now let's verify that this is in fact a vertical asymptote by evaluating the two one-sided limits,

$$\lim_{x \rightarrow \frac{10}{3}^-} \frac{7x}{(10-3x)^4} \quad \text{and} \quad \lim_{x \rightarrow \frac{10}{3}^+} \frac{7x}{(10-3x)^4}$$

In either case as $x \rightarrow \frac{10}{3}$ (from both left and right) the numerator goes to $\frac{70}{3}$.

For the one-sided limits we have the following information,

$$\begin{aligned} x \rightarrow \frac{10}{3}^- &\Rightarrow x < \frac{10}{3} &\Rightarrow \frac{10}{3} - x > 0 &\Rightarrow 10 - 3x > 0 \\ x \rightarrow \frac{10}{3}^+ &\Rightarrow x > \frac{10}{3} &\Rightarrow \frac{10}{3} - x < 0 &\Rightarrow 10 - 3x < 0 \end{aligned}$$

Now, because of the exponent on the denominator is even we can see that for either of the one-sided limits we will have,

$$(10-3x)^4 \rightarrow 0^+$$

So, in either case, in the limit, the numerator approaches a fixed positive number and the denominator is positive and increasingly small. Therefore, we will have,

$$\lim_{x \rightarrow \frac{10}{3}^-} \frac{7x}{(10-3x)^4} = \infty \quad \lim_{x \rightarrow \frac{10}{3}^+} \frac{7x}{(10-3x)^4} = \infty \quad \lim_{x \rightarrow \frac{10}{3}} \frac{7x}{(10-3x)^4} = \infty$$

Any of these limits indicate that there is in fact a vertical asymptote at $x = \frac{10}{3}$.

8. Find all the vertical asymptotes of $g(x) = \frac{-8}{(x+5)(x-9)}$.

Hint : Remember how vertical asymptotes are defined and use the examples above to help determine where they are liable to be for the given function. Once you have the locations for the possible vertical asymptotes verify that they are in fact vertical asymptotes.

Solution

Recall that vertical asymptotes will occur at $x = a$ if any of the limits (one-sided or overall limit) at $x = a$ are plus or minus infinity.

From previous examples we can see that for rational expressions vertical asymptotes will occur where there is division by zero. Therefore, it looks like we will have possible vertical asymptote at $x = -5$ and $x = 9$.

Now let's verify that these are in fact vertical asymptotes by evaluating the two one-sided limits for each of them.

Let's start with $x = -5$. We'll need to evaluate,

$$\lim_{x \rightarrow -5^-} \frac{-8}{(x+5)(x-9)} \quad \text{and} \quad \lim_{x \rightarrow -5^+} \frac{-8}{(x+5)(x-9)}$$

In either case as $x \rightarrow -5$ (from both left and right) the numerator is a constant -8.

For the one-sided limits we have the following information,

$$\begin{aligned} x \rightarrow -5^- &\Rightarrow x < -5 &\Rightarrow x + 5 < 0 \\ x \rightarrow -5^+ &\Rightarrow x > -5 &\Rightarrow x + 5 > 0 \end{aligned}$$

Also, note that for x 's close enough to -5 (which because we're looking at $x \rightarrow -5$ is safe enough to assume), we will have $x - 9 < 0$.

So, in the left-hand limit, the numerator is a fixed negative number and the denominator is positive (a product of two negative numbers) and increasingly small. Likewise, for the right-hand limit, the denominator is negative (a product of a positive and negative number) and increasingly small. Therefore, we will have,

$$\lim_{x \rightarrow -5^-} \frac{-8}{(x+5)(x-9)} = -\infty \quad \text{and} \quad \lim_{x \rightarrow -5^+} \frac{-8}{(x+5)(x-9)} = \infty$$

Now for $x = 9$. Again, the numerator is a constant -8. We also have,

$$\begin{aligned} x \rightarrow 9^- &\Rightarrow x < 9 \Rightarrow x - 9 < 0 \\ x \rightarrow 9^+ &\Rightarrow x > 9 \Rightarrow x - 9 > 0 \end{aligned}$$

Finally, for x 's close enough to 9 (which because we're looking at $x \rightarrow 9$ is safe enough to assume), we will have $x + 5 > 0$.

So, in the left-hand limit, the numerator is a fixed negative number and the denominator is negative (a product of a positive and negative number) and increasingly small. Likewise, for the right-hand limit, the denominator is positive (a product of two positive numbers) and increasingly small. Therefore, we will have,

$$\lim_{x \rightarrow 9^-} \frac{-8}{(x+5)(x-9)} = \infty \quad \text{and} \quad \lim_{x \rightarrow 9^+} \frac{-8}{(x+5)(x-9)} = -\infty$$

So, as all of these limits show we do in fact have vertical asymptotes at $x = -5$ and $x = 9$.

Section 2-7 : Limits at Infinity, Part I

1. For $f(x) = 4x^7 - 18x^3 + 9$ evaluate each of the following limits.

(a) $\lim_{x \rightarrow -\infty} f(x)$

(b) $\lim_{x \rightarrow \infty} f(x)$

(a) $\lim_{x \rightarrow -\infty} f(x)$

To do this all we need to do is factor out the largest power of x from the whole polynomial and then use basic limit properties along with **Fact 1** from this section to evaluate the limit.

$$\begin{aligned}\lim_{x \rightarrow -\infty} (4x^7 - 18x^3 + 9) &= \lim_{x \rightarrow -\infty} \left[x^7 \left(4 - \frac{18}{x^4} + \frac{9}{x^7} \right) \right] \\ &= \left(\lim_{x \rightarrow -\infty} x^7 \right) \left[\lim_{x \rightarrow -\infty} \left(4 - \frac{18}{x^4} + \frac{9}{x^7} \right) \right] = (-\infty)(4) = \boxed{-\infty}\end{aligned}$$

(b) $\lim_{x \rightarrow \infty} f(x)$

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't need to be redone here. We can pick up the problem right before we actually took the limits and then proceed.

$$\lim_{x \rightarrow \infty} (4x^7 - 18x^3 + 9) = \left(\lim_{x \rightarrow \infty} x^7 \right) \left[\lim_{x \rightarrow \infty} \left(4 - \frac{18}{x^4} + \frac{9}{x^7} \right) \right] = (\infty)(4) = \boxed{\infty}$$

2. For $h(t) = \sqrt[3]{t} + 12t - 2t^2$ evaluate each of the following limits.

(a) $\lim_{t \rightarrow -\infty} h(t)$

(b) $\lim_{t \rightarrow \infty} h(t)$

(a) $\lim_{t \rightarrow -\infty} h(t)$

To do this all we need to do is factor out the largest power of x from the whole polynomial and then use basic limit properties along with **Fact 1** from this section to evaluate the limit.

Note as well that we'll convert the root over to a fractional exponent in order to allow it to be easier to deal with. Also note that this limit is a perfectly acceptable limit because the root is a cube root and we *can* take cube roots of negative numbers! We would only have run into problems had the index on the root been an even number.

$$\begin{aligned}\lim_{t \rightarrow -\infty} \left(t^{\frac{1}{3}} + 12t - 2t^2 \right) &= \lim_{t \rightarrow -\infty} \left[t^2 \left(\frac{1}{t^{\frac{5}{3}}} + \frac{12}{t} - 2 \right) \right] \\ &= \left(\lim_{t \rightarrow -\infty} t^2 \right) \left[\lim_{t \rightarrow -\infty} \left(\frac{1}{t^{\frac{5}{3}}} + \frac{12}{t} - 2 \right) \right] = (\infty)(-2) = \boxed{-\infty}\end{aligned}$$

(b) $\lim_{t \rightarrow \infty} h(t)$

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't need to be redone here. We can pick up the problem right before we actually took the limits and then proceed.

$$\lim_{t \rightarrow \infty} \left(t^{\frac{1}{3}} + 12t - 2t^2 \right) = \left(\lim_{t \rightarrow \infty} t^2 \right) \left[\lim_{t \rightarrow \infty} \left(\frac{1}{t^{\frac{5}{3}}} + \frac{12}{t} - 2 \right) \right] = (\infty)(-2) = \boxed{-\infty}$$

3. For $f(x) = \frac{8-4x^2}{9x^2+5x}$ answer each of the following questions.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with **Fact 1** from this section to evaluate the limit.

$$\lim_{x \rightarrow -\infty} \frac{8-4x^2}{9x^2+5x} = \lim_{x \rightarrow -\infty} \frac{x^2 \left(\frac{8}{x^2} - 4 \right)}{x^2 \left(9 + \frac{5}{x} \right)} = \lim_{x \rightarrow -\infty} \frac{\frac{8}{x^2} - 4}{9 + \frac{5}{x}} = \boxed{\frac{-4}{9}}$$

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't really need to be redone here. However, it is easy enough to add them in so we'll go ahead and include them.

$$\lim_{x \rightarrow \infty} \frac{8-4x^2}{9x^2+5x} = \lim_{x \rightarrow \infty} \frac{x^2 \left(\frac{8}{x^2} - 4 \right)}{x^2 \left(9 + \frac{5}{x} \right)} = \lim_{x \rightarrow \infty} \frac{\frac{8}{x^2} - 4}{9 + \frac{5}{x}} = \boxed{\frac{-4}{9}}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

We know that there will be a horizontal asymptote for $x \rightarrow -\infty$ if $\lim_{x \rightarrow -\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x \rightarrow \infty$ if $\lim_{x \rightarrow \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote,

$$y = -\frac{4}{9}$$

for both $x \rightarrow -\infty$ and $x \rightarrow \infty$.

4. For $f(x) = \frac{3x^7 - 4x^2 + 1}{5 - 10x^2}$ answer each of the following questions.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with [Fact 1](#) from this section to evaluate the limit.

$$\lim_{x \rightarrow -\infty} \frac{3x^7 - 4x^2 + 1}{5 - 10x^2} = \lim_{x \rightarrow -\infty} \frac{x^2 \left(3x^5 - 4 + \frac{1}{x^2} \right)}{x^2 \left(\frac{5}{x^2} - 10 \right)} = \lim_{x \rightarrow -\infty} \frac{3x^5 - 4 + \frac{1}{x^2}}{\frac{5}{x^2} - 10} = \frac{-\infty}{-10} = \boxed{\infty}$$

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't really need to be redone here. However, it is easy enough to add them in so we'll go ahead and include them.

$$\lim_{x \rightarrow \infty} \frac{3x^7 - 4x^2 + 1}{5 - 10x^2} = \lim_{x \rightarrow \infty} \frac{x^2 \left(3x^5 - 4 + \frac{1}{x^2} \right)}{x^2 \left(\frac{5}{x^2} - 10 \right)} = \lim_{x \rightarrow \infty} \frac{3x^5 - 4 + \frac{1}{x^2}}{\frac{5}{x^2} - 10} = \frac{\infty}{-\infty} = \boxed{-\infty}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

We know that there will be a horizontal asymptote for $x \rightarrow -\infty$ if $\lim_{x \rightarrow -\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x \rightarrow \infty$ if $\lim_{x \rightarrow \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that this function will have **no horizontal asymptotes** since neither of the two limits are finite.

5. For $f(x) = \frac{20x^4 - 7x^3}{2x + 9x^2 + 5x^4}$ answer each of the following questions.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with [Fact 1](#) from this section to evaluate the limit.

$$\lim_{x \rightarrow -\infty} \frac{20x^4 - 7x^3}{2x + 9x^2 + 5x^4} = \lim_{x \rightarrow -\infty} \frac{x^4 \left(20 - \frac{7}{x} \right)}{x^4 \left(\frac{2}{x^3} + \frac{9}{x^2} + 5 \right)} = \lim_{x \rightarrow -\infty} \frac{20 - \frac{7}{x}}{\frac{2}{x^3} + \frac{9}{x^2} + 5} = \frac{20}{5} = \boxed{4}$$

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't really need to be redone here. However, it is easy enough to add them in so we'll go ahead and include them.

$$\lim_{x \rightarrow \infty} \frac{20x^4 - 7x^3}{2x + 9x^2 + 5x^4} = \lim_{x \rightarrow \infty} \frac{x^4 \left(20 - \frac{7}{x} \right)}{x^4 \left(\frac{2}{x^3} + \frac{9}{x^2} + 5 \right)} = \lim_{x \rightarrow \infty} \frac{20 - \frac{7}{x}}{\frac{2}{x^3} + \frac{9}{x^2} + 5} = \frac{20}{5} = \boxed{4}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

We know that there will be a horizontal asymptote for $x \rightarrow -\infty$ if $\lim_{x \rightarrow -\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x \rightarrow \infty$ if $\lim_{x \rightarrow \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote,

$$y = 4$$

for both $x \rightarrow -\infty$ and $x \rightarrow \infty$.

6. For $f(x) = \frac{x^3 - 2x + 11}{3 - 6x^5}$ answer each of the following questions.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with [Fact 1](#) from this section to evaluate the limit.

$$\lim_{x \rightarrow -\infty} \frac{x^3 - 2x + 11}{3 - 6x^5} = \lim_{x \rightarrow -\infty} \frac{x^5 \left(\frac{1}{x^2} - \frac{2}{x^4} + \frac{11}{x^5} \right)}{x^5 \left(\frac{3}{x^5} - 6 \right)} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2} - \frac{2}{x^4} + \frac{11}{x^5}}{\frac{3}{x^5} - 6} = \frac{0}{-6} = \boxed{0}$$

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't really need to be redone here. However, it is easy enough to add them in so we'll go ahead and include them.

$$\lim_{x \rightarrow \infty} \frac{x^3 - 2x + 11}{3 - 6x^5} = \lim_{x \rightarrow \infty} \frac{x^5 \left(\frac{1}{x^2} - \frac{2}{x^4} + \frac{11}{x^5} \right)}{x^5 \left(\frac{3}{x^5} - 6 \right)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2} - \frac{2}{x^4} + \frac{11}{x^5}}{\frac{3}{x^5} - 6} = \frac{0}{-6} = \boxed{0}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

We know that there will be a horizontal asymptote for $x \rightarrow -\infty$ if $\lim_{x \rightarrow -\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x \rightarrow \infty$ if $\lim_{x \rightarrow \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote,

$$y = 0$$

for both $x \rightarrow -\infty$ and $x \rightarrow \infty$.

7. For $f(x) = \frac{x^6 - x^4 + x^2 - 1}{7x^6 + 4x^3 + 10}$ answer each of the following questions.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with [Fact 1](#) from this section to evaluate the limit.

$$\lim_{x \rightarrow -\infty} \frac{x^6 - x^4 + x^2 - 1}{7x^6 + 4x^3 + 10} = \lim_{x \rightarrow -\infty} \frac{x^6 \left(1 - \frac{1}{x^2} + \frac{1}{x^4} - \frac{1}{x^6} \right)}{x^6 \left(7 + \frac{4}{x^3} + \frac{10}{x^6} \right)} = \lim_{x \rightarrow -\infty} \frac{1 - \frac{1}{x^2} + \frac{1}{x^4} - \frac{1}{x^6}}{7 + \frac{4}{x^3} + \frac{10}{x^6}} = \boxed{\frac{1}{7}}$$

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

For this part all of the mathematical manipulations we did in the first part did not depend upon the limit itself and so don't really need to be redone here. However, it is easy enough to add them in so we'll go ahead and include them.

$$\lim_{x \rightarrow \infty} \frac{x^6 - x^4 + x^2 - 1}{7x^6 + 4x^3 + 10} = \lim_{x \rightarrow \infty} \frac{x^6 \left(1 - \frac{1}{x^2} + \frac{1}{x^4} - \frac{1}{x^6} \right)}{x^6 \left(7 + \frac{4}{x^3} + \frac{10}{x^6} \right)} = \lim_{x \rightarrow \infty} \frac{1 - \frac{1}{x^2} + \frac{1}{x^4} - \frac{1}{x^6}}{7 + \frac{4}{x^3} + \frac{10}{x^6}} = \boxed{\frac{1}{7}}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

We know that there will be a horizontal asymptote for $x \rightarrow -\infty$ if $\lim_{x \rightarrow -\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x \rightarrow \infty$ if $\lim_{x \rightarrow \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote,

$$y = \frac{1}{7}$$

for both $x \rightarrow -\infty$ and $x \rightarrow \infty$.

8. For $f(x) = \frac{\sqrt{7+9x^2}}{1-2x}$ answer each of the following questions.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with [Fact 1](#) from this section to evaluate the limit.

In this case the largest power of x in the denominator is just x and so we will need to factor an x out of both the denominator and the numerator. Recall as well that this means we'll need to factor an x^2 out of the root in the numerator so that we'll have an x in the numerator when we are done.

So, let's do the first couple of steps in this process to get us started.

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{7+9x^2}}{1-2x} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 \left(\frac{7}{x^2} + 9 \right)}}{x \left(\frac{1}{x} - 2 \right)} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2} \sqrt{\frac{7}{x^2} + 9}}{x \left(\frac{1}{x} - 2 \right)} = \lim_{x \rightarrow -\infty} \frac{|x| \sqrt{\frac{7}{x^2} + 9}}{x \left(\frac{1}{x} - 2 \right)}$$

Recall from the discussion in this section that,

$$\sqrt{x^2} = |x|$$

and we *do* need to be careful with that.

Now, because we are looking at the limit $x \rightarrow -\infty$ it is safe to assume that $x < 0$. Therefore, from the definition of the absolute value we get,

$$|x| = -x$$

and the limit is then,

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{7+9x^2}}{1-2x} = \lim_{x \rightarrow -\infty} \frac{-x \sqrt{\frac{7}{x^2} + 9}}{x \left(\frac{1}{x} - 2 \right)} = \lim_{x \rightarrow -\infty} \frac{-\sqrt{\frac{7}{x^2} + 9}}{\frac{1}{x} - 2} = \frac{-\sqrt{9}}{-2} = \boxed{\frac{3}{2}}$$

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

For this part all of the mathematical manipulations we did in the first part up to dealing with the absolute value did not depend upon the limit itself and so don't really need to be redone here. So, up to that part we have,

$$\lim_{x \rightarrow \infty} \frac{\sqrt{7+9x^2}}{1-2x} = \lim_{x \rightarrow \infty} \frac{|x|\sqrt{\frac{7}{x^2}+9}}{x\left(\frac{1}{x}-2\right)}$$

In this part we are looking at the limit $x \rightarrow \infty$ and so it will be safe to assume in this part that $x > 0$. Therefore, from the definition of the absolute value we get,

$$|x| = x$$

and the limit is then,

$$\lim_{x \rightarrow \infty} \frac{\sqrt{7+9x^2}}{1-2x} = \lim_{x \rightarrow \infty} \frac{x\sqrt{\frac{7}{x^2}+9}}{x\left(\frac{1}{x}-2\right)} = \lim_{x \rightarrow \infty} \frac{\sqrt{\frac{7}{x^2}+9}}{\frac{1}{x}-2} = \frac{\sqrt{9}}{-2} = \boxed{-\frac{3}{2}}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

We know that there will be a horizontal asymptote for $x \rightarrow -\infty$ if $\lim_{x \rightarrow -\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x \rightarrow \infty$ if $\lim_{x \rightarrow \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote,

$$y = \frac{3}{2}$$

for $x \rightarrow -\infty$ and we have the horizontal asymptote,

$$y = -\frac{3}{2}$$

for $x \rightarrow \infty$.

9. For $f(x) = \frac{x+8}{\sqrt{2x^2+3}}$ answer each of the following questions.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with **Fact 1** from this section to evaluate the limit.

For the denominator we need to be a little careful. The power of x in the denominator needs to be outside of the root so it can cancel against the x 's in the numerator. The largest power of x outside of the root that we can get (and leave something we can deal with in the root) will be just x . We get this by factoring an x^2 out of the root.

So, let's do the first couple of steps in this process to get us started.

$$\lim_{x \rightarrow -\infty} \frac{x+8}{\sqrt{2x^2+3}} = \lim_{x \rightarrow -\infty} \frac{x\left(1+\frac{8}{x}\right)}{\sqrt{x^2\left(2+\frac{3}{x^2}\right)}} = \lim_{x \rightarrow -\infty} \frac{x\left(1+\frac{8}{x}\right)}{\sqrt{x^2}\sqrt{2+\frac{3}{x^2}}} = \lim_{x \rightarrow -\infty} \frac{x\left(1+\frac{8}{x}\right)}{|x|\sqrt{2+\frac{3}{x^2}}}$$

Recall from the discussion in this section that,

$$\sqrt{x^2} = |x|$$

and we *do* need to be careful with that.

Now, because we are looking at the limit $x \rightarrow -\infty$ it is safe to assume that $x < 0$. Therefore, from the definition of the absolute value we get,

$$|x| = -x$$

and the limit is then,

$$\lim_{x \rightarrow -\infty} \frac{x+8}{\sqrt{2x^2+3}} = \lim_{x \rightarrow -\infty} \frac{x\left(1+\frac{8}{x}\right)}{-x\sqrt{2+\frac{3}{x^2}}} = \lim_{x \rightarrow -\infty} \frac{1+\frac{8}{x}}{-\sqrt{2+\frac{3}{x^2}}} = \boxed{\frac{1}{-\sqrt{2}}}$$

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

For this part all of the mathematical manipulations we did in the first part up to dealing with the absolute value did not depend upon the limit itself and so don't really need to be redone here. So, up to that part we have,

$$\lim_{x \rightarrow \infty} \frac{x+8}{\sqrt{2x^2+3}} = \lim_{x \rightarrow \infty} \frac{x\left(1+\frac{8}{x}\right)}{|x|\sqrt{2+\frac{3}{x^2}}}$$

In this part we are looking at the limit $x \rightarrow \infty$ and so it will be safe to assume in this part that $x > 0$. Therefore, from the definition of the absolute value we get,

$$|x| = x$$

and the limit is then,

$$\lim_{x \rightarrow \infty} \frac{x+8}{\sqrt{2x^2+3}} = \lim_{x \rightarrow \infty} \frac{x\left(1+\frac{8}{x}\right)}{x\sqrt{2+\frac{3}{x^2}}} = \lim_{x \rightarrow \infty} \frac{1+\frac{8}{x}}{\sqrt{2+\frac{3}{x^2}}} = \boxed{\frac{1}{\sqrt{2}}}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

We know that there will be a horizontal asymptote for $x \rightarrow -\infty$ if $\lim_{x \rightarrow -\infty} f(x)$ exists and is a finite number. Likewise, we'll have a horizontal asymptote for $x \rightarrow \infty$ if $\lim_{x \rightarrow \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote,

$$y = -\frac{1}{\sqrt{2}}$$

for $x \rightarrow -\infty$ and we have the horizontal asymptote,

$$y = \frac{1}{\sqrt{2}}$$

for $x \rightarrow \infty$.

10. For $f(x) = \frac{8+x-4x^2}{\sqrt{6+x^2+7x^4}}$ answer each of the following questions.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

(a) Evaluate $\lim_{x \rightarrow -\infty} f(x)$.

To do this all we need to do is factor out the largest power of x that is in the denominator from both the denominator *and* the numerator. Then all we need to do is use basic limit properties along with **Fact 1** from this section to evaluate the limit.

For the denominator we need to be a little careful. The power of x in the denominator needs to be outside of the root so it can cancel against the x 's in the numerator. The largest power of x outside of the root that we can get (and leave something we can deal with in the root) will be just x^2 . We get this by factoring an x^4 out of the root.

So, let's do the first couple of steps in this process to get us started.

$$\begin{aligned}
 \lim_{x \rightarrow -\infty} \frac{8+x-4x^2}{\sqrt{6+x^2+7x^4}} &= \lim_{x \rightarrow -\infty} \frac{x^2 \left(\frac{8}{x^2} + \frac{1}{x} - 4 \right)}{\sqrt{x^4 \left(\frac{6}{x^4} + \frac{1}{x^2} + 7 \right)}} \\
 &= \lim_{x \rightarrow -\infty} \frac{x^2 \left(\frac{8}{x^2} + \frac{1}{x} - 4 \right)}{\sqrt{x^4} \sqrt{\frac{6}{x^4} + \frac{1}{x^2} + 7}} = \lim_{x \rightarrow -\infty} \frac{x^2 \left(\frac{8}{x^2} + \frac{1}{x} - 4 \right)}{|x^2| \sqrt{\frac{6}{x^4} + \frac{1}{x^2} + 7}}
 \end{aligned}$$

Recall from the discussion in this section that,

$$\sqrt{x^2} = |x|$$

So, in this case we'll have,

$$\sqrt{x^4} = |x^2| = x^2$$

and note that we can get rid of the absolute value bars because we know that $x^2 \geq 0$. So, let's finish the limit up.

$$\lim_{x \rightarrow -\infty} \frac{8+x-4x^2}{\sqrt{6+x^2+7x^4}} = \lim_{x \rightarrow -\infty} \frac{x^2 \left(\frac{8}{x^2} + \frac{1}{x} - 4 \right)}{x^2 \sqrt{\frac{6}{x^4} + \frac{1}{x^2} + 7}} = \lim_{x \rightarrow -\infty} \frac{\frac{8}{x^2} + \frac{1}{x} - 4}{\sqrt{\frac{6}{x^4} + \frac{1}{x^2} + 7}} = \boxed{\frac{-4}{\sqrt{7}}}$$

(b) Evaluate $\lim_{x \rightarrow \infty} f(x)$.

Unlike the previous two problems with roots in them all of the mathematical manipulations in this case did not depend upon the actual limit because we were factoring an x^2 out which will always be positive and so there will be no reason to redo all of that work.

Here is this limit (with most of the work excluded),

For this part all of the mathematical manipulations we did in the first part up to dealing with the absolute value did not depend upon the limit itself and so don't really need to be redone here. So, up to that part we have,

$$\lim_{x \rightarrow \infty} \frac{8+x-4x^2}{\sqrt{6+x^2+7x^4}} = \lim_{x \rightarrow \infty} \frac{x^2 \left(\frac{8}{x^2} + \frac{1}{x} - 4 \right)}{x^2 \sqrt{\frac{6}{x^4} + \frac{1}{x^2} + 7}} = \lim_{x \rightarrow \infty} \frac{\frac{8}{x^2} + \frac{1}{x} - 4}{\sqrt{\frac{6}{x^4} + \frac{1}{x^2} + 7}} = \boxed{\frac{-4}{\sqrt{7}}}$$

(c) Write down the equation(s) of any horizontal asymptotes for the function.

We know that there will be a horizontal asymptote for $x \rightarrow -\infty$ if $\lim_{x \rightarrow -\infty} f(x)$ exists and is a finite

number. Likewise, we'll have a horizontal asymptote for $x \rightarrow \infty$ if $\lim_{x \rightarrow \infty} f(x)$ exists and is a finite number.

Therefore, from the first two parts, we can see that we will get the horizontal asymptote,

$$y = -\frac{4}{\sqrt{7}}$$

For both $x \rightarrow -\infty$ and $x \rightarrow \infty$.

Section 2-8 : Limits At Infinity, Part II

1. For $f(x) = e^{8+2x-x^3}$ evaluate each of the following limits.

(a) $\lim_{x \rightarrow -\infty} f(x)$

(b) $\lim_{x \rightarrow \infty} f(x)$

(a) $\lim_{x \rightarrow -\infty} f(x)$

First notice that,

$$\lim_{x \rightarrow -\infty} (8 + 2x - x^3) = \infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling **Example 1** from this section, we know that because the exponent goes to infinity in the limit the answer is,

$$\lim_{x \rightarrow -\infty} e^{8+2x-x^3} = \boxed{\infty}$$

(b) $\lim_{x \rightarrow \infty} f(x)$

First notice that,

$$\lim_{x \rightarrow \infty} (8 + 2x - x^3) = -\infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling **Example 1** from this section, we know that because the exponent goes to negative infinity in the limit the answer is,

$$\lim_{x \rightarrow \infty} e^{8+2x-x^3} = \boxed{0}$$

2. For $f(x) = e^{\frac{6x^2+x}{5+3x}}$ evaluate each of the following limits.

(a) $\lim_{x \rightarrow -\infty} f(x)$

(b) $\lim_{x \rightarrow \infty} f(x)$

(a) $\lim_{x \rightarrow -\infty} f(x)$

First notice that,

$$\lim_{x \rightarrow -\infty} \frac{6x^2 + x}{5 + 3x} = -\infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling **Example 1** from this section, we know that because the exponent goes to negative infinity in the limit the answer is,

$$\lim_{x \rightarrow -\infty} e^{\frac{6x^2+x}{5+3x}} = \boxed{0}$$

(b) $\lim_{x \rightarrow \infty} f(x)$

First notice that,

$$\lim_{x \rightarrow \infty} \frac{6x^2+x}{5+3x} = \infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling **Example 1** from this section, we know that because the exponent goes to infinity in the limit the answer is,

$$\lim_{x \rightarrow \infty} e^{\frac{6x^2+x}{5+3x}} = \boxed{\infty}$$

3. For $f(x) = 2e^{6x} - e^{-7x} - 10e^{4x}$ evaluate each of the following limits.

(a) $\lim_{x \rightarrow -\infty} f(x)$

(b) $\lim_{x \rightarrow \infty} f(x)$

Hint : Remember that if there are two terms that seem to be suggesting that the function should be going in opposite directions that you'll need to factor out of the function that term that is going to infinity faster to "prove" the limit.

(a) $\lim_{x \rightarrow -\infty} f(x)$

For this limit the exponentials with positive exponents will simply go to zero and there is only one exponential with a negative exponent (which will go to infinity) and so there isn't much to do with this limit.

$$\lim_{x \rightarrow -\infty} (2e^{6x} - e^{-7x} - 10e^{4x}) = 0 - \infty - 0 = \boxed{-\infty}$$

(b) $\lim_{x \rightarrow \infty} f(x)$

Here we have two exponents with positive exponents and so both will go to infinity in the limit. However, each term has opposite signs and so each term seems to be suggesting different answers for the limit.

In order to determine which "wins out" so to speak all we need to do is factor out the term with the largest exponent and then use basic limit properties.

$$\lim_{x \rightarrow \infty} (2e^{6x} - e^{-7x} - 10e^{4x}) = \lim_{x \rightarrow \infty} [e^{6x} (2 - e^{-13x} - 10e^{-2x})] = (\infty)(2) = \boxed{\infty}$$

4. For $f(x) = 3e^{-x} - 8e^{-5x} - e^{10x}$ evaluate each of the following limits.

(a) $\lim_{x \rightarrow -\infty} f(x)$

(b) $\lim_{x \rightarrow \infty} f(x)$

Hint : Remember that if there are two terms that seem to be suggesting that the function should be going in opposite directions that you'll need to factor out of the function that term that is going to infinity faster to "prove" the limit.

(a) $\lim_{x \rightarrow -\infty} f(x)$

Here we have two exponents with negative exponents and so both will go to infinity in the limit. However, each term has opposite signs and so each term seems to be suggesting different answers for the limit.

In order to determine which "wins out" so to speak all we need to do is factor out the term with the most negative exponent and then use basic limit properties.

$$\lim_{x \rightarrow -\infty} (3e^{-x} - 8e^{-5x} - e^{10x}) = \lim_{x \rightarrow -\infty} [e^{-5x} (3e^{4x} - 8 - e^{15x})] = (\infty)(-8) = \boxed{-\infty}$$

(b) $\lim_{x \rightarrow \infty} f(x)$

For this limit the exponentials with negative exponents will simply go to zero and there is only one exponential with a positive exponent (which will go to infinity) and so there isn't much to do with this limit.

$$\lim_{x \rightarrow \infty} (3e^{-x} - 8e^{-5x} - e^{10x}) = 0 - 0 - \infty = \boxed{-\infty}$$

5. For $f(x) = \frac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}}$ evaluate each of the following limits.

(a) $\lim_{x \rightarrow -\infty} f(x)$

(b) $\lim_{x \rightarrow \infty} f(x)$

Hint : Remember that you'll need to factor the term in the denominator that is causing the denominator to go to infinity from both the numerator and denominator in order to evaluate this limit.

(a) $\lim_{x \rightarrow -\infty} f(x)$

The exponential with the negative exponent is the only term in the denominator going to infinity for this limit and so we'll need to factor the exponential with the negative exponent in the denominator from both the numerator and denominator to evaluate this limit.

$$\lim_{x \rightarrow -\infty} \frac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}} = \lim_{x \rightarrow -\infty} \frac{e^{-3x}(1 - 2e^{11x})}{e^{-3x}(9e^{11x} - 7)} = \lim_{x \rightarrow -\infty} \frac{1 - 2e^{11x}}{9e^{11x} - 7} = \frac{1 - 0}{0 - 7} = \boxed{-\frac{1}{7}}$$

(b) $\lim_{x \rightarrow \infty} f(x)$

The exponential with the positive exponent is the only term in the denominator going to infinity for this limit and so we'll need to factor the exponential with the positive exponent in the denominator from both the numerator and denominator to evaluate this limit.

$$\lim_{x \rightarrow \infty} \frac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}} = \lim_{x \rightarrow \infty} \frac{e^{8x}(e^{-11x} - 2)}{e^{8x}(9 - 7e^{-11x})} = \lim_{x \rightarrow \infty} \frac{e^{-11x} - 2}{9 - 7e^{-11x}} = \frac{0 - 2}{9 - 0} = \boxed{-\frac{2}{9}}$$

6. For $f(x) = \frac{e^{-7x} - 2e^{3x} - e^x}{e^{-x} + 16e^{10x} + 2e^{-4x}}$ evaluate each of the following limits.

(a) $\lim_{x \rightarrow -\infty} f(x)$

(b) $\lim_{x \rightarrow \infty} f(x)$

Hint : Remember that you'll need to factor the term in the denominator that is causing the denominator to go to infinity fastest from both the numerator and denominator in order to evaluate this limit.

(a) $\lim_{x \rightarrow -\infty} f(x)$

The exponentials with the negative exponents are the only terms in the denominator going to infinity for this limit and so we'll need to factor the exponential with the most negative exponent in the denominator (because it will be going to infinity fastest) from both the numerator and denominator to evaluate this limit.

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{e^{-7x} - 2e^{3x} - e^x}{e^{-x} + 16e^{10x} + 2e^{-4x}} &= \lim_{x \rightarrow -\infty} \frac{e^{-4x}(e^{-3x} - 2e^{7x} - e^{5x})}{e^{-4x}(e^{3x} + 16e^{14x} + 2)} \\ &= \lim_{x \rightarrow -\infty} \frac{e^{-3x} - 2e^{7x} - e^{5x}}{e^{3x} + 16e^{14x} + 2} = \frac{\infty - 0 - 0}{0 + 0 + 2} = \boxed{\infty} \end{aligned}$$

(b) $\lim_{x \rightarrow \infty} f(x)$

The exponentials with the positive exponents are the only terms in the denominator going to infinity for this limit and so we'll need to factor the exponential with the most positive exponent in the denominator (because it will be going to infinity fastest) from both the numerator and denominator to evaluate this limit.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{e^{-7x} - 2e^{3x} - e^x}{e^{-x} + 16e^{10x} + 2e^{-4x}} &= \lim_{x \rightarrow \infty} \frac{e^{10x}(e^{-17x} - 2e^{-7x} - e^{-9x})}{e^{10x}(e^{-11x} + 16 + 2e^{-14x})} \\ &= \lim_{x \rightarrow \infty} \frac{e^{-17x} - 2e^{-7x} - e^{-9x}}{e^{-11x} + 16 + 2e^{-14x}} = \frac{0 - 0 - 0}{0 + 16 + 0} = \boxed{0} \end{aligned}$$

7. Evaluate $\lim_{t \rightarrow -\infty} \ln(4 - 9t - t^3)$.

Solution

First notice that,

$$\lim_{t \rightarrow -\infty} (4 - 9t - t^3) = \infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling **Example 5** from this section, we know that because the argument goes to infinity in the limit the answer is,

$$\lim_{t \rightarrow -\infty} \ln(4 - 9t - t^3) = \boxed{\infty}$$

8. Evaluate $\lim_{z \rightarrow -\infty} \ln\left(\frac{3z^4 - 8}{2 + z^2}\right)$.

Solution

First notice that,

$$\lim_{z \rightarrow -\infty} \frac{3z^4 - 8}{2 + z^2} = \infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling **Example 5** from this section, we know that because the argument goes to infinity in the limit the answer is,

$$\lim_{z \rightarrow -\infty} \ln\left(\frac{3z^4 - 8}{2 + z^2}\right) = \boxed{\infty}$$

9. Evaluate $\lim_{x \rightarrow \infty} \ln\left(\frac{11 + 8x}{x^3 + 7x}\right)$.

Solution

First notice that,

$$\lim_{x \rightarrow \infty} \frac{11 + 8x}{x^3 + 7x} = 0$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Also, note that because we are evaluating the limit $x \rightarrow \infty$ it is safe to assume that $x > 0$ and so we can further say that,

$$\frac{11+8x}{x^3+7x} \rightarrow 0^+$$

Now, recalling **Example 5** from this section, we know that because the argument goes to zero from the right in the limit the answer is,

$$\lim_{x \rightarrow \infty} \ln\left(\frac{11+8x}{x^3+7x}\right) = \boxed{-\infty}$$

10. Evaluate $\lim_{x \rightarrow -\infty} \tan^{-1}(7-x+3x^5)$.

Solution

First notice that,

$$\lim_{x \rightarrow -\infty} (7-x+3x^5) = -\infty$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling **Example 7** from this section, we know that because the argument goes to negative infinity in the limit the answer is,

$$\lim_{x \rightarrow -\infty} \tan^{-1}(7-x+3x^5) = \boxed{-\frac{\pi}{2}}$$

11. Evaluate $\lim_{t \rightarrow \infty} \tan^{-1}\left(\frac{4+7t}{2-t}\right)$.

Solution

First notice that,

$$\lim_{t \rightarrow \infty} \frac{4+7t}{2-t} = -7$$

If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Then answer is then,

$$\lim_{t \rightarrow \infty} \tan^{-1}\left(\frac{4+7t}{2-t}\right) = \boxed{\tan^{-1}(-7)}$$

Do not get so used the “special case” limits that we tend to usually do in the problems at the end of a section that you decide that you must have done something wrong when you run across a problem that doesn't fall in the “special case” category.

12. Evaluate $\lim_{w \rightarrow \infty} \tan^{-1} \left(\frac{3w^2 - 9w^4}{4w - w^3} \right)$.

Solution

First notice that,

$$\lim_{w \rightarrow \infty} \frac{3w^2 - 9w^4}{4w - w^3} = \infty$$

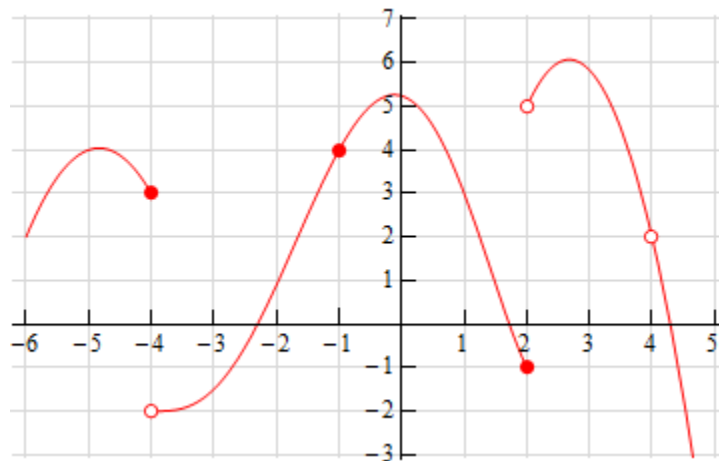
If you aren't sure about this limit you should go back to the previous section and work some of the examples there to make sure that you can do these kinds of limits.

Now, recalling [Example 7](#) from this section, we know that because the argument goes to infinity in the limit the answer is,

$$\lim_{w \rightarrow \infty} \tan^{-1} \left(\frac{3w^2 - 9w^4}{4w - w^3} \right) = \boxed{\frac{\pi}{2}}$$

Section 2-9 : Continuity

1. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



Solution

Before starting the solution recall that in order for a function to be continuous at $x = a$ both $f(a)$ and $\lim_{x \rightarrow a} f(x)$ must exist and we must have,

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Using this idea it should be fairly clear where the function is not continuous.

First notice that at $x = -4$ we have,

$$\lim_{x \rightarrow -4^-} f(x) = 3 \neq -2 = \lim_{x \rightarrow -4^+} f(x)$$

and therefore, we also know that $\lim_{x \rightarrow -4} f(x)$ doesn't exist. We can therefore conclude that $f(x)$ is **discontinuous** at $x = -4$ because the limit does not exist.

Likewise, at $x = 2$ we have,

$$\lim_{x \rightarrow 2^-} f(x) = -1 \neq 5 = \lim_{x \rightarrow 2^+} f(x)$$

and therefore, we also know that $\lim_{x \rightarrow 2} f(x)$ doesn't exist. So again, because the limit does not exist, we can see that $f(x)$ is **discontinuous** at $x = 2$.

Finally let's take a look at $x = 4$. Here we can see that,

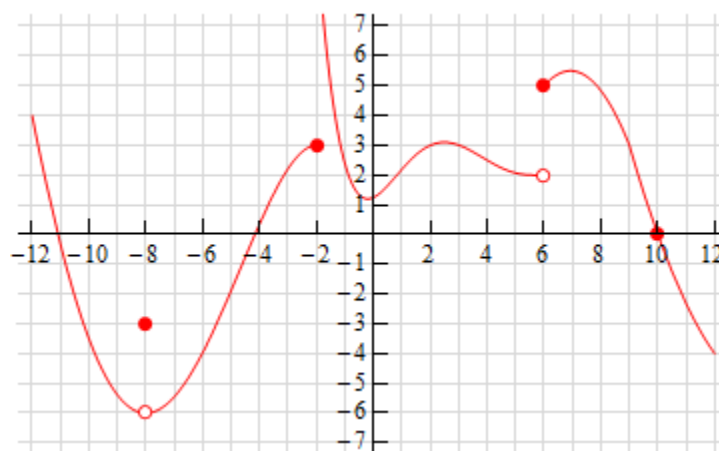
$$\lim_{x \rightarrow 4^-} f(x) = 2 = \lim_{x \rightarrow 4^+} f(x)$$

and therefore, we also know that $\lim_{x \rightarrow 4} f(x) = 2$. However, we can also see that $f(4)$ doesn't exist and so once again $f(x)$ is **discontinuous** at $x = 4$ because this time the function does not exist at $x = 4$.

All other points on this graph will have both the function and limit exist and we'll have $\lim_{x \rightarrow a} f(x) = f(a)$ and so will be continuous.

In summary then the points of discontinuity for this graph are : $x = -4$, $x = 2$ and $x = 4$.

2. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



Solution

Before starting the solution recall that in order for a function to be continuous at $x = a$ both $f(a)$ and $\lim_{x \rightarrow a} f(x)$ must exist and we must have,

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Using this idea it should be fairly clear where the function is not continuous.

First notice that at $x = -8$ we have,

$$\lim_{x \rightarrow -8^-} f(x) = -6 = \lim_{x \rightarrow -8^+} f(x)$$

and therefore, we also know that $\lim_{x \rightarrow -8} f(x) = -6$. We can also see that $f(-8) = -3$ and so we have,

$$-6 = \lim_{x \rightarrow -8} f(x) \neq f(-8) = -3$$

Because the function and limit have different values we can conclude that $f(x)$ is **discontinuous** at $x = -8$.

Next let's take a look at $x = -2$ we have,

$$\lim_{x \rightarrow -2^-} f(x) = 3 \neq \infty = \lim_{x \rightarrow -2^+} f(x)$$

and therefore, we also know that $\lim_{x \rightarrow -2} f(x)$ doesn't exist. We can therefore conclude that $f(x)$ is **discontinuous** at $x = -2$ because the limit does not exist.

Finally let's take a look at $x = 6$. Here we can see we have,

$$\lim_{x \rightarrow 6^-} f(x) = 2 \neq 5 = \lim_{x \rightarrow 6^+} f(x)$$

and therefore, we also know that $\lim_{x \rightarrow 6} f(x)$ doesn't exist. So, once again, because the limit does not exist, we can conclude that $f(x)$ is **discontinuous** at $x = 6$.

All other points on this graph will have both the function and limit exist and we'll have

$$\lim_{x \rightarrow a} f(x) = f(a) \text{ and so will be continuous.}$$

In summary then the points of discontinuity for this graph are : $x = -8$, $x = -2$ and $x = 6$.

3. Using only Properties 1- 9 from the **Limit Properties** section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a)** $x = -1$, **(b)** $x = 0$, **(c)** $x = 3$?

$$f(x) = \frac{4x + 5}{9 - 3x}$$

(a) $x = -1$

Before starting off with the solution to this part notice that we CAN NOT do what we've commonly done to evaluate limits to this point. In other words, we can't just plug in the point to evaluate the limit. Doing this implicitly assumes that the function is continuous at the point and that is what we are being asked to determine here.

Therefore, the only way for us to compute the limit is to go back to the properties from the Limit Properties section and compute the limit as we did back in that section. We won't be putting all the details here so if you need a little refresher on doing this you should go back to the problems from that section and work a few of them.

So, here we go.

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{4x + 5}{9 - 3x} = \frac{\lim_{x \rightarrow -1} (4x + 5)}{\lim_{x \rightarrow -1} (9 - 3x)} = \frac{4 \lim_{x \rightarrow -1} x + \lim_{x \rightarrow -1} 5}{\lim_{x \rightarrow -1} 9 - 3 \lim_{x \rightarrow -1} x} = \frac{4(-1) + 5}{9 - 3(-1)} = f(-1)$$

So, we can see that $\lim_{x \rightarrow -1} f(x) = f(-1)$ and so the function is **continuous** at $x = -1$.

(b) $x = 0$

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**.

Here is the work for this part.

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{4x+5}{9-3x} = \frac{\lim_{x \rightarrow 0} (4x+5)}{\lim_{x \rightarrow 0} (9-3x)} = \frac{4\lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} 5}{\lim_{x \rightarrow 0} 9 - 3\lim_{x \rightarrow 0} x} = \frac{4(0)+5}{9-3(0)} = f(0)$$

So, we can see that $\lim_{x \rightarrow 0} f(x) = f(0)$ and so the function is **continuous** at $x = 0$.

(c) $x = 3$

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**. Although there is also of course the problem here that $f(3)$ doesn't exist and so we couldn't plug in the value even if we wanted to.

This also tells us what we need to know however. As noted in the notes for this section if either the function or the limit do not exist then the function is not continuous at the point. Therefore, we can see that the function is **not continuous** at $x = 3$.

For practice you might want to verify that,

$$\lim_{x \rightarrow 3^-} f(x) = \infty \qquad \lim_{x \rightarrow 3^+} f(x) = -\infty$$

and so $\lim_{x \rightarrow 3} f(x)$ also doesn't exist.

4. Using only Properties 1- 9 from the **Limit Properties** section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a) $z = -2$, (b) $z = 0$, (c) $z = 5$** ?

$$g(z) = \frac{6}{z^2 - 3z - 10}$$

(a) $z = -2$

Before starting off with the solution to this part notice that we CAN NOT do what we've commonly done to evaluate limits to this point. In other words, we can't just plug in the point to evaluate the limit. Doing this implicitly assumes that the function is continuous at the point and that is what we are being asked to determine here.

Of course, even if we had tried to plug in the point we would have run into problems as $g(-2)$ doesn't exist and this tell us all we need to know. As noted in the notes for this section if either the function or the limit do not exist then the function is not continuous at the point. Therefore, we can see that the function is **not continuous** at $z = -2$.

For practice you might want to verify that,

$$\lim_{z \rightarrow -2^-} g(z) = \infty \qquad \lim_{z \rightarrow -2^+} g(z) = -\infty$$

and so $\lim_{z \rightarrow -2} g(z)$ also doesn't exist.

(b) $z = 0$

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**.

Therefore, because we can't just plug the point into the function, the only way for us to compute the limit is to go back to the properties from the Limit Properties section and compute the limit as we did back in that section. We won't be putting all the details here so if you need a little refresher on doing this you should go back to the problems from that section and work a few of them.

Here is the work for this part.

$$\begin{aligned}\lim_{z \rightarrow 0} g(z) &= \lim_{z \rightarrow 0} \frac{6}{z^2 - 3z - 10} = \frac{\lim_{z \rightarrow 0} 6}{\lim_{z \rightarrow 0} (z^2 - 3z - 10)} = \frac{\lim_{z \rightarrow 0} 6}{\lim_{z \rightarrow 0} z^2 - 3 \lim_{z \rightarrow 0} z - \lim_{z \rightarrow 0} 10} \\ &= \frac{6}{0^2 - 3(0) - 10} = g(0)\end{aligned}$$

So, we can see that $\lim_{z \rightarrow 0} g(z) = g(0)$ and so the function is **continuous** at $z = 0$.

(c) $z = 5$

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**. Although there is also of course the problem here that $g(5)$ doesn't exist and so we couldn't plug in the value even if we wanted to.

This also tells us what we need to know however. As noted in the notes for this section if either the function or the limit do not exist then the function is not continuous at the point. Therefore, we can see that the function is **not continuous** at $z = 5$.

For practice you might want to verify that,

$$\lim_{z \rightarrow 5^-} g(z) = -\infty \qquad \lim_{z \rightarrow 5^+} g(z) = \infty$$

and so $\lim_{z \rightarrow 5} g(z)$ also doesn't exist.

5. Using only Properties 1- 9 from the **Limit Properties** section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a) $x = 4$, (b) $x = 6$** ?

$$g(x) = \begin{cases} 2x & x < 6 \\ x-1 & x \geq 6 \end{cases}$$

(a) $x = 4$

Before starting off with the solution to this part notice that we CAN NOT do what we've commonly done to evaluate limits to this point. In other words, we can't just plug in the point to evaluate the limit. Doing this implicitly assumes that the function is continuous at the point and that is what we are being asked to determine here.

Therefore, the only way for us to compute the limit is to go back to the properties from the Limit Properties section and compute the limit as we did back in that section. We won't be putting all the details here so if you need a little refresher on doing this you should go back to the problems from that section and work a few of them.

For this part we can notice that because there are values of x on both sides of $x = 4$ in the range $x < 6$ we won't need to worry about one-sided limits here. Here is the work for this part.

$$\lim_{x \rightarrow 4} g(x) = \lim_{x \rightarrow 4} (2x) = 2 \lim_{x \rightarrow 4} x = 2(4) = g(4)$$

So, we can see that $\lim_{x \rightarrow 4} g(x) = g(4)$ and so the function is **continuous** at $x = 4$.

(b) $x = 6$

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**.

For this part we have the added complication that the point we're interested in is also the "cut-off" point of the piecewise function and so we'll need to take a look at the two one sided limits to compute the overall limit and again because we are being asked to determine if the function is continuous at this point we'll need to resort to basic limit properties to compute the one-sided limits and not just plug in the point (which assumes continuity again...).

Here is the work for this part.

$$\begin{aligned} \lim_{x \rightarrow 6^-} g(x) &= \lim_{x \rightarrow 6^-} (2x) = 2 \lim_{x \rightarrow 6^-} x = 2(6) = 12 \\ \lim_{x \rightarrow 6^+} g(x) &= \lim_{x \rightarrow 6^+} (x-1) = \lim_{x \rightarrow 6^+} x - \lim_{x \rightarrow 6^+} 1 = 6 - 1 = 5 \end{aligned}$$

So, we can see that, $\lim_{x \rightarrow 6^-} g(x) \neq \lim_{x \rightarrow 6^+} g(x)$ and so $\lim_{x \rightarrow 6} g(x)$ does not exist.

Now, as discussed in the notes for this section, in order for a function to be continuous at a point both the function and the limit must exist. Therefore, this function is **not continuous** at $x = 6$ because $\lim_{x \rightarrow 6} g(x)$ does not exist.

6. Using only Properties 1- 9 from the **Limit Properties** section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a) $t = -2$** , **(b) $t = 10$** ?

$$h(t) = \begin{cases} t^2 & t < -2 \\ t+6 & t \geq -2 \end{cases}$$

(a) $t = -2$

Before starting off with the solution to this part notice that we CAN NOT do what we've commonly done to evaluate limits to this point. In other words, we can't just plug in the point to evaluate the limit. Doing this implicitly assumes that the function is continuous at the point and that is what we are being asked to determine here.

Therefore, the only way for us to compute the limit is to go back to the properties from the Limit Properties section and compute the limit as we did back in that section. We won't be putting all the details here so if you need a little refresher on doing this you should go back to the problems from that section and work a few of them.

Also notice that for this part we have the added complication that the point we're interested in is also the "cut-off" point of the piecewise function and so we'll need to take a look at the two one sided limits to compute the overall limit and again because we are being asked to determine if the function is continuous at this point we'll need to resort to basic limit properties to compute the one-sided limits and not just plug in the point (which assumes continuity again...).

Here is the work for this part.

$$\begin{aligned} \lim_{t \rightarrow -2^-} h(t) &= \lim_{t \rightarrow -2^-} t^2 = (-2)^2 = 4 \\ \lim_{t \rightarrow -2^+} g(t) &= \lim_{t \rightarrow -2^+} (t+6) = \lim_{t \rightarrow -2^+} t + \lim_{t \rightarrow -2^+} 6 = -2 + 6 = 4 \end{aligned}$$

So, we can see that $\lim_{t \rightarrow -2^-} h(t) = \lim_{t \rightarrow -2^+} h(t) = 4$ and so $\lim_{t \rightarrow -2} h(t) = 4$.

Next, a quick computation shows us that $h(-2) = -2 + 6 = 4$ and so we can see that

$\lim_{t \rightarrow -2} h(t) = h(-2)$ and so the function is **continuous** at $t = -2$.

(b) $t = 10$

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**.

For this part we can notice that because there are values of t on both sides of $t = 10$ in the range $t \geq -2$ we won't need to worry about one-sided limits here. Here is the work for this part.

Here is the work for this part.

$$\lim_{t \rightarrow 10} h(t) = \lim_{t \rightarrow 10} (t+6) = \lim_{t \rightarrow 10} t + \lim_{t \rightarrow 10} 6 = 10 + 6 = h(10)$$

So, we can see that $\lim_{t \rightarrow 10} h(t) = h(10)$ and so the function is **continuous** at $t = 10$.

7. Using only Properties 1- 9 from the **Limit Properties** section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a)** $x = -6$, **(b)** $x = 1$?

$$g(x) = \begin{cases} 1-3x & x < -6 \\ 7 & x = -6 \\ x^3 & -6 < x < 1 \\ 1 & x = 1 \\ 2-x & x > 1 \end{cases}$$

(a) $x = -6$

Before starting off with the solution to this part notice that we CAN NOT do what we've commonly done to evaluate limits to this point. In other words, we can't just plug in the point to evaluate the limit.

Doing this implicitly assumes that the function is continuous at the point and that is what we are being asked to determine here.

Therefore, the only way for us to compute the limit is to go back to the properties from the Limit Properties section and compute the limit as we did back in that section. We won't be putting all the details here so if you need a little refresher on doing this you should go back to the problems from that section and work a few of them.

Also notice that for this part we have the added complication that the point we're interested in is also the "cut-off" point of the piecewise function and so we'll need to take a look at the two one sided limits to compute the overall limit and again because we are being asked to determine if the function is continuous at this point we'll need to resort to basic limit properties to compute the one-sided limits and not just plug in the point (which assumes continuity again...).

Here is the work for this part.

$$\begin{aligned} \lim_{x \rightarrow -6^-} g(x) &= \lim_{x \rightarrow -6^-} (1-3x) = \lim_{x \rightarrow -6^-} 1 - 3 \lim_{x \rightarrow -6^-} x = 1 - 3(-6) = 19 \\ \lim_{x \rightarrow -6^+} g(x) &= \lim_{x \rightarrow -6^+} x^3 = (-6)^3 = -216 \end{aligned}$$

So, we can see that, $\lim_{x \rightarrow -6^-} g(x) \neq \lim_{x \rightarrow -6^+} g(x)$ and so $\lim_{x \rightarrow -6} g(x)$ does not exist.

Now, as discussed in the notes for this section, in order for a function to be continuous at a point both the function and the limit must exist. Therefore, this function **is not continuous** at $x = -6$ because

$\lim_{x \rightarrow -6} g(x)$ does not exist.

(b) $x = 1$

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**.

Again, note that we are dealing with another “cut-off” point here so we’ll need to use one-sided limits again as we did in the previous part.

Here is the work for this part.

$$\begin{aligned}\lim_{x \rightarrow 1^-} g(x) &= \lim_{x \rightarrow 1^-} x^3 = 1^3 = 1 \\ \lim_{x \rightarrow 1^+} g(x) &= \lim_{x \rightarrow 1^+} (2 - x) = \lim_{x \rightarrow 1^+} 2 - \lim_{x \rightarrow 1^+} x = 2 - 1 = 1\end{aligned}$$

So, we can see that, $\lim_{x \rightarrow 1^-} g(x) = \lim_{x \rightarrow 1^+} g(x) = 1$ and so $\lim_{x \rightarrow 1} g(x) = 1$.

Next, a quick computation shows us that $g(1) = 1$ and so we can see that $\lim_{x \rightarrow 1} g(x) = g(1)$ and so the function **is continuous** at $x = 1$.

8. Determine where the following function is discontinuous.

$$f(x) = \frac{x^2 - 9}{3x^2 + 2x - 8}$$

Hint : If we have two continuous functions and form a rational expression out of them recall where the rational expression will be discontinuous. We discussed this in the Limit Properties section, although we were using the phrase “nice enough” there instead of the word “continuity”.

Solution

As noted in the hint for this problem when dealing with a rational expression in which both the numerator and denominator are continuous (as we have here since both are polynomials) the only points in which the rational expression will be discontinuous will be where we have division by zero.

Therefore, all we need to do is determine where the denominator is zero and that is fairly easy for this problem.

$$3x^2 + 2x - 8 = (3x - 4)(x + 2) = 0 \quad \Rightarrow \quad x = \frac{4}{3}, x = -2$$

The function will therefore be discontinuous at the points : $x = \frac{4}{3}$ and $x = -2$.

9. Determine where the following function is discontinuous.

$$R(t) = \frac{8t}{t^2 - 9t - 1}$$

Hint : If we have two continuous functions and form a rational expression out of them recall where the rational expression will be discontinuous. We discussed this in the Limit Properties section, although we were using the phrase “nice enough” there instead of the word “continuity”.

Solution

As noted in the hint for this problem when dealing with a rational expression in which both the numerator and denominator are continuous (as we have here since both are polynomials) the only points in which the rational expression will be discontinuous will be where we have division by zero.

Therefore, all we need to do is determine where the denominator is zero and that is fairly easy for this problem.

$$t^2 - 9t - 1 = 0 \quad \Rightarrow \quad t = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(-1)}}{2(1)} = \frac{9 \pm \sqrt{85}}{2} = -0.10977, 9.10977$$

The function will therefore be discontinuous at the points : $t = \frac{9 \pm \sqrt{85}}{2}$.

10. Determine where the following function is discontinuous.

$$h(z) = \frac{1}{2 - 4\cos(3z)}$$

Hint : If we have two continuous functions and form a rational expression out of them recall where the rational expression will be discontinuous. We discussed this in the Limit Properties section, although we were using the phrase “nice enough” there instead of the word “continuity”.

Solution

As noted in the hint for this problem when dealing with a rational expression in which both the numerator and denominator are continuous (as we have here since the numerator is just a constant and the denominator is a sum of continuous functions) the only points in which the rational expression will be discontinuous will be where we have division by zero.

Therefore, all we need to do is determine where the denominator is zero. If you don't recall how to solve equations involving trig functions you should go back to the Review chapter and take a look at the Solving Trig Equations sections there.

Here is the solution work for determining where the denominator is zero. Using our calculator we get,

$$2 - 4\cos(3z) = 0 \quad \rightarrow \quad 3z = \cos^{-1}\left(\frac{1}{2}\right) = 1.0472$$

The second angle will be in the fourth quadrant and is $2\pi - 1.0472 = 5.2360$.

The denominator will therefore be zero at,

$$\begin{array}{llll}
 3x = 1.0472 + 2\pi n & \text{OR} & 3x = 5.2360 + 2\pi n & n = 0, \pm 1, \pm 2, \dots \\
 x = 0.3491 + \frac{2\pi n}{3} & \text{OR} & x = 1.7453 + \frac{2\pi n}{3} & n = 0, \pm 1, \pm 2, \dots
 \end{array}$$

The function will therefore be discontinuous at the points,

$$x = 0.3491 + \frac{2\pi n}{3} \quad \text{OR} \quad x = 1.7453 + \frac{2\pi n}{3} \quad n = 0, \pm 1, \pm 2, \dots$$

Note as well that this was one of the few trig equations that could be solved exactly if you know your basic unit circle values. Here is the exact solution for the points of discontinuity.

$$x = \frac{\pi}{9} + \frac{2\pi n}{3} \quad \text{OR} \quad x = \frac{5\pi}{9} + \frac{2\pi n}{3} \quad n = 0, \pm 1, \pm 2, \dots$$

11. Determine where the following function is discontinuous.

$$y(x) = \frac{x}{7 - e^{2x+3}}$$

Hint : If we have two continuous functions and form a rational expression out of them recall where the rational expression will be discontinuous. We discussed this in the Limit Properties section, although we were using the phrase “nice enough” there instead of the word “continuity”.

Solution

As noted in the hint for this problem when dealing with a rational expression in which both the numerator and denominator are continuous (as we have here since the numerator is a polynomial and the denominator is a sum of two continuous functions) the only points in which the rational expression will be discontinuous will be where we have division by zero.

Therefore, all we need to do is determine where the denominator is zero and that is fairly easy for this problem.

$$7 - e^{2x+3} = 0 \rightarrow e^{2x+3} = 7 \rightarrow 2x + 3 = \ln(7) \Rightarrow x = \frac{1}{2}(\ln(7) - 3) = -0.5270$$

The function will therefore be discontinuous at : $x = \frac{1}{2}(\ln(7) - 3) = -0.5270$.

12. Determine where the following function is discontinuous.

$$g(x) = \tan(2x)$$

Hint : If we have two continuous functions and form a rational expression out of them recall where the rational expression will be discontinuous. We discussed this in the Limit Properties section, although we

were using the phrase “nice enough” there instead of the word “continuity”. And, yes we really do have a rational expression here...

Solution

As noted in the hint for this problem when dealing with a rational expression in which both the numerator and denominator are continuous the only points in which the rational expression will not be continuous will be where we have division by zero.

Also, writing the function as,

$$g(x) = \frac{\sin(2x)}{\cos(2x)}$$

we can see that we really do have a rational expression here. Therefore, all we need to do is determine where the denominator (*i.e.* cosine) is zero. If you don’t recall how to solve equations involving trig functions you should go back to the Review chapter and take a look at the Solving Trig Equations sections there.

Here is the solution work for determining where the denominator is zero. Using our basic unit circle knowledge we know where cosine will be zero so we have,

$$2x = \frac{\pi}{2} + 2\pi n \quad \text{OR} \quad 2x = \frac{3\pi}{2} + 2\pi n \quad n = 0, \pm 1, \pm 2, \dots$$

The denominator will therefore be zero, and the function will be discontinuous, at,

$$x = \frac{\pi}{4} + \pi n \quad \text{OR} \quad x = \frac{3\pi}{4} + \pi n \quad n = 0, \pm 1, \pm 2, \dots$$

13. Use the Intermediate Value Theorem to show that $25 - 8x^2 - x^3 = 0$ has at least one root in the interval $[-2, 4]$. Note that you are NOT asked to find the solution only show that at least one must exist in the indicated interval,

Hint : The hardest part of these problems for most students is just getting started.

First, you need to determine the value of “ M ” that you need to use and then actually use the Intermediate Value Theorem. So, go back to the IVT and compare the conclusions of the theorem and it should be pretty obvious what the M should be and then just check that the hypothesis (*i.e.* the “requirements” of the theorem) are met and you’ll pretty much be done.

Solution

Okay, let’s start off by defining,

$$f(x) = 25 - 8x^2 - x^3 \quad \& \quad M = 0$$

The problem is then asking us to show that there is a c in $[-2, 4]$ so that,

$$f(c) = 0 = M$$

but this is exactly the second conclusion of the Intermediate Value Theorem. So, let's see that the "requirements" of the theorem are met.

First, the function is a polynomial and so is continuous everywhere and in particular is continuous on the interval $[-2, 4]$. Note that this IS a requirement that MUST be met in order to use the IVT and it is the one requirement that is most often overlooked. If we don't have a continuous function the IVT simply can't be used.

Now all that we need to do is verify that M is between the function values as the endpoints of the interval. So,

$$f(-2) = 1 \qquad f(4) = -167$$

Therefore, we have,

$$f(4) = -167 < 0 < 1 = f(-2)$$

So, by the Intermediate Value Theorem there must be a number c such that,

$$-2 < c < 4 \qquad \& \qquad f(c) = 0$$

and we have shown what we were asked to show.

14. Use the Intermediate Value Theorem to show that $w^2 - 4 \ln(5w + 2) = 0$ has at least one root in the interval $[0, 4]$. Note that you are NOT asked to find the solution only show that at least one must exist in the indicated interval,

Hint : The hardest part of these problems for most students is just getting started.

First, you need to determine the value of " M " that you need to use and then actually use the Intermediate Value Theorem. So, go back to the IVT and compare the conclusions of the theorem and it should be pretty obvious what the M should be and then just check that the hypothesis (*i.e.* the "requirements" of the theorem) are met and you'll pretty much be done.

Solution

Okay, let's start off by defining,

$$f(w) = w^2 - 4 \ln(5w + 2) \qquad \& \qquad M = 0$$

The problem is then asking us to show that there is a c in $[-2, 4]$ so that,

$$f(c) = 0 = M$$

but this is exactly the second conclusion of the Intermediate Value Theorem. So, let's see that the "requirements" of the theorem are met.

First, the function is a sum of a polynomial (which is continuous everywhere) and a natural logarithm (which is continuous on $w > -\frac{2}{5}$ - *i.e.* where the argument is positive) and so is continuous on the

interval $[0, 4]$. Note that this IS a requirement that MUST be met in order to use the IVT and it is the one requirement that is most often overlooked. If we don't have a continuous function the IVT simply can't be used.

Now all that we need to do is verify that M is between the function values as the endpoints of the interval. So,

$$f(0) = -2.7726 \qquad f(4) = 3.6358$$

Therefore, we have,

$$f(0) = -2.7726 < 0 < 3.6358 = f(4)$$

So, by the Intermediate Value Theorem there must be a number c such that,

$$0 < c < 4 \qquad \& \qquad f(c) = 0$$

and we have shown what we were asked to show.

15. Use the Intermediate Value Theorem to show that $4t + 10e^t - e^{2t} = 0$ has at least one root in the interval $[1, 3]$. Note that you are NOT asked to find the solution only show that at least one must exist in the indicated interval,

Hint : The hardest part of these problems for most students is just getting started.

First, you need to determine the value of " M " that you need to use and then actually use the Intermediate Value Theorem. So, go back to the IVT and compare the conclusions of the theorem and it should be pretty obvious what the M should be and then just check that the hypothesis (*i.e.* the "requirements" of the theorem) are met and you'll pretty much be done.

Solution

Okay, let's start off by defining,

$$f(t) = 4t + 10e^t - e^{2t} \qquad \& \qquad M = 0$$

The problem is then asking us to show that there is a c in $[-2, 4]$ so that,

$$f(c) = 0 = M$$

but this is exactly the second conclusion of the Intermediate Value Theorem. So, let's see that the "requirements" of the theorem are met.

First, the function is a sum and difference of a polynomial and two exponentials (all of which are continuous everywhere) and so is continuous on the interval $[1, 3]$. Note that this IS a requirement that MUST be met in order to use the IVT and it is the one requirement that is most often overlooked. If we don't have a continuous function the IVT simply can't be used.

Now all that we need to do is verify that M is between the function values at the endpoints of the interval. So,

$$f(1) = 23.7938$$

$$f(3) = -190.5734$$

Therefore, we have,

$$f(3) = -190.5734 < 0 < 23.7938 = f(1)$$

So, by the Intermediate Value Theorem there must be a number c such that,

$$1 < c < 3 \quad \& \quad f(c) = 0$$

and we have shown what we were asked to show.

Section 2-10 : The Definition of the Limit

1. Use the definition of the limit to prove the following limit.

$$\lim_{x \rightarrow 3} x = 3$$

Step 1

First, let's just write out what we need to show.

Let $\varepsilon > 0$ be any number. We need to find a number $\delta > 0$ so that,

$$|x - 3| < \varepsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta$$

This problem can look a little tricky since the two inequalities both involve $|x - 3|$. Just keep in mind that the first one is really $|f(x) - L| < \varepsilon$ where $f(x) = x$ and $L = 3$ and the second is really $0 < |x - a| < \delta$ where $a = 3$.

Step 2

In this case, despite the "trickiness" of the statement we need to prove in Step 1, this is really a very simple problem.

We need to determine a δ that will allow us to prove the statement in Step 1. However, because both inequalities involve exactly the same absolute value statement so all we need to do is choose $\delta = \varepsilon$.

Step 3

So, let's see if this works.

Start off by first assuming that $\varepsilon > 0$ is any number and choose $\delta = \varepsilon$. We can now assume that

$$0 < |x - 3| < \delta = \varepsilon \quad \Rightarrow \quad 0 < |x - 3| < \varepsilon$$

However, if we just look at the right portion of the double inequality we see that this assumption tells us that,

$$|x - 3| < \varepsilon$$

which is exactly what we needed to show given our choice of δ .

Therefore, according to the definition of the limit we have just proved that,

$$\lim_{x \rightarrow 3} x = 3$$

2. Use the definition of the limit to prove the following limit.

$$\lim_{x \rightarrow -1} (x + 7) = 6$$

Step 1

First, let's just write out what we need to show.

Let $\varepsilon > 0$ be any number. We need to find a number $\delta > 0$ so that,

$$|(x + 7) - 6| < \varepsilon \quad \text{whenever} \quad 0 < |x - (-1)| < \delta$$

Or, with a little simplification this becomes,

$$|x + 1| < \varepsilon \quad \text{whenever} \quad 0 < |x + 1| < \delta$$

Step 2

This problem is very similar to Problem 1 from this point on.

We need to determine a δ that will allow us to prove the statement in Step 1. However, because both inequalities involve exactly the same absolute value statement all we need to do is choose $\delta = \varepsilon$.

Step 3

So, let's see if this works.

Start off by first assuming that $\varepsilon > 0$ is any number and choose $\delta = \varepsilon$. We can now assume that,

$$0 < |x - (-1)| < \delta = \varepsilon \quad \Rightarrow \quad 0 < |x + 1| < \varepsilon$$

This gives,

$$\begin{aligned} |(x + 7) - 6| &= |x + 1| && \text{simplify things up a little} \\ &< \varepsilon && \text{using the information we got by assuming } \delta = \varepsilon \end{aligned}$$

So, we've shown that,

$$|(x + 7) - 6| < \varepsilon \quad \text{whenever} \quad 0 < |x - (-1)| < \varepsilon$$

and so by the definition of the limit we have just proved that,

$$\lim_{x \rightarrow -1} (x + 7) = 6$$

3. Use the definition of the limit to prove the following limit.

$$\lim_{x \rightarrow 2} x^2 = 4$$

Step 1

First, let's just write out what we need to show.

Let $\varepsilon > 0$ be any number. We need to find a number $\delta > 0$ so that,

$$|x^2 - 4| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta$$

Step 2

Let's start with a little simplification of the first inequality.

$$|x^2 - 4| = |(x + 2)(x - 2)| = |x + 2||x - 2| < \varepsilon$$

We have the $|x - 2|$ we expect to see but we also have an $|x + 2|$ that we'll need to deal with.

Step 3

To deal with the $|x + 2|$ let's first assume that

$$|x - 2| < 1$$

As we noted in a similar example in the notes for this section this is a legitimate assumption because the limit is $x \rightarrow 2$ and so x 's will be getting very close to 2. Therefore, provided x is close enough to 2 we will have $|x - 2| < 1$.

Starting with this assumption we get that,

$$-1 < x - 2 < 1 \quad \rightarrow \quad 1 < x < 3$$

If we now add 2 to all parts of this inequality we get,

$$3 < x + 2 < 5$$

Noticing that $3 > 0$ we can see that we then also know that $x + 2 > 0$ and so provided $|x - 2| < 1$ we will have $x + 2 = |x + 2|$.

All this means is that, provided $|x - 2| < 1$, we will also have,

$$|x+2| = x+2 < 5 \quad \rightarrow \quad |x+2| < 5$$

This in turn means that we have,

$$|x+2||x-2| < 5|x-2| \quad \text{because } |x+2| < 5$$

Therefore, if we were to further assume, for some reason, that we wanted $5|x-2| < \varepsilon$ this would tell us that,

$$|x-2| < \frac{\varepsilon}{5}$$

Step 4

Okay, even though it doesn't seem like it we actually have enough to make a choice for δ .

Given any number $\varepsilon > 0$ let's chose

$$\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$$

Again, this means that δ will be the smaller of the two values which in turn means that,

$$\delta \leq 1 \quad \text{AND} \quad \delta \leq \frac{\varepsilon}{5}$$

Now assume that $0 < |x-2| < \delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$.

Step 5

So, let's see if this works.

Given the assumption $0 < |x-2| < \delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$ we know two things. First, we know that $|x-2| < \frac{\varepsilon}{5}$. Second, we also know that $|x-2| < 1$ which in turn implies that $|x+2| < 5$ as we saw in Step 3.

Now, let's do the following,

$$\begin{aligned} |x^2 - 4| &= |x+2||x-2| && \text{factoring} \\ &< 5|x-2| && \text{because we know } |x+2| < 5 \\ &< 5\left(\frac{\varepsilon}{5}\right) && \text{because we know } |x-2| < \frac{\varepsilon}{5} \\ &= \varepsilon \end{aligned}$$

So, we've shown that,

$$|x^2 - 4| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \min \left\{ 1, \frac{\varepsilon}{5} \right\}$$

and so by the definition of the limit we have just proved that,

$$\lim_{x \rightarrow 2} x^2 = 4$$

4. Use the definition of the limit to prove the following limit.

$$\lim_{x \rightarrow -3} (x^2 + 4x + 1) = -2$$

Step 1

First, let's just write out what we need to show.

Let $\varepsilon > 0$ be any number. We need to find a number $\delta > 0$ so that,

$$|x^2 + 4x + 1 - (-2)| < \varepsilon \quad \text{whenever} \quad 0 < |x - (-3)| < \delta$$

Simplifying this a little gives,

$$|x^2 + 4x + 3| < \varepsilon \quad \text{whenever} \quad 0 < |x + 3| < \delta$$

Step 2

Let's start with a little simplification of the first inequality.

$$|x^2 + 4x + 3| = |(x+1)(x+3)| = |x+1||x+3| < \varepsilon$$

We have the $|x+3|$ we expect to see but we also have an $|x+1|$ that we'll need to deal with.

Step 3

To deal with the $|x+1|$ let's first assume that

$$|x+3| < 1$$

As we noted in a similar example in the notes for this section this is a legitimate assumption because the limit is $x \rightarrow -3$ and so x 's will be getting very close to -3. Therefore, provided x is close enough to -3 we will have $|x+3| < 1$.

Starting with this assumption we get that,

$$-1 < x + 3 < 1 \quad \rightarrow \quad -4 < x < -2$$

If we now add 1 to all parts of this inequality we get,

$$-3 < x + 1 < -1$$

Noticing that $-1 < 0$ we can see that we then also know that $x + 1 < 0$ and so provided $|x + 3| < 1$ we will have $|x + 1| = -(x + 1)$. Also, from the inequality above we see that,

$$1 < -(x + 1) < 3$$

All this means is that, provided $|x + 3| < 1$, we will also have,

$$|x + 1| = -(x + 1) < 3 \quad \rightarrow \quad |x + 1| < 3$$

This in turn means that we have,

$$|x + 1||x + 3| < 3|x + 3| \quad \text{because } |x + 1| < 3$$

Therefore, if we were to further assume, for some reason, that we wanted $3|x + 3| < \varepsilon$ this would tell us that,

$$|x + 3| < \frac{\varepsilon}{3}$$

Step 4

Okay, even though it doesn't seem like it we actually have enough to make a choice for δ .

Given any number $\varepsilon > 0$ let's chose

$$\delta = \min \left\{ 1, \frac{\varepsilon}{3} \right\}$$

Again, this means that δ will be the smaller of the two values which in turn means that,

$$\delta \leq 1 \quad \text{AND} \quad \delta \leq \frac{\varepsilon}{3}$$

Now assume that $0 < |x + 3| < \delta = \min \left\{ 1, \frac{\varepsilon}{3} \right\}$.

Step 5

So, let's see if this works.

Given the assumption $0 < |x+3| < \delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$ we know two things. First, we know that $|x+3| < \frac{\varepsilon}{3}$. Second, we also know that $|x+3| < 1$ which in turn implies that $|x+1| < 3$ as we saw in Step 3.

Now, let's do the following,

$$\begin{aligned} |x^2 + 4x + 3| &= |x+1||x+3| && \text{factoring} \\ &< 3|x+3| && \text{because we know } |x+1| < 3 \\ &< 3\left(\frac{\varepsilon}{3}\right) && \text{because we know } |x+3| < \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

So, we've shown that,

$$|x^2 + 4x + 3| < \varepsilon \quad \text{whenever} \quad 0 < |x+3| < \min\left\{1, \frac{\varepsilon}{3}\right\}$$

and so by the definition of the limit we have just proved that,

$$\lim_{x \rightarrow -3} (x^2 + 4x + 1) = -2$$

5. Use the definition of the limit to prove the following limit.

$$\lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = \infty$$

Step 1

First, let's just write out what we need to show.

Let $M > 0$ be any number. We need to find a number $\delta > 0$ so that,

$$\frac{1}{(x-1)^2} > M \quad \text{whenever} \quad 0 < |x-1| < \delta$$

Step 2

Let's do a little rewrite the first inequality above a little bit.

$$\frac{1}{(x-1)^2} > M \quad \rightarrow \quad (x-1)^2 < \frac{1}{M} \quad \rightarrow \quad |x-1| < \frac{1}{\sqrt{M}}$$

From this it looks like we can choose $\delta = \frac{1}{\sqrt{M}}$.

Step 3

So, let's see if this works.

We'll start by assuming that $M > 0$ is any number and chose $\delta = \frac{1}{\sqrt{M}}$. We can now assume that,

$$0 < |x-1| < \delta = \frac{1}{\sqrt{M}} \quad \Rightarrow \quad 0 < |x-1| < \frac{1}{\sqrt{M}}$$

So, if we start with the second inequality we get,

$$\begin{aligned} |x-1| &< \frac{1}{\sqrt{M}} \\ |x-1|^2 &< \frac{1}{M} && \text{squaring both sides} \\ (x-1)^2 &< \frac{1}{M} && \text{because } |x-1|^2 = (x-1)^2 \\ \frac{1}{(x-1)^2} &> M && \text{rewriting things a little bit} \end{aligned}$$

So, we've shown that,

$$\frac{1}{(x-1)^2} > M \quad \text{whenever} \quad 0 < |x-1| < \frac{1}{\sqrt{M}}$$

and so by the definition of the limit we have just proved that,

$$\lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = \infty$$

6. Use the definition of the limit to prove the following limit.

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

Step 1

First, let's just write out what we need to show.

Let $N < 0$ be any number. Remember that because our limit is going to negative infinity here we need N to be negative. Now, we need to find a number $\delta > 0$ so that,

$$\frac{1}{x} < N \quad \text{whenever} \quad -\delta < x - 0 < 0$$

Step 2

Let's do a little rewrite on the first inequality above to get,

$$\frac{1}{x} < N \quad \rightarrow \quad x > \frac{1}{N}$$

Now, keep in mind that N is negative and so $\frac{1}{N}$ is also negative. From this it looks like we can choose $\delta = -\frac{1}{N}$. Again, because N is negative this makes δ positive, which we need!

Step 3

So, let's see if this works.

We'll start by assuming that $N < 0$ is any number and chose $\delta = -\frac{1}{N}$. We can now assume that,

$$-\delta < x - 0 < 0 \quad \Rightarrow \quad \frac{1}{N} < x < 0$$

So, if we start with the second inequality we get,

$$\begin{aligned} x &> \frac{1}{N} \\ \frac{1}{x} &< N \end{aligned} \quad \text{rewriting things a little bit}$$

So, we've shown that,

$$\frac{1}{x} < N \quad \text{whenever} \quad \frac{1}{N} < x < 0$$

and so by the definition of the limit we have just proved that,

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

7. Use the definition of the limit to prove the following limit.

$$\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$$

Step 1

First, let's just write out what we need to show.

Let $\varepsilon > 0$ be any number. We need to find a number $M > 0$ so that,

$$\left| \frac{1}{x^2} - 0 \right| < \varepsilon \quad \text{whenever} \quad x > M$$

Or, with a little simplification this becomes,

$$\left| \frac{1}{x^2} \right| < \varepsilon \quad \text{whenever} \quad x > M$$

Step 2

Let's start with the inequality on the left and do a little rewriting on it.

$$\left| \frac{1}{x^2} \right| < \varepsilon \quad \rightarrow \quad \frac{1}{|x|^2} < \varepsilon \quad \rightarrow \quad |x|^2 > \frac{1}{\varepsilon} \quad \rightarrow \quad |x| > \frac{1}{\sqrt{\varepsilon}}$$

From this it looks like we can choose $M = \frac{1}{\sqrt{\varepsilon}}$

Step 3

So, let's see if this works.

Start off by first assuming that $\varepsilon > 0$ is any number and choose $M = \frac{1}{\sqrt{\varepsilon}}$. We can now assume that,

$$x > \frac{1}{\sqrt{\varepsilon}}$$

Starting with this inequality we get,

$$\begin{array}{ll} x > \frac{1}{\sqrt{\varepsilon}} & \\ \frac{1}{x} < \sqrt{\varepsilon} & \text{do a little rewrite} \\ \frac{1}{x^2} < \varepsilon & \text{square both sides} \\ \left| \frac{1}{x^2} \right| < \varepsilon & \text{because } \frac{1}{x^2} = \left| \frac{1}{x^2} \right| \end{array}$$

So, we've shown that,

$$\left| \frac{1}{x^2} - 0 \right| < \varepsilon \quad \text{whenever} \quad x > M$$

and so by the definition of the limit we have just proved that,

$$\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$$
