5.2 | Unit Circle: Sine and Cosine Functions

Learning Objectives

In this section, you will:

5.2.1 Find function values for the sine and cosine of $30°$ or $\left(\frac{\pi}{6}\right)$, $45°$ or $\left(\frac{\pi}{4}\right)$ and $60°$ or $\left(\frac{\pi}{3}\right)$.

5.2.2 Identify the domain and range of sine and cosine functions.

5.2.3 Use reference angles to evaluate trigonometric functions.

Figure 5.28 The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.

Finding Function Values for the Sine and Cosine

To define our trigonometric functions, we begin by drawing a unit circle, a circle centered at the origin with radius 1, as shown in **Figure 5.29**. The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a unit circle, $s = t$.

Recall that the x- and y-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y). The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.

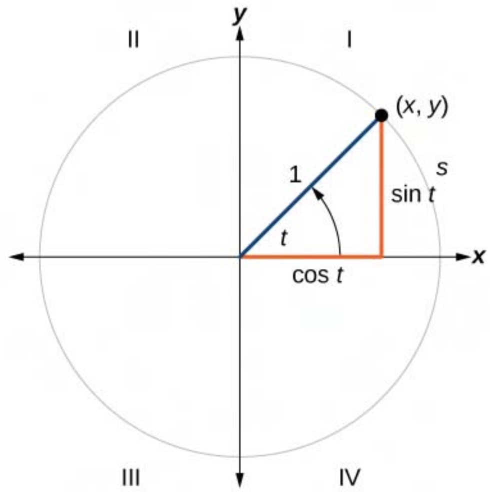


Figure 5.29 Unit circle where the central angle is t radians

Unit Circle

A **unit circle** has a center at $(0, 0)$ and radius 1 . In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle 1.

Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions

Now that we have our unit circle labeled, we can learn how the (x, y) coordinates relate to the arc length and angle. The **sine function** relates a real number t to the y-coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t. In **Figure 5.29**, the sine is equal to y. Like all functions, the sine function has an input and an output. Its input is the measure of the angle; its output is the y-coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the x-value of the endpoint on the unit circle of an arc of length t. In **Figure 5.30**, the cosine is equal to x.

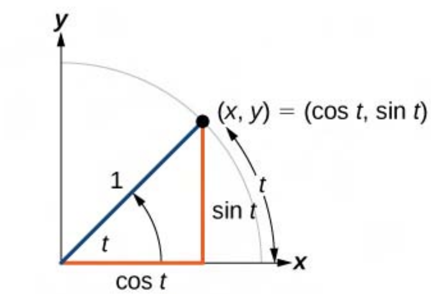


Figure 5.30

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

Sine and Cosine Functions

If t is a real number and a point (x, y) on the unit circle corresponds to an angle of t, then

$$\cos t = x \tag{5.6}$$
$$\sin t = y \tag{5.7}$$

How To: **Given a point P (x, y) on the unit circle corresponding to an angle of t, find the sine and cosine.**

1. The sine of t is equal to the y-coordinate of point P : $\sin t = y$.
2. The cosine of t is equal to the x-coordinate of point P : $\cos t = x$.

Example 5.12

Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t, as shown in **Figure 5.31**. Find $\cos(t)$ and $\sin(t)$.

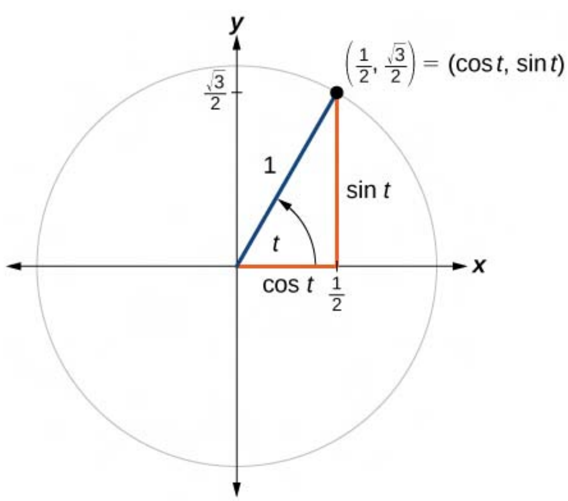


Figure 5.31

Solution

We know that $\cos t$ is the x-coordinate of the corresponding point on the unit circle and $\sin t$ is the y-coordinate of the corresponding point on the unit circle. So:

$$x = \cos t = \frac{1}{2}$$

$$y = \sin t = \frac{\sqrt{3}}{2}$$

5.12 A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in **Figure 5.32**. Find $\cos t$ and $\sin t$.

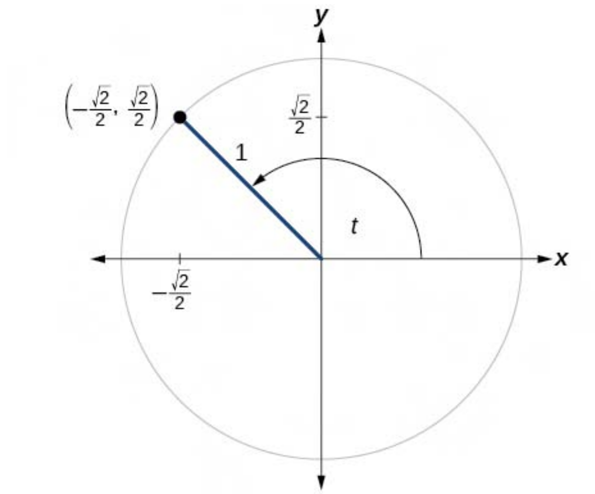


Figure 5.32

Finding Sines and Cosines of Angles on an Axis

For quadrantal angles, the corresponding point on the unit circle falls on the *x*- or *y*-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Example 5.13

Calculating Sines and Cosines along an Axis

Find $\cos(90°)$ and $\sin(90°)$.

Solution

Moving $90°$ counterclockwise around the unit circle from the positive *x*-axis brings us to the top of the circle, where the (x, y) coordinates are (0, 1), as shown in **Figure 5.33**.

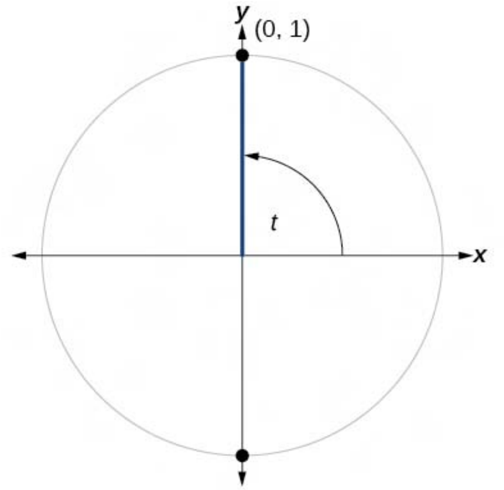


Figure 5.33

Using our definitions of cosine and sine,

$$x = \cos t = \cos(90°) = 0$$
$$y = \sin t = \sin(90°) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

Try It **5.13** Find cosine and sine of the angle π.

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See Figure 5.34.

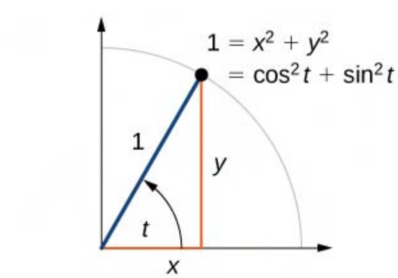


Figure 5.34

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

Pythagorean Identity

The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1 \tag{5.8}$$

How To: Given the sine of some angle t and its quadrant location, find the cosine of t.

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the *x*-values in the quadrant where t is located.

Example 5.14

Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Solution

If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See **Figure 5.35**.

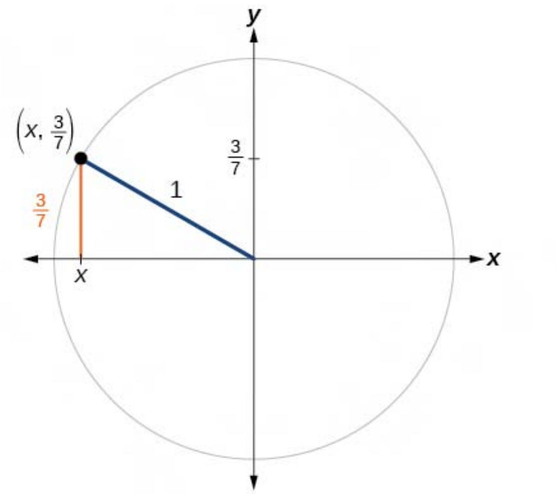


Figure 5.35

Substituting the known value for sine into the Pythagorean Identity,

$$\begin{aligned} \cos^2(t) + \sin^2(t) &= 1 \\ \cos^2(t) + \frac{9}{49} &= 1 \\ \cos^2(t) &= \frac{40}{49} \\ \cos(t) &= \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7} \end{aligned}$$

Because the angle is in the second quadrant, we know the *x*-value is a negative real number, so the cosine is also negative. So $\cos(t) = -\frac{2\sqrt{10}}{7}$

Try It **5.14** If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees. We can also calculate sines and cosines of the special angles using the Pythagorean Identity and our knowledge of triangles.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of $45°$ or $\frac{\pi}{4}$, as shown in **Figure 5.36**. A $45° - 45° - 90°$ triangle is an isosceles triangle, so the *x*- and *y*-coordinates of the corresponding point on the circle are the same. Because the *x*- and *y*-values are the same, the sine and cosine values will also be equal.

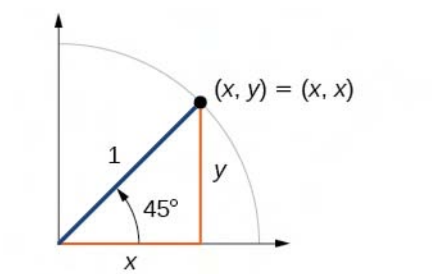


Figure 5.36

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first quadrantal angle. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y $(y = x)$, and a radius = 1. See **Figure 5.37**.

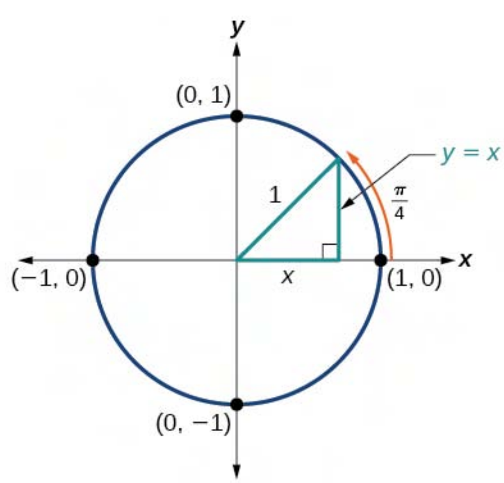


Figure 5.37

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

Substituting $y = x,$ we get

$$x^2 + x^2 = 1$$

Combining like terms we get

$$2x^2 = 1$$

And solving for $x,$ we get

$$\begin{aligned} x^2 &= \frac{1}{2} \\ x &= \pm\frac{1}{\sqrt{2}} \end{aligned}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}.$

At $t = \frac{\pi}{4}$ or 45 degrees,

$$\begin{aligned} &(x,\ y) = (x,\ x) = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right) \\ &x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}} \\ &\cos t = \frac{1}{\sqrt{2}},\ \sin t = \frac{1}{\sqrt{2}} \end{aligned}$$

If we then rationalize the denominators, we get

$$\begin{aligned} \cos t &= \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \\ \sin t &= \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

Therefore, the $(x,\ y)$ coordinates of a point on a circle of radius 1 at an angle of $45°$ are $\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right).$

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of $30°,$ or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of $30°,$ and another at an angle of $-30°,$ as shown in **Figure 5.38**. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be $60°,$ as shown in **Figure 5.39**.

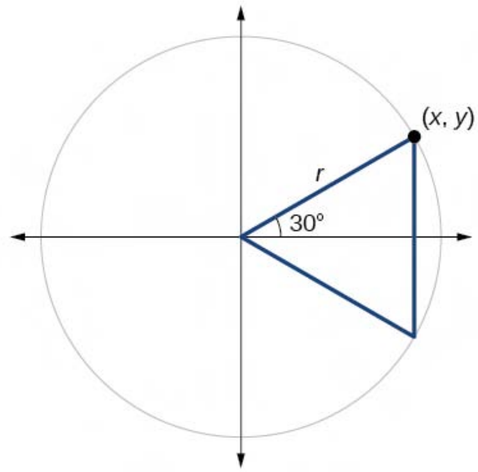


Figure 5.38

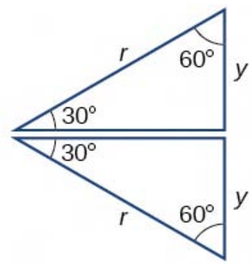


Figure 5.39

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}r$$

And since $r = 1$ in our unit circle,

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1)\\ &= \frac{1}{2}\end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\begin{aligned}\cos^2\frac{\pi}{6} + \sin^2\left(\frac{\pi}{6}\right) &= 1\\ \cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 &= 1\\ \cos^2\left(\frac{\pi}{6}\right) &= \frac{3}{4} && \text{Use the square root property.}\\ \cos\left(\frac{\pi}{6}\right) &= \frac{\pm\sqrt{3}}{\pm\sqrt{4}} = \frac{\sqrt{3}}{2} && \text{Since } y \text{ is positive, choose the positive root.}\end{aligned}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of $30°$ are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. At $t = \frac{\pi}{3}$ $(60°)$, the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in **Figure 5.40**. Angle A has measure $60°$. At point B, we draw an angle ABC with measure of $60°$. We know the angles in a triangle sum to $180°$, so the measure of angle C is also $60°$. Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

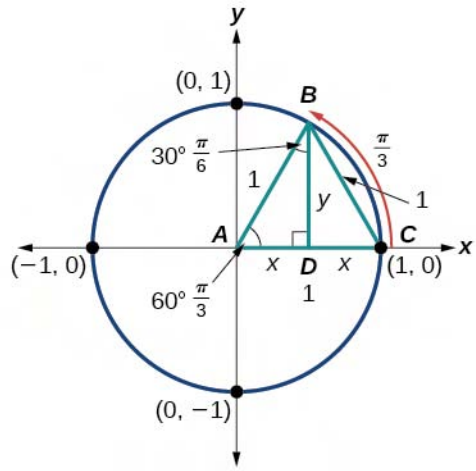


Figure 5.40

The measure of angle ABD is 30°. So, if double, angle ABC is 60°. BD is the perpendicular bisector of AC, so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the x-coordinate of point B, which is at the intersection of the 60° angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2 + y^2 = 1$$

Substituting $x = \frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$

Solving for y, we get

$$\begin{aligned} \frac{1}{4} + y^2 &= 1 \\ y^2 &= 1 - \frac{1}{4} \\ y^2 &= \frac{3}{4} \\ y &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

Since $t = \frac{\pi}{3}$ has the terminal side in quadrant I where the y-coordinate is positive, we choose $y = \frac{\sqrt{3}}{2}$, the positive value.

At $t = \frac{\pi}{3}$ (60°), the (x, y) coordinates for the point on a circle of radius 1 at an angle of 60° are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$\begin{aligned} (x, y) &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ x &= \frac{1}{2}, y = \frac{\sqrt{3}}{2} \\ \cos t &= \frac{1}{2}, \sin t = \frac{\sqrt{3}}{2} \end{aligned}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. **Table 5.2** summarizes these values.

Angle	0	$\frac{\pi}{6}$, or 30	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Table 5.2

Figure 5.41 shows the common angles in the first quadrant of the unit circle.

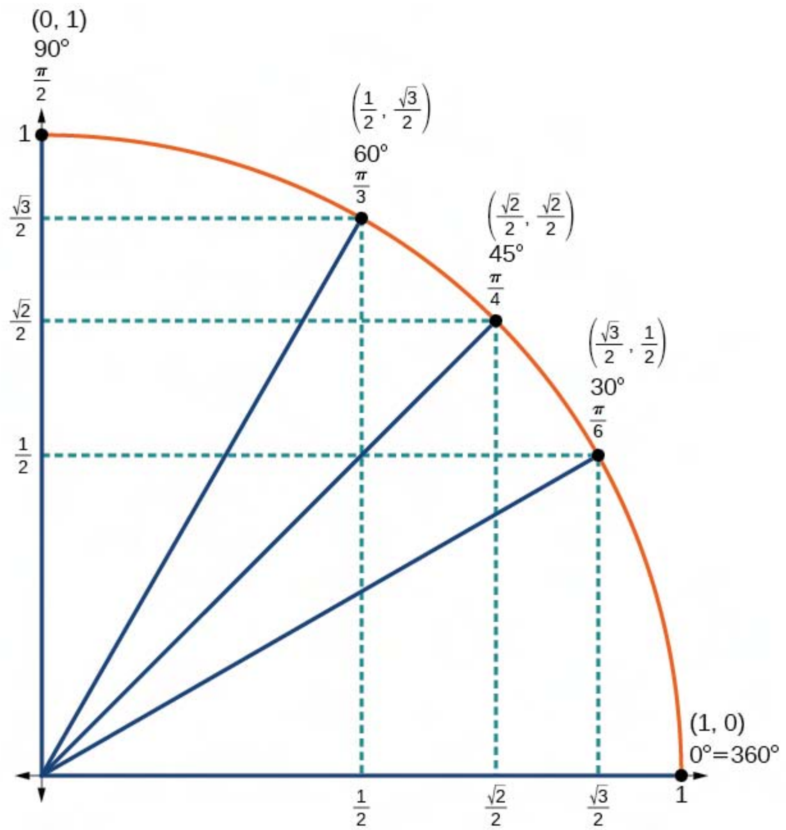


Figure 5.41

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the special angles, we turn to a computer or calculator. **Be aware**: Most calculators can be set into “degree” or “radian” mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To: **Given an angle in radians, use a graphing calculator to find the cosine.**

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

Example 5.15

Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\frac{5\pi}{3}\right)$ using a graphing calculator or computer.

Solution

Enter the following keystrokes:

COS(5 × π ÷ 3) ENTER

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

Analysis

We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of $20°$, for example, by including the conversion factor to radians as part of the input:

SIN(20 × π ÷ 180) ENTER

Try It **5.15** Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of $x, y,$ and $r,$ there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive *x*-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the unit circle, as shown in **Figure 5.42**. The bounds of the *x*-coordinate are $[-1, 1]$. The bounds of the *y*-coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

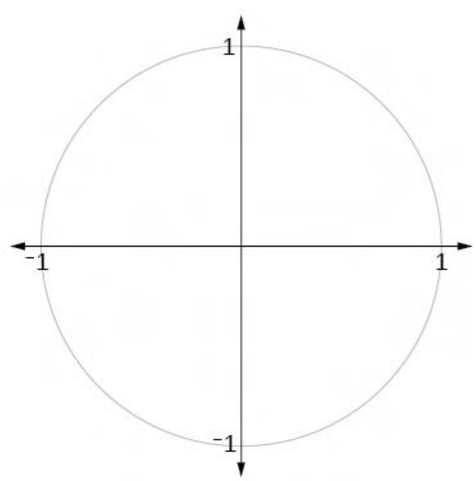


Figure 5.42

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the *y*-coordinate on the unit circle, the other angle with the same sine will share the same *y*-value, but have the opposite *x*-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same *x*-value but will have the opposite *y*-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in **Figure 5.43**, angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$$\begin{aligned} \sin(t) &= \sin(\alpha) \quad &&\text{and} \quad \cos(t) = -\cos(\alpha) \\ \sin(t) &= -\sin(\beta) \quad &&\text{and} \quad \cos(t) = \cos(\beta) \end{aligned}$$

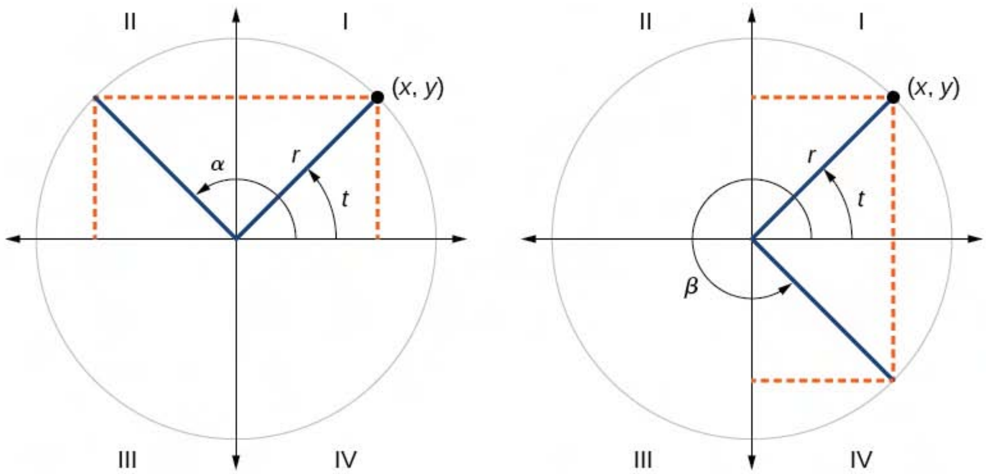


Figure 5.43

Recall that an angle's reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and $90°$, or 0 and $\frac{\pi}{2}$ radians. As we can see from **Figure 5.44**, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

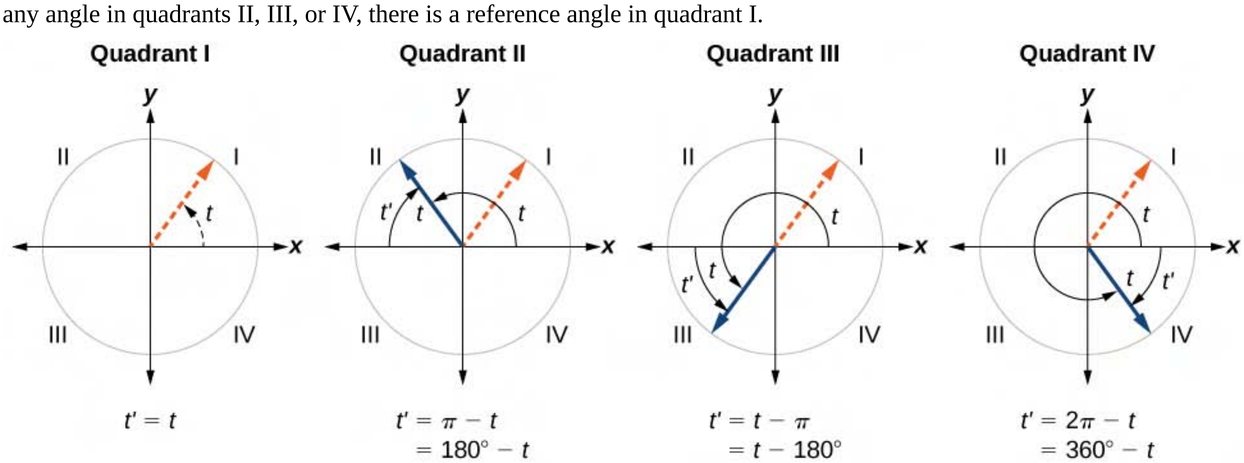


Figure 5.44

How To: Given an angle between 0 and 2π, find its reference angle.

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180° - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360° - t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Example 5.16

Finding a Reference Angle

Find the reference angle of $225°$ as shown in Figure 5.45.

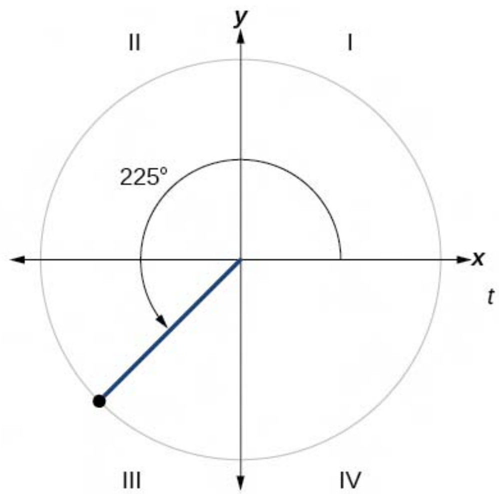


Figure 5.45

Solution

Because $225°$ is in the third quadrant, the reference angle is

$$|(180° - 225°)| = |-45°| = 45°$$

Try It 5.16 Find the reference angle of $\frac{5\pi}{3}$.

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a negative angle. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the reference angle of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x-values in that quadrant. The sine will be positive or negative depending on the sign of the y-values in that quadrant.

Using Reference Angles to Find Cosine and Sine

Angles have cosines and sines with the same absolute value as their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To: **Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle.**

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

Example 5.17

Using Reference Angles to Find Sine and Cosine

a. Using a reference angle, find the exact value of $\cos(150°)$ and $\sin(150°)$.

b. Using the reference angle, find $\cos\frac{5\pi}{4}$ and $\sin\frac{5\pi}{4}$.

Solution

a. 150° is located in the second quadrant. The angle it makes with the x-axis is 180° − 150° = 30°, so the reference angle is 30°.
This tells us that 150° has the same sine and cosine values as 30°, except for the sign. We know that

$$\cos(30°) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(30°) = \frac{1}{2}.$$

Since 150° is in the second quadrant, the x-coordinate of the point on the circle is negative, so the cosine value is negative. The y-coordinate is positive, so the sine value is positive.

$$\cos(150°) = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(150°) = \frac{1}{2}$$

b. $\frac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$. The cosine and sine of $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$.
In the third quadrant, both x and y are negative, so:

$$\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$

Try It **5.17**

a. Use the reference angle of $315°$ to find $\cos(315°)$ and $\sin(315°)$.

b. Use the reference angle of $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$.

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in **Figure 5.46**. Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

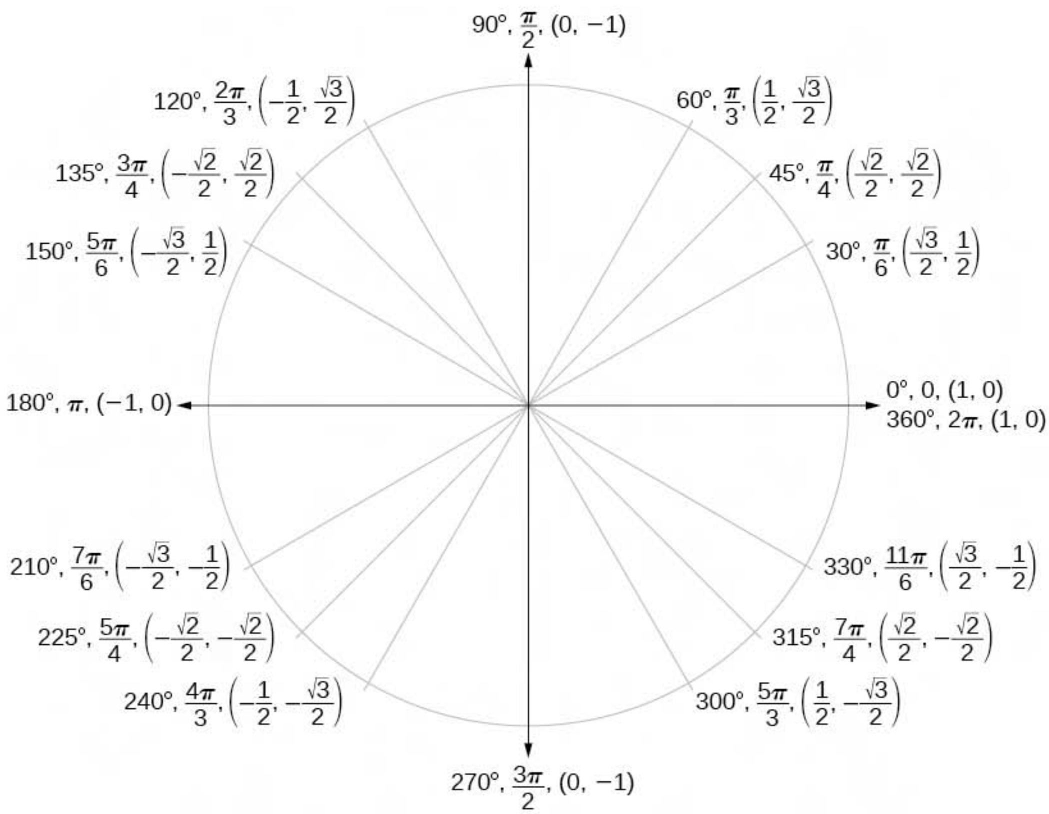


Figure 5.46 Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the identities

$$x = \cos t$$
$$y = \sin t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the reference angle, and give them the signs corresponding to the y- and x-values of the quadrant.

How To: **Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point.**

1. Find the reference angle by measuring the smallest angle to the x-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Example 5.18

Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution

We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the x-axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π.

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$$

Next, we will find the cosine and sine of the reference angle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\begin{aligned}\cos\left(\frac{7\pi}{6}\right) &= -\frac{\sqrt{3}}{2}\\ \sin\left(\frac{7\pi}{6}\right) &= -\frac{1}{2}\end{aligned}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos\theta$ and $y = \sin\theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Try It **5.18** Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Access these online resources for additional instruction and practice with sine and cosine functions.

- **Trigonometric Functions Using the Unit Circle (http://openstaxcollege.org/l/trigunitcir)**
- **Sine and Cosine from the Unit Circle (http://openstaxcollege.org/l/sincosuc)**
- **Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six (http://openstaxcollege.org/l/sincosmult)**
- **Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four (http://openstaxcollege.org/l/sincosmult4)**
- **Trigonometric Functions Using Reference Angles (http://openstaxcollege.org/l/trigrefang)**

5.2 EXERCISES

Verbal

75. Describe the unit circle.

76. What do the x- and y-coordinates of the points on the unit circle represent?

77. Discuss the difference between a coterminal angle and a reference angle.

78. Explain how the cosine of an angle in the second quadrant differs from the cosine of its reference angle in the unit circle.

79. Explain how the sine of an angle in the second quadrant differs from the sine of its reference angle in the unit circle.

Algebraic

For the following exercises, use the given sign of the sine and cosine functions to find the quadrant in which the terminal point determined by t lies.

80. $\sin(t) < 0$ and $\cos(t) < 0$

81. $\sin(t) > 0$ and $\cos(t) > 0$

82. $\sin(t) > 0$ and $\cos(t) < 0$

83. $\sin(t) < 0$ and $\cos(t) > 0$

For the following exercises, find the exact value of each trigonometric function.

84. $\sin\frac{\pi}{2}$

85. $\sin\frac{\pi}{3}$

86. $\cos\frac{\pi}{2}$

87. $\cos\frac{\pi}{3}$

88. $\sin\frac{\pi}{4}$

89. $\cos\frac{\pi}{4}$

90. $\sin\frac{\pi}{6}$

91. $\sin\pi$

92. $\sin\frac{3\pi}{2}$

93. $\cos\pi$

94. $\cos 0$

95. $\cos\frac{\pi}{6}$

96. $\sin 0$

Numeric

For the following exercises, state the reference angle for the given angle.

97. $240°$

98. $-170°$

99. $100°$

100. $-315°$

101. $135°$

102. $\frac{5\pi}{4}$

103. $\frac{2\pi}{3}$

104. $\frac{5\pi}{6}$

105. $\frac{-11\pi}{3}$

106. $\frac{-7\pi}{4}$

107. $\frac{-\pi}{8}$

For the following exercises, find the reference angle, the quadrant of the terminal side, and the sine and cosine of each angle. If the angle is not one of the angles on the unit circle, use a calculator and round to three decimal places.

108. $225°$

109. $300°$

110. $320°$

111. $135°$

112. $210°$

113. $120°$

114. $250°$

115. $150°$

116. $\frac{5\pi}{4}$

117. $\frac{7\pi}{6}$

118. $\frac{5\pi}{3}$

119. $\frac{3\pi}{4}$

120. $\frac{4\pi}{3}$

121. $\frac{2\pi}{3}$

122. $\frac{5\pi}{6}$

123. $\frac{7\pi}{4}$

For the following exercises, find the requested value.

124. If $\cos(t) = \frac{1}{7}$ and t is in the 4^{th} quadrant, find $\sin(t)$.

125. If $\cos(t) = \frac{2}{9}$ and t is in the 1^{st} quadrant, find $\sin(t)$.

126. If $\sin(t) = \frac{3}{8}$ and t is in the 2^{nd} quadrant, find $\cos(t)$.

127. If $\sin(t) = -\frac{1}{4}$ and t is in the 3^{rd} quadrant, find $\cos(t)$.

128. Find the coordinates of the point on a circle with radius 15 corresponding to an angle of $220°$.

129. Find the coordinates of the point on a circle with radius 20 corresponding to an angle of $120°$.

130. Find the coordinates of the point on a circle with radius 8 corresponding to an angle of $\frac{7\pi}{4}$.

131. Find the coordinates of the point on a circle with radius 16 corresponding to an angle of $\frac{5\pi}{9}$.

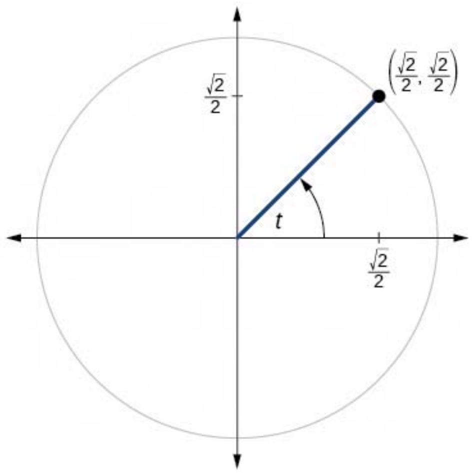
132. State the domain of the sine and cosine functions.

133. State the range of the sine and cosine functions.

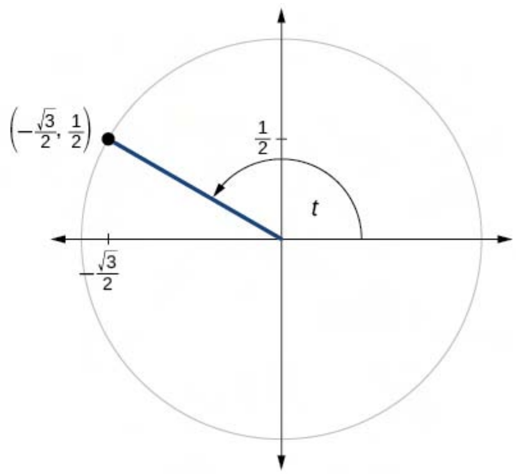
Graphical

For the following exercises, use the given point on the unit circle to find the value of the sine and cosine of t.

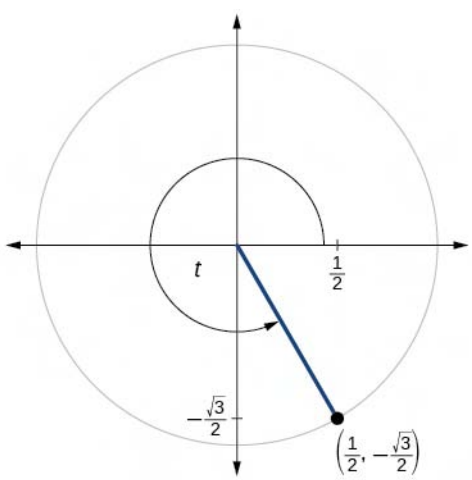
134.


$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\sqrt{2}}{2}$
t
$\frac{\sqrt{2}}{2}$

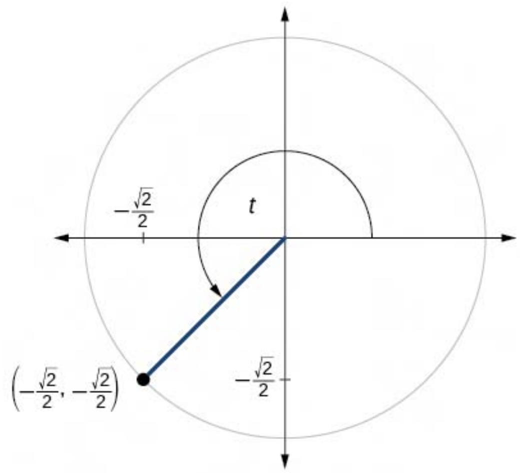
135.


$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{1}{2}$
t
$-\frac{\sqrt{3}}{2}$

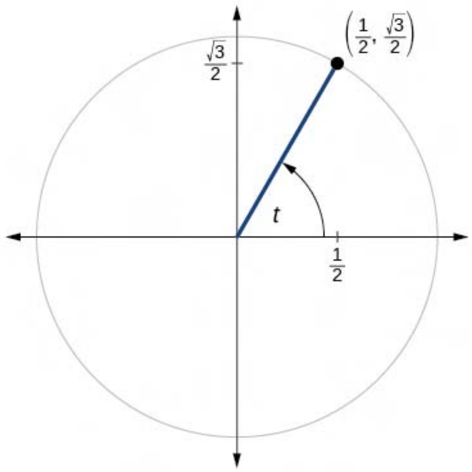
136.


t
$\frac{1}{2}$
$-\frac{\sqrt{3}}{2}$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

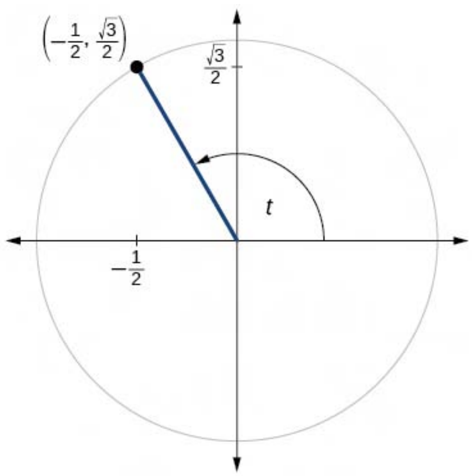
137.


$-\frac{\sqrt{2}}{2}$
t
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$-\frac{\sqrt{2}}{2}$

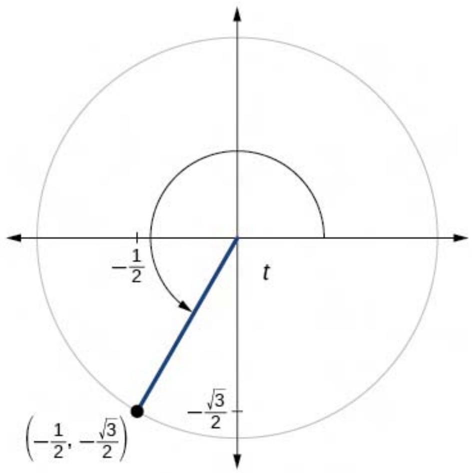
138.


$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\sqrt{3}}{2}$
t
$\frac{1}{2}$

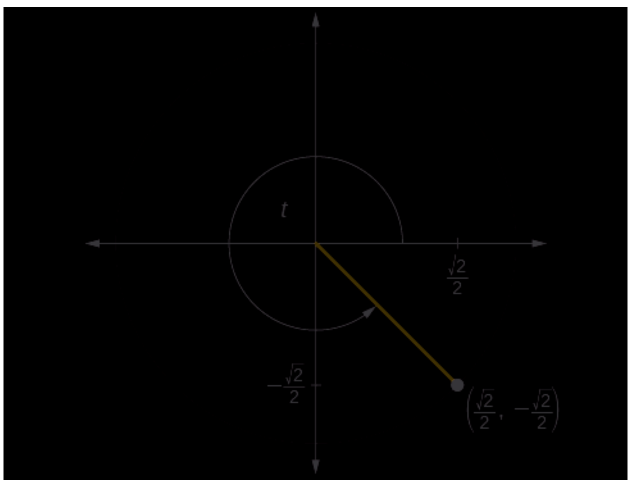
139.


$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\sqrt{3}}{2}$
t
$-\frac{1}{2}$

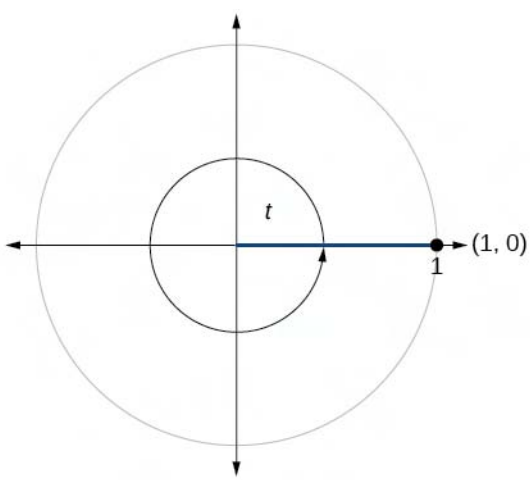
140.


$-\frac{1}{2}$
t
$-\frac{\sqrt{3}}{2}$
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

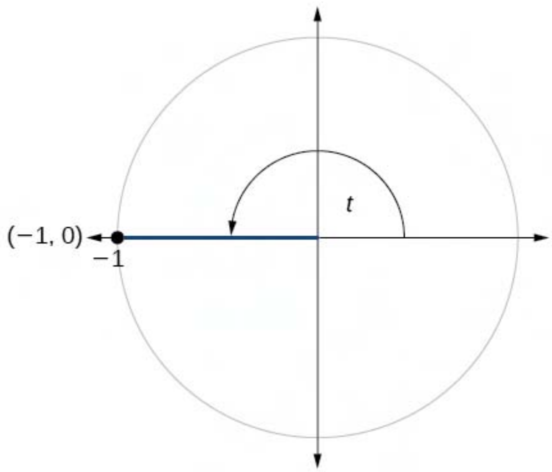
141.


t
$\frac{\sqrt{2}}{2}$
$-\frac{\sqrt{2}}{2}$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

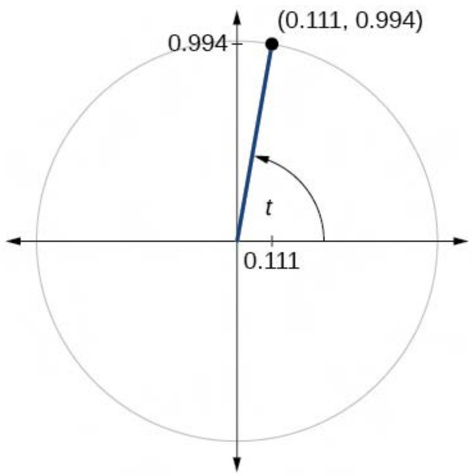
142.


t
(1, 0)
1

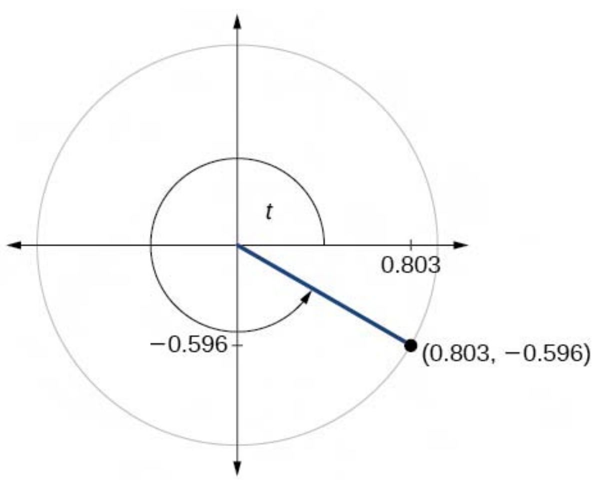
143.



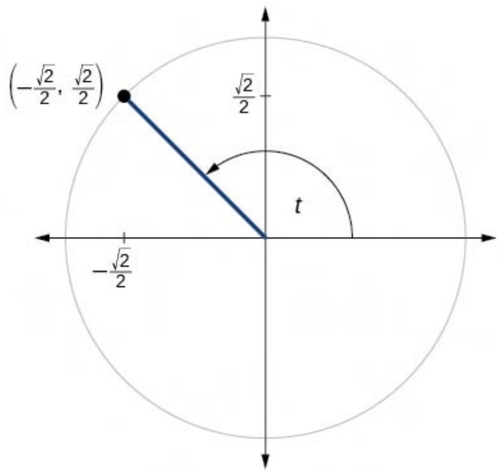
144.



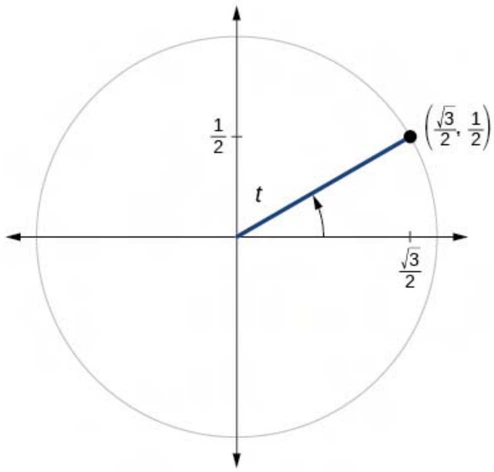
145.



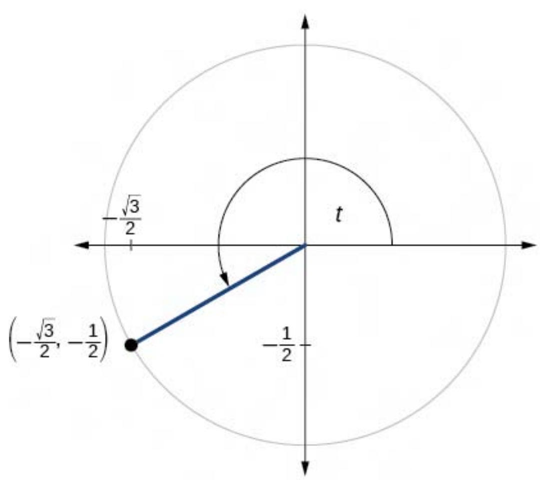
146.


$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\sqrt{2}}{2}$
t
$-\frac{\sqrt{2}}{2}$

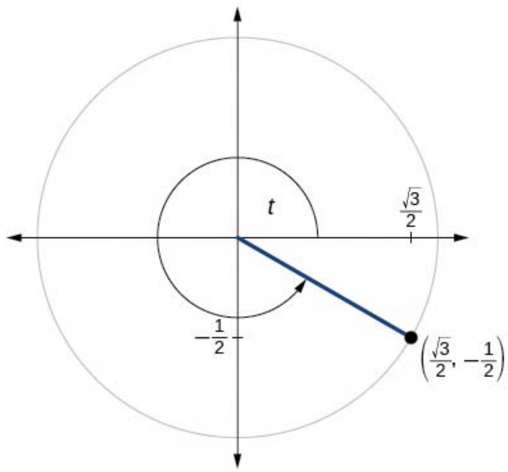
147.


$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{1}{2}$
t
$\frac{\sqrt{3}}{2}$

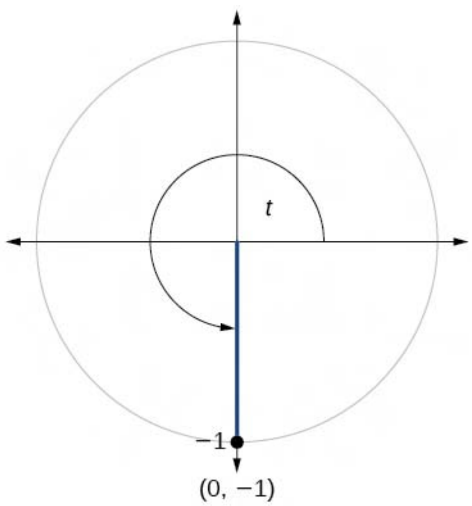
148.


$-\frac{\sqrt{3}}{2}$
t
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$-\frac{1}{2}$

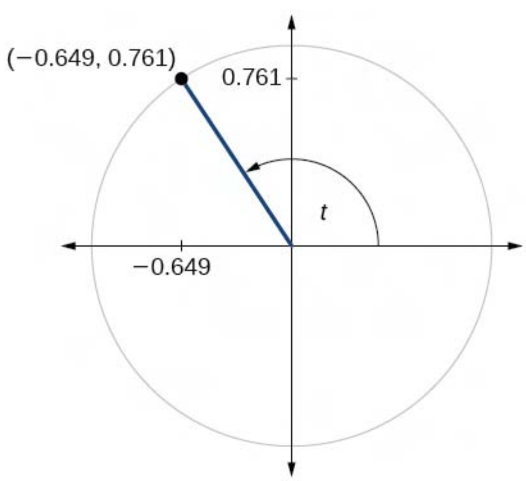
149.


t
$\frac{\sqrt{3}}{2}$
$-\frac{1}{2}$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

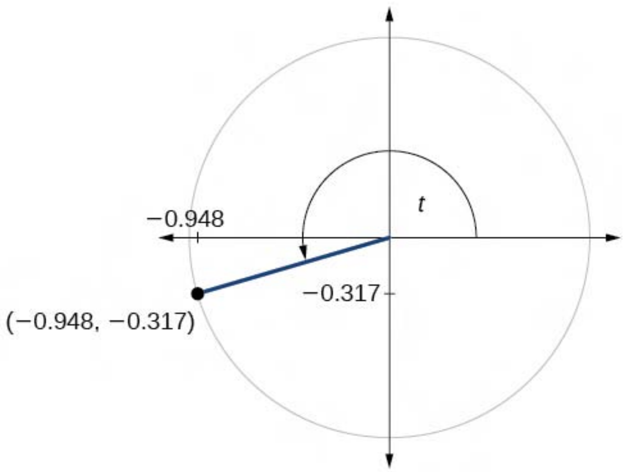
150.


t
−1
(0, −1)

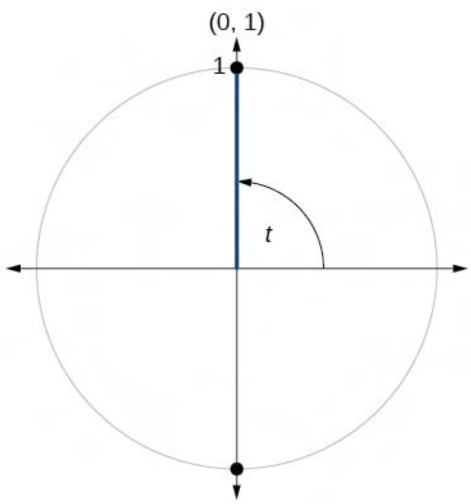
151.


(−0.649, 0.761)
0.761
t
−0.649

152.



153.



Technology

For the following exercises, use a graphing calculator to evaluate.

154. $\sin \frac{5\pi}{9}$

155. $\cos \frac{5\pi}{9}$

156. $\sin \frac{\pi}{10}$

157. $\cos \frac{\pi}{10}$

158. $\sin \frac{3\pi}{4}$

159. $\cos \frac{3\pi}{4}$

160. $\sin 98°$

161. $\cos 98°$

162. $\cos 310°$

163. $\sin 310°$

Extensions

164. $\sin\left(\frac{11\pi}{3}\right)\cos\left(\frac{-5\pi}{6}\right)$

165. $\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right)$

166. $\sin\left(-\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{2}\right)$

167. $\sin\left(\frac{-9\pi}{4}\right)\cos\left(\frac{-\pi}{6}\right)$

168. $\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{-\pi}{3}\right)$

169. $\sin\left(\frac{7\pi}{4}\right)\cos\left(\frac{-2\pi}{3}\right)$

170. $\cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{2\pi}{3}\right)$

171. $\cos\left(\frac{-\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)$

172. $\sin\left(\frac{-5\pi}{4}\right)\sin\left(\frac{11\pi}{6}\right)$

173. $\sin(\pi)\sin\left(\frac{\pi}{6}\right)$

Real-World Applications

For the following exercises, use this scenario: A child enters a carousel that takes one minute to revolve once around. The child enters at the point $(0, 1)$, that is, on the due north position. Assume the carousel revolves counter clockwise.

174. What are the coordinates of the child after 45 seconds?

175. What are the coordinates of the child after 90 seconds?

176. What is the coordinates of the child after 125 seconds?

177. When will the child have coordinates $(0.707, -0.707)$ if the ride lasts 6 minutes? (There are multiple answers.)

178. When will the child have coordinates $(-0.866, -0.5)$ if the ride last 6 minutes?

5.3 | The Other Trigonometric Functions

Learning Objectives

In this section, you will:

5.3.1 Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.

5.3.2 Use reference angles to evaluate the trigonometric functions secant, cosecant, tangent, and cotangent.

5.3.3 Use properties of even and odd trigonometric functions.

5.3.4 Recognize and use fundamental identities.

5.3.5 Evaluate trigonometric functions with a calculator.

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $\frac{1}{12}$ or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

To define the remaining functions, we will once again draw a unit circle with a point (x, y) corresponding to an angle of t, as shown in Figure 5.47. As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.

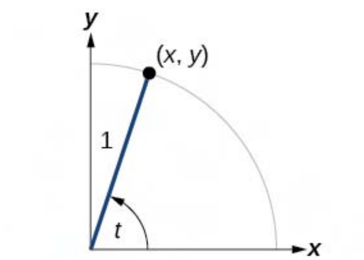


Figure 5.47

The first function we will define is the tangent. The **tangent** of an angle is the ratio of the y-value to the x-value of the corresponding point on the unit circle. In Figure 5.47, the tangent of angle t is equal to $\frac{y}{x}, x \neq 0$. Because the y-value is equal to the sine of t, and the x-value is equal to the cosine of t, the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}, \cos t \neq 0$. The tangent function is abbreviated as tan. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In Figure 5.47, the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}, x \neq 0$. The secant function is abbreviated as sec.
- The **cotangent** function is the reciprocal of the tangent function. In Figure 5.47, the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}, y \neq 0$. The cotangent function is abbreviated as cot.
- The **cosecant** function is the reciprocal of the sine function. In Figure 5.47, the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}, y \neq 0$. The cosecant function is abbreviated as csc.

Tangent, Secant, Cosecant, and Cotangent Functions

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x},\ x \neq 0$$
$$\sec t = \frac{1}{x},\ x \neq 0$$
$$\csc t = \frac{1}{y},\ y \neq 0$$
$$\cot t = \frac{x}{y},\ y \neq 0$$

Example 5.19

Finding Trigonometric Functions from a Point on the Unit Circle

The point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is on the unit circle, as shown in **Figure 5.48**. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

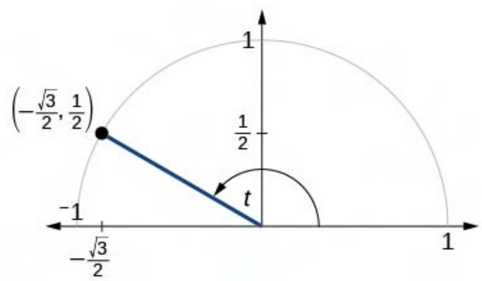


Figure 5.48

Solution

Because we know the (x, y) coordinates of the point on the unit circle indicated by angle t, we can use those coordinates to find the six functions:

$$\sin t = y = \frac{1}{2}$$
$$\cos t = x = -\frac{\sqrt{3}}{2}$$
$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$
$$\sec t = \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$
$$\csc t = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$
$$\cot t = \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$

5.19 The point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is on the unit circle, as shown in Figure 5.49. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

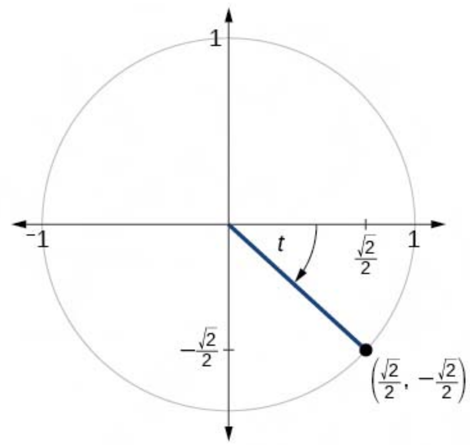


Figure 5.49

Example 5.20

Finding the Trigonometric Functions of an Angle

Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$ when $t = \frac{\pi}{6}$.

Solution

We have previously used the properties of equilateral triangles to demonstrate that $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\begin{aligned}
\tan \frac{\pi}{6} &= \frac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{6}} \\
&= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}
\end{aligned}$$

$$\begin{aligned}
\sec \frac{\pi}{6} &= \frac{1}{\cos \frac{\pi}{6}} \\
&= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}
\end{aligned}$$

$$\csc \frac{\pi}{6} = \frac{1}{\sin \frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot\frac{\pi}{6} = \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}$$
$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

Try It **5.20** Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$ when $t = \frac{\pi}{3}$.

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x equal to the cosine and y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in **Table 5.3**.

Angle	**0**	$\frac{\pi}{6}$, **or 30°**	$\frac{\pi}{4}$, **or 45°**	$\frac{\pi}{3}$, **or 60°**	$\frac{\pi}{2}$, **or 90°**
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Table 5.3

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the reference angle formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by x- and y-values in the original quadrant. **Figure 5.50** shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "A Smart Trig Class." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive. In quadrant II, "**S**mart," only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, "**T**rig," only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lass," only **c**osine and its reciprocal function, secant, are positive.

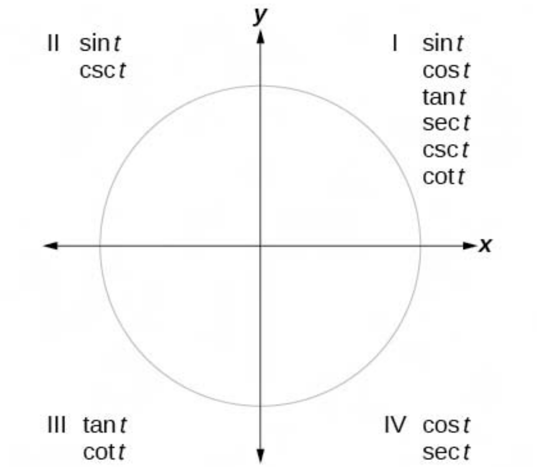


Figure 5.50

How To:

Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions.

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Example 5.21

Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Solution

The angle between this angle's terminal side and the x-axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both x and y are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$$\cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2},\ \sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2},\ \tan\left(-\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{3}$$
$$\sec\left(-\frac{5\pi}{6}\right) = -\frac{2\sqrt{3}}{3},\ \csc\left(-\frac{5\pi}{6}\right) = -2,\ \cot\left(-\frac{5\pi}{6}\right) = \sqrt{3}$$

Try It

5.21 Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard.

Consider the function $f(x) = x^2$, shown in **Figure 5.51**. The graph of the function is symmetrical about the *y*-axis. All along the curve, any two points with opposite *x*-values have the same function value. This matches the result of calculation: $(4)^2 = (-4)^2$, $(-5)^2 = (5)^2$, and so on. So $f(x) = x^2$ is an even function, a function such that two inputs that are opposites have the same output. That means $f(-x) = f(x)$.

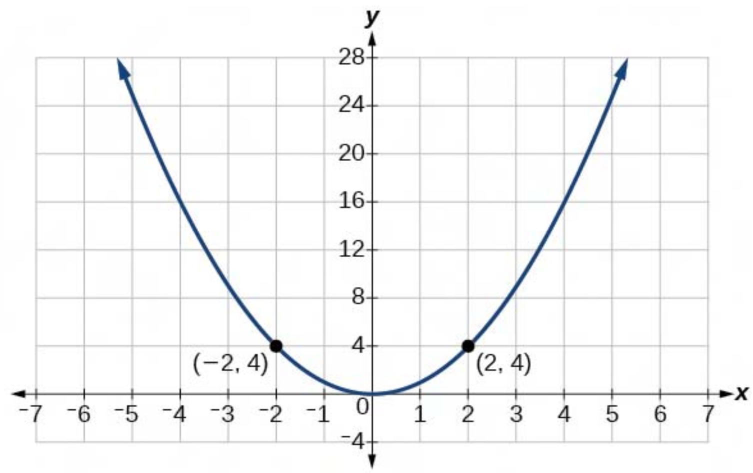


Figure 5.51 The function $f(x) = x^2$ is an even function.

Now consider the function $f(x) = x^3$, shown in **Figure 5.52**. The graph is not symmetrical about the *y*-axis. All along the graph, any two points with opposite *x*-values also have opposite *y*-values. So $f(x) = x^3$ is an odd function, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.

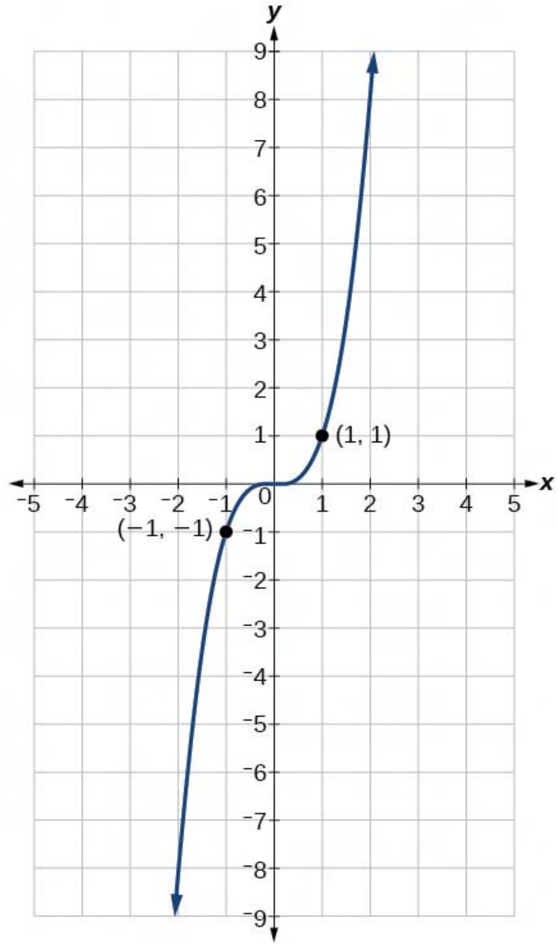


Figure 5.52 The function $f(x) = x^3$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a unit circle with a positive and a negative angle, as in **Figure 5.53**. The sine of the positive angle is y. The sine of the negative angle is $-y$. The sine function, then, is an odd function. We can test each of the six trigonometric functions in this fashion. The results are shown in **Table 5.4**.

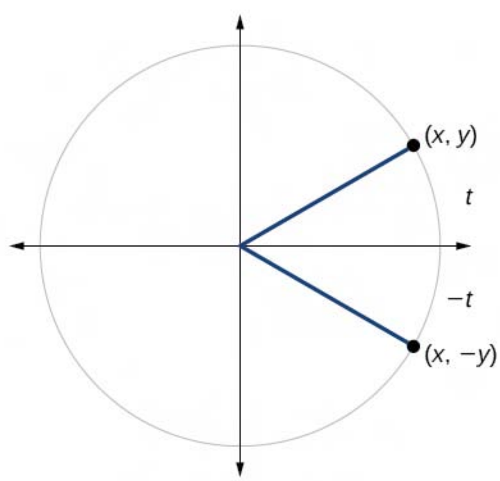


Figure 5.53

$\sin t = y$ $\sin(-t) = -y$ $\sin t \neq \sin(-t)$	$\cos t = x$ $\cos(-t) = x$ $\cos t = \cos(-t)$	$\tan(t) = \frac{y}{x}$ $\tan(-t) = -\frac{y}{x}$ $\tan t \neq \tan(-t)$
$\sec t = \frac{1}{x}$ $\sec(-t) = \frac{1}{x}$ $\sec t = \sec(-t)$	$\csc t = \frac{1}{y}$ $\csc(-t) = \frac{1}{-y}$ $\csc t \neq \csc(-t)$	$\cot t = \frac{x}{y}$ $\cot(-t) = \frac{x}{-y}$ $\cot t \neq cot(-t)$

Table 5.4

Even and Odd Trigonometric Functions

An even function is one in which $f(-x) = f(x)$.

An odd function is one in which $f(-x) = -f(x)$.

Cosine and secant are even:

$$\cos(-t) = \cos t$$
$$\sec(-t) = \sec t$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t$$
$$\tan(-t) = -\tan t$$
$$\csc(-t) = -\csc t$$
$$\cot(-t) = -\cot t$$

Example 5.22

Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t is 2, what is the secant of $-t$?

Solution

Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Try It **5.22** If the cotangent of angle t is $\sqrt{3},$ what is the cotangent of $-t$?

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

Fundamental Identities

We can derive some useful identities from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t} \tag{5.9}$$

$$\sec t = \frac{1}{\cos t} \tag{5.10}$$

$$\csc t = \frac{1}{\sin t} \tag{5.11}$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t} \tag{5.12}$$

Example 5.23

Using Identities to Evaluate Trigonometric Functions

a. Given $\sin(45°) = \frac{\sqrt{2}}{2}, \cos(45°) = \frac{\sqrt{2}}{2},$ evaluate $\tan(45°)$.

b. Given $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}, \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2},$ evaluate $\sec\left(\frac{5\pi}{6}\right)$.

Solution

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a.
$$\begin{aligned} \tan(45°) &= \frac{\sin(45°)}{\cos(45°)} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1 \end{aligned}$$

b.
$$\begin{aligned} \sec\left(\frac{5\pi}{6}\right) &= \frac{1}{\cos\left(\frac{5\pi}{6}\right)} \\ &= \frac{1}{-\frac{\sqrt{3}}{2}} \\ &= \frac{-2\sqrt{3}}{1} \\ &= \frac{-2}{\sqrt{3}} \\ &= -\frac{2\sqrt{3}}{3} \end{aligned}$$

Try It **5.23** Evaluate $\csc\left(\frac{7\pi}{6}\right)$.

Example 5.24

Using Identities to Simplify Trigonometric Expressions

Simplify $\frac{\sec t}{\tan t}$.

Solution

We can simplify this by rewriting both functions in terms of sine and cosine.

$$\begin{aligned} \frac{\sec t}{\tan t} &= \frac{\frac{1}{\cos t}}{\frac{\sin t}{\cos t}} && \text{To divide the functions, we multiply by the reciprocal.} \\ &= \frac{1}{\cos t}\frac{\cos t}{\sin t} && \text{Divide out the cosines.} \\ &= \frac{1}{\sin t} && \text{Simplify and use the identity.} \\ &= \csc t \end{aligned}$$

By showing that $\frac{\sec t}{\tan t}$ can be simplified to $\csc t$, we have, in fact, established a new identity.

$$\frac{\sec t}{\tan t} = \csc t$$

Try It **5.24** Simplify $\tan t(\cos t)$.

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternative forms of the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\begin{aligned} \frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} &= \frac{1}{\cos^2 t} \\ 1 + \tan^2 t &= \sec^2 t \end{aligned}$$

The other form is obtained by dividing both sides by $\sin^2 t$:

$$\begin{aligned} \frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} &= \frac{1}{\sin^2 t} \\ \cot^2 t + 1 &= \csc^2 t \end{aligned}$$

Alternate Forms of the Pythagorean Identity

$$1 + \tan^2 t = \sec^2 t$$
$$\cot^2 t + 1 = \csc^2 t$$

Example 5.25

Using Identities to Relate Trigonometric Functions

If $\cos(t) = \frac{12}{13}$ and t is in quadrant IV, as shown in **Figure 5.54**, find the values of the other five trigonometric functions.

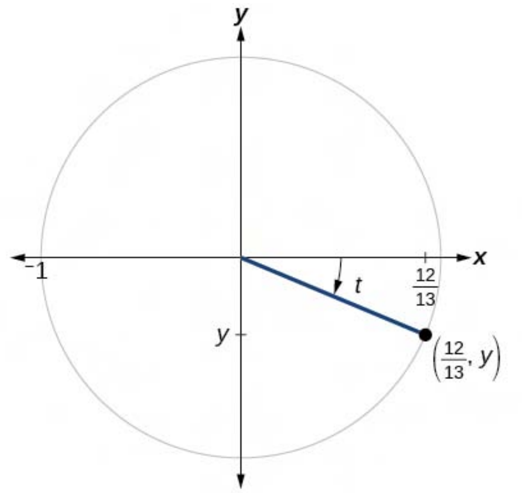


Figure 5.54

Solution

We can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1,$ and the remaining functions by relating them to sine and cosine.

$$\begin{aligned}
\left(\frac{12}{13}\right)^2 + \sin^2 t &= 1 \\
\sin^2 t &= 1 - \left(\frac{12}{13}\right)^2 \\
\sin^2 t &= 1 - \frac{144}{169} \\
\sin^2 t &= \frac{25}{169} \\
\sin t &= \pm\sqrt{\frac{25}{169}} \\
\sin t &= \pm\frac{\sqrt{25}}{\sqrt{169}} \\
\sin t &= \pm\frac{5}{13}
\end{aligned}$$

The sign of the sine depends on the y-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the y-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12}$$
$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{12}{13}} = \frac{13}{12}$$
$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5}$$
$$\cot t = \frac{1}{\tan t} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5}$$

Try It **5.25** If $\sec(t) = -\frac{17}{8}$ and $0 < t < \pi,$ find the values of the other five functions.

As we discussed in the chapter opening, a function that repeats its values in regular intervals is known as a periodic function. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or $2\pi,$ will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x+4) = f(x).$ This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

Period of a Function

The **period** P of a repeating function f is the number representing the interval such that $f(x+P) = f(x)$ for any value of $x.$

The period of the cosine, sine, secant, and cosecant functions is $2\pi.$

The period of the tangent and cotangent functions is $\pi.$

Example 5.26

Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t based on **Figure 5.55**.

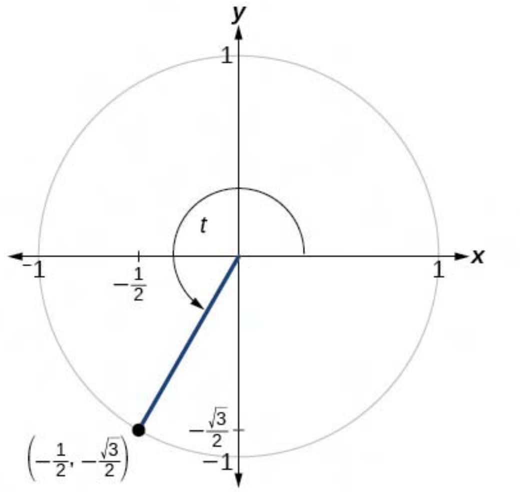


Figure 5.55

Solution

$$\sin t = y = -\frac{\sqrt{3}}{2}$$
$$\cos t = x = -\frac{1}{2}$$
$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$
$$\sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -2$$
$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$
$$\cot t = \frac{1}{\tan t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

5.26 Find the values of the six trigonometric functions of angle t based on **Figure 5.56**.

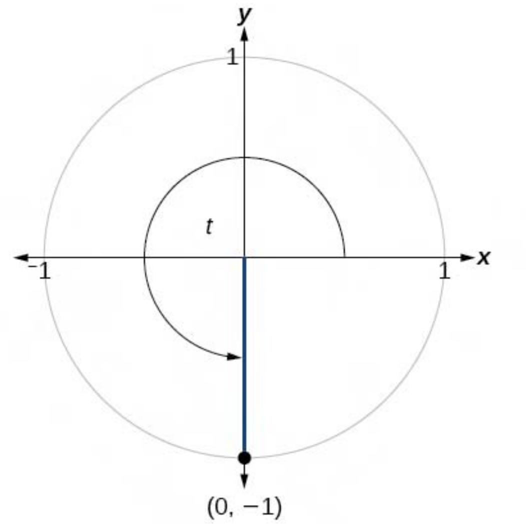


Figure 5.56

Example 5.27

Finding the Value of Trigonometric Functions

If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$, find $\sec(t)$, $\csc(t)$, $\tan(t)$, $\cot(t)$.

Solution

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} - \frac{2\sqrt{3}}{3}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

5.27 If $\sin(t) = \frac{\sqrt{2}}{2}$ and $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sec(t)$, $\csc(t)$, $\tan(t)$, and $\cot(t)$.

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the TAN key. For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of $30°,$ we could press

$$\text{(for a scientific calculator):}\frac{1}{30\times\frac{\pi}{180}\text{COS}}$$

or

$$\text{(for a graphing calculator): } \frac{1}{\cos\left(\frac{30\pi}{180}\right)}$$

How To: Given an angle measure in radians, use a scientific calculator to find the cosecant.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Enter the value of the angle inside parentheses.
4. Press the SIN key.
5. Press the = key.

How To: Given an angle measure in radians, use a graphing utility/calculator to find the cosecant.

1. If the graphing utility has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Press the SIN key.
4. Enter the value of the angle inside parentheses.
5. Press the ENTER key.

Example 5.28

Evaluating the Secant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}.$

Solution

For a scientific calculator, enter information as follows:

$$1 \,/\, (\, 5 \times \pi \,/\, 7 \,)\ \text{SIN} =$$

$$\csc\left(\frac{5\pi}{7}\right) \approx 1.279$$

Try It 5.28 Evaluate the cotangent of $-\frac{\pi}{8}.$

Access these online resources for additional instruction and practice with other trigonometric functions.

- **Determining Trig Function Values (http://openstaxcollege.org/l/trigfuncval)**
- **More Examples of Determining Trig Functions (http://openstaxcollege.org/l/moretrigfun)**
- **Pythagorean Identities (http://openstaxcollege.org/l/pythagiden)**
- **Trig Functions on a Calculator (http://openstaxcollege.org/l/trigcalc)**

5.3 EXERCISES

Verbal

179. On an interval of $[0, 2\pi)$, can the sine and cosine values of a radian measure ever be equal? If so, where?

180. What would you estimate the cosine of π degrees to be? Explain your reasoning.

181. For any angle in quadrant II, if you knew the sine of the angle, how could you determine the cosine of the angle?

182. Describe the secant function.

183. Tangent and cotangent have a period of π. What does this tell us about the output of these functions?

Algebraic

For the following exercises, find the exact value of each expression.

184. $\tan\frac{\pi}{6}$

185. $\sec\frac{\pi}{6}$

186. $\csc\frac{\pi}{6}$

187. $\cot\frac{\pi}{6}$

188. $\tan\frac{\pi}{4}$

189. $\sec\frac{\pi}{4}$

190. $\csc\frac{\pi}{4}$

191. $\cot\frac{\pi}{4}$

192. $\tan\frac{\pi}{3}$

193. $\sec\frac{\pi}{3}$

194. $\csc\frac{\pi}{3}$

195. $\cot\frac{\pi}{3}$

For the following exercises, use reference angles to evaluate the expression.

196. $\tan\frac{5\pi}{6}$

197. $\sec\frac{7\pi}{6}$

198. $\csc\frac{11\pi}{6}$

199. $\cot \frac{13\pi}{6}$

200. $\tan \frac{7\pi}{4}$

201. $\sec \frac{3\pi}{4}$

202. $\csc \frac{5\pi}{4}$

203. $\cot \frac{11\pi}{4}$

204. $\tan \frac{8\pi}{3}$

205. $\sec \frac{4\pi}{3}$

206. $\csc \frac{2\pi}{3}$

207. $\cot \frac{5\pi}{3}$

208. $\tan 225°$

209. $\sec 300°$

210. $\csc 150°$

211. $\cot 240°$

212. $\tan 330°$

213. $\sec 120°$

214. $\csc 210°$

215. $\cot 315°$

216. If $\sin t = \frac{3}{4},$ and t is in quadrant II, find $\cos t,$ $\sec t,$ $\csc t,$ $\tan t,$ $\cot t.$

217. If $\cos t = -\frac{1}{3},$ and t is in quadrant III, find $\sin t,$ $\sec t,$ $\csc t,$ $\tan t,$ $\cot t.$

218. If $\tan t = \frac{12}{5},$ and $0 \le t < \frac{\pi}{2},$ find $\sin t,$ $\cos t,$ $\sec t,$ $\csc t,$ and $\cot t.$

219. If $\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2},$ find $\sec t,$ $\csc t,$ $\tan t,$ and $\cot t.$

220. If $\sin 40° \approx 0.643$ $\cos 40° \approx 0.766$ $\sec 40°,$ $\csc 40°,$ $\tan 40°,$ and $\cot 40°.$

221. If $\sin t = \frac{\sqrt{2}}{2},$ what is the $\sin(-t)$?

222.

If $\cos t = \frac{1}{2}$, what is the $\cos(-t)$?

223. If $\sec t = 3.1$, what is the $\sec(-t)$?

224. If $\csc t = 0.34$, what is the $\csc(-t)$?

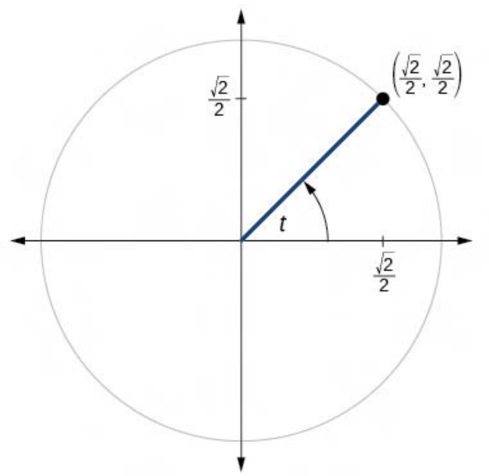
225. If $\tan t = -1.4$, what is the $\tan(-t)$?

226. If $\cot t = 9.23$, what is the $\cot(-t)$?

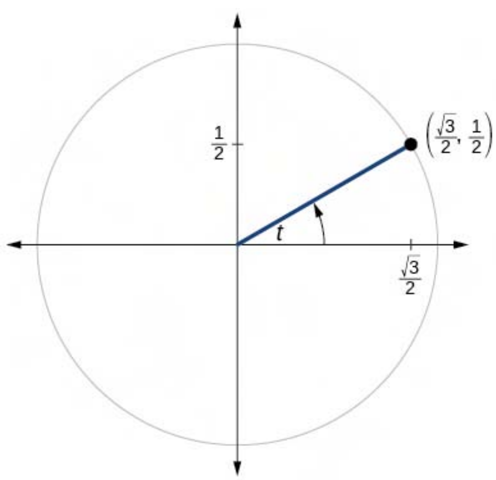
Graphical

For the following exercises, use the angle in the unit circle to find the value of the each of the six trigonometric functions.

227.



228.



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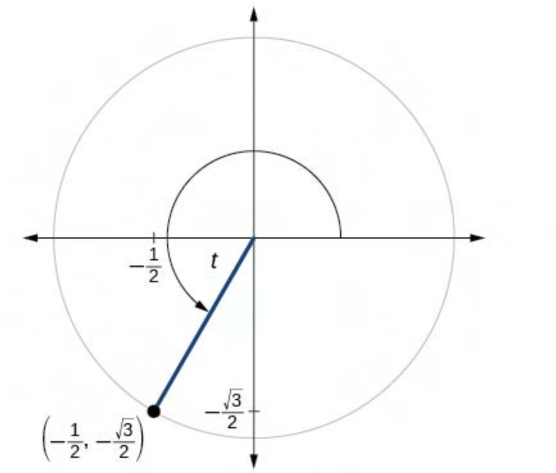


Technology

For the following exercises, use a graphing calculator to evaluate.

230. $\csc \frac{5\pi}{9}$

231. $\cot \frac{4\pi}{7}$

232. $\sec \frac{\pi}{10}$

233. $\tan \frac{5\pi}{8}$

234. $\sec \frac{3\pi}{4}$

235. $\csc \frac{\pi}{4}$

236. $\tan 98°$

237. $\cot 33°$

238. $\cot 140°$

239. $\sec 310°$

Extensions

For the following exercises, use identities to evaluate the expression.

240. If $\tan(t) \approx 2.7,$ and $\sin(t) \approx 0.94,$ find $\cos(t).$

241. If $\tan(t) \approx 1.3,$ and $\cos(t) \approx 0.61,$ find $\sin(t).$

242. If $\csc(t) \approx 3.2,$ and $\cos(t) \approx 0.95,$ find $\tan(t).$

243. If $\cot(t) \approx 0.58,$ and $\cos(t) \approx 0.5,$ find $\csc(t).$

244.

Determine whether the function $f(x) = 2\sin x \cos x$ is even, odd, or neither.

245. Determine whether the function $f(x) = 3\sin^2 x \cos x + \sec x$ is even, odd, or neither.

246. Determine whether the function $f(x) = \sin x - 2\cos^2 x$ is even, odd, or neither.

247. Determine whether the function $f(x) = \csc^2 x + \sec x$ is even, odd, or neither.

For the following exercises, use identities to simplify the expression.

248. $\csc t \tan t$

249. $\frac{\sec t}{\csc t}$

Real-World Applications

250. The amount of sunlight in a certain city can be modeled by the function $h = 15\cos\left(\frac{1}{600}d\right),$ where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on February 10, the 42nd day of the year. State the period of the function.

251. The amount of sunlight in a certain city can be modeled by the function $h = 16\cos\left(\frac{1}{500}d\right),$ where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on September 24, the 267th day of the year. State the period of the function.

252. The equation $P = 20\sin(2\pi t) + 100$ models the blood pressure, $P,$ where t represents time in seconds. (a) Find the blood pressure after 15 seconds. (b) What are the maximum and minimum blood pressures?

253. The height of a piston, $h,$ in inches, can be modeled by the equation $y = 2\cos x + 6,$ where x represents the crank angle. Find the height of the piston when the crank angle is $55°$.

254. The height of a piston, $h,$ in inches, can be modeled by the equation $y = 2\cos x + 5,$ where x represents the crank angle. Find the height of the piston when the crank angle is $55°$.

5.4 | Right Triangle Trigonometry

Learning Objectives

In this section, you will:

5.4.1 Use right triangles to evaluate trigonometric functions.
5.4.2 Find function values for $30°\left(\frac{\pi}{6}\right)$, $45°\left(\frac{\pi}{4}\right)$, and $60°\left(\frac{\pi}{3}\right)$.
5.4.3 Use cofunctions of complementary angles.
5.4.4 Use the definitions of trigonometric functions of any angle.
5.4.5 Use right triangle trigonometry to solve applied problems.

We have previously defined the sine and cosine of an angle in terms of the coordinates of a point on the unit circle intersected by the terminal side of the angle:

$$\cos t = x$$
$$\sin t = y$$

In this section, we will see another way to define trigonometric functions using properties of right triangles.

Using Right Triangles to Evaluate Trigonometric Functions

In earlier sections, we used a unit circle to define the trigonometric functions. In this section, we will extend those definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. Figure 5.57 shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x, y) to the x-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

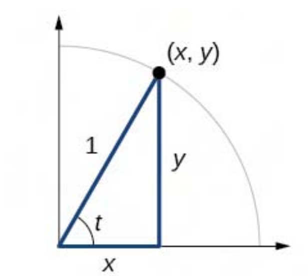


Figure 5.57

We know

$$\cos t = \frac{x}{1} = x$$

Likewise, we know

$$\sin t = \frac{y}{1} = y$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x, y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x, we will call the side between the given angle and the right angle the **adjacent side** to angle t. (Adjacent means "next to.") Instead of y, we will call the side most distant from the given angle the **opposite side** from angle t. And instead of 1, we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure 5.58.

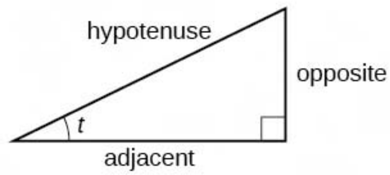


Figure 5.58 The sides of a right triangle in relation to angle t.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

How To: **Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle.**

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent is the ratio of the opposite side to the adjacent side.

Example 5.29

Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in **Figure 5.59**, find the value of $\cos\alpha$.

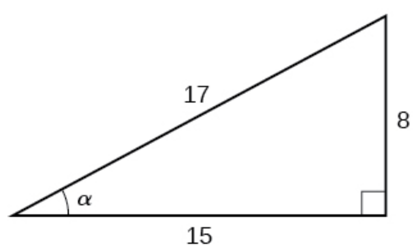


Figure 5.59

Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so:

$$\begin{aligned}\cos(\alpha) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{15}{17}\end{aligned}$$

Try It **5.29** Given the triangle shown in **Figure 5.60**, find the value of $\sin t$.

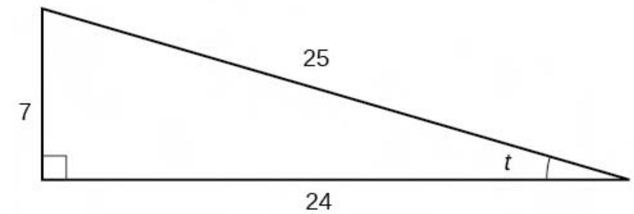


Figure 5.60

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in **Figure 5.61**. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

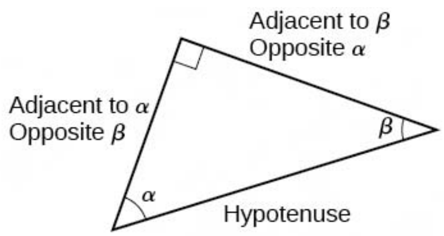


Figure 5.61 The side adjacent to one angle is opposite the other.

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

How To: **Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles.**

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example 5.30

Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in **Figure 5.62**, evaluate $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

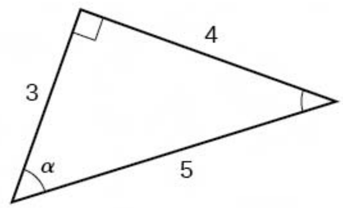


Figure 5.62

Solution

$$\sin \alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos \alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan \alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3}$$
$$\sec \alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3}$$
$$\csc \alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4}$$
$$\cot \alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

Try It **5.30** Using the triangle shown in Figure 5.63, evaluate $\sin t, \quad \cos t, \quad \tan t, \quad \sec t, \quad \csc t,$ and $\cot t$.

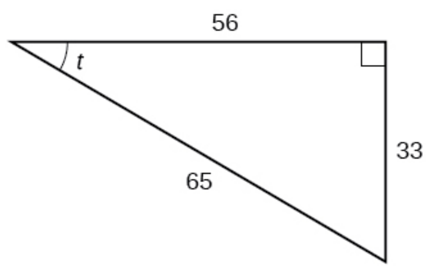


Figure 5.63

Finding Trigonometric Functions of Special Angles Using Side Lengths

We have already discussed the trigonometric functions as they relate to the special angles on the unit circle. Now, we can use those relationships to evaluate triangles that contain those special angles. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of $30°, \quad 60°,$ and $45°,$ however, remember that when dealing with right triangles, we are limited to angles between $0°$ and $90°$.

Suppose we have a $30°, 60°, 90°$ triangle, which can also be described as a $\frac{\pi}{6}, \quad \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45°, 45°, 90°$ triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in Figure 5.64.

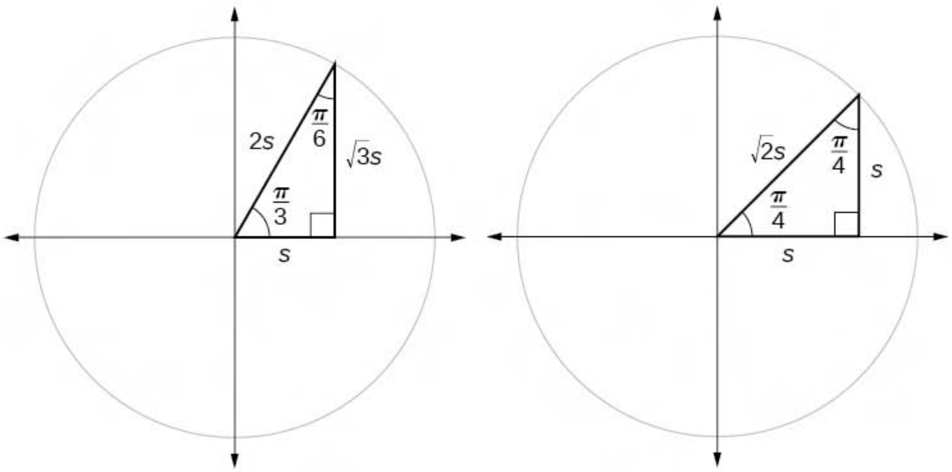


Figure 5.64 Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

How To:

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure 5.64 for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 5.31

Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3},$ using side lengths.

Solution

$$\sin\left(\frac{\pi}{3}\right) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$

$$\sec\left(\frac{\pi}{3}\right) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$

$$\csc\left(\frac{\pi}{3}\right) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot\left(\frac{\pi}{3}\right) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Try It

5.31 Find the exact value of the trigonometric functions of $\frac{\pi}{4},$ using side lengths.

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$.

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

See **Figure 5.65**

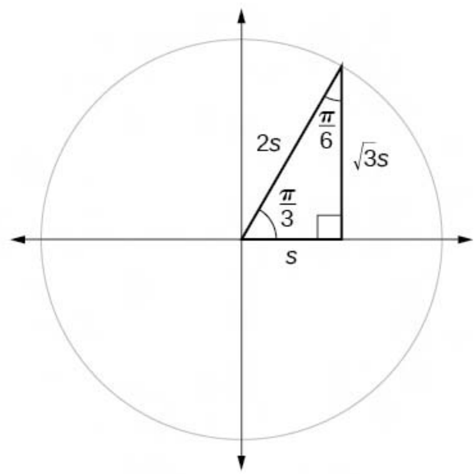
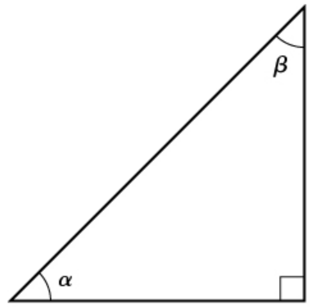


Figure 5.65 The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from **Figure 5.65**, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{6}\right)$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{6}\right)$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in **Figure 5.66**.



$$\sin\alpha = \cos\beta$$
$$\sin\beta = \cos\alpha$$

Figure 5.66 Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin\left(\frac{\pi}{2} - t\right) = \frac{5}{13}$ as well.

Cofunction Identities

The cofunction identities in radians are listed in **Table 5.5**.

$\cos t = \sin\left(\frac{\pi}{2} - t\right)$	$\sin t = \cos\left(\frac{\pi}{2} - t\right)$
$\tan t = \cot\left(\frac{\pi}{2} - t\right)$	$\cot t = \tan\left(\frac{\pi}{2} - t\right)$
$\sec t = \csc\left(\frac{\pi}{2} - t\right)$	$\csc t = \sec\left(\frac{\pi}{2} - t\right)$

Table 5.5

How To:

Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 5.32

Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $\left(\cos\frac{\pi}{2} - t\right)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin t = \cos\left(\frac{\pi}{2} - t\right).$$

So

$$\cos\left(\frac{\pi}{2} - t\right) = \frac{5}{12}.$$

Try It

5.32 If $\csc\left(\frac{\pi}{6}\right) = 2,$ find $\sec\left(\frac{\pi}{3}\right).$

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

How To:

Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides.

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 5.33

Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in **Figure 5.67**.

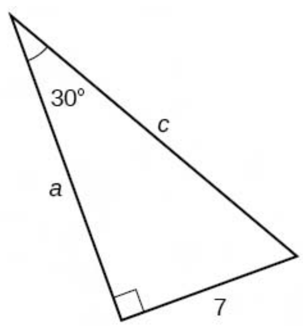


Figure 5.67

Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30°) = \frac{7}{a}$$

We rearrange to solve for a.

$$\begin{aligned} a &= \frac{7}{\tan(30°)} \\ &= 12.1 \end{aligned}$$

We can use the sine to find the hypotenuse.

$$\sin(30°) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$\begin{aligned} c &= \frac{7}{\sin(30°)} \\ &= 14 \end{aligned}$$

Try It

5.33 A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See **Figure 5.68**.

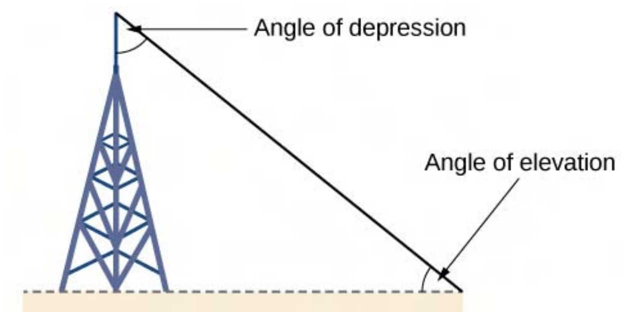


Figure 5.68

How To:

Given a tall object, measure its height indirectly.

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 5.34

Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of $57°$ between a line of sight to the top of the tree and the ground, as shown in **Figure 5.69**. Find the height of the tree.

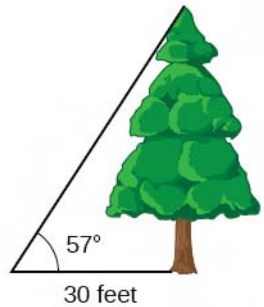


Figure 5.69

Solution

We know that the angle of elevation is $57°$ and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of $57°$, letting h be the unknown height.

$$\begin{aligned} \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} & \\ \tan(57°) &= \frac{h}{30} & \text{Solve for } h. \\ h &= 30\tan(57°) & \text{Multiply.} \\ h &\approx 46.2 & \text{Use a calculator.} \end{aligned}$$

The tree is approximately 46 feet tall.

Try It

5.34 How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

Access these online resources for additional instruction and practice with right triangle trigonometry.

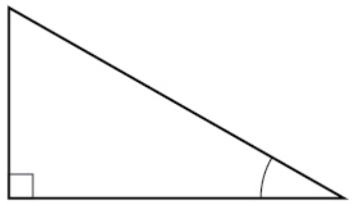
- **Finding Trig Functions on Calculator (http://openstaxcollege.org/l/findtrigcal)**
- **Finding Trig Functions Using a Right Triangle (http://openstaxcollege.org/l/trigrttri)**
- **Relate Trig Functions to Sides of a Right Triangle (http://openstaxcollege.org/l/reltrigtri)**
- **Determine Six Trig Functions from a Triangle (http://openstaxcollege.org/l/sixtrigfunc)**
- **Determine Length of Right Triangle Side (http://openstaxcollege.org/l/rttriside)**

Visit **this website (http://openstaxcollege.org/l/PreCalcLPC05)** for additional practice questions from Learningpod.

5.4 EXERCISES

Verbal

255. For the given right triangle, label the adjacent side, opposite side, and hypotenuse for the indicated angle.



256. When a right triangle with a hypotenuse of 1 is placed in the unit circle, which sides of the triangle correspond to the *x*- and *y*-coordinates?

257. The tangent of an angle compares which sides of the right triangle?

258. What is the relationship between the two acute angles in a right triangle?

259. Explain the cofunction identity.

Algebraic

For the following exercises, use cofunctions of complementary angles.

260. $\cos(34°) = \sin(\_\_°)$

261. $\cos\left(\frac{\pi}{3}\right) = \sin(\_\_\_)$

262. $\csc(21°) = \sec(\_\_\_°)$

263. $\tan\left(\frac{\pi}{4}\right) = \cot(\_\_)$

For the following exercises, find the lengths of the missing sides if side a is opposite angle $A,$ side b is opposite angle $B,$ and side c is the hypotenuse.

264. $\cos B = \frac{4}{5}, a = 10$

265. $\sin B = \frac{1}{2}, \quad a = 20$

266. $\tan A = \frac{5}{12}, b = 6$

267. $\tan A = 100, b = 100$

268. $\sin B = \frac{1}{\sqrt{3}}, \quad a = 2$

269. $a = 5, \quad \measuredangle A = 60°$

270. $c = 12, \quad \measuredangle A = 45°$

Graphical

For the following exercises, use **Figure 5.70** to evaluate each trigonometric function of angle A.

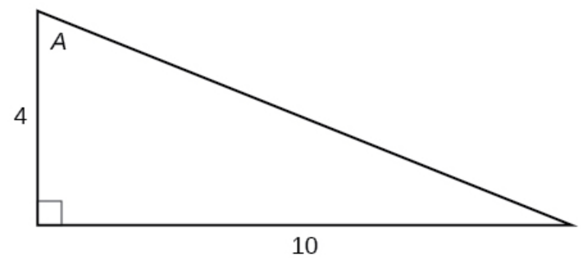


Figure 5.70

271. $\sin A$

272. $\cos A$

273. $\tan A$

274. $\csc A$

275. $\sec A$

276. $\cot A$

For the following exercises, use **Figure 5.71** to evaluate each trigonometric function of angle A.

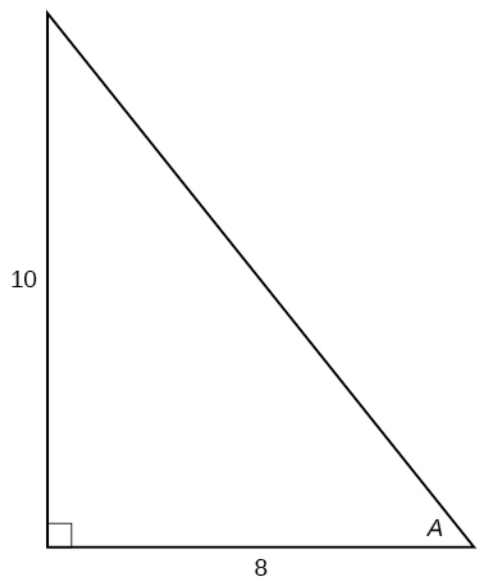


Figure 5.71

277. $\sin A$

278. $\cos A$

279. $\tan A$

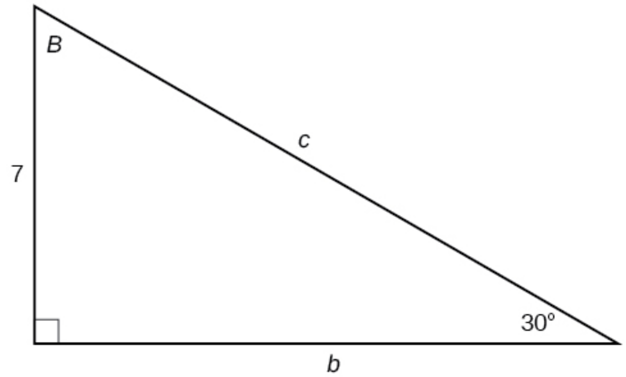
280. $\csc A$

281. $\sec A$

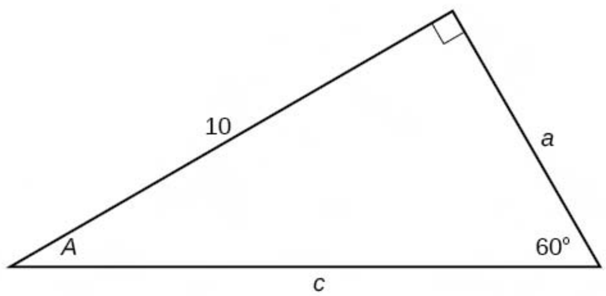
282. $\cot A$

For the following exercises, solve for the unknown sides of the given triangle.

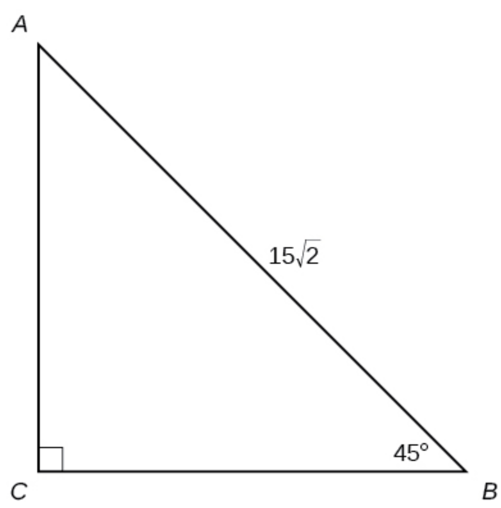
283.



284.



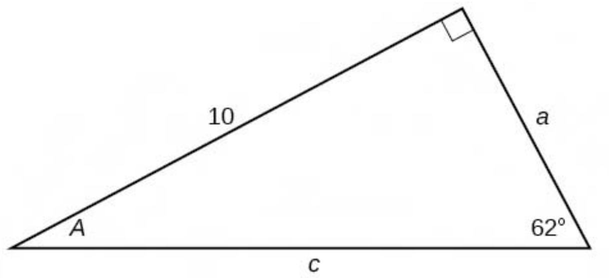
285.



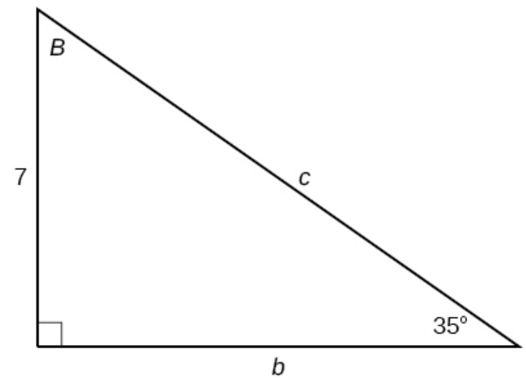
Technology

For the following exercises, use a calculator to find the length of each side to four decimal places.

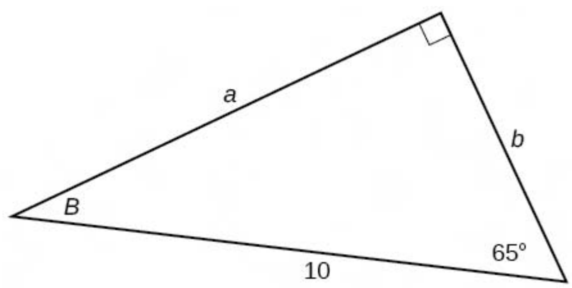
286.



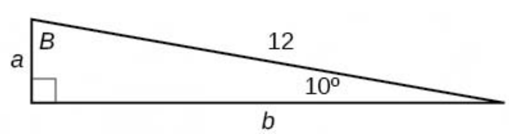
287.



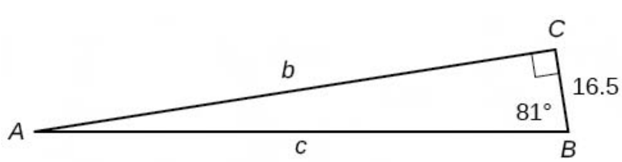
288.



289.



290.



291. $b = 15, \quad \measuredangle B = 15^\circ$

292. $c = 200, \quad \measuredangle B = 5^\circ$

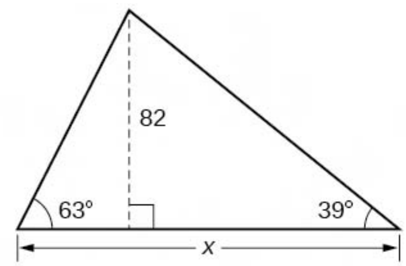
293. $c = 50, \quad \measuredangle B = 21^\circ$

294. $a = 30, \quad \measuredangle A = 27°$

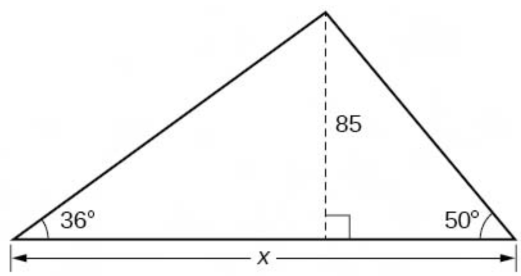
295. $b = 3.5, \quad \measuredangle A = 78°$

Extensions

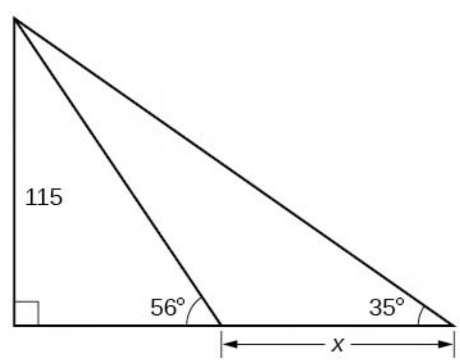
296. Find x.



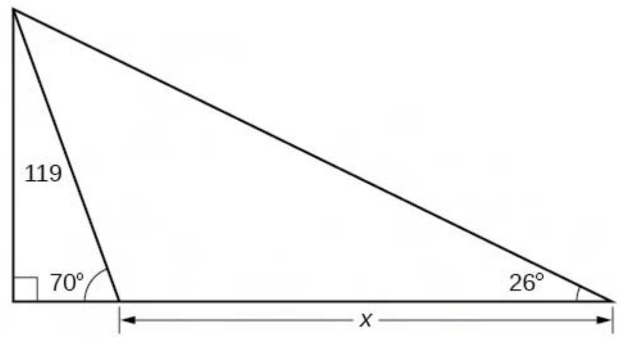
297. Find x.



298. Find x.



299. Find x.



300. A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is $36°$, and that the angle of depression to the bottom of the tower is $23°$. How tall is the tower?

301. A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is $43°$, and that the angle of depression to the bottom of the tower is $31°$. How tall is the tower?

302. A 200-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is $15°$, and that the angle of depression to the bottom of the tower is $2°$. How far is the person from the monument?

303. A 400-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is $18°$, and that the angle of depression to the bottom of the tower is $3°$. How far is the person from the monument?

304. There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be $40°$. From the same location, the angle of elevation to the top of the antenna is measured to be $43°$. Find the height of the antenna.

305. There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be $36°$. From the same location, the angle of elevation to the top of the lightning rod is measured to be $38°$. Find the height of the lightning rod.

Real-World Applications

306. A 33-ft ladder leans against a building so that the angle between the ground and the ladder is $80°$. How high does the ladder reach up the side of the building?

307. A 23-ft ladder leans against a building so that the angle between the ground and the ladder is $80°$. How high does the ladder reach up the side of the building?

308. The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

309. The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

310. Assuming that a 370-foot tall giant redwood grows vertically, if I walk a certain distance from the tree and measure the angle of elevation to the top of the tree to be $60°$, how far from the base of the tree am I?

CHAPTER 5 REVIEW

KEY TERMS

adjacent side in a right triangle, the side between a given angle and the right angle

angle of depression the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

angle the union of two rays having a common endpoint

angular speed the angle through which a rotating object travels in a unit of time

arc length the length of the curve formed by an arc

area of a sector area of a portion of a circle bordered by two radii and the intercepted arc; the fraction $\frac{\theta}{2\pi}$ multiplied by the area of the entire circle

cosecant the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y},\ y \neq 0$

cosine function the x-value of the point on a unit circle corresponding to a given angle

cotangent the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y},\ y \neq 0$

coterminal angles description of positive and negative angles in standard position sharing the same terminal side

degree a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

hypotenuse the side of a right triangle opposite the right angle

identities statements that are true for all values of the input on which they are defined

initial side the side of an angle from which rotation begins

linear speed the distance along a straight path a rotating object travels in a unit of time; determined by the arc length

measure of an angle the amount of rotation from the initial side to the terminal side

negative angle description of an angle measured clockwise from the positive x-axis

opposite side in a right triangle, the side most distant from a given angle

Pythagorean Identity a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

period the smallest interval P of a repeating function f such that $f(x+P) = f(x)$

positive angle description of an angle measured counterclockwise from the positive x-axis

quadrantal angle an angle whose terminal side lies on an axis

radian measure the ratio of the arc length formed by an angle divided by the radius of the circle

radian the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

ray one point on a line and all points extending in one direction from that point; one side of an angle

reference angle the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

secant the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x},\ x \neq 0$

sine function the y-value of the point on a unit circle corresponding to a given angle

standard position the position of an angle having the vertex at the origin and the initial side along the positive x-axis

tangent the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x},\ x \neq 0$

terminal side the side of an angle at which rotation ends

unit circle a circle with a center at $(0,\ 0)$ and radius 1.

vertex the common endpoint of two rays that form an angle

KEY EQUATIONS

arc length	$s = r\theta$
area of a sector	$A = \frac{1}{2}\theta r^2$
angular speed	$\omega = \frac{\theta}{t}$
linear speed	$v = \frac{s}{t}$
linear speed related to angular speed	$v = r\omega$

Cosine	$\cos t = x$
Sine	$\sin t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$
Tangent function	$\tan t = \frac{\sin t}{\cos t}$
Secant function	$\sec t = \frac{1}{\cos t}$
Cosecant function	$\csc t = \frac{1}{\sin t}$
Cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Cofunction Identities

$$\cos t = \sin\left(\frac{\pi}{2} - t\right)$$
$$\sin t = \cos\left(\frac{\pi}{2} - t\right)$$
$$\tan t = \cot\left(\frac{\pi}{2} - t\right)$$
$$\cot t = \tan\left(\frac{\pi}{2} - t\right)$$
$$\sec t = \csc\left(\frac{\pi}{2} - t\right)$$
$$\csc t = \sec\left(\frac{\pi}{2} - t\right)$$

KEY CONCEPTS

5.1 Angles

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive x-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive x-axis and then place the terminal side according to the fraction of a full rotation the angle represents. See **Example 5.1.**
- In addition to degrees, the measure of an angle can be described in radians. See **Example 5.2.**
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta^R}{\pi}$. See **Example 5.3** and **Example 5.4**.
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting 360° or 2π. See **Example 5.5** and **Example 5.6**.
- Coterminal angles can be found using radians just as they are for degrees. See **Example 5.7.**
- The length of a circular arc is a fraction of the circumference of the entire circle. See **Example 5.8.**
- The area of sector is a fraction of the area of the entire circle. See **Example 5.9.**
- An object moving in a circular path has both linear and angular speed.
- The angular speed of an object traveling in a circular path is the measure of the angle through which it turns in a unit of time. See **Example 5.10.**
- The linear speed of an object traveling along a circular path is the distance it travels in a unit of time. See **Example 5.11.**

5.2 Unit Circle: Sine and Cosine Functions

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the x-value of the endpoint. See **Example 5.12**.
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See **Example 5.13**.
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See **Example 5.14**.
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See **Example 5.15**.
- The domain of the sine and cosine functions is all real numbers.

- The range of both the sine and cosine functions is $[-1,\ 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the *x*- and *y*-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, $t,$ formed by the terminal side of the angle t and the horizontal axis. See **Example 5.16**.
- Reference angles can be used to find the sine and cosine of the original angle. See **Example 5.17**.
- Reference angles can also be used to find the coordinates of a point on a circle. See **Example 5.18**.

5.3 The Other Trigonometric Functions

- The tangent of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle. See **Example 5.19**.
- Trigonometric functions can also be found from an angle. See **Example 5.20**.
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See **Example 5.21**.
- A function is said to be even if $f(-x)=f(x)$ and odd if $f(-x)=-f(x)$.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See **Example 5.22**.
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See **Example 5.23** and **Example 5.24**.
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See **Example 5.25**.
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f is the smallest interval such that $f(x+P)=f(x)$ for any value of x.
- The values of trigonometric functions of special angles can be found by mathematical analysis.
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software. See **Example 5.28**.

5.4 Right Triangle Trigonometry

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See **Example 5.29**.
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See **Example 5.30**.
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See **Example 5.31**.
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See **Example 5.32**.
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See **Example 5.33**.
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See **Example 5.34**.

CHAPTER 5 REVIEW EXERCISES

Angles

For the following exercises, convert the angle measures to degrees.

311. $\frac{\pi}{4}$

312. $-\frac{5\pi}{3}$

For the following exercises, convert the angle measures to radians.

313. -210°

314. 180°

315. Find the length of an arc in a circle of radius 7 meters subtended by the central angle of 85°.

316. Find the area of the sector of a circle with diameter 32 feet and an angle of $\frac{3\pi}{5}$ radians.

For the following exercises, find the angle between 0° and 360° that is coterminal with the given angle.

317. 420°

318. $-80°$

For the following exercises, find the angle between 0 and 2π in radians that is coterminal with the given angle.

319. $-\frac{20\pi}{11}$

320. $\frac{14\pi}{5}$

For the following exercises, draw the angle provided in standard position on the Cartesian plane.

321. -210°

322. 75°

323. $\frac{5\pi}{4}$

324. $-\frac{\pi}{3}$

325. Find the linear speed of a point on the equator of the earth if the earth has a radius of 3,960 miles and the earth rotates on its axis every 24 hours. Express answer in miles per hour.

326. A car wheel with a diameter of 18 inches spins at the rate of 10 revolutions per second. What is the car's speed in miles per hour?

Unit Circle: Sine and Cosine Functions

327. Find the exact value of $\sin\frac{\pi}{3}$.

328. Find the exact value of $\cos\frac{\pi}{4}$.

329. Find the exact value of $\cos\pi$.

330. State the reference angle for $300°$.

331. State the reference angle for $\frac{3\pi}{4}$.

332. Compute cosine of $330°$.

333. Compute sine of $\frac{5\pi}{4}$.

334. State the domain of the sine and cosine functions.

335. State the range of the sine and cosine functions.

The Other Trigonometric Functions

For the following exercises, find the exact value of the given expression.

336. $\cos\frac{\pi}{6}$

337. $\tan\frac{\pi}{4}$

338. $\csc\frac{\pi}{3}$

339. $\sec\frac{\pi}{4}$

For the following exercises, use reference angles to evaluate the given expression.

340. $\sec\frac{11\pi}{3}$

341. $\sec 315°$

342. If $\sec(t) = -2.5$, what is the $\sec(-t)$?

343. If $\tan(t) = -0.6,$ what is the $\tan(-t)$?

344. If $\tan(t) = \frac{1}{3},$ find $\tan(t - \pi)$.

345. If $\cos(t) = \frac{\sqrt{2}}{2},$ find $\sin(t + 2\pi)$.

346. Which trigonometric functions are even?

347. Which trigonometric functions are odd?

Right Triangle Trigonometry

For the following exercises, use side lengths to evaluate.

348. $\cos\frac{\pi}{4}$

349. $\cot\frac{\pi}{3}$

350. $\tan\frac{\pi}{6}$

351. $\cos\left(\frac{\pi}{2}\right) = \sin(\_\_°)$

352. $\csc(18°) = \sec(\_\_°)$

For the following exercises, use the given information to find the lengths of the other two sides of the right triangle.

353. $\cos B = \frac{3}{5},\ a = 6$

354. $\tan A = \frac{5}{9},\ b = 6$

For the following exercises, use **Figure 5.72** to evaluate each trigonometric function.

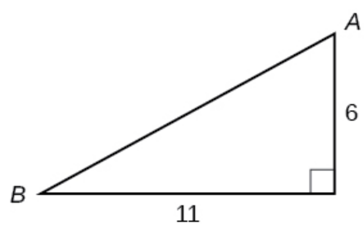


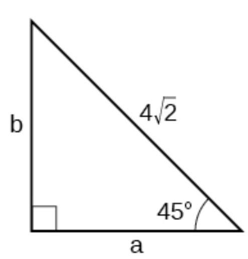
Figure 5.72

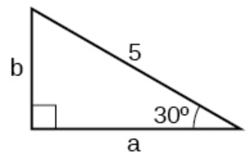
355. $\sin A$

356. $\tan B$

For the following exercises, solve for the unknown sides of the given triangle.

357.





358.

359. A 15-ft ladder leans against a building so that the angle between the ground and the ladder is $70°$. How high does the ladder reach up the side of the building?

360. The angle of elevation to the top of a building in Baltimore is found to be 4 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

CHAPTER 5 PRACTICE TEST

361. Convert $\frac{5\pi}{6}$ radians to degrees.

362. Convert $-620°$ to radians.

363. Find the length of a circular arc with a radius 12 centimeters subtended by the central angle of $30°$.

364. Find the area of the sector with radius of 8 feet and an angle of $\frac{5\pi}{4}$ radians.

365. Find the angle between $0°$ and $360°$ that is coterminal with $375°$.

366. Find the angle between 0 and 2π in radians that is coterminal with $-\frac{4\pi}{7}$.

367. Draw the angle $315°$ in standard position on the Cartesian plane.

368. Draw the angle $-\frac{\pi}{6}$ in standard position on the Cartesian plane.

369. A carnival has a Ferris wheel with a diameter of 80 feet. The time for the Ferris wheel to make one revolution is 75 seconds. What is the linear speed in feet per second of a point on the Ferris wheel? What is the angular speed in radians per second?

370. Find the exact value of $\sin\frac{\pi}{6}$.

371. Compute sine of $240°$.

372. State the domain of the sine and cosine functions.

373. State the range of the sine and cosine functions.

374. Find the exact value of $\cot\frac{\pi}{4}$.

375. Find the exact value of $\tan\frac{\pi}{3}$.

376. Use reference angles to evaluate $\csc\frac{7\pi}{4}$.

377. Use reference angles to evaluate $\tan 210°$.

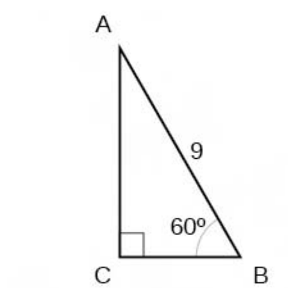
378. If $\csc t = 0.68$, what is the $\csc(-t)$?

379. If $\cos \mathrm{t} = \frac{\sqrt{3}}{2}$, find $\cos(t - 2\pi)$.

380. Which trigonometric functions are even?

381. Find the missing angle: $\cos\left(\frac{\pi}{6}\right) = \sin(\underline{\quad})$

382. Find the missing sides of the triangle ABC : $\sin B = \frac{3}{4}$, $c = 12$



383. Find the missing sides of the triangle.

384. The angle of elevation to the top of a building in Chicago is found to be 9 degrees from the ground at a distance of 2000 feet from the base of the building. Using this information, find the height of the building.