CHAPTER

2 Limits and Continuity

The yield on the June to August rice crop in Laos depends on the amount of rainfall, which is usually between 0.5 and 0.8 meter over this three-month growing season. More rain produces a higher yield, but the benefits diminish as the amount of rain increases. Crop yield in normal years can be approximated by

$$C(r) = -2.1 + 8.7r - 3.7r^2$$

metric tons per hectare, where r is the total rainfall, measured in meters. When the total rainfall is about 0.7 meter, how sensitive is the crop yield to a small increase in the amount of rain? Section 2.4 can help answer this question.

CHAPTER 2 Overview

The concept of limit is one of the fundamental building blocks of calculus, enabling us to describe with precision how change in one variable affects change in another variable.

In this chapter, we show how to define and calculate limits of function values. The calculation rules are straightforward, and most of the limits we need can be found by substitution, graphical investigation, numerical approximation, algebra, or some combination of these.

One of the uses of limits lies in building a careful definition of continuity. Continuous functions arise frequently in scientific work because they model such an enormous range of natural behavior and because they have special mathematical properties.

2.1 Rates of Change and Limits

You will be able to interpret, estimate, and determine limits of function values.

- Interpretation and expression of limits using correct notation
- Estimation of limits using numerical and graphical information
- Limits of sums, differences, products, quotients, and composite functions
- Interpretation and expression of one-sided limits
- The Squeeze Theorem

Average and Instantaneous Speed

The average speed of a moving body during an interval of time is found by dividing the change in distance or position by the change in time. More precisely, if $y = f(t)$ is a distance or position function of a moving body at time t, then the **average rate of change** (or **average speed**) is the ratio

$$\frac{\Delta y}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t},$$

where the elapsed time is the interval from t to $t + \Delta t$, or simply Δt, and the distance traveled during this time interval is $f(t + \Delta t) - f(t)$. It is also common to use the letter h instead of Δt to denote the elapsed time, in which case the average rate of change can be written

$$\frac{\Delta y}{\Delta t} = \frac{f(t + h) - f(t)}{h}.$$

Free Fall

Near the surface of the earth, all bodies fall with the same constant acceleration. The distance a body falls after it is released from rest is a constant multiple of the square of the time fallen. At least, that is what happens when a body falls in a vacuum, where there is no air to slow it down. The square-of-time rule also holds for dense, heavy objects like rocks, ball bearings, and steel tools during the first few seconds of fall through air, before the velocity builds up to where air resistance begins to matter. When air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way the body falls *free fall.*

EXAMPLE 1 Finding an Average Speed

A rock breaks loose from the top of a tall cliff. What is its average speed during the first 2 seconds of fall?

SOLUTION

Experiments show that a dense solid object dropped from rest to fall freely near the surface of the earth will fall

$$y = 16t^2$$

feet in the first t seconds. The average speed of the rock over any given time interval is the distance traveled, Δy, divided by the length of the interval Δt. For the first 2 seconds of fall, from $t = 0$ to $t = 2$, we have

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\,\frac{\text{ft}}{\text{sec}}.$$

Now Try Exercise 1.

The speed of a falling rock is always increasing. If we know the position as a function of time, we can calculate average speed over any given interval of time. But we can also talk about its **instantaneous speed** or **instantaneous rate of change,** the speed at one instant of time. As we will see after the next example, we need the idea of *limit* to make precise what we mean by instantaneous rate of change.

TABLE 2.1
Average Speeds over Short Time Intervals Starting at $t = 2$

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$$

Length of Time Interval, h (sec)	Average Speed for Interval $\Delta y/\Delta t$ (ft/sec)
1	80
0.1	65.6
0.01	64.16
0.001	64.016
0.0001	64.0016
0.00001	64.00016

EXAMPLE 2 Finding an Instantaneous Speed

Find the speed of the rock in Example 1 at the instant $t = 2$.

SOLUTION

We can calculate the average speed of the rock over the interval from time $t = 2$ to any slightly later time $t = 2 + h$ as

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}. \quad (1)$$

We cannot use this formula to calculate the speed at the exact instant $t = 2$ because that would require taking $h = 0$, and $0/0$ is undefined. However, we can get a good idea of what is happening at $t = 2$ by evaluating the formula at values of h *close* to 0. When we do, we see a clear pattern (Table 2.1). As h approaches 0, the average speed approaches the limiting value 64 ft/sec.

If we expand the numerator of Equation 1 and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16(4 + 4h + h^2) - 64}{h}$$

$$= \frac{64h + 16h^2}{h} = 64 + 16h$$

For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $64 + 16h$ ft/sec. We can now see why the average speed has the limiting value $64 + 16(0) = 64$ ft/sec as h approaches 0. ***Now Try Exercise 3.***

Formal Definition of Limit

The formal definition of a limit is given in Appendix A, pp. 583–589. This appendix also illustrates how the formal definition is applied and how it leads to the *Properties of Limits* given in Theorem 1.

Definition of Limit

As in the preceding example, most limits of interest in the real world can be viewed as numerical limits of values of functions. And this is where a graphing utility and calculus come in. A calculator can suggest the limits, and calculus can give the mathematics for confirming the limits analytically.

Limits give us a language for describing how the outputs of a function behave as the inputs approach some particular value. In Example 2, the average speed was not defined at $h = 0$ but approached the limit 64 as h approached 0. We were able to see this numerically and to confirm it algebraically by eliminating h from the denominator. But we cannot always do that. For instance, we can see both graphically and numerically (Figure 2.1) that the values of $f(x) = (\sin x)/x$ approach 1 as x approaches 0.

We cannot eliminate the x from the denominator of $(\sin x)/x$ to confirm the observation algebraically. We need to use a theorem about limits to make that confirmation, as you will see in Exercise 77.

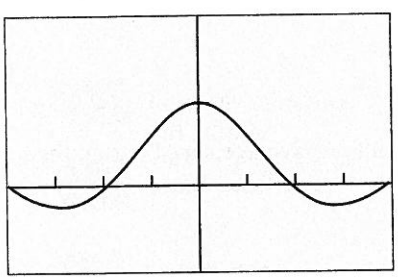
The sentence $\lim_{x \to c} f(x) = L$ is read, "The limit of f of x as x approaches c equals L." The notation means that we can force $f(x)$ to be as close to L as we wish simply by restricting the distance between x and c, *but not allowing x to equal c.*

We saw in Example 2 that $\lim_{h \to 0}(64 + 16h) = 64$.

As suggested in Figure 2.1,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Because we need to distinguish between what happens at c and what happens near c, the value or existence of the limit as $x \to c$ *never* depends on how the function may or may not be defined at c. This is illustrated in Figure 2.2. The function f has limit 2 as $x \to 1$ even though f is not defined at 1. The function g has limit 2 as $x \to 1$ even though $g(1) \neq 2$. The function h is the only one whose limit as $x \to 1$ equals its value at $x = 1$.



$[-2\pi, 2\pi]$ by $[-1, 2]$

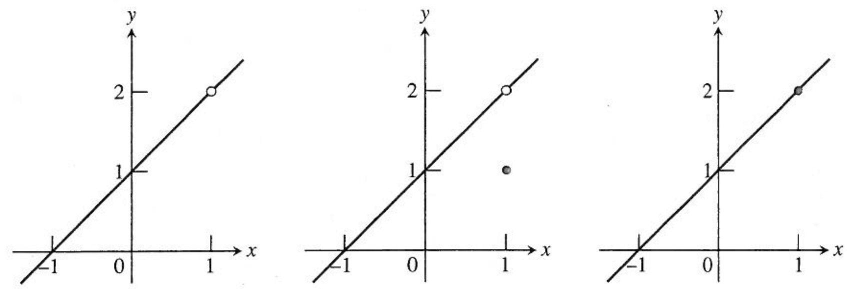
(a)

X	Y1	
−.3	.98507	
−.2	.99335	
−.1	.99833	
0	ERROR	
.1	.99833	
.2	.99335	
.3	.98507	

Y1 = sin(X)/X

(b)

Figure 2.1 (a) A graph and (b) table of values for $f(x) = (\sin x)/x$ that suggest the limit of f as x approaches 0 is 1.



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

Figure 2.2 $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = 2.$

Properties of Limits

By applying six basic facts about limits, we can calculate many unfamiliar limits from limits we already know. For instance, from knowing that

$$\lim_{x \to c} (k) = k \quad \text{Limit of the function with constant value } k$$

and

$$\lim_{x \to c} (x) = c, \quad \text{Limit of the identity function at } x = c$$

we can calculate the limits of all polynomial and rational functions. The facts are listed in Theorem 1.

THEOREM 1 Properties of Limits

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \text{ then}$$

1. *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = L + M$

   The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:* $\lim_{x \to c} (f(x) - g(x)) = L - M$

   The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:* $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$

   The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$

   The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \; M \neq 0$

   The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

continued

6. *Power Rule:* If r and s are integers, $s \neq 0$, then

$$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number and $L > 0$ if s is even.

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number and $L > 0$ if s is even.

Here are some examples of how Theorem 1 can be used to find limits of polynomial and rational functions.

Using Analytic Methods

We remind the student that *unless otherwise stated* all examples and exercises are to be done using analytic algebraic methods *without* the use of graphing calculators or computer algebra systems.

EXAMPLE 3 Using Properties of Limits

Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$, and the properties of limits to find the following limits.

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$ **(b)** $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

SOLUTION

(a)
$$\lim_{x \to c} (x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$$ Sum and Difference Rules

$$= c^3 + 4c^2 - 3$$ Product and Constant Multiple Rules

(b)
$$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)}$$ Quotient Rule

$$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$$ Sum and Difference Rules

$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$ Product Rule

Now Try Exercises 5 and 6.

Example 3 shows the remarkable strength of Theorem 1. From the two simple observations that $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$, we can immediately work our way to limits of polynomial functions and most rational functions using substitution.

THEOREM 2 Polynomial and Rational Functions

1. If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is any polynomial function and c is any real number, then

$$\lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

2. If $f(x)$ and $g(x)$ are polynomials and c is any real number, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \quad \text{provided that } g(c) \neq 0.$$

EXAMPLE 4 Using Theorem 2

(a) $\lim_{x \to 3} [x^2(2 - x)] = (3)^2(2 - 3) = -9$

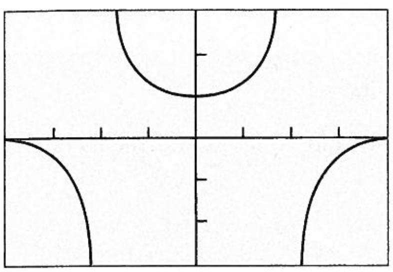
(b) $\lim_{x \to 2} \frac{x^2 + 2x + 4}{x + 2} = \frac{(2)^2 + 2(2) + 4}{2 + 2} = \frac{12}{4} = 3$

Now Try Exercises 9 and 11.

As with polynomials, limits of many familiar functions can be found by substitution at points where they are defined. This includes trigonometric functions, exponential and logarithmic functions, and composites of these functions. Feel free to use these properties.

EXAMPLE 5 Using the Product Rule

Determine $\lim_{x \to 0} \frac{\tan x}{x}$.



$[-\pi, \pi]$ by $[-3, 3]$

Figure 2.3 The graph of

$$f(x) = (\tan x)/x$$

suggests that $f(x) \to 1$ as $x \to 0$. (Example 5)

SOLUTION

The graph of $f(x) = (\tan x)/x$ in Figure 2.3 suggests that the limit exists and is about 1. Using the analytic result of Exercise 77, we have

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) \qquad \tan x = \frac{\sin x}{\cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} \qquad \text{Product Rule}$$

$$= 1 \cdot \frac{1}{\cos 0} = 1 \cdot \frac{1}{1} = 1$$

Now Try Exercise 33.

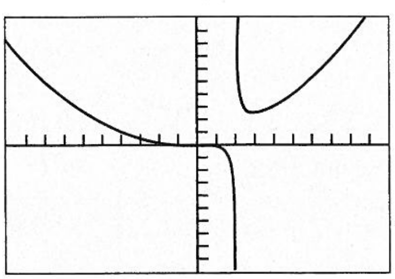
Sometimes we can use a graph to discover that limits do not exist, as illustrated by Example 6.

EXAMPLE 6 Exploring a Nonexistent Limit

Use a graph to explore whether

$$\lim_{x \to 2} \frac{x^3 - 1}{x - 2}$$

exists.



$[-10, 10]$ by $[-100, 100]$

Figure 2.4 The graph of

$$f(x) = (x^3 - 1)/(x - 2).$$

(Example 6)

SOLUTION

Notice that the denominator is 0 when x is replaced by 2, so we cannot use substitution to determine the limit. The graph in Figure 2.4 of $f(x) = (x^3 - 1)/(x - 2)$ strongly suggests that as $x \to 2$ from either side, the absolute values of the function values get very large. This, in turn, suggests that the limit does not exist.

Now Try Exercise 35.

One-Sided and Two-Sided Limits

Sometimes we need to distinguish between what happens to the function just to the right of c and just to the left. To do this, we call the limit of f as x approaches c from the right the **right-hand limit** of f at c and the limit as x approaches c from the left the **left-hand limit** of f at c. Here is the notation we use:

right-hand: $\lim_{x \to c^+} f(x)$ *The limit of f as x approaches c from the right.*

left-hand: $\lim_{x \to c^-} f(x)$ *The limit of f as x approaches c from the left.*

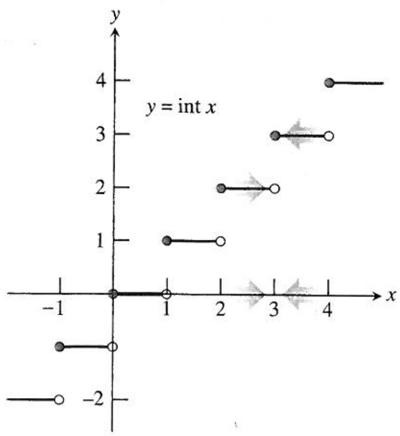


Figure 2.5 At each integer, the greatest integer function $y = \text{int } x$ has different right-hand and left-hand limits. (Example 7)

On the Far Side

If f is not defined to the left of $x = c$, then f does not have a left-hand limit at c. Similarly, if f is not defined to the right of $x = c$, then f does not have a right-hand limit at c.

EXAMPLE 7 Function Values Approach Two Numbers

The greatest integer function $f(x) = \text{int } x$ has different right-hand and left-hand limits at each integer, as we can see in Figure 2.5. For example,

$$\lim_{x \to 3^+} \text{int } x = 3 \quad \text{and} \quad \lim_{x \to 3^-} \text{int } x = 2.$$

The limit of $\text{int } x$ as x approaches an integer n from the right is n, while the limit as x approaches n from the left is $n - 1$. ***Now Try Exercises 37 and 38.***

The greatest integer function, which appears as $\text{int } x$ on most calculators, is known to mathematicians as the **floor function,** written $\lfloor x \rfloor$, where we use only the bottom horizontal parts of the brackets to indicate that we go *down* until we reach an integer. You should be able to recognize and use either notation.

We sometimes call $\lim_{x \to c} f(x)$ the **two-sided limit** of f at c to distinguish it from the *one-sided* right-hand and left-hand limits of f at c. Theorem 3 shows how these limits are related.

THEOREM 3 One-Sided and Two-Sided Limits

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbols,

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L.$$

Thus, the greatest integer function $f(x) = \text{int } x$ of Example 7 does not have a limit as $x \to 3$ even though each one-sided limit exists.

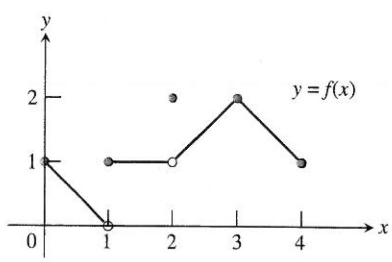


Figure 2.6 The graph of the function

$$f(x) = \begin{cases} -x + 1, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x - 1, & 2 < x \le 3 \\ -x + 5, & 3 < x \le 4. \end{cases}$$

(Example 8)

EXAMPLE 8 Exploring Right- and Left-Hand Limits

All the following statements about the function $y = f(x)$ graphed in Figure 2.6 are true.

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x \to 1^+} f(x) = 1$,

f has no limit as $x \to 1$. (The right- and left-hand limits at 1 are not equal, so $\lim_{x \to 1} f(x)$ does not exist.)

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1$,

$\lim_{x \to 2^+} f(x) = 1$,

$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 2 = f(3) = \lim_{x \to 3} f(x)$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1$.

At noninteger values of c between 0 and 4, f has a limit as $x \to c$.

Now Try Exercise 43.

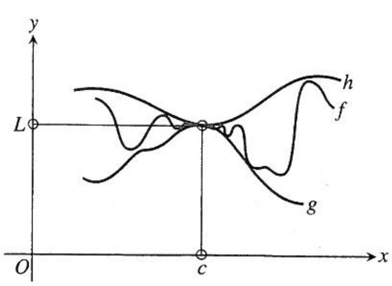


Figure 2.7 Squeezing f between g and h creates a bottleneck around the point (c, L). If we keep x close to c, the bottleneck forces $f(x)$ to be close to L.

Squeeze Theorem

If we cannot find a limit directly, we may be able to find it indirectly with the Squeeze Theorem. The theorem refers to a function f whose values are squeezed between the values of two other functions, g and h. If g and h have the same limit as $x \to c$, then f has that limit too, as suggested by Figure 2.7.

THEOREM 4 The Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ for all $x \neq c$ in some interval about c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

EXAMPLE 9 Using the Squeeze Theorem

Show that $\lim_{x \to 0} [x^2 \sin(1/x)] = 0$.

SOLUTION

We know that the values of the sine function lie between −1 and 1. So, it follows that

$$\left| x^2 \sin \frac{1}{x} \right| = |x^2| \cdot \left| \sin \frac{1}{x} \right| \leq |x^2| \cdot 1 = x^2$$

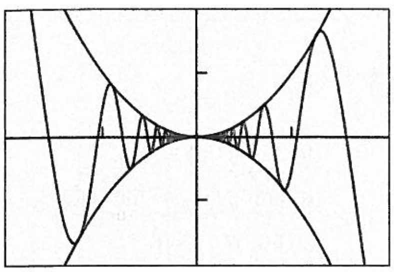
and

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2.$$

Because $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0$, the Squeeze Theorem gives

$$\lim_{x \to 0} \left(x^2 \sin \frac{1}{x} \right) = 0.$$

The graphs in Figure 2.8 support this result. ***Now Try Exercise 65.***



[−0.2, 0.2] by [−0.02, 0.02]

Figure 2.8 The graphs of $y_1 = x^2$, $y_2 = x^2 \sin(1/x)$, and $y_3 = -x^2$. Notice that $y_3 \leq y_2 \leq y_1$. (Example 9)

Quick Review 2.1 *(For help, go to Section 1.2.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–4, find $f(2)$.

1. $f(x) = 2x^3 - 5x^2 + 4$

2. $f(x) = \dfrac{4x^2 - 5}{x^3 + 4}$

3. $f(x) = \sin\left(\pi \dfrac{x}{2}\right)$

4. $f(x) = \begin{cases} 3x - 1, & x < 2 \\ \dfrac{1}{x^2 - 1}, & x \geq 2 \end{cases}$

In Exercises 5–8, write the inequality in the form $a < x < b$.

5. $|x| < 4$

6. $|x| < c^2$

7. $|x - 2| < 3$

8. $|x - c| < d^2$

In Exercises 9 and 10, write the fraction in reduced form.

9. $\dfrac{x^2 - 3x - 18}{x + 3}$

10. $\dfrac{2x^2 - x}{2x^2 + x - 1}$

Section 2.1 Exercises

In Exercises 1–4, an object dropped from rest from the top of a tall building falls $y = 16t^2$ feet in the first t seconds.

1. Find the average speed during the first 3 seconds of fall.

2. Find the average speed during the first 4 seconds of fall.

3. Find the speed of the object at $t = 3$ seconds and confirm your answer algebraically.

4. Find the speed of the object at $t = 4$ seconds and confirm your answer algebraically.

In Exercises 5 and 6, use $\lim_{x\to c} k = k$, $\lim_{x\to c} x = c$, and the properties of limits to find the limit.

5. $\lim_{x\to c} (2x^3 - 3x^2 + x - 1)$

6. $\lim_{x\to c} \dfrac{x^4 - x^3 + 1}{x^2 + 9}$

In Exercises 7–14, determine the limit by substitution.

7. $\lim_{x\to -1/2} 3x^2 (2x - 1)$

8. $\lim_{x\to -4} (x + 3)^{2016}$

9. $\lim_{x\to 1} (x^3 + 3x^2 - 2x - 17)$

10. $\lim_{y\to 2} \dfrac{y^2 + 5y + 6}{y + 2}$

11. $\lim_{y\to -3} \dfrac{y^2 + 4y + 3}{y^2 - 3}$

12. $\lim_{x\to 1/2} \text{int } x$

13. $\lim_{x\to -2} (x - 6)^{2/3}$

14. $\lim_{x\to 2} \sqrt{x + 3}$

In Exercises 15–20, complete the following tables and state what you believe $\lim_{x\to 0} f(x)$ to be.

(a)

x	−0.1	−0.01	−0.001	−0.0001	...
$f(x)$	?	?	?	?	

(b)

x	0.1	0.01	0.001	0.0001	...
$f(x)$	?	?	?	?	

15. $f(x) = \dfrac{x^2 + 6x + 2}{x + 1}$

16. $f(x) = \dfrac{x^2 - x}{x}$

17. $f(x) = x \sin \dfrac{1}{x}$

18. $f(x) = \sin \dfrac{1}{x}$

19. $f(x) = \dfrac{10^x - 1}{x}$

20. $f(x) = x \sin (\ln |x|)$

In Exercises 21–24, explain why you cannot use substitution to determine the limit. Find the limit if it exists.

21. $\lim_{x\to -2} \sqrt{x - 2}$

22. $\lim_{x\to 0} \dfrac{1}{x^2}$

23. $\lim_{x\to 0} \dfrac{|x|}{x}$

24. $\lim_{x\to 0} \dfrac{(4 + x)^2 - 16}{x}$

In Exercises 25–34, explore the limit graphically. Confirm algebraically.

25. $\lim_{x\to 1} \dfrac{x - 1}{x^2 - 1}$

26. $\lim_{t\to 2} \dfrac{t^2 - 3t + 2}{t^2 - 4}$

27. $\lim_{x\to 0} \dfrac{5x^3 + 8x^2}{3x^4 - 16x^2}$

28. $\lim_{x\to 0} \dfrac{\frac{1}{2 + x} - \frac{1}{2}}{x}$

29. $\lim_{x\to 0} \dfrac{(2 + x)^3 - 8}{x}$

30. $\lim_{x\to 0} \dfrac{\sin 2x}{x}$

31. $\lim_{x\to 0} \dfrac{\sin x}{2x^2 - x}$

32. $\lim_{x\to 0} \dfrac{x + \sin x}{x}$

33. $\lim_{x\to 0} \dfrac{\sin^2 x}{x}$

34. $\lim_{x\to 5} \dfrac{x^3 - 125}{x - 5}$

In Exercises 35 and 36, use a graph to explore whether the limit exists.

35. $\lim_{x\to 1} \dfrac{x^2 - 4}{x - 1}$

36. $\lim_{x\to 2} \dfrac{x + 1}{x^2 - 4}$

In Exercises 37–42, determine the limit.

37. $\lim_{x\to 0^+} \text{int } x$

38. $\lim_{x\to 0^-} \text{int } x$

39. $\lim_{x\to 0.01} \text{int } x$

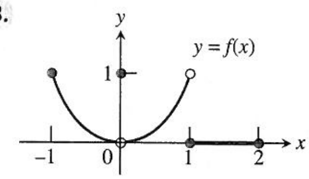
40. $\lim_{x\to 2^-} \text{int } x$

41. $\lim_{x\to 0^+} \dfrac{x}{|x|}$

42. $\lim_{x\to 0^-} \dfrac{x}{|x|}$

In Exercises 43 and 44, which of the statements are true about the function $y = f(x)$ graphed there, and which are false?

43.



(a) $\lim_{x\to -1^+} f(x) = 1$

(b) $\lim_{x\to 0^-} f(x) = 0$

(c) $\lim_{x\to 0^-} f(x) = 1$

(d) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

(e) $\lim_{x\to 0} f(x)$ exists

(f) $\lim_{x\to 0} f(x) = 0$

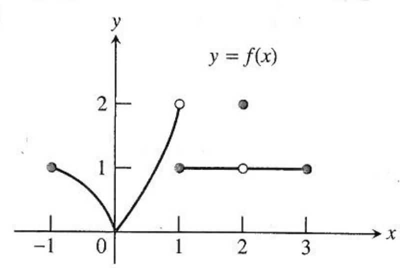
(g) $\lim_{x\to 0} f(x) = 1$

(h) $\lim_{x\to 1} f(x) = 1$

(i) $\lim_{x\to 1} f(x) = 0$

(j) $\lim_{x\to 2^-} f(x) = 2$

44.



(a) $\lim_{x\to -1^+} f(x) = 1$

(b) $\lim_{x\to 2} f(x)$ does not exist.

(c) $\lim_{x\to 2} f(x) = 2$

(d) $\lim_{x\to 1^-} f(x) = 2$

(e) $\lim_{x\to 1^+} f(x) = 1$

(f) $\lim_{x\to 1} f(x)$ does not exist.

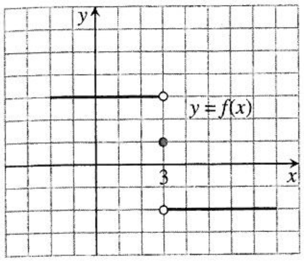
(g) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

(h) $\lim_{x\to c} f(x)$ exists at every c in $(-1, 1)$.

(i) $\lim_{x\to c} f(x)$ exists at every c in $(1, 3)$.

In Exercises 45–50, use the graph to estimate the limits and value of the function, or explain why the limits do not exist.

45.



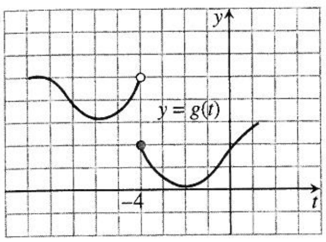
(a) $\lim_{x\to 3^-} f(x)$

(b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to 3} f(x)$

(d) $f(3)$

46.



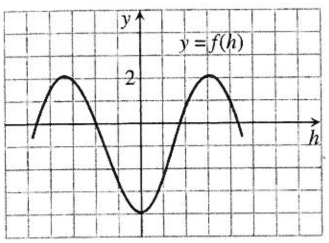
(a) $\lim_{t\to -4^-} g(t)$

(b) $\lim_{t\to -4^+} g(t)$

(c) $\lim_{t\to -4} g(t)$

(d) $g(-4)$

47.



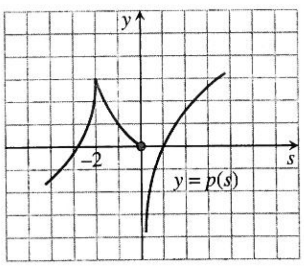
(a) $\lim_{h\to 0^-} f(h)$

(b) $\lim_{h\to 0^+} f(h)$

(c) $\lim_{h\to 0} f(h)$

(d) $f(0)$

48.



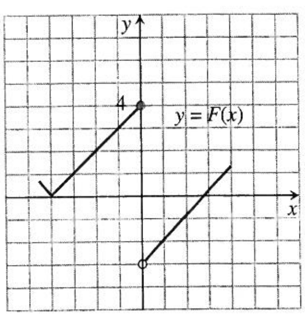
(a) $\lim_{s\to -2^-} p(s)$

(b) $\lim_{s\to -2^+} p(s)$

(c) $\lim_{s\to -2} p(s)$

(d) $p(-2)$

49.



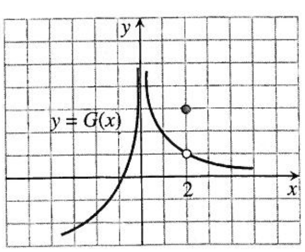
(a) $\lim_{x\to 0^-} F(x)$

(b) $\lim_{x\to 0^+} F(x)$

(c) $\lim_{x\to 0} F(x)$

(d) $F(0)$

50.



(a) $\lim_{x\to 2^-} G(x)$

(b) $\lim_{x\to 2^+} G(x)$

(c) $\lim_{x\to 2} G(x)$

(d) $G(2)$

In Exercises 51–54, match the function with the table.

51. $y_1 = \dfrac{x^2 + x - 2}{x - 1}$

52. $y_1 = \dfrac{x^2 - x - 2}{x - 1}$

53. $y_1 = \dfrac{x^2 - 2x + 1}{x - 1}$

54. $y_1 = \dfrac{x^2 + x - 2}{x + 1}$

X	Y1
.7	−.4765
.8	−.3111
.9	−.1526
1	0
1.1	.14762
1.2	.29091
1.3	.43043
X = .7	

(a)

X	Y1
.7	7.3667
.8	10.8
.9	20.9
1	ERROR
1.1	−18.9
1.2	−8.8
1.3	−5.367
X = .7	

(b)

X	Y1
.7	2.7
.8	2.8
.9	2.9
1	ERROR
1.1	3.1
1.2	3.2
1.3	3.3
X = .7	

(c)

X	Y1
.7	−.3
.8	−.2
.9	−.1
1	ERROR
1.1	.1
1.2	.2
1.3	.3
X = .7	

(d)

In Exercises 55 and 56, determine the limit.

55. Assume that $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = 3$.

(a) $\lim_{x\to 4} (g(x) + 3)$ (b) $\lim_{x\to 4} x f(x)$

(c) $\lim_{x\to 4} g^2(x)$ (d) $\lim_{x\to 4} \dfrac{g(x)}{f(x) - 1}$

56. Assume that $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$.

(a) $\lim_{x\to b} (f(x) + g(x))$ (b) $\lim_{x\to b} (f(x) \cdot g(x))$

(c) $\lim_{x\to b} 4\, g(x)$ (d) $\lim_{x\to b} \dfrac{f(x)}{g(x)}$

In Exercises 57–60, complete parts (a), (b), and (c) for the piecewise-defined function.

(a) Draw the graph of f.

(b) Determine $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$.

(c) **Writing to Learn** Does $\lim_{x\to c} f(x)$ exist? If so, what is it? If not, explain.

57. $c = 2$, $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$

58. $c = 2$, $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$

59. $c = 1$, $f(x) = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \ge 1 \end{cases}$

60. $c = -1$, $f(x) = \begin{cases} 1 - x^2, & x \ne -1 \\ 2, & x = -1 \end{cases}$

In Exercises 61–64, complete parts (a)–(d) for the piecewise-defined function.

(a) Draw the graph of f.

(b) At what points c in the domain of f does $\lim_{x\to c} f(x)$ exist?

(c) At what points c does only the left-hand limit exist?

(d) At what points c does only the right-hand limit exist?

61. $f(x) = \begin{cases} \sin x, & -2\pi \le x < 0 \\ \cos x, & 0 \le x \le 2\pi \end{cases}$

62. $f(x) = \begin{cases} \cos x, & -\pi \le x < 0 \\ \sec x, & 0 \le x \le \pi \end{cases}$

63. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

64. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1, \text{ or } x > 1 \end{cases}$

In Exercises 65–68, find the limit graphically. Use the Squeeze Theorem to confirm your answer.

65. $\lim_{x\to 0} x \sin x$

66. $\lim_{x\to 0} x^2 \sin x$

67. $\lim_{x\to 0} x^2 \sin \dfrac{1}{x^2}$

68. $\lim_{x\to 0} x^2 \cos \dfrac{1}{x^2}$

69. ***Free Fall*** A water balloon dropped from a window high above the ground falls $y = 4.9t^2$ m in t sec. Find the balloon's

(a) average speed during the first 3 sec of fall.

(b) speed at the instant $t = 3$.

70. ***Free Fall on a Small Airless Planet*** A rock released from rest to fall on a small airless planet falls $y = gt^2$ m in t sec, g a constant. Suppose that the rock falls to the bottom of a crevasse 20 m below and reaches the bottom in 4 sec.

(a) Find the value of g.

(b) Find the average speed for the fall.

(c) With what speed did the rock hit the bottom?

Standardized Test Questions

71. True or False If $\lim_{x\to c^-} f(x) = 2$ and $\lim_{x\to c^+} f(x) = 2$, then $\lim_{x\to c} f(x) = 2$. Justify your answer.

72. True or False $\lim_{x\to 0} \dfrac{x + \sin x}{x} = 2$. Justify your answer.

In Exercises 73–76, use the following function.

$$f(x) = \begin{cases} 2 - x, & x \le 1 \\ \dfrac{x}{2} + 1, & x > 1 \end{cases}$$

73. Multiple Choice What is the value of $\lim_{x\to 1^-} f(x)$?

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

74. Multiple Choice What is the value of $\lim_{x\to 1^+} f(x)$?

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

75. Multiple Choice What is the value of $\lim_{x\to 1} f(x)$?

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

76. Multiple Choice What is the value of $f(1)$?

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

77. Group Activity To prove that $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$ when θ is measured in radians, the plan is to show that the right- and left-hand limits are both 1.

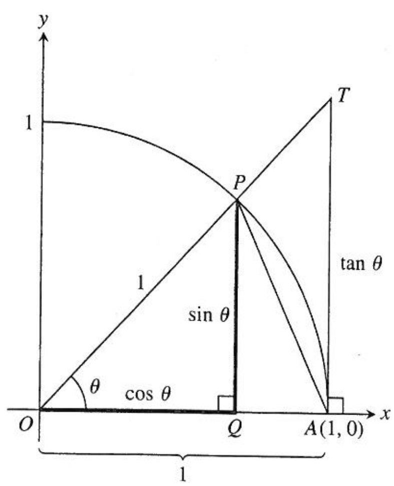
(a) To show that the right-hand limit is 1, explain why we can restrict our attention to $0 < \theta < \pi/2$.

(b) Use the figure to show that

$$\text{area of } \Delta OAP = \frac{1}{2}\sin\theta,$$

$$\text{area of sector } OAP = \frac{\theta}{2},$$

$$\text{area of } \Delta OAT = \frac{1}{2}\tan\theta.$$



(c) Use part (b) and the figure to show that for $0 < \theta < \pi/2$,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

(d) Show that for $0 < \theta < \pi/2$ the inequality of part (c) can be written in the form

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

(e) Show that for $0 < \theta < \pi/2$ the inequality of part (d) can be written in the form

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

(f) Use the Squeeze Theorem to show that

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$