AP Calculus
WS 8.1 Separable Differential Equations

Name: Key

two ways to work both these problems
just the k values will be different

Solve the differential equation by separation of variables.

1. $\dfrac{dy}{dx} = 2x^3y$ $\boxed{y = ke^{\frac{x^4}{2}}}$

$\frac{dy}{y} = 2x^3 dx$

$\int \frac{dy}{y} = \int 2x^3 dx$

$\ln y = \frac{1}{4} \cdot 2x^4 + C$

$e^{\ln y} = e^{\frac{1}{2}x^4 + C}$

$y = e^{\frac{1}{2}x^4} \cdot e^C$

$K = e^C$ or $A = e^C$

2. $\dfrac{dy}{dx} = 2y - 4$

① $\frac{dy}{2y-4} = dx$

$\frac{1}{2}\int \frac{dy}{y-2} = \int dx$

$\frac{1}{2}\ln|y-2| = x + C$

$e^{\ln\sqrt{y-2}} = e^{x+C}$

$\sqrt{y-2} = Ke^x$

$\boxed{y = (K_1e^x)^2 + 2}$

② $u = 2y - 4$

$du = 2dy$

$\frac{1}{2}du = dy$

$\frac{1}{2}\int \frac{du}{u} = \int dx$

$\frac{1}{2}\ln u = x + C$

$e^{\ln\sqrt{u}} = e^{x+C}$

$\sqrt{u} = Ke^x$

$2y - 4 = (Ke^x)^2$

$\boxed{y = \frac{(K_2e^x)^2 + 4}{2}}$

3. $(2+x)y' = 3y$

$(2+x)\frac{dy}{dx} = 3y$

$\int \frac{dy}{3y} = \int \frac{dx}{2+x}$

$\frac{1}{3}\ln 3y = \ln(2+x) + C$

$e^{\ln 3y^{\frac{1}{3}}} = e^{\ln(2+x) + C}$

$\sqrt[3]{3y} = K(2+x)$

$\boxed{y = \frac{[K(2+x)]^3}{3}}$

4. $yy' = \sin x$

$\int y\,dy = \int \sin x\,dx$

$\frac{1}{2}y^2 = -\cos x + C$

$y^2 = 2(-\cos x + C)$

$y = \pm\sqrt{2(-\cos x + C)}$

5. $\sqrt{1-4x^2}\,y' = x$

$dy = \frac{x}{\sqrt{1-4x^2}}\,dx$

$u = 1 - 4x^2$

$du = -8x\,dx$

$-\frac{1}{8}du = x\,dx$

$\int dy = -\frac{1}{8}\int u^{-\frac{1}{2}}du$

$y = -\frac{1}{8} \cdot 2u^{\frac{1}{2}} + C$

$\boxed{y = -\frac{1}{4}\sqrt{1-4x^2} + C}$

6. $y\ln x - xy' = 0$

$-x\frac{dy}{dx} = -y\ln x$ $\quad u = \ln x$, $du = \frac{1}{x}dx$

$\frac{dy}{y} = \frac{\ln x}{x}dx$

$\ln y = \frac{1}{2}u^2 + C$

$y = e^{\frac{1}{2}(\ln x)^2 + C}$

$\boxed{y = Ke^{\frac{1}{2}(\ln x)^2}}$

Find the particular solution that satisfies the given initial condition.

7. $yy' - e^x = 0$, $y(0) = 4$

$\int y\,dy = \int e^x dx$

$\frac{1}{2}y^2 = e^x + C$

$y^2 = 2(e^x + C)$

$(4)^2 = 2(e^0 + C)$

$16 = 2(1 + C)$

$8 = 1 + C$

$C = 7$

$\Rightarrow y^2 = 2(e^x + 7)$

$y = \pm\sqrt{2(e^x + 7)}$

$\boxed{= \sqrt{2e^x + 14}}$

8. $\sqrt{x} + \sqrt{y}\,y' = 0$, $y(1) = 4$

$y^{\frac{1}{2}}dy = -x^{\frac{1}{2}}dx$

$\frac{2}{3}y^{\frac{3}{2}} = -\frac{2}{3}x^{\frac{3}{2}} + C$

$\frac{2}{3}(\sqrt{4})^3 = -\frac{2}{3}(\sqrt{1})^3 + C$

$\frac{16}{3} = -\frac{2}{3} + C$

$C = 6$

$\rightarrow \frac{2}{3}y^{\frac{3}{2}} = -\frac{2}{3}x^{\frac{3}{2}} + \frac{18}{3}$

$y^{\frac{3}{2}} = -x^{\frac{3}{2}} + 9$

$\boxed{y = \left(-x^{\frac{3}{2}} + 9\right)^{\frac{2}{3}}}$

9. $\dfrac{dy}{dx} = \dfrac{x^2}{y}$ when y = -5 and x = 3

$\int y\,dy = \int x^2 dx$

$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C$

$\frac{1}{2}(-5)^2 = \frac{1}{3}(3)^3 + C$

$\frac{25}{2} - 9 = C$

$C = \frac{7}{2}$

$\frac{1}{2}y^2 = \frac{1}{3}x^3 + \frac{7}{2}$

$y^2 = \frac{2}{3}x^3 + 7$

$\boxed{y = -\sqrt{\frac{2}{3}x^3 + 7}}$

10. $\dfrac{dy}{dx} = 6x^2y$ when x = 0 and y = 4

$\int \frac{dy}{y} = \int 6x^2 dx$

$\ln y = 2x^3 + C$

$y = Ke^{2x^3}$

$4 = Ke^0$

$K = 4$

$\Rightarrow \boxed{y = 4e^{2x^3}}$

11. $\dfrac{dy}{dx} = -xy^2$, $y(1) = -0.25$

$\frac{dy}{y^2} = -x\,dx$

$\int y^{-2}dy = \int -x\,dx$

$-\frac{1}{y} = -\frac{1}{2}x^2 + C$

$-\frac{1}{-\frac{1}{4}} = -\frac{1}{2}(1)^2 + C$

$4 = -\frac{1}{2} + C$

$\frac{9}{2} = C$

$\Rightarrow -\frac{1}{y} = -\frac{1}{2}x^2 + \frac{9}{2}$

$-\frac{1}{y} = \frac{-x^2 + 9}{2}$

$\boxed{y = \frac{-2}{-x^2 + 9}}$

12. $y' = \dfrac{1+x}{\sqrt{y}}$, $y(2) = 9$

$\int y^{\frac{1}{2}}dy = \int (1+x)dx$

$\frac{2}{3}y^{\frac{3}{2}} = x + \frac{1}{2}x^2 + C$

$\frac{2}{3}(\sqrt{9})^3 = 2 + \frac{1}{2}(2)^2 + C$

$18 = 2 + 2 + C$

$C = 14$

$\frac{2}{3}y^{\frac{3}{2}} = x + \frac{1}{2}x^2 + 14$

$y^{\frac{3}{2}} = \frac{3}{2}x + \frac{3}{4}x^2 + 21$

$\Rightarrow \boxed{y = \left(\frac{3}{2}x + \frac{3}{4}x^2 + 21\right)^{\frac{2}{3}}}$

13. Which of the following is the solution to the differential equation $\frac{dy}{dx} = 2\sin x$ with the initial condition $y(\pi) = 1$?

$\int dy = \int 2\sin x \, dx$
$y = -2\cos x + C$
$1 = -2\cos\pi + C$
$1 = 2 + C$
$C = -1$
$y = -2\cos x - 1$

(A) $y = 2\cos x + 3$ (B) $y = 2\cos x - 1$
(C) $y = -2\cos x + 3$ (D) $y = -2\cos x + 1$ (E) $y = -2\cos x - 1$

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14. If a function $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = -4y$ and $f(0) = 6$, then $f(x) =$

$\int \frac{dy}{y} = \int -4\,dx$
$e^{\ln y} = e^{-4x + C}$
$y = Ke^{-4x}$
$6 = K$
$y = 6e^{-4x}$

(A) $-2x^2 + 6$ (B) $-\frac{x}{4} + 6$
(C) $6e^{-4x}$ (D) $e^{-4x} + 5$ (E) $-\frac{1}{4}\ln\left(x + e^{-24}\right)$

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15. The solution to the differential equation $\frac{dy}{dx} = \frac{x}{\cos y}$ with the initial condition $y(1) = 0$ is

$\int \cos y\, dy = \int x\, dx$
$\sin y = \frac{1}{2}x^2 + C$
$\sin 0 = \frac{1}{2} + C$
$C = -\frac{1}{2}$
$\sin y = \frac{1}{2}x^2 - \frac{1}{2}$
$y = \sin^{-1}\left(\frac{1}{2}x^2 - \frac{1}{2}\right)$

(A) $y = \sin^{-1}\left(\frac{x^2 - 1}{2}\right)$ (B) $y = \sin^{-1}\left(\frac{x^2}{2}\right)$
(C) $y = \cos^{-1}\left(x^2 - 2\right)$ (D) $y = \ln[\cos(x-1)]$ (E) $y = \ln(\sin x)$

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16. A particle moves along the y-axis with velocity given by $v(t) = 4(y+5)t^3$ and an initial velocity of $y(0) = 8$. Which of the following is an expression for the position of the particle over time $y(t)$?

$\frac{dy}{dt} = 4(y+5)t^3$
$\int \frac{dy}{y+5} = \int 4t^3\, dt$
$\ln|y+5| = t^4 + C$
$y + 5 = Ke^{t^4}$
$8 + 5 = K$
$y + 5 = 13e^{t^4}$
$y = 13e^{t^4} - 5$

(A) $y(t) = 8e^{t^4} - 5$ (B) $y(t) = 13e^{t^4} - 5$
(C) $y(t) = \ln|8t^4 - 5|$ (D) $y(t) = 13t^4 - 5$ (E) $y(t) = 12t^2 - 8$

A calf that weighs 60 pounds at birth gains weight at the rate of $\frac{dw}{dt} = (1200 - w)$ where w is weight in pounds and t is time in years.

(A) Find an expression that gives the weight, W , of the calf over time t .

$\frac{dw}{1200 - w} = dt$
$-\ln|1200 - w| = t + C$
$e^{\ln(1200-w)^{-1}} = e^{t+C}$
$(1200 - w)^{-1} = Ke^t$
$\frac{1}{1200 - w} = Ke^t$
$\frac{1}{1200 - 60} = K$
$K = \frac{1}{1140}$
$\frac{1}{1200 - w} = \frac{1}{1140}e^t$
$1200 - w = 1140e^{-t}$
$w = 1200 - 1140e^{-t}$
$\boxed{w = 1200 - \frac{1140}{e^t}}$

(B) If the animal is sold when its weight reaches 800 pounds, find the time of sale.

$800 = 1200 - \frac{1140}{e^t}$
$-400 = -\frac{1140}{e^t}$
$e^t = \frac{1140}{400}$
$\boxed{t = \ln\left(\frac{1140}{400}\right) = 1.047 \text{ years}}$