Solutions

class: 1, 2, 3, 4

Differential Equations

A differential equation in x and y is an equation that involves x, y, and derivatives of y. Previously we have solved two types of differential equations---those of the form $y' = f(x)$ and $y'' = f(x)$. Now we will learn to solve a more general type of differential equation called a **variables separable** differential equation. The strategy used to solve these is to rewrite the equation so that each variable occurs on only one side of the equation. This strategy is called **separation of variables**.

Ex.1 Given the differential equation $\frac{dy}{dx} = \frac{2x}{y}$, solve for y as a function of x.

$y\,dy = 2x\,dx$

$\frac{y^2}{2} + C_1 = x^2 + C_2$

$\frac{y^2}{2} = x^2 + C_3$

$y^2 = 2x^2 + C_4$

$y(x) = \pm\sqrt{2x^2 + C_4} \qquad y \neq 0$

HW pg
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Your answer is called the **general solution**. What if you were given the initial condition $y(1) = -3$ and asked to find the particular solution?

must use $-\sqrt{\ }$
$y<0$

$y(1) = -3$

$(-3)^2 = (-\sqrt{2(1)^2 + C_4})^2$

$9 = 2 + C_4$

$7 = C_4$

$\boxed{y(x) = -\sqrt{2x^2 + 7}}$

Ex. 2 Solve for y as a function of x.

$y' = x^2\sqrt{y}$ and $y(3) = 4$

$\frac{dy}{dx} = x^2\sqrt{y}$

$\frac{dy}{\sqrt{y}} = \frac{x^2\sqrt{y}\,dx}{\sqrt{y}}$

$\frac{dy}{\sqrt{y}} = x^2\,dx$

$\int y^{-1/2}\,dy = \int x^2\,dx$

$\frac{y^{1/2}}{1/2} = \frac{x^3}{3} + C$

$(y^{1/2})^2 = \left(\frac{x^3}{6} + C_1\right)^2$

$\boxed{y = \left(\frac{x^3}{6} + C_1\right)^2}$

$y(3) = 4$

$4 = \left(\frac{3^3}{6} + C_1\right)^2$

$2 = \left(\frac{27}{6} + C_1\right)$ or $-2 = \frac{27}{6} + C_1$

$2 - \frac{9}{2} = C_1$ $\qquad -2 - \frac{9}{2} = C_1$

$-\frac{5}{2} = C_1$ $\qquad -\frac{20}{9} = C_1$

$\boxed{y(x) = \left(\frac{x^3}{6} - \frac{5}{2}\right)^2}$ $\qquad \boxed{y(x) = \left(\frac{x^3}{6} - \frac{20}{9}\right)^2}$

Ex. 3 $\frac{dy}{dx} = 4xy^2$ and $y(2) = 1$

$$\frac{dy}{y^2} = \frac{4x\,y^2 dx}{y^2}$$

$$\frac{dy}{y^2} = 4x\,dx$$

$$\int y^{-2} dy = \int 4x\,dx$$

$$\frac{y^{-1}}{-1} = \frac{4x^2}{2} + C_1$$

$$(y^{-1})^{-1} = (-2x^2 + C_2)^{-1}$$

$$\boxed{y(x) = \frac{1}{-2x^2 + C_2}}$$

$$y(2) = 1$$

$$1 = \frac{1}{-2(2)^2 + C_2}$$

$$1 = -8 + C_2$$

$$9 = C_2$$

$$\boxed{y(x) = \frac{1}{-2x^2 + 9}} \qquad \boxed{x \neq \pm\sqrt{9/2}}$$

$$-2x^2 + 9 \neq 0 \text{ or}$$

$$9 \neq 2x^2$$

$$\pm\sqrt{\frac{9}{2}} \neq x$$

CK $\frac{dy}{dx} = \frac{-1}{\underbrace{(-2x^2+9)^2}_{-y^2}} (-4x)$

$$= -y^2(-4x) = 4xy^2 \checkmark$$

Differential Equations

Ex. 1 $\dfrac{dy}{dx} = 9x^2 y$ and $y(0) = -2$

$$\int \frac{dy}{y} = \int 9x^2\,dx$$

$$\ln|y| = \frac{9x^3}{3} + C$$

$$\ln|y| = 3x^3 + C$$

$$e^{\ln|y|} = e^{3x^3 + C}$$

$$y = \pm e^C e^{3x^3}$$

$$\boxed{y(x) = A e^{3x^3}} \quad A = \pm e^C$$

$$y(0) = -2$$

$$-2 = A e^{3(0)^3}$$

$$-2 = A e^0$$

$$-2 = A$$

$$\boxed{y(x) = -2e^{3x^3}}$$

ck

$$\frac{dy}{dx} = \underbrace{-2e^{3x^3}}_{y}(9x^2)$$

$$\frac{dy}{dx} = 9x^2 y \checkmark$$

To check your general solution on your TI-89: deSolve $\left(y' = 9x^2 \cdot y,\ x,\ y\right)$

To check your particular solution on your TI-89: deSolve $\left(y' = 9x^2 \cdot y \text{ and } y(0) = -2, x, y\right)$

Note: "deSolve" is in F3; the word "and" is in the Catalog. When you choose "and", it gives a space, then "and" followed by another space, which is what you need.

Ex. 2 $y' = \dfrac{y+3}{x^2}$, $x \neq 0$, and $y(2) = 1$

$$\frac{dy}{dx} = \frac{y+3}{x^2}$$

$$\frac{dy}{y+3} = \frac{dx}{x^2}$$

* plug in I.C here

$$\ln|y+3| = -x^{-1} + C$$

$$e^{\ln|y+3|} = e^{-\frac{1}{x} + C}$$

$$y + 3 = A e^{-\frac{1}{x}}$$

$$\boxed{y(x) = A e^{-\frac{1}{x}} - 3}$$

$$A = \pm e^C$$

$$y(2) = 1$$

$$1 = C_1 e^{-\frac{1}{2}} - 3$$

$$4 = C_1 e^{-\frac{1}{2}}$$

$$\frac{4}{e^{-1/2}} = C_1$$

$$4e^{1/2} = C_1$$

$$y(x) = 4e^{1/2} e^{-1/x} - 3$$

$$\boxed{y(x) = 4e^{-1/x + 1/2} - 3} \quad \boxed{x \neq 0}$$

$$\ln|1+3| = -(2)^{-1} + C$$

$$\ln(4) = -\frac{1}{2} + C$$

$$C = \ln(4) + \frac{1}{2}$$

$$\ln|y+3| = -x^{-1} + \ln(4) + \frac{1}{2}$$

$$y + 3 = e^{-\frac{1}{x} + \ln 4 + 1/2}$$

$$y(x) = 4e^{-\frac{1}{x} + \frac{1}{2}} - 3$$

Ex. 3 $\frac{dy}{dx} = 4x^3y + 20x^3$ and $y(1) = -3$

$\frac{dy}{dx} = 4x^3(y+5)$

$\frac{dy}{y+5} = 4x^3dx$

* plugin I.C.

$\ln|y+5| = x^4 + C$

$e^{\ln|y+5|} = e^{x^4+C}$

$y+5 = Ae^{x^4}$ $A = \pm e^C$

$\boxed{y(x) = Ae^{x^4} - 5}$

$y(1) = -3$

$-3 = Ae^{(1)^4} - 5$

$-3 = Ae - 5$

$2 = Ae$

$2e^{-1} = A$

$y(x) = 2e^{-1}e^{x^4} - 5$

$\boxed{y(x) = 2e^{x^4-1} - 5}$

Ex. 4 Consider the differential equation $\frac{dy}{dx} = \frac{x}{y}$. Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-3) = -1$, and state its domain.

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C$

$y^2 = x^2 + C_1$

$y = \pm\sqrt{x^2 + C_1}$

$y(-3) = -1$ $y<0$

$-1 = -\sqrt{(-3)^2 + C_1}$

$1 = 9 + C_1$

$-8 = C_1$

$\boxed{y(x) = -\sqrt{x^2 - 8}}$

$x^2 - 8 > 0$

$x^2 > 8$

$x > \sqrt{8}$

$x < -\sqrt{8}$

$(-\infty, -\sqrt{8})$

$(\sqrt{8}, \infty)$

1997 AB6/BC6

Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $\frac{dv}{dt} = -2v - 32$, with initial condition $v(0) = -50$.

(a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

(b) Terminal velocity is defined as $\lim_{t \to \infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

(c) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

a) $dv = (-2v-32)dt$

$\int \frac{dv}{-2v-32} = \int dt$

* plug in IC

$\frac{\ln|-2v-32|}{-2} = t + C$

$e^{\ln(-2v-32)} = e^{(-2t+C)}$

$-2v-32 = C_2 e^{-2t}$

$-2v = C_2 e^{-2t} + 32$

$v(t) = C_3 e^{-2t} - 16$

$v(0) = -50$

$-50 = C_3 e^{(0)} - 16$

$-50 = C_3 - 16$

$-34 = C_3$

$\boxed{v(t) = -34e^{-2t} - 16}$

b) $\lim_{t\to\infty} v(t) =$

$\lim_{t\to\infty} -34e^{-2t} - 16 = \boxed{-16 \text{ Ft/s}}$

c. $20 = -34e^{-2t} - 16$

$36 = -34e^{-2t}$

$\ln(-1.058824) = \cancel{\ln} \cancel{e}^{(-2t)}$

↳ cant take ln of (-) numb.

speed = 20 when $v(t) = \pm 20$

$-20 = -34e^{-2t} - 16$

$\ln(0.117647) = \ln e^{(-2t)}$

$-2.14007 = -2t$

$\boxed{t = 1.070 \text{ s}}$

5

1997 AB6/BC6
Solution

Calculator active

(a) $\dfrac{dv}{dt} = -2v - 32 = -2(v+16)$

$\dfrac{dv}{v+16} = -2dt$

$\ln|v+16| = -2t + A$

$|v+16| = e^{-2t+A} = e^{A}e^{-2t}$ → $v+16 = \pm e^{A} e^{-2t}$

Let $C = \pm e^{A}$

$v + 16 = Ce^{-2t}$

$-50 + 16 = Ce^{0};\ C = -34$

$v = -34e^{-2t} - 16$

(b) $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty}\left(-34e^{-2t} - 16\right) = -16$

(c) $v(t) = -34e^{-2t} - 16 = -20$

$e^{-2t} = \dfrac{2}{17};\ t = -\dfrac{1}{2}\ln\left(\dfrac{2}{17}\right) = 1.070$

CALCULUS BC
WORKSHEET 1 ON DIFFERENTIAL EQUATIONS

HW
#1-3, 5, 9

Work the following on **notebook paper**. Do not use your calculator.

Solve for y as a function of x.

1. $\dfrac{dy}{dx}=\dfrac{x-3}{y}$ and $y(2)=-5$

(1) $\int y\,dy=\int(x-3)\,dx$
$\frac{y^2}{2}=\frac{x^2}{2}-3x+C$
$y^2=x^2-6x+C_1$
$y=\sqrt{x^2-6x+C_1}$

$y(2)=-5$
$-5=\sqrt{2^2-6(2)+C_1}$
$25=4-12+C_1$
$C_1=33$
$\boxed{y(x)=\sqrt{2x^2-6x+33}}$

2. $y'=2x\sqrt{y}$ and $y(2)=25$

(2) $\int\frac{dy}{y^{1/2}}=\int 2x\,dx$
$2y^{1/2}=x^2+C$
$y=(\frac{x^2}{2}+C_1)^2$

$y(2)=25=(\frac{2^2}{2}+C_1)^2$
$5=2+C_1$
$C_1=3$
$\boxed{y(x)=(\frac{x^2}{2}+3)^2}$

3. $\dfrac{dy}{dx}=4y^2\sec^2(2x)$ and $y\left(\dfrac{\pi}{8}\right)=1$

(3) $\frac{1}{4}y^{-2}dy=\sec^2(2x)\,dx$
$-\frac{1}{4}y^{-1}=\frac{1}{2}\tan(2x)+C_1$
$y(x)=(-2\tan(2x)+C_2)^{-1}$

$y(\frac{\pi}{8})=1=(-2(\tan(2\frac{\pi}{8})+C_2)^{-1}$
$1=(-2+C_2)^{-1}$
$3=C_2$
$\boxed{y(x)=(-2\tan(2x)+3)^{-1}}$

4. $xy\dfrac{dy}{dx}=\ln x$ and $y(1)=2$

(4) $y\,dy=\frac{\ln x}{x}\,dx$
$\frac{y^2}{2}=$

5. $y'=2x\sec y$ and $y(2)=-\dfrac{\pi}{2}$

(5) $\int\cos y\,dy=\int 2x\,dx$
$\sin y=x^2+C$
$y=\sin^{-1}(x^2+C)$

$y(2)=-\frac{\pi}{2}=\sin^{-1}(2^2+C)$
$-1=4+C$
$C=-5$
$\boxed{y(x)=\sin^{-1}(x^2-5)}$

6. $y'-xe^y=2e^y$ and $y(0)=0$

(6) $\frac{dy}{dx}=e^y(x+2)$
$\int e^{-y}dy=\int(x+2)\,dy$
$-e^{-y}=\frac{x^2}{2}+2x+C_1$
$e^{-y}=-\frac{x^2}{2}-2x+C_2$
$-y=\ln(-\frac{x^2}{2}-2x+C_2)$
$y=-\ln(-\frac{x^2}{2}-2x+C_2)$

$y(0)=0$
$0=\ln(-\frac{0^2}{2}-2\cdot0+C)$
$0=\ln(C)$
$C=1$
$\boxed{y(x)=-\ln(-\frac{x^2}{2}-2x+1)}$

7. $\dfrac{dy}{dx}=2xy^3\sin\left(x^2\right)$ and $y(0)=-1$

8. $\dfrac{dy}{dx}=\dfrac{1}{y^2}$ and $y(0)=4$

9. Find a curve in the xy-plane that passes through the point (0, 3) and whose tangent line at a point (x, y) has slope $\dfrac{2x}{y^2}$.

(7) $\int y^{-3}dy=\int 2x\sin x^2\,dx$
$u=x^2$
$du=2x$
$-\frac{y^{-2}}{2}=\int\sin u\,du$
$-\frac{y^{-2}}{2}=-\cos x^2+C$
$y^{-2}=2\cos x^2+C_1$
$y(x)=\pm(2\cos x^2+C_1)^{-1/2}$
$y(0)=-1$
$-1=-\frac{1}{(2\cos 0+C_1)^{1/2}}$
$-1=\frac{1}{(2(1)+C_1)^{1/2}}$
$-1=\frac{1}{(2+C_1)^{1/2}}$
$(2+C_1)^2=1$
$2+C_1=1$
$C_1=-1$
$y(x)=-(2\cos x^2-1)^{-1/2}$

(8) $\int y^2dy=\int dx$
$\frac{y^3}{3}=x+C$
$y^3=3x+C_1$
$y=(3x+C_1)^{1/3}$
$y(0)=4$
$4=(3\cdot0+C_1)^{1/3}$
$4=C^{1/3}$
$64=C$
$\boxed{y(x)=(3x+64)^{1/3}}$

(9) $\frac{dy}{dx}=\frac{2x}{y^2}$
$\int y^2dy=\int 2x\,dx$
$\frac{y^3}{3}=x^2+C$
$y^3=3x^2+C_1$
$y=(3x^2+C_1)^{1/3}$
$3=(0+C_1)^{1/3}$
$27=C_1$
$\boxed{y(x)=(3x^2+27)^{1/3}}$

7 6

CALCULUS BC

WORKSHEET 2 ON DIFFERENTIAL EQUATIONS

Work the following on **notebook paper**. Do not use your calculator.

Solve for y as a function of x.

1. $\dfrac{dy}{dx} = 6x^2 y$ and $y(0) = 4$

2. $\dfrac{dy}{dx} = \dfrac{1+x}{\sqrt{y}}$ and $y(2) = 9$

3. $\dfrac{dy}{dx} = (y-2)^2 \cos(3x)$ and $y\left(\dfrac{\pi}{2}\right) = 3$

4. $\dfrac{dy}{dx} = 3x + xy$ and $y(4) = -2$

5. $\dfrac{dy}{dx} = \dfrac{y-3}{x^2}$, $x \neq 0$, and $y(4) = 0$

6. $\dfrac{dy}{dx} = x^2 y + 2x^2$, $y(-1) = 4$

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7. If $\dfrac{dy}{dx} = y\cos x$ and $y = 3$ when $x = 0$, then $y = ?$

8. Find an equation of the curve that satisfies $\dfrac{dy}{dx} = 4x^3 y$ and whose y-intercept is 7.

Multiple choice. Solve. All steps must be shown.

9. If $\dfrac{dy}{dx} = 2y^2$ and if $y = -1$ when $x = 1$, then when $x = 2$, $y =$

(A) $-\dfrac{2}{3}$ (B) $-\dfrac{1}{3}$ (C) 0 (D) $\dfrac{1}{3}$ (E) $\dfrac{2}{3}$

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10. Consider the differential equation $\dfrac{dy}{dx} = xy^3$. Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = -1$, and state its domain.

11. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{y+5}{x}$, $x \neq 0$. Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-3) = 1$, and state its domain.

(1) $\frac{dy}{dx} = 6x^2y \quad y(0)=4$

$\int \frac{dy}{y} = \int 6x^2 dx$

$e^{\ln y} = e^{\left(\frac{6x^3}{3} + C\right)}$

$y = c_1 e^{2x^3}$

$y(0) = 4$

$4 = c_1 e^0$

$4 = c_1$

$\boxed{y(x) = 4e^{2x^3}}$

(2) $\frac{dy}{dx} = \frac{1+x}{\sqrt{y}}$

$\int y^{1/2} dy = \int (1+x)\,dx$

$\frac{2}{3}y^{3/2} = x + \frac{x^2}{2} + C$

$y^{3/2} = \frac{3}{2}x + \frac{3}{4}x^2 + C_1$

$y(x) = \left(\frac{3}{4}x^2 + \frac{3}{2}x + C_1\right)^{2/3}$

$y(2) = 9$

$9 = \left(\frac{3}{4}2^2 + \frac{3}{2}(2) + C_1\right)^{2/3}$

$9 = (3 + 3 + C_1)^{2/3}$

$9^{3/2} = 6 + C_1$

$27 - 6 = C_1$

$C_1 = 21$

$\boxed{y(x) = \left(\frac{3}{4}x^2 + \frac{3}{2}x + 21\right)^{2/3}}$

3) $\frac{dy}{dx} = (y-2)^2\cos(3x)$

$(y-2)^{-2}dy = \int \cos(3x)\,dx$

$-(y-2)^{-1} = \frac{\sin 3x}{3} + C$

$(y-2)^{-1} = -\frac{\sin 3x}{3} + C_1$

$y - 2 = \left(-\frac{\sin 3x}{3} + C_1\right)^{-1}$

$y(x) = \left(-\frac{\sin 3x}{3} + C_1\right)^{-1} + 2$

$y\left(\frac{\pi}{2}\right) = 3$

$3 = \left(-\frac{\sin\frac{3\pi}{2}}{3} + C_1\right)^{-1} + 2$

$1 = 1/3 + C_1$

$2/3 = C_1$

$\boxed{y(x) = \left(-\frac{\sin 3x}{3} + 2/3\right)^{-1} + 2}$

(4) $\frac{dy}{dx} = 3x + xy$

$\frac{dy}{dx} = (3+y)x$

$\int \frac{dy}{3+y} = \int x\,dx$

$\ln(3+y) = \frac{x^2}{2} + C$

$3 + y = C_1 e^{\frac{x^2}{2}}$

$y(x) = C_1 e^{\frac{x^2}{2}} - 3$

$y(4) = -2$

$-2 = C_1 e^{\frac{4^2}{2}} - 3$

$1 = C_1 e^8$

$e^{-8} = C_1$

$y(x) = e^{-8}e^{\frac{x^2}{2}} - 3$

$\boxed{y(x) = e^{x^2/2 - 8} - 3}$

(5) $\frac{dy}{dx} = \frac{y-3}{x^2}$

$\int \frac{dy}{y-3} = \int x^{-2}dx$

$\ln(y-3) = -x^{-1} + C$

$y - 3 = C_1 e^{-\frac{1}{x}}$

$y(x) = C_1 e^{-\frac{1}{x}} + 3$

$y(4) = 0$

$0 = C_1 e^{-\frac{1}{4}} + 3$

$-3 = C_1 e^{-\frac{1}{4}}$

$-3e^{1/4} = C_1$

$y(x) = -3e^{1/4}e^{-1/x}$

$\boxed{y(x) = -3e^{-1/x + 1/4} + 3}$

(6) $\frac{dy}{dx} = x^2(y+2)$

$\int \frac{dy}{y+2} = \int x^2 dx$

$\ln(y+2) = \frac{x^3}{3} + C$

$y + 2 = C_1 e^{x^3/3}$

$y(x) = C_1 e^{x^3/3} - 2$

$y(-1) = 4$

$4 = C_1 e^{(-1)^3/3} - 2$

$6 = C_1 e^{-1/3}$

$6e^{1/3} = C_1$

$y(x) = 6e^{1/3}e^{x^3/3} - 2$

$\boxed{y(x) = 6e^{x^3/3 - 1/3} - 2}$

(7) $\frac{dy}{dx} = y\cos x$

$\int \frac{dy}{y} = \int \cos x\,dy$

$\ln y = \sin x + C$

$y = C_1 e^{\sin x}$

$y(0) = 3$

$3 = C_1 e^{\sin 0}$

$3 = C_1$

$\boxed{y(x) = 3e^{\sin x}}$

(8) $\frac{dy}{dx} = 4x^3y$

$\int \frac{dy}{y} = \int 4x^3 dx$

$\ln y = x^4 + C$

$e^{\ln y} = e^{x^4 + C}$

$y(x) = C_1 e^{x^4}$

$y(0) = 7$

$7 = C_1 e^0$

$7 = C_1$

$\boxed{y(x) = 7e^{x^4}}$

(9) $\frac{dy}{y^2} = 2dx$

$-y^{-1} = 2x + C_1$

$y^{-1} = -2x + C_2$

$y(x) = (-2x + C_2)^{-1}$

$y(1) = -1$

$y(1) = -1 = (-2(1) + C_2)^{-1}$

$-1 = -2 + C_2$

$1 = C_2$

$y(x) = (-2x+1)^{-1}$

$y(2) = (-2(2)+1)^{-1} = (-3)^{-1} = \boxed{-\frac{1}{3}}$ E

(10) $\frac{dy}{dx} = xy^3$

$\frac{dy}{y^3} = x\,dx$

$-\frac{y^{-2}}{2} = \frac{x^2}{2} + C$

$-y^{-2} = x^2 + C_1$

$y^{-1} = -x^2 + C_2$

$y(x) = (-x^2 + C_2)^{-1}$

$y(2) = -1$

$-1 = (-2^2 + C_2)^{-1}$

$-1 = -4 + C_2$

$3 = C_2$

$\boxed{y(x) = (-x^2 + 3)^{-1}}$

(11) $\frac{dy}{y+5} = \frac{dx}{x}$

$\ln(y+5) = \ln x + C$

$y + 5 = Ce^x$

$y(x) = Ce^x - 5$

$y(-3) = 1$

$1 = Ce^{-3} - 5$

$6e^3 = C$

$\boxed{y(x) = 6e^{x+3} - 5}$

$x \neq 0$

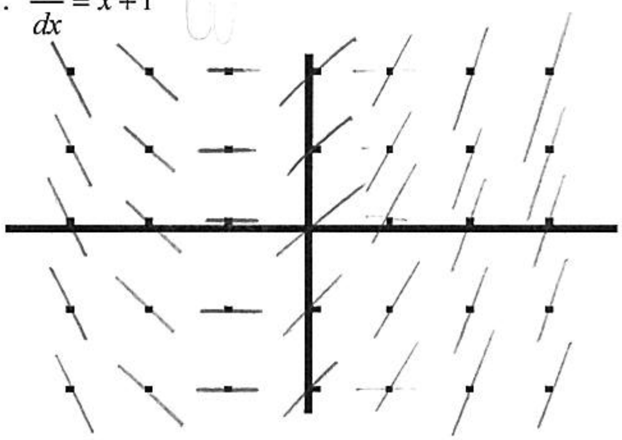
Name______________________________ Date____________________ Period_____

Worksheet 5.2—Slope Fields
Show all work when applicable.

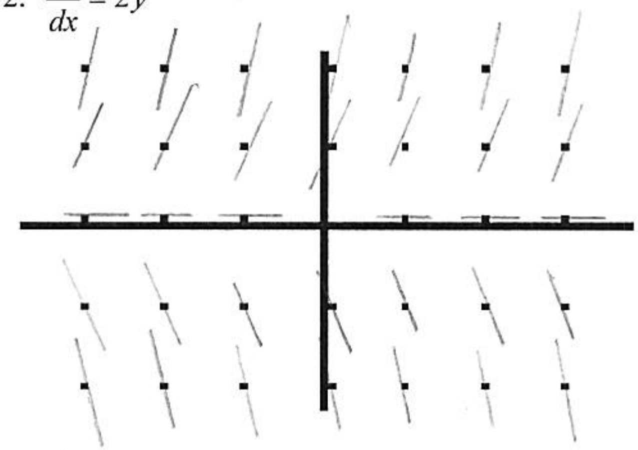
Short Answer and Free Response:

Draw a slope field for each of the following differential equations.

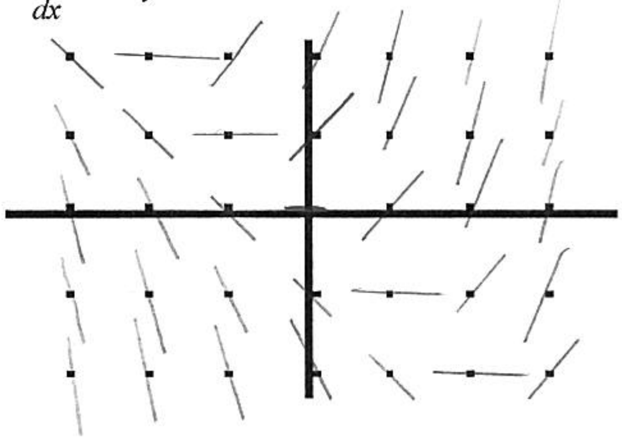
1. $\dfrac{dy}{dx} = x + 1$



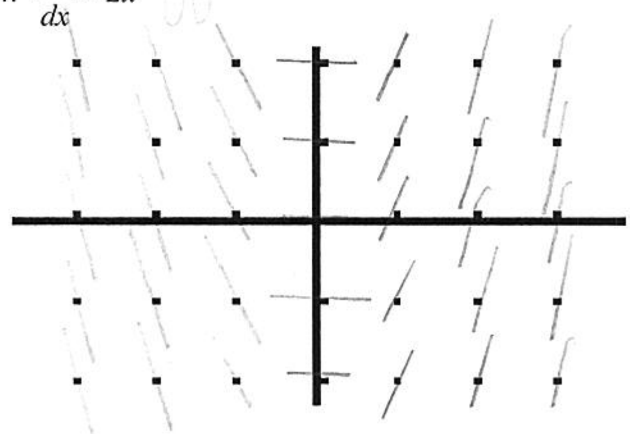
2. $\dfrac{dy}{dx} = 2y$



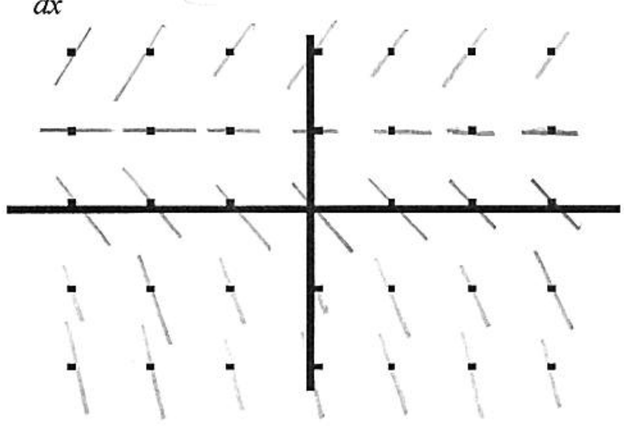
3. $\dfrac{dy}{dx} = x + y$



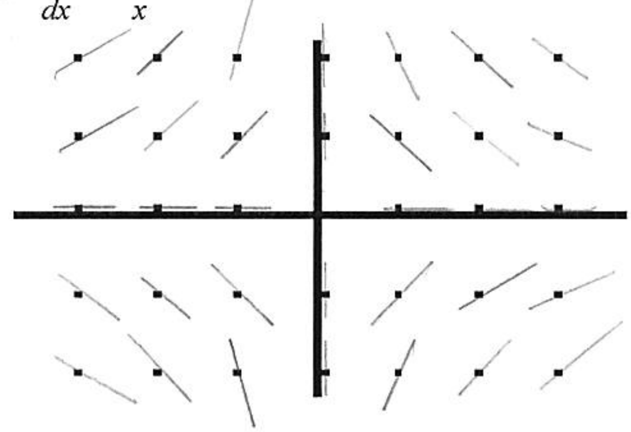
4. $\dfrac{dy}{dx} = 2x$



5. $\dfrac{dy}{dx} = y - 1$



6. $\dfrac{dy}{dx} = -\dfrac{y}{x}$

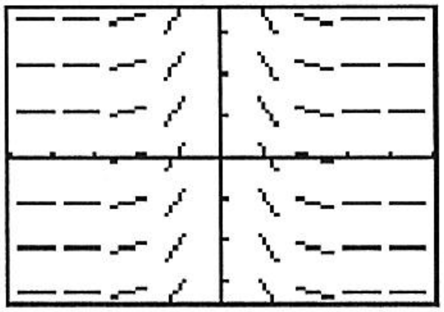


7

For 7 – 12, match each slope field with the **equation** that the slope field could represent.

7.

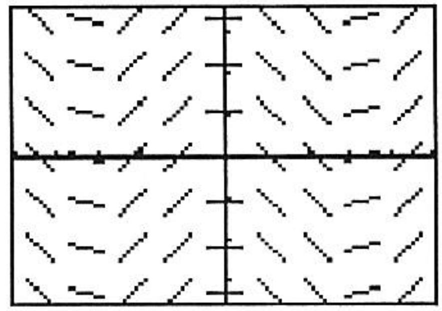
E $y = \frac{1}{x^2}$



8.

G

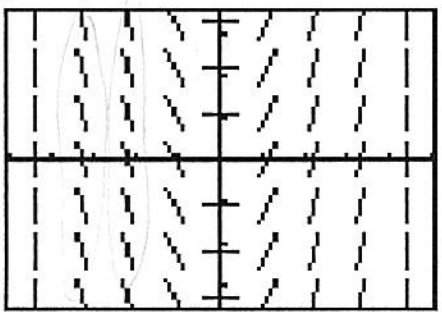
$y = \cos x$



9.

C

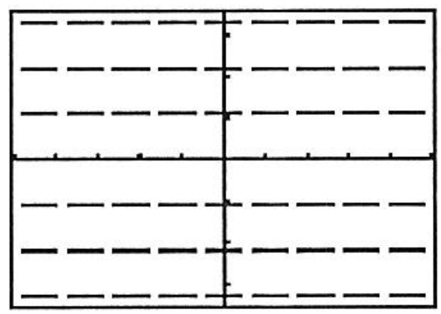
$y = x^2$



10.

A

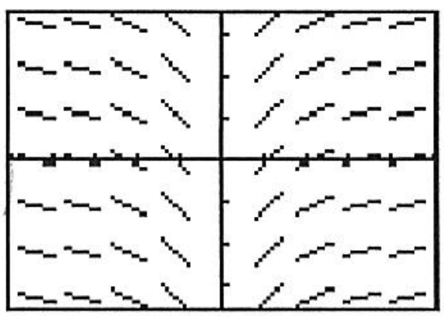
$y = 1$



11.

H

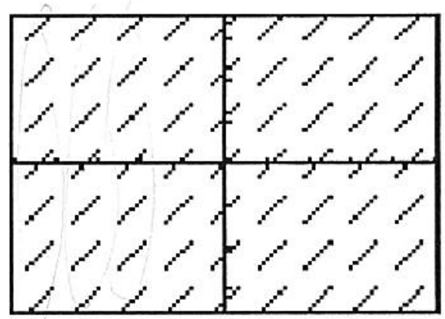
$y = \ln|x|$



12.

B

$y = x$



(A) $y = 1$ (B) $y = x$ (C) $y = x^2$

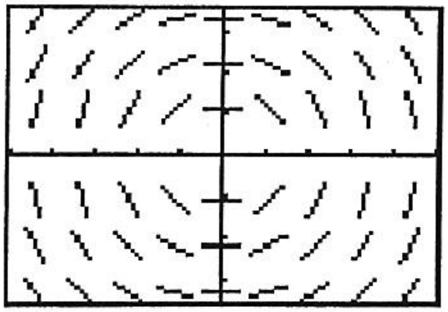
(D) $y = \frac{1}{6}x^3$ (E) $y = \frac{1}{x^2}$ (F) $y = \sin x$

(G) $y = \cos x$ (H) $y = \ln|x|$

8

For 13 – 16, match the slope fields with their differential equations.

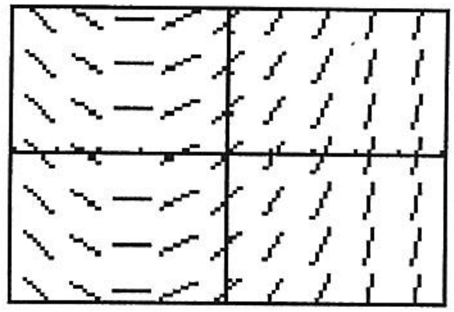
13.



$x=0 \rightarrow \frac{dy}{dx}=0$

D

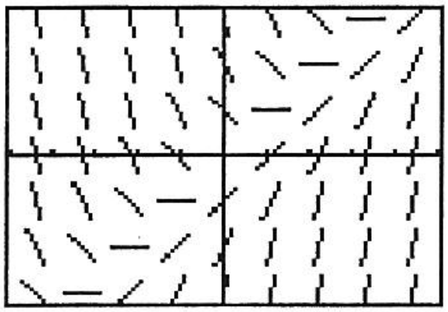
14.



A

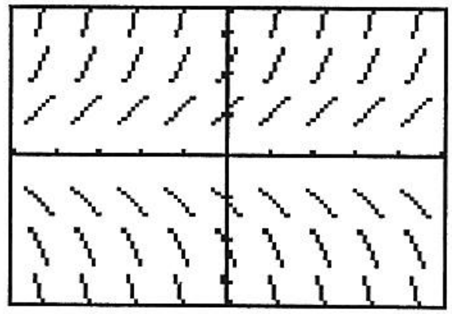
only x

15.



B

16.



C

only y

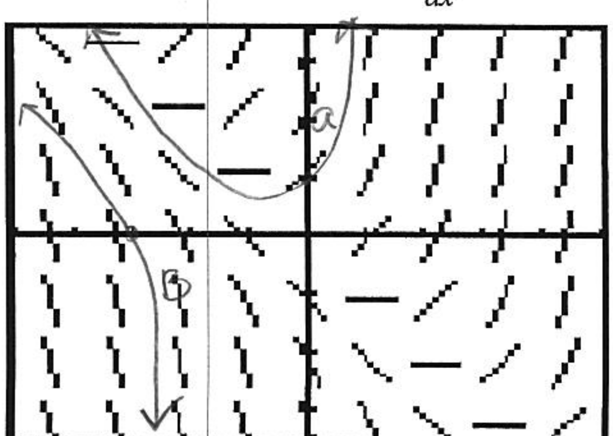
(A) $\frac{dy}{dx}=\frac{1}{2}x+1$

(B) $\frac{dy}{dx}=x-y$

(C) $\frac{dy}{dx}=y$

(D) $\frac{dy}{dx}=-\frac{x}{y}$

17. The calculator-drawn slope field for the differential equation $\frac{dy}{dx}=x+y$ is shown in the figure below.



(a) Sketch the solution curve through the point $(0,1)$.

(b) Sketch the solution curve through the point $(-3,0)$.

(c) Approximate $y(-3.1)$ using the equation of the tangent line to $y=f(x)$ at the point $(-3,0)$.

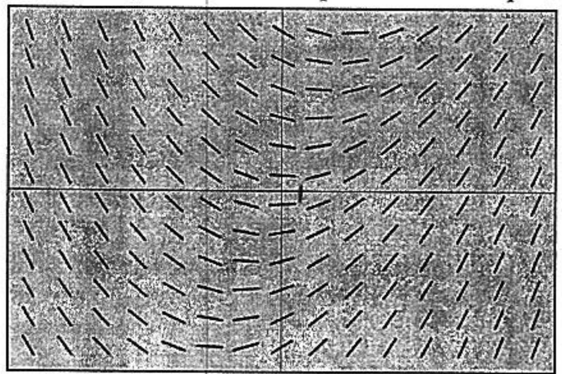
$\left.\frac{dy}{dx}\right|_{(-3,0)} = -3+0=-3$

$y-0=-3(x+3)$

$y=-3x-9$

$y(-3.1)=-3(-3.1)-9 = \boxed{0.3}$

23. Which of the following differential equations would produce the slope field shown below?



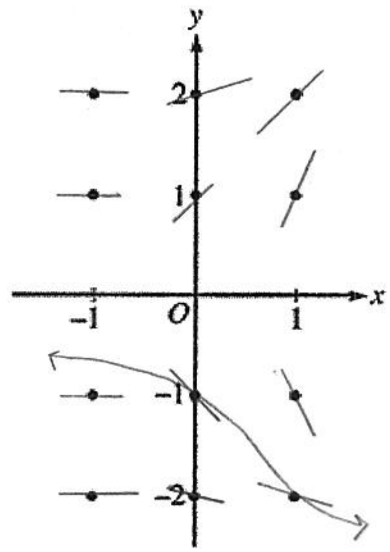
$\frac{dy}{dx} = 0$ about when $x = 0$

(A) $\frac{dy}{dx} = y - 3x$ (B) $\frac{dy}{dx} = y - \frac{x}{3}$ (C) $\frac{dy}{dx} = y + \frac{x}{3}$ (D) $\frac{dy}{dx} = x + \frac{y}{3}$ (E) $\frac{dy}{dx} = x - \frac{y}{3}$

24. AP 2010B-5 (No Calculator)

Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.

(a) On the axes provided at right, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1 < x < 1$, sketch the solution curve that passes through the point $(0, -1)$.



(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane for which $y \neq 0$. Describe all points in the xy-plane, $y \neq 0$, for which $\frac{dy}{dx} = -1$.

$-1 = \frac{x+1}{y}$

$-y = x+1$

$y = -x-1 \rightarrow$ point in x-y plane, $y \neq 0$ on this line

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$.

$\int y\,dy = \int (x+1)\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + x + C_1$

$y^2 = x^2 + 2x + C_2$

$y = \pm\sqrt{x^2 + 2x + C_2}$

$-2 = -\sqrt{0 + C^2}$

$4 = C_2$

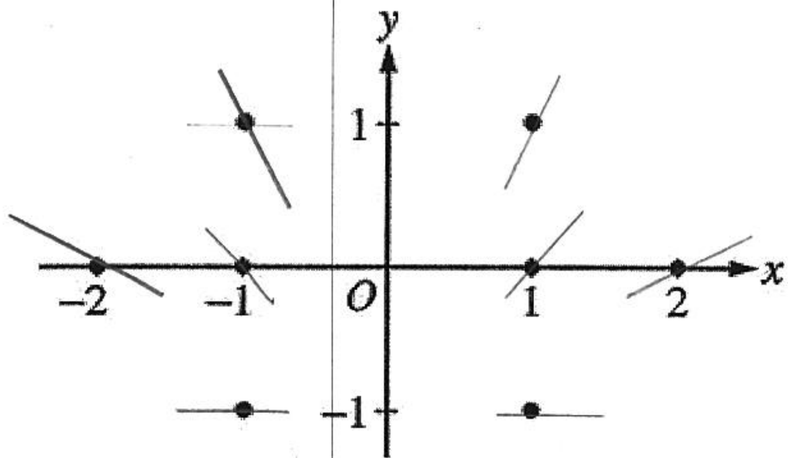
$\boxed{y = -\sqrt{x^2 + 2x + 4}}$

10

25. AP 2006-5 (No Calculator)

Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

$$\int \frac{dy}{1+y} = \int \frac{dx}{x}$$

$$e^{\ln|1+y|} = e^{\ln|x| + C_1}$$

$$|1+y| = e^{\ln|x|} e^{C_1}$$

$$1+y = A e^{\ln|x|}$$

$$1+y = A|x|$$

$$y = A|x| - 1$$

$$1 = A|-1| - 1$$

$$2 = A$$

$$\boxed{y = 2|x| - 1} \quad \boxed{\mathbb{R}}$$

11

Law of Exponential Change

For exponential growth functions, the more you have, the more you get. For exponential decay functions, the less you have, the less you lose. Quantities that grow/decrease by a factor or a percentage at regular intervals, are exponential. This can be stated equivalently as:

The rate of change of a quantity is directly proportional to that quantity itself.

Mathematically, we state this as $\frac{dy}{dt} = ky$, where k is either a growth ($k > 0$) or decay ($k < 0$) constant.

Example 6:

Solve the separable differential equation $\frac{dy}{dt} = ky$ → quantity y changes at a rate proportional to itself

$$\frac{dy}{y} = k\,dt$$

$$\ln|y| = kt + c$$

$$|y| = Ce^{kt}$$

$$y = Ce^{kt}$$

constant of proportionality

initial amount

MEMORIZE. MEMORIZE. MEMORIZE.

If $\frac{dy}{dt} = ky$, then $y = Ce^{kt}$, where C is the initial amount present (y-intercept of the graph).

Example 7:
The population of bacteria in a culture increased from 400 to 1600 in three hours. Assuming that the Bacteria population, P, grows according to the rate equation $\frac{dP}{dt} = kP$, where t is the time in hours.

(a) Find the value of k

$P(t) = Ce^{kt}$

$400 = Ce^{k(0)}$

$400 = C$

$P(t) = 400e^{k(3)}$

$\frac{1600}{400} = e^{3k}$

$\ln(4) = 3k$

$\frac{\ln 4}{3} = k$

$= \frac{1}{3}\ln 4 = \ln 4^{1/3}$

$= \frac{1}{3}\ln 2^2 = \frac{2}{3}\ln 2$

(b) How fast is the population of bacteria increasing when the population is 3000?

$\frac{dP}{dt} = \frac{\ln 4}{3}(3000)$

$= 1000\ln 4$ bacteria/hr

Finish Strong!

Example 12:
AP 2012-5
The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

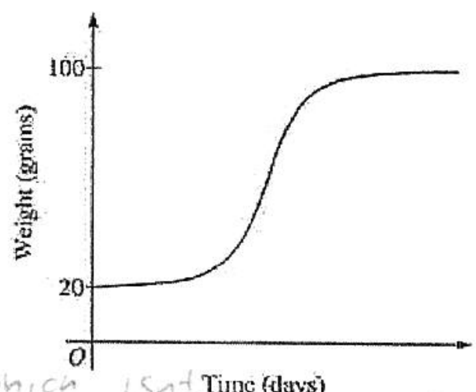
$\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(100-40) = 12$ gr/day

$\left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(100-70) = 6$ gr/day

$\frac{dB}{dt}$ decreases as the bird gains weight

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain whey the graph of B cannot resemble the following graph.

$$\frac{d^2B}{dt^2} = -\frac{1}{5}\left(\frac{dB}{dt}\right) = -\frac{1}{5}\left(\frac{1}{5}(100-B)\right) = -\frac{1}{25}(100-B)$$



since $B(t) < 100$, $\frac{d^2B}{dt^2} < 0$

for $20 \le B < 100$

so $B(t)$ conc down, which isn't true in graph above

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0)20$.

$$\frac{dB}{100-B} = \frac{1}{5}\,dt$$

$$-\ln|100-B| = \frac{1}{5}t + C$$

$$(100-B)^{-1} = Ae^{1/5\,t}$$

$$100 - B = Ae^{-1/5\,t}$$

$$B(t) = 100 - Ae^{-\frac{1}{5}t}$$

$$20 = 100 - Ae^{0}$$

$$A = 80$$

don't need ||, $B<100$

$$\boxed{B(t) = 100 - 80e^{-\frac{1}{5}t}}$$

Example: [1988 AP Calculus BC #43] Bacteria in a certain culture increase at rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

A) $\frac{3\ln 3}{\ln 2}$ B) $\frac{2\ln 3}{\ln 2}$ C) $\frac{\ln 3}{\ln 2}$ D) $\ln\left(\frac{27}{2}\right)$ E) $\ln\left(\frac{9}{2}\right)$

$\frac{dB}{dt} = KB$ $B(t) = B_0 e^{Kt}$

$2B_0 = B_0 e^{3K}$

$\frac{\ln(2)}{3} = K$

$3B_0 = B_0 e^{\frac{\ln(2)}{3}t}$

$\ln(3) = \frac{\ln(2)}{3}t$

Example: [AP Calculus 1993 AB #42] A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, then how much will the puppy weigh when it is 3 months old?

A) 4.2 pounds B) 4.6 pounds C) 4.8 pounds D) 5.6 pounds E) 6.5 pounds

$3.5 = 2e^{K(2)}$

$\frac{\ln\left(\frac{3.5}{2}\right)}{2} = K$

$B = 2 \cdot e^{\frac{\ln\left(\frac{3.5}{2}\right)}{2}(3)}$

Example: [1993 AP Calculus BC #38] During a certain epidemic, the number of people that are infected at any time increases at rate proportional to the number of people that are infected at that time. If 1,000 people are infected when the epidemic is first discovered, and 1,200 are infected 7 days later, how many people are infected 12 days after the epidemic is first discovered?

A) 343 B) 1,343 C) 1,367 D) 1,400 E) 2,057

$1200 = 1000e^{7K}$

$K = \frac{\ln(1.2)}{7}$

$P = 1000e^{\frac{\ln(1.2)}{7}(12)}$

Example: [1998 AP Calculus AB #84] Population y grows according to the equation $\frac{dy}{dt} = ky$, where k is a constant and t is measured in years. If the population doubles every 10 years, then the value of k is

A) 0.069 B) 0.200 C) 0.301 D) 3.322 E) 5.000

$2P_0 = P_0 e^{10K}$

$\frac{\ln(2)}{10} = K$

Notecards from Section 6.4: Derivation of an exponential function