Limit Comparison Test

A useful method for demonstrating the convergence or divergence of an improper integral is comparison to an improper integral with a simpler integrand. However, often a direct comparison to a simple function does not yield the inequality we need. For example, consider the following improper integral:

$$\int_1^\infty \frac{x}{x^2+\sqrt{x}+1}\,dx.$$

Estimating the degree, we see that $\frac{x}{x^2+\sqrt{x}+1} \approx \frac{1}{x}$ and we expect the improper integral to diverge. If we plot the functions, we find that

$$\frac{x}{x^2+\sqrt{x}+1} \leq \frac{1}{x} \qquad x \geq 1,$$

so that we cannot directly compare our integral to that of $1/x$ to show it diverges. One trick is to find some constant C so that

$$C\frac{1}{x} \leq \frac{x}{x^2+\sqrt{x}+1} \qquad x \geq 1.$$

The value of C, in practice, has no effect on our conclusion and takes work to find. The *limit comparison test* is a result which makes precise the notion of two functions growing at the same rate and reduces the process of finding some constant C to the computation of a single, often easy limit.

Theorem 1 (Limit Comparison Test)**.** *Suppose $f(x), g(x) > 0$ are positive, continuous functions defined on $[a, b)$ such that*

$$\lim_{x\to b} \frac{f(x)}{g(x)} = c \neq 0, \infty,$$

then $\int_a^b f(x)\,dx$ converges exactly when $\int_a^b g(x)\,dx$ converges.

When we use this test, we let b be either ∞ or an infinite discontinuity of both functions. The hypothesis that $\lim_{x\to b} \frac{f(x)}{g(x)}$ converges to a constant not equal to 0 or ∞ is saying that the functions $f(x)$ and $g(x)$ have the same growth rate at b Before proving this theorem, let's see how it applies to the above example.

Example 2. Let $f(x) = \frac{x}{x^2+\sqrt{x}+1}$ and consider again the improper integral $\int_1^\infty f(x)\,dx$. We estimated that $f(x)$ grows like $g(x) = 1/x$ as $x \to \infty$ previously, and this is how we choose our functions $f(x), g(x)$. Note that both functions are positive on $[1, \infty)$. To apply the limit comparison test, we must compute a limit.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x/(x^2+\sqrt{x}+1)}{1/x} = \lim_{x\to\infty} \frac{x^2}{x^2+\sqrt{x}+1} = 1,$$

since the numerator and denominator have the same degree the limit is simply the ratio of the x^2 coefficients. Because our limit converged to a finite, non-zero constant, we may conclude that since $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{X}\,dx$ diverges, so must $\int_1^\infty f(x)\,dx$.

We now give a proof of the limit comparison test and then conclude with a couple examples.

Proof. Since our functions are both positive, the limit c must also be positive. We may choose x_0 close to b so that for $x > x_0$ we have

$$-c/2 < \frac{f(x)}{g(x)} - c < c/2 \Longrightarrow 0 < c/2 < \frac{f(x)}{g(x)} < 3c/2. \tag{1}$$

For any y in the interval $[x_0, b)$ we now have

$$\int_{x_0}^{y} f(x)\,dx = \int_{x_0}^{y} \left(\frac{f(x)}{g(x)}\right) g(x)\,dx, \tag{2}$$

and then by (1) we deduce

$$0 < (c/2)\int_{x_0}^{y} g(x)\,dx < \int_{x_0}^{y} \left(\frac{f(x)}{g(x)}\right) g(x)\,dx < (3c/2)\int_{x_0}^{y} g(x)\,dx.$$

Taking limits as $y \to b$ and using (2) we have

$$0 < (c/2)\int_{x_0}^{b} g(x)\,dx < \int_{x_0}^{b} f(x)\,dx < (3c/2)\int_{x_0}^{b} g(x)\,dx.$$

Now we apply comparison: if $\int_{x_0}^{b} g(x)\,dx$ converges, then the above inequalities show that $\int_{x_0}^{b} f(x)\,dx$ does as well (here we are using $f(x), g(x) > 0$.) If $\int_{x_0}^{b} g(x)\,dx$ diverges, then the second inequality shows $\int_{x_0}^{b} f(x)\,dx$ diverges as well. We assumed both functions were continuous on $[a, b)$ so integrating from x_0 instead of a does not affect convergence. Hence, $\int_{a}^{b} f(x)\,dx$ and $\int_{a}^{b} g(x)\,dx$ either both converge or both diverge. □

Example 3. Let $f(x) = \frac{x^2+x+1}{(x^2-1)^{1/3}}$ and suppose we wish to determine the convergence of

$$\int_{1}^{2} \frac{x^2 + x + 1}{(x^2 - 1)^{1/3}}\,dx.$$

This is an improper integral because the denominator vanishes at $x = 1$. Notice that limit comparison test applies when our integral is improper at the the first limit of integration, since exchanging limits of integration only changes the value of the integral by a sign hence does not affect convergence. We should always first see if we can compute an anti-derivative by hand, although that appears difficult in this case.

To determine the correct function $g(x)$ to compare with $f(x)$ we must be careful: what's important is *to what degree the denominator vanishes at 1*. The numerator does not vanish at 1, and factoring the denominator as $(x - 1)^{1/3}(x + 1)^{1/3}$ we see that the denominator vanishes to the 1/3 power. So, we choose $g(x) = 1/(x - 1)^{1/3}$ for limit comparison. To apply the test, we check that $f(x), g(x) > 0$ on $(1, 2]$ and then compute the following limit:

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{(x^2 + x + 1)/(x^2 - 1)^{1/3}}{1/(x - 1)^{1/3}} = \lim_{x\to 1} \frac{x^2 + x + 1}{(x + 1)^{1/3}} = 3/2^{1/3}.$$

The last limit was computed by evaluating at $x = 1$ since after simplifying we see that the quotient $f(x)/g(x)$ is continuous at $x = 1$. Hence, the limit comparison test applies and we are now reduced to determining whether the convergence of

$$\int_1^2 \frac{1}{(x-1)^{1/3}}\,dx.$$

We do a u-substitution with $u = x - 1$ to change this to the familiar improper integral

$$\int_0^1 \frac{1}{u^{1/3}}\,du,$$

which we know converges because $p = 1/3 < 1$. Hence, we conclude that $\int_1^2 \frac{x^2+x+1}{(x^2-1)^{1/3}}\,dx$ also converges by the limit comparison test.

Example 4. The limit comparison test does not work for every problem. For example, consider $f(x) = \frac{5-2\sin(x)}{x^{3/2}}$ and suppose we wish to determine the convergence of $\int_1^\infty f(x)\,dx$. Since the sine function is bounded, we estimate that $f(x) \approx 1/x^{3/2}$ as $x \to \infty$. Both functions are positive on $[1, \infty)$ so we then try to compute the limit necessary to apply the limit comparison test:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{(5-2\sin(x))/(x^{3/2})}{1/x^{3/2}} = \lim_{x\to\infty} 5 - 2\sin(x).$$

The final limit does not converge because the sine function has no limit as $x \to \infty$. Thus, the test fails and we must resort to a direct comparison.

Example 5. We conclude with an example where finding $g(x)$ is trickier. Let $f(x) = \frac{\sin(x)}{x^{3/2}}$ and determine the convergence of $\int_0^{\pi/2} f(x)\,dx$. The denominator of $f(x)$ vanishes at $x = 0$, although so does the numerator—its not even clear whether the integral is improper! However, the limit comparison test does not assume the integrals are improper so we can proceed in ignorance. How do we determine the function $g(x)$? We need to estimate to what degree $f(x)$ vanishes at zero. When we discussed L'hopital's rule, one of the first limits we used it to compute was

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

We may interpret this limit as saying the $\sin(x)$ grows like x at 0. Then we approximate

$$\frac{\sin(x)}{x^{3/2}} \approx \frac{x}{x^{3/2}} = \frac{1}{x^{1/2}} \qquad \text{as } x \to 0,$$

which suggests that we use $g(x) = 1/x^{1/2}$ for limit comparison. Both functions $f(x), g(x)$ are positive on $(0, \pi/2]$ although we should assert this carefully since there is a sine function in $f(x)$. Now we compute

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\sin(x)/x^{3/2}}{1/x^{1/2}} = \lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Thus, the problem has been reduced to determining the convergence of $\int_0^{\pi/2} \frac{1}{x^{1/2}}\,dx$, which does converge because $p = 1/2 < 1$. Hence, by the limit comparison test, $\int_0^{\pi/2} f(x)\,dx$ converges.