Preliminary Questions

1. State whether the integral converges or diverges:

(a) $\int_1^\infty x^{-3}\,dx$ (b) $\int_0^1 x^{-3}\,dx$

(c) $\int_1^\infty x^{-2/3}\,dx$ (d) $\int_0^1 x^{-2/3}\,dx$

2. Is $\int_0^{\pi/2} \cot x\,dx$ an improper integral? Explain.

3. Find a value of $b > 0$ that makes $\int_0^b \frac{1}{x^2-4}\,dx$ an improper integral.

4. Which comparison would show that $\int_0^\infty \frac{dx}{x+e^x}$ converges?

5. Explain why it is not possible to draw any conclusions about the convergence of $\int_1^\infty \frac{e^{-x}}{x}\,dx$

by comparing with the integral $\int_1^\infty \frac{dx}{x}$.

1. Which of the following integrals is improper? Explain your answer, but do not evaluate the integral.

(a) $\int_0^2 \frac{dx}{x^{1/3}}$ (b) $\int_1^\infty \frac{dx}{x^{0.2}}$ (c) $\int_{-1}^\infty e^{-x}\,dx$

(d) $\int_0^1 e^{-x}\,dx$ (e) $\int_0^{\pi/2} \sec x\,dx$ (f) $\int_0^\infty \sin x\,dx$

(g) $\int_0^1 \sin x\,dx$ (h) $\int_0^1 \frac{dx}{\sqrt{3-x^2}}$ (i) $\int_1^\infty \ln x\,dx$

(j) $\int_0^3 \ln x\,dx$

2. Let $f(x) = x^{-4/3}$.

(a) Evaluate $\int_1^R f(x)\,dx$.

(b) Evaluate $\int_1^\infty f(x)\,dx$ by computing the limit

$$\lim_{R\to\infty} \int_1^R f(x)\,dx$$

3. Prove that $\int_1^\infty x^{-2/3}\,dx$ diverges by showing that

$$\lim_{R\to\infty} \int_1^R x^{-2/3}\,dx = \infty$$

4. Determine whether $\int_0^3 \frac{dx}{(3-x)^{3/2}}$ converges by computing

$$\lim_{R\to 3-} \int_0^R \frac{dx}{(3-x)^{3/2}}$$

In Exercises 5–40, determine whether the improper integral converges and, if so, evaluate it.

5. $\int_1^\infty \frac{dx}{x^{19/20}}$ **6.** $\int_1^\infty \frac{dx}{x^{20/19}}$ **7.** $\int_{-\infty}^4 e^{0.0001t}\,dt$ **8.** $\int_{20}^\infty \frac{dt}{t}$

9. $\int_0^5 \frac{dx}{x^{20/19}}$ **10.** $\int_0^5 \frac{dx}{x^{19/20}}$ **11.** $\int_0^4 \frac{dx}{\sqrt{4-x}}$ **12.** $\int_5^6 \frac{dx}{(x-5)^{3/2}}$

In Exercises 5–40, determine whether the improper integral converges and, if so, evaluate it.

13. $\int_2^\infty x^{-3}\,dx$ **14.** $\int_0^\infty \frac{dx}{(x+1)^3}$ **15.** $\int_{-3}^\infty \frac{dx}{(x+4)^{3/2}}$

16. $\int_2^\infty e^{-2x}\,dx$ **17.** $\int_0^1 \frac{dx}{x^{0.2}}$ **18.** $\int_2^\infty x^{-1/3}\,dx$ **19.** $\int_4^\infty e^{-3x}\,dx$

21. $\int_{-\infty}^0 e^{3x}\,dx$ **22.** $\int_1^2 \frac{dx}{(x-1)^2}$ **23.** $\int_1^3 \frac{dx}{\sqrt{3-x}}$ **24.** $\int_{-2}^4 \frac{dx}{(x+2)^{1/3}}$

25. $\int_0^\infty \frac{dx}{1+x}$ **26.** $\int_{-\infty}^0 xe^{-x^2}\,dx$ **27.** $\int_0^\infty \frac{x\,dx}{(1+x^2)^2}$

28. $\int_3^6 \frac{x\,dx}{\sqrt{x-3}}$ **29.** $\int_0^\infty e^{-x}\cos x\,dx$ **30.** $\int_1^\infty xe^{-2x}\,dx$

31. $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ **32.** $\int_0^1 \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$ **33.** $\int_1^\infty \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$

34. $\int_0^{\pi/2} \sec\theta\,d\theta$ **35.** $\int_0^\infty \sin x\,dx$ **36.** $\int_0^{\pi/2} \tan x\,dx$

37. $\int_0^1 \ln x\,dx$ **38.** $\int_1^2 \frac{dx}{x\ln x}$ **39.** $\int_0^1 \frac{\ln x}{x^2}\,dx$ **40.** $\int_1^\infty \frac{\ln x}{x^2}\,dx$

41. Let $I = \int_4^\infty \frac{dx}{(x-2)(x-3)}$.

(a) Show that for $R > 4$,

$$\int_4^R \frac{dx}{(x-2)(x-3)} = \ln\left|\frac{R-3}{R-2}\right| - \ln\frac{1}{2}$$

(b) Then show that $I = \ln 2$.

43. Evaluate $I = \int_0^1 \frac{dx}{x(2x+5)}$ or state that it diverges.

44. Evaluate $I = \int_2^\infty \frac{dx}{(x+3)(x+1)^2}$ or state that it diverges.

In Exercises 45–48, determine whether the doubly infinite improper integral converges and, if so, evaluate it.

45. $\int_{-\infty}^\infty \frac{x\,dx}{1+x^2}$ **46.** $\int_{-\infty}^\infty e^{-|x|}\,dx$ **47.** $\int_{-\infty}^\infty xe^{-x^2}\,dx$ **48.** $\int_{-\infty}^\infty \frac{dx}{(x^2+1)^{3/2}}$

49. Define $J = \int_{-1}^1 \frac{dx}{x^{1/3}}$ as the sum of the two improper integrals $\int_{-1}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}}$.

Show that J converges and that $J=0$

50. Determine whether $J = \int_{-1}^1 \frac{dx}{x^2}$ converges

51. For which values of a does $\int_0^\infty e^{ax}\,dx$ converge?

52. Show that $\int_0^1 \frac{dx}{x^p}$ converges if $p < 1$ and diverges if $p \geq 1$.

53. Sketch the region under the graph of $f(x) = \frac{1}{1+x^2}$ for $-\infty < x < \infty$, and show that its area is π.

54. Show that $\frac{1}{\sqrt{x^4+1}} \leq \frac{1}{x^2}$ for all x, and use this to prove that $\int_1^\infty \frac{dx}{\sqrt{x^4+1}}$ converges.

55. Show that $\int_1^\infty \frac{dx}{x^3+4}$ converges by comparing with $\int_1^\infty x^{-3}\,dx$.

56. Show that $\int_2^\infty \frac{dx}{x^3-4}$ converges by comparing with $\int_2^\infty 2x^{-3}\,dx$.

57. Show that $0 \leq e^{-x^2} \leq e^{-x}$ for $x \geq 1$ (Figure 1). Use the Comparison Test to show that $\int_0^\infty e^{-x^2}\,dx$ converges. *Hint:* It suffices (why?) to make the comparison for $x \geq 1$ because

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

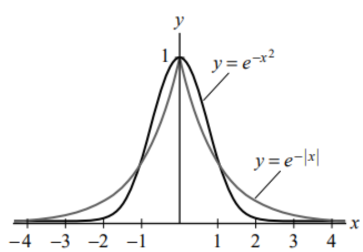

FIGURE 1 Comparison of $y = e^{-|x|}$ and $y = e^{-x^2}$.

58. Prove that $\int_{-\infty}^\infty e^{-x^2}\,dx$ converges by comparing with $\int_{-\infty}^\infty e^{-|x|}\,dx$ (Figure 1).

59. Show that $\int_1^\infty \frac{1-\sin x}{x^2}\,dx$ converges.

60. Let $a > 0$. Recall that $\lim\limits_{x\to\infty} \dfrac{x^a}{\ln x} = \infty$

(a) Show that $x^a > 2\ln x$ for all x sufficiently large.

(b) Show that $e^{-x^a} < x^{-2}$ for all x sufficiently large.

(c) Show that $\displaystyle\int_1^\infty e^{-x^a}\,dx$ converges.

In Exercises 61–74, use the Comparison Test to determine whether or not the integral converges.

61. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^5+2}}\,dx$ **62.** $\displaystyle\int_1^\infty \frac{dx}{(x^3+2x+4)^{1/2}}$

63. $\displaystyle\int_3^\infty \frac{dx}{\sqrt{x}-1}$ **64.** $\displaystyle\int_0^5 \frac{dx}{x^{1/3}+x^3}$ **65.** $\displaystyle\int_1^\infty e^{-(x+x^{-1})}\,dx$

66. $\displaystyle\int_0^1 \frac{|\sin x|}{\sqrt{x}}\,dx$ **67.** $\displaystyle\int_0^1 \frac{e^x}{x^2}\,dx$ **68.** $\displaystyle\int_1^\infty \frac{1}{x^4+e^x}\,dx$

69. $\displaystyle\int_0^1 \frac{1}{x^4+\sqrt{x}}\,dx$ **70.** $\displaystyle\int_1^\infty \frac{\ln x}{\sinh x}\,dx$ **71.** $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x^{1/3}+x^3}}$

72. $\displaystyle\int_0^\infty \frac{dx}{(8x^2+x^4)^{1/3}}$ **73.** $\displaystyle\int_0^\infty \frac{dx}{(x+x^2)^{1/3}}$ **74.** $\displaystyle\int_0^\infty \frac{dx}{xe^x+x^2}$

75. Define $J = \displaystyle\int_0^\infty \frac{dx}{x^{1/2}(x+1)}$ as the sum of the two improper integrals

$$\int_0^1 \frac{dx}{x^{1/2}(x+1)} + \int_1^\infty \frac{dx}{x^{1/2}(x+1)}$$

Use the Comparison Test to show that J converges.

76. Determine whether $J = \int_0^\infty \frac{dx}{x^{3/2}(x+1)}$ converges