Kuta Worksheet: Implicit Differentiation

① $2x^3 = 2y^2 + 5$

$\frac{d}{dx}(2x^3) = \frac{d}{dx}(2y^2+5)$

$6x^2 = 4y\frac{dy}{dx} + 0$

$6x^2 = 4y\frac{dy}{dx}$

$\frac{6x^2}{4y} = \frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = \frac{3x^2}{2y}}$

③ $5y^2 = 2x^3 - 5y$

$\frac{d}{dx}[5y^2] = \frac{d}{dx}[2x^3 - 5y]$

$10y\frac{dy}{dx} = 6x^2 - 5\frac{dy}{dx}$

$10y\frac{dy}{dx} + 5\frac{dy}{dx} = 6x^2$

$\frac{dy}{dx}(10y+5) = 6x^2$

$\boxed{\frac{dy}{dx} = \frac{6x^2}{10y+5}}$

⑤ $5x^3 = -3xy + 2$ — Need product rule

$\frac{d}{dx}[5x^3] = \frac{d}{dx}[-3xy+2]$

$15x^2 = \frac{d}{dx}[-3x]\cdot[y] + \frac{d}{dx}[y]\cdot[-3x] + 0$

$15x^2 = -3y + 1\frac{dy}{dx}\cdot[-3x]$

$15x^2 = -3y - 3x\frac{dy}{dx}$

$15x^2 + 3y = -3x\frac{dy}{dx}$

$\frac{15x^2+3y}{-3x} = \frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = \frac{-5x^2-y}{x}}$

⑦ $3x^2y^2 = 4x^2 - 4xy$

$\frac{d}{dx}[3x^2y^2] = \frac{d}{dx}[4x^2 - 4xy]$

need product rule

Difference & Product Rule

$\frac{d}{dx}[3x^2]\cdot y^2 + \frac{d}{dx}[y^2]\cdot 3x^2 = \frac{d}{dx}[4x^2] + \frac{d}{dx}[-4xy]$

$6xy^2 + 2y\frac{dy}{dx}\cdot 3x^2 = 8x + \frac{d}{dx}[-4x]\cdot y + \frac{d}{dx}[y]\cdot(-4x)$

$6xy^2 + 6x^2y\frac{dy}{dx} = 8x - 4y + 1\frac{dy}{dx}\cdot(-4x)$

$6xy^2 + 6x^2y\frac{dy}{dx} = 8x - 4y - 4x\frac{dy}{dx}$

$6xy^2 + 4y - 8x = -4x\frac{dy}{dx} - 6x^2y\frac{dy}{dx}$

$6xy^2 + 4y - 8x = \frac{dy}{dx}(-4x - 6x^2y)$

$$\frac{6xy^2 + 4y - 8x}{-4x - 6x^2y} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3xy^2 + 2y - 4x}{-2x - 3x^2y}$$

(9) $2x^3 = (3xy+1)^2$ Method ① Do NOT expand Power

$$\frac{d}{dx}(2x^3) = \frac{d}{dx}(3xy+1)^2$$

$$6x^2 = \underbrace{2[3xy+1]\cdot[3xy+1]\frac{d}{dx}}_{\text{chain Rule}}$$

$$6x^2 = [6xy+2]\cdot\left[\frac{d}{dx}(3xy)+0\right]$$

$$6x^2 = [6xy+2]\left[\frac{d}{dx}(3x)\cdot y + \frac{d}{dx}(y)\cdot 3x\right]$$

$$6x^2 = [6xy+2]\left[3y + 1\frac{dy}{dx}\cdot 3x\right]$$

$$6x^2 = (6xy+2)\left[3y + 3x\frac{dy}{dx}\right]$$ * ↑ without "FOIL"

$$\frac{6x^2}{6xy+2} = 3y + 3x\frac{dy}{dx}$$

$$\frac{6x^2}{6xy+2} - 3y = 3x\frac{dy}{dx}$$

$$\frac{1}{3x}\left[\frac{6x^2}{6xy+2}\right] - \frac{1}{3x}[3y] = \frac{dy}{dx}$$

$$\boxed{\frac{2x}{6xy+2} - \frac{y}{x} = \frac{dy}{dx}}$$

9 Method (1) Cont

From * $6x^2 = (6xy+2)\left[3y + 3x\frac{dy}{dx}\right]$

Expand

$$6x^2 = 18xy^2 + 6y + 18x^2y\frac{dy}{dx} + 6x\frac{dy}{dx}$$

$$6x^2 - 18xy^2 - 6y = 18x^2y\frac{dy}{dx} + 6x\frac{dy}{dx}$$

$$6x^2 - 18xy^2 - 6y = \frac{dy}{dx}(18x^2y + 6x)$$

$$\boxed{\frac{6x^2 - 18xy^2 - 6y}{18x^2y + 6x} = \frac{dy}{dx}}$$

$$\frac{6}{d} \cdot \frac{x^2 - 3xy^2 - y}{3x^2y + x} = \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2 - 3xy^2 - y}{3x^2y + x}}$$

Problem (9) Method (1)

Note $\frac{dy}{dx} = \frac{2x}{6xy+2} + -\frac{y}{x} = \frac{2x^2}{6x^2y+2x} + \frac{-y(6xy+2)}{6x^2y+2x}$

$$= \frac{2x^2 - 6xy^2 - 2y}{6x^2y + 2x}$$

$$= \frac{2}{2} \cdot \frac{x^2 - 3xy^2 - y}{3x^2y + x}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2 - 3xy^2 - y}{3x^2y + x}}$$

⑨ Method ② Expand Power 15/

$$2x^3 = (3xy+1)^2$$

$$2x^3 = (3xy+1)(3xy+1)$$

$$2x^3 = 9x^2y^2 + 6xy + 1$$

$$\frac{d}{dx}[2x^3] = \frac{d}{dx}[9x^2y^2 + 6xy + 1]$$

$$6x^2 = \frac{d}{dx}[9x^2]\cdot y^2 + \frac{d}{dx}[y^2]\cdot 9x^2 + \frac{d}{dx}[6x]\cdot y + \frac{d}{dx}[y]\cdot 6x$$

$$6x^2 = 18x y^2 + 2y\frac{dy}{dx}\cdot 9x^2 + 6y + 1\frac{dy}{dx}\cdot 6x$$

$$6x^2 = 18xy^2 + 6y + 18x^2y\frac{dy}{dx} + 6x\frac{dy}{dx}$$

$$6x^2 - 18xy^2 - 6y = 18x^2y\frac{dy}{dx} + 6x\frac{dy}{dx}$$

$$6x^2 - 18xy^2 - 6y = (18x^2y + 6x)\frac{dy}{dx}$$

$$\boxed{\frac{6x^2 - 18xy^2 - 6y}{18x^2y + 6x} = \frac{dy}{dx}}$$

$$\frac{6}{6}\cdot\frac{x^2 - 3xy^2 - y}{3x^2y + x} = \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2 - 3y^2 - y}{3x^2y + x}}$$

(11) $\sin 2x^2y^3 = 3x^3 + 1$

$$\frac{d}{dx}\left(\sin(2x^2y^3)\right) = \frac{d}{dx}(3x^3+1)$$

Chain Rule & Implicit Differentiation

$$\cos(2x^2y^3)\cdot\frac{d}{dx}(2x^2y^3) = 9x^2 + 0$$

Product Rule

$$\cos(2x^2y^3)\cdot\left[\frac{d}{dx}(2x^2)\cdot y^3 + \frac{d}{dx}[y^3]\cdot 2x^2\right] = 9x^2$$

$$\cos(2x^2y^3)\cdot\left[4xy^3 + 3y^2\cdot\frac{dy}{dx}\cdot 2x^2\right] = 9x^2$$

$$\cos(2x^2y^3)\cdot\left[4xy^3 + 6x^2y^2\frac{dy}{dx}\right] = 9x^2$$

Method (1) Do not expand

$$\left[4xy^3 + 6x^2y^2\frac{dy}{dx}\right] = \frac{9x^2}{\cos(2x^2y^3)}$$

$$6x^2y^2\frac{dy}{dx} = \frac{9x^2}{\cos(2x^2y^3)} - 4xy^3$$

$$\frac{dy}{dx} = \frac{9x^2}{\cos(2x^2y^3)}\cdot\frac{1}{6x^2y^2} - \frac{4xy^3}{1}\cdot\frac{1}{6x^2y^2}$$

(11) Method (1) cont

$$\frac{dy}{dx} = \frac{9x^2}{6x^2y^2} \cdot \frac{1}{\cos(2x^2y^3)} - \frac{4xy^3}{6x^2y^2}$$

$$= \frac{3}{y^2} \cdot \frac{1}{\cos(2x^2y^3)} - \frac{2y}{3x}$$

$$\boxed{\frac{dy}{dx} = \frac{3}{y^2\cos(2x^2y^3)} - \frac{2y}{3x}}$$

Note $\boxed{\frac{dy}{dx} = \frac{9x^2}{6x^2y^2\cos(2x^2y^3)} - \frac{4xy^3}{6x^2y^2}}$

would be acceptable to AP!!

Method (2) from * (Expand)

$$\cos(2x^2y^3)\cdot\left[4xy^3 + 6x^2y^2\frac{dy}{dx}\right] = 9x^2$$

$$4xy^3\cos(2x^2y^3) + 6x^2y^2\cos(2x^2y^3)\frac{dy}{dx} = 9x^2$$

$$6x^2y^2\cos(2x^2y^3)\frac{dy}{dx} = 9x^2 - 4xy^3\cos(2x^2y^3)$$

$$\boxed{\frac{dy}{dx} = \frac{9x^2 - 4xy^3\cos(2x^2y^3)}{6x^2y^2\cos(2x^2y^3)}}$$

$$\boxed{\frac{dy}{dx} = \frac{9x - 4y^3\cos(2x^2y^3}{\ldots}}$$

(13) $4y^2 + 2 = 3x^2$

$\frac{d}{dx}[4y^2 + 2] = \frac{d}{dx}[3x^2]$

$8y\frac{dy}{dx} + 0 = 6x$

$8y\frac{dy}{dx} = 6x$

$\frac{dy}{dx} = \frac{6x}{8y}$

$$\boxed{\frac{dy}{dx} = \frac{3x}{4y}}$$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{3x}{4y}\right]$ ← Quotient Rule

$= \frac{\frac{d}{dx}[3x] \cdot 4y - \frac{d}{dx}[4y] \cdot 3x}{(4y)^2}$

$= \frac{3 \cdot 4y - 4\frac{dy}{dx} \cdot 3x}{16y^2}$

$\frac{d^2y}{dx^2} = \frac{12y - 12x\frac{dy}{dx}}{16y^2}$

13 cont) Recall $\frac{dy}{dx} = \frac{3x}{4y}$

$$\frac{d^2y}{dx^2} = \frac{12y - 12x\frac{dy}{dx}}{16y^2}$$

$$= \frac{12y - 12x\left(\frac{3x}{4y}\right)}{16y^2}$$

$$= \frac{12y - \frac{36x^2}{4y}}{16y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12y - \frac{9x^2}{y}}{16y^2} \longrightarrow \frac{12y - \frac{9x^2}{y}}{16y^2} \cdot \frac{y}{y}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{12y^2 - 9x^2}{16y^3}}$$

15 Method ③ Solve for y 1st

$$\frac{3x^2}{4y} = \frac{x}{1}$$

$$3x^2 = 4xy$$

$$\frac{3x^2}{4x} = y$$

$$\frac{3x}{4} = y$$

$$\frac{d}{dx}\left[\frac{3x}{4}\right] = \frac{d}{dx}[y]$$

$$\boxed{\frac{3}{4} = \frac{dy}{dx}}$$

Method ① $\frac{dy}{dx} = \frac{6xy - 4y^2}{3x^2}$ we know $y = \frac{3}{4}x$

$$= \frac{6x\left(\frac{3}{4}x\right) - 4\left(\frac{3}{4}x\right)^2}{3x^2} = \frac{\frac{18}{4}x^2 - 4\left(\frac{9x^2}{16}\right)}{3x^2}$$

$$= \frac{\left(\frac{18}{4} - \frac{36}{16}\right)x^2}{(3)\,x^2} = \frac{18}{12} - \frac{36}{48} = \frac{3}{2} - \frac{3}{4}$$

$$= \frac{3}{4}$$

1S method 2 Eliminate "fraction" 1st

$$4y\left[\frac{3x^2}{4y}\right] = \left[x\right]4y$$

$$3x^2 = 4xy$$

Product Rule

$$\frac{d}{dx}\left[3x^2\right] = \frac{d}{dx}\left[4xy\right]$$

$$6x = \frac{d}{dx}[4x]\cdot y + \frac{d}{dx}[y]\cdot 4x$$

$$6x = 4y + 1\frac{dy}{dx}\cdot 4x$$

$$6x = 4y + 4x\frac{dy}{dx}$$

$$6x - 4y = 4x\frac{dy}{dx}$$

$$\frac{6x-4y}{4x} = \frac{dy}{dx}$$

$$\boxed{\frac{3x-2y}{2x} = \frac{dy}{dx}}$$

(15) $\frac{3x^2}{4y} = x$

Strategy ① As is Implicit differentiation.

$$\frac{d}{dx}\left[\frac{3x^2}{4y}\right] = \frac{d}{dx}[x]$$

Quotient Rule

$$\frac{\frac{d}{dx}[3x^2)\cdot 4y - \frac{d}{dx}[4y]\cdot 3x^2}{(4y)^2} = 1$$

$$\frac{6x \cdot 4y - 4\cdot\frac{dy}{dx}\cdot 3x^2}{16y^2} = 1$$

$$\frac{24xy - 12x^2\frac{dy}{dx}}{16y^2} = 1$$

$$24xy - 12x^2\frac{dy}{dx} = 16y^2$$

$$24xy - 16y^2 = 12x^2\frac{dy}{dx}$$

$$\boxed{\frac{24xy - 16y^2}{12x^2} = \frac{dy}{dx} = \frac{6xy - 4y^2}{3x^2}}$$

15 cont

Method 2 $\frac{dy}{dx} = \frac{3x-2y}{2x}$ we know $y = \frac{3}{4}x$

$$= \frac{3x-2\left(\frac{3}{4}x\right)}{2x}$$

$$= \frac{3x-\frac{6}{4}x}{2x}$$

$$= \left(\frac{3-\frac{3}{2}}{2}\right)\frac{x}{x}$$

$$= \frac{3}{2}-\frac{3}{4}$$

$$= \frac{3}{4}\checkmark$$

All methods yield "same" answer