1) $\int_1^{\infty} \frac{1}{\sqrt[3]{x}}\,dx = \int_1^{\infty} \frac{1}{x^{1/3}}\,dx$

this is improper because ∞ as a boundary

so $\int_1^{\infty} \frac{1}{x^{1/3}}\,dx = \lim_{A \to \infty^-} \int_1^{A} \frac{1}{x^{1/3}}\,dx$

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first find $\int \frac{1}{x^{1/3}}\,dx = \int x^{-1/3}\,dx$

$= \frac{3}{2} x^{2/3} + C$

$\int_1^{\infty} f(x)\,dx = \lim_{A \to \infty^-} \left[\frac{3}{2} A^{2/3} - \frac{3}{2}(1)^{2/3} \right]$

since $\frac{3}{2} A^{2/3} \to \infty$ as $A \to \infty$

$\frac{3}{2} A^{2/3} - \frac{3}{2}(1)^{2/3} \to \infty$ &

$\lim_{A \to \infty^-} \left[\frac{3}{2} A^{2/3} - \frac{3}{2}(1)^{2/3} \right] = \infty$

$\therefore \int_1^{\infty} f(x)\,dx$ diverges

(2) $\int_1^{\infty} \frac{1}{x}\,dx$ Since ∞ is a boundary this is improper

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{A \to \infty^-} \int_1^A f(x)\,dx$$

Note $\int \frac{1}{x}\,dx = \ln|x| + C$

$$= \lim_{A \to \infty^-} [\ln A - \ln 1]$$

$$= \lim_{A \to \infty^-} [\ln A - 0]$$

$$= \lim_{A \to \infty^-} [\ln A]$$

Since $\ln A \to \infty$ as $A \to \infty$

$$= \lim_{A \to \infty^-} [\ln A] = \infty$$

$\therefore \int_1^{\infty} \frac{1}{x}\,dx$ diverges

(3) $\int_1^{\infty} \frac{1}{x^4}\,dx = \int_1^{\infty} x^{-4}\,dx$

since ∞ is a bounday this is improper

So $\int_1^{\infty} f(x)\,dx = \lim_{A\to\infty^-} \int_1^{A} f(x)$

we need $\int x^{-4}\,dx = -\frac{1}{3}x^{-3} + C$

$$= \lim_{A\to\infty^-}\left[-\frac{1}{3}A^{-3} + \frac{1}{3}(1)^{-3}\right]$$

$$= \lim_{A\to\infty^-}\left[-\frac{1}{3}A^{-3} + \frac{1}{3}\right]$$

$$= \lim_{A\to\infty^-}\left[-\frac{1}{3A^3} + \frac{1}{3}\right]$$

Since $A^3 \to \infty$ as $A \to \infty^-$

$\frac{-1}{3A^3} \to 0^-$ as $A \to -\infty$

$$= \lim_{A\to\infty^-}\left[-\frac{1}{3A^3} + \frac{1}{3}\right] = 0 + \frac{1}{3} = \frac{1}{3}$$

$\boxed{\text{So } \int_1^{\infty} f(x)\,dx \text{ converges to } \frac{1}{3}}$

Solutions

(4) $\int_2^6 \frac{1}{x^2-2x-15}\,dx$ this is improper because $x \in [2,6]$ has $x=5$ within it

if $f(x) = \frac{1}{x^2-2x-15} = \frac{1}{(x-5)(x+3)}$

then $f(x)$ is undefined at $x=5$

$$\int_2^6 f(x)\,dx = \int_2^6 \frac{1}{(x-5)(x+3)}\,dx = \int_2^5 f(x)\,dx + \int_5^6 f(x)\,dx$$

we need to use partial fractions here

$$\frac{1}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3} \rightarrow 1 = A(x+3) + B(x-5)$$

$\rightarrow$ Let $x=5$

$1 = A(8) + B(0)$

$1 = 8A$

$\boxed{A = \frac{1}{8}}$

$\rightarrow$ Let $x=-3$

$1 = A(0) + B(-8)$

$1 = -8B$

$\boxed{B = \frac{-1}{8}}$

so $\boxed{f(x) = \frac{1}{x^2-2x-15} = \frac{\frac{-1}{8}}{x+3} + \frac{\frac{1}{8}}{x-5}}$

④ cont

$$\int_2^5 f(x)\,dx = \lim_{A\to 5^-} \int_2^A f(x)\,dx$$

First find Indefinite Integral for $\int f(x)\,dx$

$$\int \frac{\frac{1}{8}}{x+3} + \frac{-\frac{1}{8}}{x-5}\,dx = \frac{1}{8}\int \frac{1}{x+3}\,dx - \frac{1}{8}\int \frac{1}{x-5}\,dx$$

$$= \frac{1}{8}\ln|x+3| - \frac{1}{8}\ln|x-5| + C$$

So $$\int_2^5 f(x)\,dx = \lim_{A\to 5^-}\left(\frac{1}{8}\ln|A+3| - \frac{1}{8}\ln|A-5| - \left[\frac{1}{8}\ln|5+3| - \frac{1}{8}\ln|5-5|\right]\right.$$

$$= \lim_{A\to 5^-}\left[\frac{1}{8}\ln|A+3| - \frac{1}{8}\ln|A-5| - \frac{1}{8}\ln|8| + \frac{1}{8}\ln|0|\right]$$

Since $\frac{1}{8}\ln|0|$ DNE we can say

$\int_2^5 f(x)\,dx = \text{DNE}$ $\therefore$ $\int_2^6 f(x)\,dx$ diverges

5 cont

$$\int_0^4 f(x)\,dx = \lim_{A\to 4^-}\left[-2(4-A)^{1/2}+4\right]$$

Since $-2(4-A)^{1/2} \to 0$ as $A \to 4^-$

$$= \lim_{A\to 4^-}\left[-2(4-A)^{1/2}+4\right] = 0+4$$

$$= 4$$

$\boxed{\therefore \int_0^4 f(x)\,dx \text{ converges to } 4}$

(5) $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \int_0^4 \frac{1}{(4-x)^{1/2}}\,dx$

$f(x) = \frac{1}{(4-x)^{1/2}}$ $x \in [0,4]$ but $x \neq 4$

So this is an improper integral

$\int_0^4 f(x)\,dx = \lim_{A \to 4^-} \int_0^A f(x)\,dx$

Find Indefinite Integral $\int \frac{1}{(4-x)^{1/2}}\,dx$ 1st

$\int (4-x)^{-1/2}\,dx$

Let $w = (4-x)$

$dw = -1\,dx$

$\to -\int (4-x)^{-1/2} -1\,dx$

$\to -1 \int w^{-1/2}\,dw$

$-1(2)\,w^{1/2} + C$

$\boxed{-2(4-x)^{1/2} + C}$

So $\int_0^4 f(x)\,dx = \lim_{A \to 4^-} \int_0^A f(x)\,dx$

$= \lim_{A \to 4^-} \left[-2(4-A)^{1/2} + 2(4-0)^{1/2}\right]$

$= \lim_{A \to 4^-} \left[-2(4-A)^{1/2} + 2(2)\right]$

$= \lim_{A \to 4^-} \left[-2(4-A)^{1/2} + 4\right]$

(6) $\int_{-\infty}^{0} xe^x\,dx$ this is improper because $-\infty$ is a boundary

so $\int_{-\infty}^{0} xe^x\,dx = \lim\limits_{A \to -\infty^+} \int_A^0 f(x)\,dx$

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we need Indefinite Integral 1st

$\int f(x)dx$ requires integration by parts

	D	I
+	x	e^x
−	1	e^x
+	0	e^x
−		

$\boxed{\int xe^x dx = xe^x - 1e^x + C}$

table or DI method

OR $w = x$ $\quad dv = e^x\,dx$

$dw = 1\,dx \quad v = e^x$

$\int f(x)dx = wv - \int v\,dw = xe^x - \int e^x\,1dx$

$= xe^x - e^x + C$

---

so $\int_{-\infty}^{0} f(x)\,dx = \lim\limits_{A \to -\infty^+} \int_A^0 f(x)$

$= \lim\limits_{A \to -\infty^+} \left[0e^0 - e^0 - \left[Ae^A - e^A\right]\right]$

(6) cont $\int_{-\infty}^{0} f(x)dx = \lim_{A\to-\infty} \int_{A}^{0} f(x)$

$$= \lim_{A\to-\infty^+} \left[0e^0 - e^0 - \left[Ae^A - e^A\right]\right]$$

$$= \lim_{A\to-\infty^+} \left[0 - 1 - \left[e^A(A-1)\right]\right]$$

$$= \lim_{A\to-\infty^+} \left[-1 - e^A(A-1)\right]$$

Since $e^A \to \frac{1}{e^{|A|}}$ as $A \to -\infty$

& $e^{|A|} \to \infty$ as $A \to -\infty$

& $\frac{1}{e^{|A|}} \to \frac{1}{\infty}$ as $A \to -\infty$

& $\frac{1}{\infty} \to 0$

$$\lim_{A\to-\infty^+} \left[-1 - e^A(A-1)\right] = -1 - 0$$
$$= -1$$

$\therefore \int_{-\infty}^{0} f(x)dx$ converges to -1

(7) $\int_{-\infty}^{\infty} \frac{x}{4x^2+8}\,dx$ this is improper because $-\infty$ & ∞ are boundary values

So $\int_{-\infty}^{\infty} \frac{x}{4x^2+8}\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$

First we need $\int \frac{x}{4x^2+8}\,dx$

Let $w = 4x^2+8$
$dw = 8x\,dx$ $= \frac{1}{8}\int \frac{8x}{4x^2+8}\,dx$

$= \frac{1}{8}\int \frac{1}{w}\,dw$

$= \frac{1}{8}\ln|w| + C = \frac{1}{8}\ln|4x^2+8| + C$

$\int_{-\infty}^{0} f(x) = \lim_{A\to-\infty^+} \int_{A}^{0} f(x)\,dx$

$= \lim_{A\to-\infty^+} \left[\frac{1}{8}\ln|4(0)^2+8| - \frac{1}{8}\ln|4A^2+8|\right]$

$= \lim_{A\to-\infty^+} \left[\frac{1}{8}\ln|8| - \frac{1}{8}\ln|4A^2+8|\right]$

Since $4A^2+8 \to \infty$ as $A \to -\infty$

$-\frac{1}{8}\ln|4A^2+8| \to -\infty$

$\therefore \int_{-\infty}^{0} f(x)\,dx$ diverges & $\int_{-\infty}^{\infty} f(x)\,dx$ also diverges